Uniqueness of Stationary Equilibrium Payoffs in Coalitional Bargaining

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Abstract

We study a model of sequential bargaining in which, in each period before an agreement is reached, a proposer is randomly selected, the proposer suggests a division of a pie of size one, each other agent either approves or rejects the proposal, and the proposal is implemented if the set of approving agents is a winning coalition for the proposer. We show that stationary equilibrium outcomes of a coalitional bargaining game are unique. This generalizes Eraslan (2002) insofar as: (a) there are no restrictions on the structure of sets of winning coalitions; (b) different proposers may have different sets of winning coalitions; (c) there may be a positive probability that no proposer is selected.

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1 Introduction

Baron and Ferejohn (1989) study a model in which a group of n risk neutral agents divide a fixed pie. In each period a "proposer" is selected randomly, the proposer suggests a division of the pie, and this division is implemented if it is approved by an effective majority of the agents. Otherwise the process is repeated in the next period. If agreement is reach in period t, each agent's payoff is the fraction of the pie she receives, multiplied by δ^t, where $0 < \delta < 1$ is a common discount factor. They show that when an effective majority is any set of k agents, where $1 \leq k \leq n$, stationary equilibrium expected utilities are unique when the model is symmetric in the sense that all agents have the same "recognition probability" (probability of being selected as the proposer). In these circumstances agreement is reached with probability one in the first period.

For the application motivating Baron and Ferejohn (bargaining among parties in a legislature or parliament) it is natural to suppose that recognition probabilities differ across agents, with larger parties typically having higher recognition probabilities. In a committee it is natural to suppose that the chair's recognition probability is higher than the recognition probabilities of other members. For several years it was unknown whether there could be multiple stationary equilibria yielding different expected utilities when recognition probabilites or discount factors differ across agents. Eraslan (2002) resolved this problem, showing that, even with unequal recognition probabilities and discount factors, there is a single vector of expected utilities common to all stationary equilibria.

Her analysis is restricted to k-majority rule for $1 \leq k \leq n$, but in legislative settings it is also natural to allow different agents to have different weights in the voting over approval of a proposal. This direction of generalization is also of interest from the point of view of other applications. In corporate bankruptcies governed by Chapter 11, the voting over approval of a proposed reorganization is asymmetric with respect to different seniority classes of debt, and creditors who are owed more money have greater power. Other examples are described in the next section.

Here we show that, under more general conditions than those considered by Eraslan (2002), there is a unique vector of expected payoffs that is generated by all of the game's stationary equilibria. Specifically, in addition to allowing different agents to have different recognition probabilities and discount factors, we allow the set of winning coalitions to be arbitrary, and to depend on the proposer, and we allow the sum of the recognition probabilities to be less than one.

Our argument has an interesting mathematical structure. Roughly, the fixed point index assigns an integer to each compact set of fixed points that has a neighborhood containing no other fixed points. (These are the unions of connected components when there are finitely many components.) When the domain is convex, for any partition of the set of fixed points into such sets, the sum of the indices of the sets must be one. We demonstrate uniqueness by establishing that each component of the set of fixed points of the relevant correspondence has a neighborhood that has no other fixed points, and that its index is one. Consequently the set of fixed points must consist of a single component. As has been noted (cf. p. 615 of Mas-Colell et al. (1995)) this method of proving uniqueness is widely applicable, but in all earlier cases elementary methods were also available.

The organization of the remainder is as follows. Section 2 describes some of the extensive literature descended from Baron and Ferejohn (1989) as it relates to our work. Section 3 presents the axioms that characterize the fixed point index, along with their relevant consequences. The model, and the stationary equilibrium concept, are explained formally in Section 4. Section 6 passes from the definition of stationary equilibrium to a fixed point characterization of the contuation values, and Section 7 proves the main result. At the heart of the proof is a technical result asserting that a certain matrix is positive definite; this is proved in Section 8. Some possible topics for further research on this topic are sketched in Section 10.

2 Related Literature

Since Baron and Ferejohn's paper, variations of their model have been applied to a large number of issues in political science. Baron (1989) considers a model in which some coalition controls the chair, so that only members of the coalition are allowed to propose. Responding to Austen-Smith and Banks (1988), in Baron (1991) the actors have preferences over a space of policies, with each agent's utility function depending on the distance from her blisspoint. It is shown that agreement is always reached in the first period, and that there is a tendency in the direction of centrist policies. In particular, the median voting rule is a limiting special case. In Baron (1996) these results are extended to a hybrid model in which there is both a policy choice and a division of benefits. Persson (1998) applies Baron-Ferejohn style bargaining in a setting where the legislature must choose an overall level of taxation and an allocation of the revenues to the various districts.

There are several papers that study models with multiple stages in which the Baron-Ferejohn setup is applied in at least one phase. In Baron (1998) and Diermeier and Feddersen (1998) there are two phases, with formation of a government in the first phase and voting on a proposal in the second phase. Both phases are governed by a bargaining model similar to the one described above. Bennedsen and Feldmann (2002) develop a multistage model in which Baron-Ferejohn bargaining is preceeded by lobbying activity. The model in McKelvey and Riezman (1992) alternates between election periods in which legislators are chosen and legislative periods in which a Baron-Ferejohn style procedure is used to select a policy.

The applications mentioned above already take one far beyond the case in which passage of a proposal requires unanimous assent. Other authors have considered applications that go beyond the legislative or parliamentary setting that was the original motivation of Baron and Ferejohn (1989). Chari et al. (1997), McCarty (2000b), and McCarty (2000a) study models in which one of the agents has powers modelled on the US Presidency. Ansolabehere et al. (2003b) considers a bicameral legislature. Winter (1996) studies a model based on the UN Security Council in which some actors have veto rights, so that in order to pass a proposal must have the support of all veto agents as well as an absolute majority of all agents.

Several papers consider variations on the Baron and Ferejohn model that aim at providing theories of which coalitions will form in equilibrium. Calvert and Dietz (1996) allow agents' preferences to depend on the shares received by others, so that coalitions between agents who are mutually sympathetic become natural. Banks and Duggan (2000) and Banks and Duggan (2003) consider a very general model in which the space of outcomes can be any convex compact set and the utility functions are concave but otherwise unrestricted. Cardona and Ponsati (2005) analyze a model in which the agents bargain over a one dimensional poicy variable. Jackson and Moselle (2002) study a model that includes both a choice of a one dimensional policy variable and a division of a pie, investigating the intuition that agents with similar idealogical views are natural coalition partners, while Battaglini and Coate (2005) use such a model to investigate sources of inefficiency in legislative policy making. The models in Kalandrakis (2004) and Battaglini and Coate (2005) are dynamic, with the outcome in one period determining a status quo for the next period that will continue to be implemented until a new agreement replaces it.

The Baron-Ferejohn has been the subject of a number of empirical and experimental studies. Ansolabehere et al. (2003a) study the allocation of ministries to coalition partners in postwar European parliamentary governments. Adachi and Watanabe (2004) study the allocation to ministries to factions of Japan's Liberal Democratic Party. Diermeier and Merlo (2004) use European data to investigate whether formateurs are selected randomly, with probabilities proportional to seat share, as in Baron (1991), or deterministically in order of seat share, as in Austen-Smith and Banks (1988).

As these examples suggest, the methodology pioneered by Baron and Ferejohn (1989) is emerging as an important tool for addressing a central issue of political science: the relationship between the rules governing political institutions and the outcomes they produce. Perhaps the main alternatives would be concepts from cooperative game theory such as the Shapley value (applied to simple games[1]) or the power indices due to Banzhaf (Banzhaf (1965, 1968)) Deegan and Packel (1972) and Johnston (1978). In contrast with those concepts, the Baron-Ferejohn model has explicit noncooperative foundations. Certainly those foundations are open to question in some applications, but the framework invites additional work in the form of alternative models, whereas cooperative concepts seem to be less susceptible to variations that respond to such critiques.

From the point of view of applications mentioned above, the value of our result is, perhaps, obvious. In particular, as a general rule, empirical research depends on (or is at least greatly eased by) the model producing a unique outcome for any vector of parameters. In relation to some of the papers discussed above (McCarty (2000b), Snyder et al. (2001), Ansolabehere et al. (2003b)) it strengthens the work by providing uniqueness results that were not available to the authors, or allowing uniqueness to be proved under weaker hypotheses. For instance, Snyder et al. (2001) (pp. 14–15) give an illustrative proof of uniqueness for a particular example, and omit (to save space) proofs of uniqueness for similar examples appearing in that paper. In addition, they consider only equilibria that are symmetric, in that identical agents have the same continuation values, but our result implies that the unique equilibrium must be symmetric. In another case (Winter (1996)) it allows immediate generalization of the model and/or the result. Finally, the literature contains models (e.g., McCarty (2000a)) that would become instances of our framework after small modifications.

We now describe the history of uniqueness results for the Baron-Ferejohn model, explaining how our model goes beyond earlier results. As Baron and Ferejohn point out, for unanimity rule there is a unique stationary equilibrium. They also point out that under k-majority rule there are a

[1] A *cooperative game with transferable utility* consists of a set of agents $I = \{1, \dots, n\}$ together with a specification of a payoff $v(S)$ for each coalition. The game is *simple* if $v(S)$ is always either 0 or 1.

continuum of stationary equilibria, but that they all have the same vector of continuation utilities for all agents. Specifically, each agent has the same total probability of being included in another proposer's coalition, but there is considerable flexibility as to which proposers include which other agents.

Subsequent papers concerned with uniqueness of equilibrium expected payoffs include Eraslan (2002), Norman (2002), Yan (2002), Cho and Duggan (2003), and Yıldırım (2004). Norman (2002) shows that equilibrium payoffs may fail to be unique when there are finitely many bargaining periods. Generalizing Okada (1993), Yan (2002) studies the bargaining protocol analyzed here, with general recognition probabilities, applied to a TU game with a nonempty core, showing that an allocation in the core is realized as the vector of continuation values if and only if it coincides with the vector of recognition probabilities, in which case there are no other stationary equilibrium payoffs. Since we study simple games, which rarely have a nonempty core, the overlap of her results with ours is small. Cho and Duggan (2003) and Cardona and Ponsati (2005) consider models in which the space of outcomes is one dimensional, modelling policy concerns rather than private rewards. Cho and Duggan (2003) establish uniqueness when utility functions are quadratic and provide an example with multiple equilibria when the utility functions are not quadratic. Cardona and Ponsati (2005) establish uniqueness in the case of unanimity and asymptotic uniqueness as the discount factor goes to one. Yıldırım (2004) studies a model in which agents can influence the probability of becoming the proposer by exerting effort, demonstrating uniqueness under unanimity rule and k-majority rule.

Merlo and Wilson (1995) and Eraslan and Merlo (2002) study a generalization of the Baron-Ferejohn model in which the size of the pie varies stochastically; Merlo and Wilson (1995) demonstrate uniqueness under unanimity rule, and Eraslan and Merlo (2002) demonstrate nonuniqueness under majority rule.

3 The Fixed Point Index

Let $D \subset \mathbb{R}^m$ be a nonempty compact convex set. Between Brouwer's (1910) proof of his fixed point theorem and the middle of the Twentieth Century, there emerged a theory of a fixed point index that assigns an integer to each closed set of fixed points of an upper semicontinuous convex valued correspondence $F : D \to D$ that is isolated, in the sense of having a neighborhood containing no other fixed points. More generally, an *index admissable correspondence* is an upper semicontinuous convex valued correspondence

$F : \overline{U} \to D$ where $U \subset D$ is open and F has no fixed points in $\partial U := \overline{U} \setminus U$. Let $\mathcal{C}$ be the set of index admissable correspondences.

Proposition 1. *There is a unique function $\Lambda : \mathcal{C} \to \mathbb{Z}$ satisfying the following conditions:*

(A) (Normalization) If $c : D \to D$ is a constant function, then $\Lambda(c) = 1$.

(B) (Additivity) If $F : \overline{U} \to D$ is index admissable, $U_1, \dots, U_r \subset U$ are pairwise disjoint, and F has no fixed points in $\overline{U} \setminus (U_1 \cup \dots \cup U_r)$, then

$$\Lambda(F) = \sum_{i=1}^{r} \Lambda(F|_{\overline{U}_i}).$$

(C) (Homotopy) If $h : \overline{U} \times [0,1] \to D$ is a homotopy (i.e., a continuous function) such that for each $0 \leq t \leq 1$, $h_t := h(\cdot, t) : \overline{U} \to D$ is an index admissable function, then $\Lambda(h_0) = \Lambda(h_1)$.

(D) (Continuity) If $F : \overline{U} \to D$ is index admissable, then there exists a neighborhood $V \subset \overline{U} \times D$ of the graph of F such that $\Lambda(F') = \Lambda(F)$ whenever $F' : \overline{U} \to D$ is an upper semicontinuous convex valued correspondence whose graph is contained in V.

We will not prove this here. Brown (1971) is a standard reference for the fixed point index for continuous functions. The extension of the index to convex valued correspondences can be accomplished using either algebraic topology, extending the work of Eilenberg and Montgomery (1946), or by exploiting the fact that upper semicontinuous functions can be approximated, in a suitable sense, by continuous functions. This approach was developed in connection with degree theory in Cellina (1969b,a) and Cellina and Lasota (1969), and is explained in McLennan (1989).

It can be shown (e.g., McLennan (1989)) that $\Lambda(F) = 1$ whenever $F : D \to D$ is an upper semicontinuous correspondence. This will also not be proved here. (Since any two continuous functions are homotopic, its validity for functions follows from Normalization and Homotopy; the main difficulty is to show that any u.s.c.c.v. correspondence can be approximated by a continuous function, so that its validity for correspondences follows from Continuity.) In particular, for any partition of the set of fixed points into isolated sets, the sum of the indices of the sets must be one.

Our strategy for proving uniqueness is to show that each fixed point of the relevant correspondence is contained in a connected component of the set of fixed points that is isolated and has index one. Since, by Additivity, the sum of the indices of the components is one, there must be exactly one connected component. We also show that all equilibria in each connected component give rise to the same vector of continuation payoffs.

In principle this strategy for proving uniqueness can be applied in any setting in which a unique fixed point, or connected component of the set of fixed points, is obtained from an economic model. In spite of this, we know of no other proof of uniqueness that applies this method: *to the best of our knowledge (cf. p. 615 of Mas-Colell et al. (1995)) this is the first setting in which uniqueness can be obtained in this manner even though more direct methods (e.g., the contraction mapping theorem) are not (so far as we can tell) applicable.*

4 The Model

Let the set of agents be $N := \{1, \dots, n\}$. These agents are bargaining over the division of a pie of size 1. In each period a proposer is selected randomly. Let the probability that i is selected to be the proposer be p_i, and let p_0 be the probability that no proposer is selected, so

$$p = (p_1, \dots, p_n)$$

is a vector of nonnegative numbers that sum to $1 - p_0 \leq 1$.

For each proposer i there is a set $\mathcal{S}_i = \{S_{i1}, \dots, S_{iK_i}\}$ of subsets of N that are sufficient to pass a proposal made by i. Such sets are called *winning coalitions* for i. We assume that $\mathcal{S}_i \neq \emptyset$ for all i, and that $i \neq S_{ik}$ for all $k = 1, \dots, K_i$. (That is, we adopt the convention that the proposer is never a member of the winning coalition; of course this is insubstantial, in that it would be logically equivalent to assume that the proposer is always a member of the winning coalition.) We assume that $S' \in \mathcal{S}_i$ whenever $S \in \mathcal{S}_i$ and $S \subset S' \subset N \setminus \{i\}$.

The proposer i suggests a division of the pie. There is then a secret ballot concerning whether to accept the proposed division. If the set of agents voting in favor is an element of $\mathcal{S}_i$, then the proposal is implemented, and the game ends. Otherwise the process is repeated in the next period. The utility for agent i resulting from being awarded d_i in period t is $\delta_i^t d_i$ where

$$\delta = (\delta_1, \dots, \delta_n) \in (0,1)^n$$

is a vector of discount factors.

We only consider stationary equilibria. (It has been well known since Baron and Ferejohn (1989) that in general there can be a continuum of nonstationary equilibrium giving different continuation values.) For each $i = 1, \dots, n$ let $\Pi_i := \Delta(N)$ be the space of possible divisions of the pie, interpreted as the set of proposals that i can make when she is the proposer. A *stationary strategy* for agent i is a pair (π_i, ρ_i) where $\pi_i \in \Delta(\Pi_i)$ and $\rho_i : \bigcup_{j \neq i} \Pi_j \to \Delta(\{Y, N\})$ is measurable. Thus π_i describes the behavior of i when she is the proposer as a probability distribution over the possible proposals, and ρ_i describes her behavior when she is responding to proposals by other agents. For $j \neq i$ let $\rho_i(d; j)$ be the probability that agent i votes to accept d when it is proposed by agent j. For the time being we fix

$$(\pi, \rho) = ((\pi_1, \rho_1), \dots, (\pi_n, \rho_n)).$$

These strategies induce expected payoffs at various points in the play in period zero. Let $v_i = v_i(\pi, \rho)$ be agent i's expected payoff prior to the selection of a proposer. Let $w_i = w_i(\pi, \rho)$ be agent i's expected payoff in the event that she is selected as the proposer. Let $w_i(d) = w_i(\pi, \rho; d)$ be her expected payoff if she proposes $d \in \Pi_i$. The probability that a proposal by i of d is accepted is the sum over winning coalitions $S \in \mathcal{S}_i$ of the probability that the set of voters voting in favor is S. Let

$$\alpha_i(d) = \alpha_i(\rho; d) := \sum_{S \in \mathcal{S}_i} \Big(\prod_{j \in S} \rho_j(d; i) \Big) \Big(\prod_{j \in N \setminus (S \cup \{i\})} (1 - \rho_j(d; i)) \Big)$$

be this probability. Then

$$w_i(d) = \alpha_i(d) d_i + (1 - \alpha_i(d)) \delta_i v_i.$$

Consequently

$$w_i = \int_{\Pi_i} [\alpha_i(d) d_i + (1 - \alpha_i(d)) \delta_i v_i] \, d\pi_i. \tag{1}$$

Finally, the expected payoff prior to selection of a proposer must satisfy the condition

$$v_i = p_0 \delta_i v_i + p_i w_i + \sum_{j \neq i} p_j \int_{\Pi_j} [\alpha_j(d) d_i + (1 - \alpha_j(d)) \delta_i v_i] \, d\pi_j. \tag{2}$$

For any given (π, ρ), the system of linear equations in the variables (v_i, w_i) given by (1) and (2) has a unique solution. Specifically, substituting the right hand side of (1) for w_i in (2) gives a linear equation

$$v_i = p_0 \delta_i v_i + \sum_{j=1}^{n} p_j \int_{\Pi_j} [\alpha_j(d) d_i + (1 - \alpha_j(d)) \delta_i v_i] \, d\pi_j \tag{3}$$

in the variable v_i with a coefficient on the right hand side that is less than $1 - p_i$, since it is a sum of probabilities of disjoint events, some of which are multiplied by 0 or δ_i.

The vector (π, ρ) is a *stationary equilibrium* if, for each $i = 1, \ldots, n$:

(i) $\pi_i(\text{argmax } w_i(d)) = 1$;

(ii) for all $j \neq i$ and all $d \in \Pi_j$,

$$\rho_i(d; j) = \begin{cases} N, & d_i < \delta_i v_i \\ Y, & d_i \geq \delta_i v_i. \end{cases}$$

As in many other voting models, absent an assumption that agents vote sincerely there can be perverse equilibria in which no voter is pivotal. By passing to a more demanding solution concept, such as trembling hand perfection, one can eliminate such equilibria, but the technical details are not of interest here. Similarly, our assumption that agents vote in favor when they are indifferent can be justified, here and in many other bargaining models, by arguing that the proposer can offer slightly more than the continuation value to each member of the targetted winning coalition. In combination with a device to rule out equilibria in which agents vote perversely because they are not pivotal, this argument implies that the only equilibria have the proposer offering exactly the continuation value, which is accepted with probability one. Again, the techniques involved in such arguments are tedious and not of interest here.

5 Analysis

Let (π, ρ) be a stationary equilibrium, and let $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$ be the vectors of numbers satisfying (1) and (2). In this section we establish certain properties of equilibria that typically hold in noncooperative models of bargaining: proposers always make proposals that are just barely acceptable, and these proposals are always accepted, so that agreement is reached in the first period. In addition we show that an agent has a positive expected payoff if and only if that agent's recognition probability is positive.

We begin with some technical results that establish useful properties of v and w. Let $q_i := \int_{\Pi_i} \alpha_i(d)\, d\pi_i$ be the probability that an agreement is reached when i is the proposer.

Lemma 1. $\sum_i v_i \leq 1$ *with equality if and only if* $\sum_i p_i q_i = 1$, *i.e.,* $p_0 = 0$ *and* $q_i = 1$ *for all* i.

Proof. Summing equation (3) over i, and recognizing that the total payout is one whenever an agreement is reached, gives

$$\sum_{i=1}^{n} v_i = \sum_{j=1}^{n} p_j q_j + \Big(1 - \sum_{j=1}^{n} p_j q_j\Big) \sum_{i=1}^{n} \delta_i v_i.$$

Thus $\sum_{i=1}^{n} v_i$ is a weighted average of 1 and a number that is strictly less than itself if there is any i with $v_i > 0$. $\square$

For each i let

$$\xi_i(v) := \min_{S_{ik} \in \mathcal{S}_i} \sum_{j \in S_{ik}} \delta_j v_j$$

be the minimal cost of inducing agreement, and let

$$\mathcal{S}_i^*(v) = \{\, S_{ik} \in \mathcal{S}_i : \sum_{j \in S_{ik}} \delta_j v_j = \xi_i(v) \,\}$$

be the set of coalitions that achieve the minimum.

Lemma 2. *For each* i, $w_i = 1 - \xi_i(v)$. *Consequently agreement is reached in the first period with probability one after the proposer* i *offers* $\delta_j v_j$ *to each element of some* $\mathcal{S}_i^*(v)$.

Proof. For any $S_{ik} \in \mathcal{S}_i^*(v)$, agent i can propose the allocation that gives $1 - \xi_i(v)$ to himself, $\delta_j v_j$ to each agent $j \in S_{ik}$, and 0 to all other agents. This will certainly be accepted according to (ii). Thus

$$w_i \geq 1 - \xi_i(v) \geq 1 - \sum_{j \neq i} \delta_j v_j \geq 1 - \sum_{j \neq i} v_j \geq v_i \geq \delta_i v_i.$$

But there is no proposal that will ever be accepted that gives agent i more than this, and in fact $w_i > \delta_i v_i$ since it is impossible to have equality in every one of the inequalities above. Clearly the claims follow from this. $\square$

For each i and each $S_{ik} \in \mathcal{S}$ let $\tilde{S}_{ik} \in [0,1]^n$ be the vector whose j^{th} coordinate is 1 or 0 according to whether $j \in S_{ik}$. Let $\mathcal{Y}_i$ is the convex hull of $\{\, \tilde{S}_{ik} : S_{ik} \in \mathcal{S}_i \,\}$, and let $\mathcal{Y} := \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_n$. We treat elements of $\mathcal{Y}$ as $n \times n$ matrices with row i being the element of $\mathcal{Y}_i$.

Let η_{ik} be the probability that i proposes the allocation in which each member of S_{ik} receives $\delta_j v_j$ and i receives $1 - \sum_{j \in S_{ik}} \delta_j v_j$. Let $y_i :=$

$\sum_k \eta_{ik}\tilde{S}_{ik}$, so y_{ij} is the probability, conditional on i being the proposer, that j is included in the minimal winning coalition. Lemma 2 implies that $y_i \in \mathcal{Y}_i$. Since i assigns all probability to cheapest coalitions,

$$\sum_{j\neq i} y_{ij}\delta_j v_j = \xi_i(v) = \min_{\tilde{y}\in\mathcal{Y}_i} \sum_{j\neq i} \tilde{y}_{ij}\delta_j v_j. \tag{4}$$

Note that the numbers η_{ik} embody one aspect of indeterminacy: there may be many vectors η_i that yield a a particular y_i, but the equilibrium conditions depend only on y_i.

Let $Y \in \mathcal{Y}$ be the matrix whose i^{th} row is y_i, and let

$$m = m(Y) := Y^T p$$

be the vector whose i^{th} component $m_i := \sum_{j\neq i} p_j y_{ji}$ is the probability that i is offered her continuation value when another agent is the proposer. Given the behavior described in Lemma 2, equation (2) implies that

$$v_i = (m_i + p_0)\delta_i v_i + p_i(1 - \sum_{j\neq i} y_{ij}\delta_j v_j). \tag{5}$$

Proposition 2. *For any stationary equilibrium (π, ρ) the pair (v, Y), derived in the manner explained above, satisfies (4) and (5) for all i. Conversely, if $v \in [0,1]^n$ and $Y \in \mathcal{Y}$ satisfy (4) and (5) for all i, then they are derived from a stationary equilibrium.*

Proof. The explanation to this point constitutes a proof that (4) and (5) are satisfied if (v, Y) is derived from a stationary equilibrium. Given $(v, Y) \in [0,1]^n \times \mathcal{Y}$ satisfying (4) and (5), one can find $\eta \in \prod_i \Delta(\mathcal{S}_i^*(v))$ such that $y_i = \sum_k \eta_{ik}\tilde{S}_{ik}$ for each i. Given such an η, it is straightforward to construct a stationary equilibrium in which each agent i makes a minimally acceptable offer to S_{ik} with probability η_{ik} when she is the proposer, and votes to accept any proposal that offers at least v_i. □

Thus the analysis of the expected payoffs resulting from stationary equilibria reduces to the study of the solutions of the system (4) and (5). It may seem that one can reduce to an even smaller system of variables. Let

$$\mathcal{M} := \sum_j p_j \mathcal{Y}_j = \{\, p_1 y_1 + \cdots + p_n y_n : Y \in \mathcal{Y} \,\}.$$

Then a necessary and sufficient set of conditions for the pair $(v, m) \in [0,1]^n \times \mathcal{M}$ to be derived from a stationary equilibrium is that m minimizes $\sum_j m'_j \delta_j v_j$ over $m' \in \mathcal{M}$ and

$$v_i = (m_i + p_0)\delta_i v_i + p_i(1 - \xi_i(v)) \tag{6}$$

holds for each i. Unfortunately our analysis cannot be conducted in terms of the simpler system of variables (v, m) because Y is needed to describe the relationship between $(\xi_1(v), \ldots, \xi_n(v))$ and m.

The result above has an important consequence: all bargaining power is ultimately derived from one's recognition probability, though of course it may be amplified by membership in minimal winning coalitions or diminished by impatience. A more general and complete result along these lines is given by Kalandrakis (2005).

Corollary 1. *For all i, if $p_i > 0$, then $v_i > 0$, and if $p_i = 0$, then $v_i = 0$.*

Proof. When $p_i = 0$, (5) implies that $v_i = 0$. If $p_i > 0$, then Lemmas 1 and 2 imply that $v_i \geq p_i(1 - \sum_{j \neq i} \delta_j v_j) > 0$. □

More generally, any player i with $p_i = 0$ is a "dummy" and has no effect on the set of stationary equilibria. Specifically, given a stationary equilibrium, there is a corresponding stationary equilibrium of the game obtained by eliminating i from the list of agents, and from all minimal winning coalitions. Conversely, an equilibrium of the reduced game may be construed as an equilibrium of the game that includes i. For this reason there is effectively no loss of generality in assuming that $p_i > 0$ for all i, and we shall do so since it simplifies the argument in certain ways.

6 A Fixed Point Formulation

At this point we introduce some notational conventions that will simplify the algebra to come. In general we will denote $n \times n$ matrices by capital letters, with the usual understanding that the corresponding lower case letter is used to denote the entries of the matrix. In addition, given a vector denoted by a lower case symbol, the corresponding upper case letter will denote the diagonal matrix whose diagonal entries are the components of the vector. Specifically, this treatment will be applied to p, v, δ, and $m = Y^T p$. We say that P (for example) is the *diagonalization* of p.

Let $\mathbf{1}$ be the $n \times n$ identity matrix. By diagonalizing m, p, and δ, equation (5) can be rewritten as

$$v = \big(M + p_0 \mathbf{1} - PY\big)\Delta v + p.$$

Rearranging this gives $A(Y)v = p$ where, for $Y \in \mathcal{Y}$,

$$A(Y) := \mathbf{1} - \big(M + p_0 \mathbf{1} - PY\big)\Delta.$$

A *nonsingular M-matrix* is a square matrix that has positive entries on the diagonal and nonpositive off-diagonal entries, and is dominant diagonal, meaning that for each column, the diagonal entry is greater than the negation of the sum of the off-diagonal elements.

Lemma 3. *For each $Y \in \mathcal{Y}$, $H(Y) := A(Y) - p\delta^T$ is a nonsingular M-matrix.*

Proof. The entries of $H(Y)$ are

$$h_{ij}(Y) = \begin{cases} 1 - (m_i + p_0 + p_i)\delta_i, & j = i, \\ p_i(y_{ij} - 1)\delta_j, & j \neq i. \end{cases}$$

Of course $p_i y_{ij} < 1$, so the off diagonal entries of $H(Y)$ are negative, and for each j we have $-\sum_{i \neq j} h_{ij}(Y) = (1 - p_0 - p_j - m_j)\delta_j < h_{jj}(Y)$. □

Lemma 4. *For each $Y \in \mathcal{Y}$, $A(Y)$ is invertible.*

Proof. Theorem 2.3 of Chapter 6 of Berman (1979) states that a nonsingular M-matrix is invertible, and the entries of its inverse are nonnegative. It follows that $1 + \delta^T H(Y)^{-1} p \neq 0$, so the Sherman-Morrison formula[2] implies that $A(Y) = H(Y) + p\delta^T$ is invertible. □

For $Y \in \mathcal{Y}$ let

$$\nu(Y) := A(Y)^{-1} p = \left[\mathbf{1} - (M + p_0 \mathbf{1} - PY)\Delta\right]^{-1} p$$

be the vector of continuation values resulting from the matrix of inclusion probabilities Y.

For each i and $v \in [0,1]^n$ let

$$\mathcal{Y}_i^*(v) := \operatorname{argmin}_{y_i \in \mathcal{Y}_i} \sum_{j \neq i} y_{ij} \delta_j v_j,$$

so that $\mathcal{Y}_i^*(v)$ is the set of elements of $\mathcal{Y}_i$ satisfying (4). For $Y \in \mathcal{Y}$ let $\mathcal{Y}^*(v) = \mathcal{Y}_1^*(v) \times \cdots \times \mathcal{Y}_n^*(v)$. Let $F : \mathcal{Y} \to \mathcal{Y}$ be the correspondence

$$F(Y) := \mathcal{Y}^*(\nu(Y)).$$

Then F is an upper semicontinuous convex valued correspondence with a convex domain. We have shown that the vector v of continuation values associated with an equilibrium is $\nu(Y)$ for some fixed point Y of F.

Our main result is:

[2] If A is a nonsingular $n \times n$ matrix, $u, v \in \mathbb{R}^n$ are column vectors, and $\lambda := v^T A^{-1} u \neq -1$, then the formula $(A + uv^T)^{-1} = A^{-1} - A^{-1} uv^T A^{-1}/(1 + \lambda)$ can be verified by multiplying the right hand side by $A + uv^T$. (Cf., p. 124 of Meyer (2001).)

Theorem 1. *There is a unique vector v such that $v = \nu(Y)$ for all fixed points Y of F.*

Once v is determined, the vector m of probabilities of being invited to join other proposers' coalitions is determined by (6). Thus:

Corollary 2. *There is a unique vector m such that $m = Y^T p$ for all fixed points Y of F.*

An important, if rather obvious, consequence of Theorem 1 is that the equilibrium continuation values respect any symmetry of the given data. Let G be the group of such symmetries: an element of G is a bijection $g : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that $p_{g(i)} = p_i$ and $\delta_{g(i)} = \delta_i$ for all i, and $\{ g(j) : j \in S_{ik} \} \in \mathcal{S}_{g(i)}$ for all i and $S_{ik} \in \mathcal{S}_i$.

Corollary 3. *For all fixed points Y of F, all $g \in G$, and all i, $\nu_{g(i)}(Y) = \nu_i(Y)$.*

Proof. By transforming all objects and equations in the appropriate way, for the given g, one can show that there is a fixed point Y' of F such that $\nu_{g(i)}(Y') = \nu_i(Y)$ for all i, and Theorem 1 implies that $\nu(Y) = \nu(Y')$. □

7 The Proof of Theorem 1

In the remainder we study a particular fixed point Y of F. Let

$$\mathcal{E} := m^{-1}(m(Y)),$$

and set $v := \nu(Y)$. In the next lemma we show that any other point in $\mathcal{E}$, say Y', is a fixed point of F with $\nu(Y') = v$. Theorem 1 follows once we show that $\mathcal{E}$ is the entire set of fixed points of F, which is the agenda of the remainder of the paper.

In order to express the relevant concepts, which are derived from linear algebra, we must expand our perspective. Let $\mathcal{J}_i$ be the affine hull[3] of $\mathcal{Y}_i^*(v)$, and let $\mathcal{J} := \mathcal{J}_1 \times \cdots \times \mathcal{J}_n$. Concretely, $\mathcal{J}_i$ is the set of y_i' in the affine hull of $\mathcal{Y}_i$ that satisfy $y_i' \Delta v = y_i \Delta v$. Note that if T is a vector (actually matrix) that is parallel or tangent to $\mathcal{J}$, then $T \Delta v = 0$.

[3]The *affine hull* of a set $S \subset \mathbb{R}^m$ is the smallest affine subspace that contains S. Equivalently, it is the set of all weighted sums $\sum_{h=1}^k \alpha_h s_h$ where $s_1, \ldots, s_k \in S$ and $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$ with $\alpha_1 + \cdots + \alpha_k = 1$. Such a weighted sum is said to be an *affine combination* of elements of S. If S is its own affine hull, then we say that S is an *affine subspace* of $\mathbb{R}^m$. The affine subspaces of $\mathbb{R}^m$ are the translates of the linear subspaces.

The function m is the restriction of a linear function to $\mathcal{Y}$, and all linear functions that agree with m on $\mathcal{Y}$ agree on all of $\mathcal{J}$, so there is a well defined extension $\mu : \mathcal{J} \to \mathbb{R}^n$ of m to $\mathcal{J}$. Let $\hat{\mathcal{E}} := \mu^{-1}(m(Y))$. Then $\hat{\mathcal{E}}$ is an affine subspace of $\mathcal{J}$. Clearly $\mathcal{E} = \hat{\mathcal{E}} \cap \mathcal{Y}$.

Lemma 5. *Each $Y' \in \mathcal{E}$ is a fixed point of F, and $\nu(Y') = v$.*

Proof. The definition of $\mathcal{E}$ gives $m(Y') = m(Y)$. The definition of $\mathcal{J}$ gives $Y'\Delta v = Y\Delta v$. Together these facts imply that (5) holds with Y' in place of Y, so $\nu(Y') = v$.

Since $Y \in \mathcal{Y}^*(v)$, each row y_i of Y minimizes $\sum_{j \neq i} y_{ij}\delta_j v_j$ in $\mathcal{Y}_i$. Multiplying by p_i and summing over i gives

$$m(Y) = \min_{\tilde{Y} \in \mathcal{Y}} \sum_j m_j(\tilde{Y})\delta_j v_j.$$

Since $m(Y') = m(Y)$ and $Y' \in \mathcal{Y}$, Y' also solves the minimization problem on the right hand side, so each row y'_i of Y' minimizes $\sum_{j \neq i} y'_{ij}\delta_j v_j$ in $\mathcal{Y}_i$. That is, $Y' \in \mathcal{Y}^*(v)$.

We conclude that $Y' \in \mathcal{Y}^*(\nu(Y'))$. □

Since m is linear and $\mathcal{Y}$ is convex, $\mathcal{E}$ is convex, and in particular it is connected. The proof that $\mathcal{E}$ is the entire set of fixed points of F is a matter of showing that the set of fixed points has only one connected component, and that $\mathcal{E}$ has a neighborhood with no other fixed points of F, so it is a connected component and must therefore be the entire set of fixed points.

The next two results express the intuitive reason one expects uniqueness: as one changes the probabilities assigned to the various coalitions, the continuation values of agents who are included more frequently increase, while the continuation values of agents who are included less frequently decrease, so that a change in any direction makes continued change in that direction increasingly expensive.

The vector space of vectors (actually matrices) parallel or tangent to $\mathcal{J}$ can be expressed as the direct sum of the vectors that are parallel to $\hat{\mathcal{E}}$ and $\mathcal{Z}$, the vector space of matrices that are parallel to $\mathcal{J}$ and orthogonal to $\hat{\mathcal{E}}$. Then $\mathcal{Z}$ is complementary to $\hat{\mathcal{E}}$ in the sense that $(\tilde{Y} - \tilde{Y}') \perp Z$ for all $\tilde{Y}, \tilde{Y}' \in \hat{\mathcal{E}}$ and all $Z \in \mathcal{Z}$, and each point $\tilde{Y} \in \mathcal{J}$ has a unique representation of the form $\tilde{Y} = \tilde{Y}_0 + Z$ where $\tilde{Y}_0 \in \hat{\mathcal{E}}$ and $Z \in \mathcal{Z}$. The decomposition $\mathcal{J} = \hat{\mathcal{E}} + \mathcal{Z}$ allows us to give a precise expression of the principle described above: motion in $\hat{\mathcal{E}}$ has no effect on m, and consequently no effect on v, but the principle holds with strict inequality for directions of motion that are not parallel to $\hat{\mathcal{E}}$.

In some contexts it is appropriate to treat $\mathcal{Z}$ as a linear subspace of a space of matrices, but in what follows we will sometimes treat its elements as vectors, especially when we evaluate a derivative of a function on $\mathcal{Y}$ in a direction given by an element of $\mathcal{Z}$. Choosing an arbitrary linear coordinate system for $\mathcal{Z}$, let ζ^Z (or simply ζ if Z is clear from context) denote the vector of coordinates of Z in this coordinate system.

Lemma 6. *$p^T Z \Delta \frac{d\nu}{dY}(Y)\zeta^Z > 0$ for all $Y \in \mathcal{E}$ and nonzero $Z \in \mathcal{Z}$.*

The proof is given in Section 8.

Lemma 7. *There is a neighborhood $U \subset \hat{\mathcal{E}}$ of $\mathcal{E}$ and a neighborhood $V \subset \mathcal{Z}$ of the origin such that $p^T Z\Delta(\nu(\tilde{Y}_0 + Z) - \nu(\tilde{Y}_0)) > 0$ for all $\tilde{Y}_0 \in U$ and all nonzero $Z \in V$.*

Proof. Note that, by Lemma 4, ν is a C^∞ function in a neighborhood of $\mathcal{Y}$. Therefore Lemma 6 implies that there is a neighborhood $W \subset \mathcal{J}$ of $\mathcal{E}$ such that for some $\varepsilon > 0$, $p^T Z\Delta \frac{d\nu}{dY}(\tilde{Y})\zeta^Z > \varepsilon$ for all $\tilde{Y} \in W$ and all Z in the unit sphere in $\mathcal{Z}$. Since $\mathcal{E}$ is compact we may choose a neighborhood $U \subset \hat{\mathcal{E}}$ of the $\mathcal{E}$ and a convex neighborhood $V \subset \mathcal{Z}$ of the origin such that $U + V \subset W$. If $\tilde{Y} = \tilde{Y}_0 + Z$ where $\tilde{Y}_0 \in U$ and $0 \neq Z \in \mathcal{Z}$, then

$$\begin{aligned} p^T Z\Delta(\nu(\tilde{Y}_0 + Z) - \nu(\tilde{Y}_0)) &= p^T Z\Delta\Big(\int_0^1 \frac{d\nu}{dY}(\tilde{Y}_0 + sZ)\zeta^Z\, ds\Big) \\ &= \int_0^1 p^T Z\Delta\frac{d\nu}{dY}(\tilde{Y}_0 + sZ)\zeta^Z\, ds \\ &= \|Z\|^2 \int_0^1 p^T \tfrac{Z}{\|Z\|}\Delta\frac{d\nu}{dY}(\tilde{Y}_0 + sZ)\tfrac{\zeta^Z}{\|\zeta^Z\|}\, ds \\ &> \varepsilon\|Z\|^2. \end{aligned}$$

□

The results above will be applied in the analysis of the fixed points of a one parameter family of correspondences

$$H_t := \mathcal{Y} \to \mathcal{Y}^*(v) \subset \mathcal{Y} \quad (0 \leq t \leq 1).$$

Define $H : \mathcal{Y} \times [0,1] \to \mathcal{Y}$ by letting $H(\tilde{Y}, t) = \prod_i H_i(\tilde{Y}, t)$ where

$$H_i(\tilde{Y}, t) := \operatorname{argmin}_{\hat{y}_i \in \mathcal{Y}_i^*(v)} \hat{y}_i\Big((1-t)\Delta\nu(\tilde{Y}) + t\frac{d\nu}{dY}(Y)\zeta^Z\Big)$$

and $\tilde{Y} = \tilde{Y}_0 + Z$. For each t let $H_t := H(\cdot, t)$. By continuity, for Y' near $\mathcal{E}$ the cost minimizing inclusion probabilities in $\mathcal{Y}_i$ are contained in $\mathcal{Y}_i^*(v)$, so H_0 agrees with F near $\mathcal{E}$.

Lemma 8. *$\mathcal{E}$ is the entire set of fixed points of H_1.*

Proof. Consider $\tilde{Y} = \tilde{Y}_0 + Z \in \mathcal{Y}^*(v)$. If $\tilde{Y} \in \mathcal{E}$, then $Z = 0$ and in particular $\zeta^Z = 0$, which implies that $H_1(\tilde{Y}, 1) = \mathcal{Y}^*(v)$. Therefore $\mathcal{E}$ is contained in the set of fixed points of H_1.

If $\tilde{Y} \notin \mathcal{E}$, then $Z \neq 0$, and Lemma 6 gives

$$p^T Z \Delta \frac{d\nu}{dY}(Y)\zeta^Z > 0.$$

Since $\tilde{Y} = \tilde{Y}_0 + Z$, we have

$$p^T \tilde{Y} \Delta \frac{d\nu}{dY}(Y)\zeta^Z > p^T \tilde{Y}_0 \Delta \frac{d\nu}{dY}(Y)\zeta^Z.$$

The definition of $\hat{\mathcal{E}}$ gives $p^T \tilde{Y}_0 = p^T Y$, so that

$$p^T \tilde{Y}_0 \Delta \frac{d\nu}{dY}(Y)\zeta^Z = p^T Y \Delta \frac{d\nu}{dY}(Y)\zeta^Z.$$

But $p^T \tilde{Y} = \sum_i p_i \tilde{y}_i$ and $p^T Y = \sum_i p_i y_i$, so it follows that there is some i such that

$$\tilde{y}_i \Delta \frac{d\nu}{dY}(Y)\zeta^Z > y_i \Delta \frac{d\nu}{dY}(Y)\zeta^Z.$$

This implies that $\tilde{y}_i \notin H_i(\tilde{Y}, 1)$ and thus $\tilde{Y} \notin H(\tilde{Y}, 1)$. □

Lemma 9. *Let $U \subset \hat{\mathcal{E}}$ and $V \subset \mathcal{Z}$ be neighborhoods of $\mathcal{E}$ and the origin respectively such that*

$$p^T Z \Delta(\nu(\tilde{Y}_0 + Z) - \nu(\tilde{Y}_0)) > 0 \tag{7}$$

for all $\tilde{Y}_0 \in U$ and all nonzero $Z \in V$. Let $W \subset \mathcal{Y}$ be a neighborhood of $\mathcal{E}$ such that $W \cap \mathcal{Y}^*(v) \subset U + V$. For each $0 \leq t \leq 1$, the set of fixed points of H_t in W is $\mathcal{E}$.

The proof is given in Section 9.

Since, for all t, $\mathcal{E}$ is the set of fixed points of H_t in W, the Continuity property of the index implies that the index of $\mathcal{E}$, as a set of fixed points of H_t, is constant as a function of t. Since $\mathcal{E}$ is the entire set of fixed points of H_1, the index of $\mathcal{E}$ as a set of fixed points of H_1 is one. Therefore the index of $\mathcal{E}$ as a set of fixed points of H_0 is one. Since H_0 agrees with F on a neighborhood of $\mathcal{E}$, the index of $\mathcal{E}$ as a set of fixed points of F is one.

Consider what we have done. We started with an arbitrary fixed point Y of F. We showed that $\mathcal{E} := m^{-1}(m(Y))$ is the component of the set of fixed points of F that contains Y, and that the index of $\mathcal{E}$ is one. Since Y was arbitrary, it follows that the index of every component of the set of fixed points of F is one. Since $\Lambda_{\mathcal{Y}}(F) = 1$, Additivity implies that there can be only one such component, so $\mathcal{E}$ is the entire set of equilibria, as desired.

8 The Proof of Lemma 6

Recall that we are given Y, a fixed point of F, and our goal is to show that $p^T Z \Delta \frac{d\nu}{dY}(Y)\zeta^Z > 0$ for all nonzero $Z \in \mathcal{Z}$. Let B be the matrix, with respect to the coordinate system we chose for $\mathcal{Z}$, of the derivative at Y of the restriction of ν to $\mathcal{Z}$: $\frac{d\nu}{dY}(Y)\zeta^Z = B\zeta^Z$ for $Z \in \mathcal{Z}$. The function $Z \mapsto Z^T p$ is linear, so there is a matrix Q such that $Z^T p = Q\zeta^Z$. Then

$$p^T Z \Delta \frac{d\nu}{dY}(Y)\zeta^Z = (\zeta^Z)^T Q^T \Delta B \zeta^Z$$

for all $Z \in \mathcal{Z}$. We need to show that $Q^T \Delta B$ is positive definite.

We first solve for B. The definition of ν implies that, for any $\tilde{Y}$,

$$A(\tilde{Y})\nu(\tilde{Y}) = \big[\mathbf{1} - (\tilde{M} + p_0\mathbf{1} - P\tilde{Y})\Delta\big]\nu(\tilde{Y}) = p. \tag{8}$$

where $\tilde{M}$ is the diagonalization of $\tilde{m} = \tilde{Y}^T p$. Replacing $\tilde{Y}$ with $Y + Z$, differentiating with respect to a component ζ_j^Z of ζ^Z, and evaluating at $Z = 0$, gives

$$Ab^j + \big[-\Delta Q^j + P\frac{\partial \tilde{Y}}{\partial \zeta_j}(Y)\Delta\big]v = 0.$$

(Here $A = A(Y)$, $v = \nu(Y)$, b^j and q^j are the j^{th} columns of B and Q respectively, and Q^j is the diagonalization of q^j.) We have $\frac{\partial \tilde{Y}}{\partial \zeta_j}(Y)\Delta v = 0$ because each proposer is indifferent, at the vector of continuation values v, between all coalitions chosen at $Y + Z$ for any $Z \in \mathcal{Z}$. Therefore

$$Ab^j = \Delta Q^j v.$$

Recognizing that $Q^j v = V q^j$ (this is a general fact about multiplying a vector by a matrix obtained by diagonalizing another vector) and $\Delta V = V\Delta$ (multiplication of matrices obtained by diagonalization is commutative) leads to $Ab^j = V\Delta q^j$. This is true for each j, so $AB = V\Delta Q$, or $B = A^{-1}V\Delta Q$. (Lemma 4 states that A is invertible.)

Therefore

$$Q^T \Delta B = Q^T \Delta A^{-1} V \Delta Q.$$

A square matrix is positive definite if and only if the sum of the matrix and its transpose is positive definite. Consequently it suffices to show that

$$Q^T\Delta\big(A^{-1}V + V(A^{-1})^T\big)\Delta Q = Q^T\Delta A^{-1}\big(VA^T + AV\big)(A^{-1})^T\Delta Q$$

is positive definite.

For $\gamma \in \mathbb{R}$ let

$$G(\gamma) := VA^T + AV - \gamma p v^T - \gamma v p^T.$$

Recall that Q is the matrix of the derivative of $Z^T p$, and vectors in $\mathcal{Z}$ represent changes in the probabilities assigned to coalitions that all have the same expense for each of the proposers, so Δv is orthogonal to each of the columns of Q. Therefore $Q^T\Delta v = 0$. Note that $Av = p$ and thus $A^{-1}p = v$, so

$$Q^T\Delta A^{-1}\gamma p v^T(A^{-1})^T\Delta Q = \gamma\big(Q^T\Delta v\big)v^T(A^{-1})^T\Delta Q = 0.$$

Similarly,

$$Q^T\Delta A^{-1}\gamma v p^T(A^{-1})^T\Delta Q = \gamma Q^T\Delta A^{-1}v\big(v^T\Delta Q\big) = 0.$$

Therefore, for any γ,

$$Q^T\Delta\big(A^{-1}V + V(A^{-1})^T\big)\Delta Q = Q^T\Delta A^{-1}G(\gamma)(A^{-1})^T\Delta Q.$$

Since $\mathcal{Z}$ is orthogonal to the kernel of the map $Y \mapsto Y^T p$, and Q is the matrix of the restriction of this map to $\mathcal{Z}$, the columns of Q are linearly independent. In addition A is nonsingular, and Δ is a diagonal matrix with nonzero diagonal entries, so it suffices to find a number γ such that $G(\gamma)$ is positive definite. Let

$$\bar{\gamma} = \max_i \max\Big\{\delta_i, \frac{p_i + v_i - 2(1 - \delta_i(m_i(Y) + p_0))v_i}{p_i + v_i - 2p_i v_i}\Big\}.$$

(The denominator is positive because $1 > p_i, v_i > 0$, so that $p_i, v_i > p_i v_i$.)

It is known that a nonsingular M-matrix has positive principal minors definite[4] so a symmetric nonsingular M-matrix is positive definite. Therefore it suffices to establish:

[4]The principal minors of an nonsingular M-matrix are also nonsingular M-matrices, so it suffices to show that the determinant of an nonsingular M-matrix is positive. But any nonsingular M matrix can be deformed continuously into a diagonal matrix with positive diagonal entries by shrinking the off diagonal terms, and this deformation stays in the space of nonsingular M-matrices.

Lemma 10. *$G(\bar{\gamma})$ is a symmetric nonsingular M-matrix.*

Proof. Since $m_i(Y)$ is the probability of being in another proposer's coalition, $m_i(Y) \leq 1 - p_0 - p_i$, so that $m_i(Y) + p_0 \leq 1 - p_i$ and $\delta_i(m_i(Y) + p_0) < 1 - p_i$ and $p_i < 1 - \delta_i(m_i(Y) + p_0)$. It follows that $\bar{\gamma} < 1$. Also, note that

$$a_{ij} = \begin{cases} 1 - \delta_i(m_i(Y) + p_0), & j = i, \\ p_i \delta_j y_{ij}, & j \neq i. \end{cases}$$

Claim 1: $G(\bar{\gamma})$ is symmetric.

This follows immediately from the definition.

Claim 2: The diagonal entries of $G(\overline{\gamma})$ are positive.

Note that

$$\begin{aligned} g_{ii}(\overline{\gamma}) &= 2a_{ii}v_i - 2\bar{\gamma}p_i v_i \\ &= 2(1 - \delta_i(m_i(Y) + p_0))v_i - 2\bar{\gamma}p_i v_i \\ &> 2p_i v_i(1 - \bar{\gamma}) > 0 \end{aligned}$$

since $(1 - \delta_i(m_i(Y) + p_0)) > p_i > \bar{\gamma}p_i$.

Claim 3: The off-diagonal entries of $G(\bar{\gamma})$ are nonpositive.

Observe that

$$g_{ij}(\overline{\gamma}) = a_{ij}v_j + a_{ji}v_i - \bar{\gamma}p_i v_j - \bar{\gamma}p_j v_i \leq 0$$

since $a_{ij} \leq p_i \delta_i \leq p_i \bar{\gamma}$ for all i, $j \neq i$.

Claim 4: $G(\overline{\gamma})$ is dominant diagonal.

For any n-vectors x and y we have $XY = YX$ because X and Y are diagonal matrices, and it is also clear that $Xy = Yx$. Replacing $M(Y)\Delta$ in (8) by $\Delta M(Y)$, then multiplying both sides by $\mathbf{e}^T$, where $\mathbf{e} = (1, \dots, 1) \in \mathbb{R}^n$, and recalling that $m(Y) = Y^T p$ by definition, we obtain

$$\begin{aligned} 1 - p_0 &= \big(\mathbf{e}^T - p_0\delta^T - \delta^T M(Y) + p^T Y \Delta\big)\nu(Y) \\ &= \big(\mathbf{e}^T - p_0\delta^T - m(Y)^T \Delta + p^T Y \Delta\big)\nu(Y) \\ &= \big(\mathbf{e}^T - p_0\delta^T\big)\nu(Y). \end{aligned}$$

Since $\delta_i < 1$ for all i, this implies that

$$1 - p_0 \geq \big(\mathbf{e}^T - p_0\mathbf{e}^T\big)\nu(Y) = (1 - p_0)\mathbf{e}^T \nu(Y).$$

In particular, $\mathbf{e}^T\nu(Y) \leq 1$.

We begin with the following observations. First, for each i we have

$$\sum_{j\neq i} a_{ij}v_j = \sum_{j\neq i} p_i y_{ij}\delta_j v_j = p_i(1-w_i)$$

and

$$\sum_{j\neq i} a_{ji}v_i = (\sum_{j\neq i} y_{ji})\delta_i v_i = m_i(Y)\delta_i v_i.$$

Therefore

$$\begin{aligned}
\sum_{j\neq i} |g_{ij}(\overline{\gamma})| &= -\sum_{j\neq i} g_{ij}(\overline{\gamma}) \\
&= -\sum_{j\neq i} \big(a_{ij}v_j + a_{ji}v_i - \overline{\gamma}p_i v_j - \overline{\gamma}p_j v_i\big) \\
&= -(p_i - p_i w_i) - m_i(Y)\delta_i v_i + \overline{\gamma}p_i(\sum_{j\neq i} v_j) + \overline{\gamma}v_i(\sum_{j\neq i} p_j)
\end{aligned}$$

Applying the inequality $\mathbf{e}^T\nu(\tilde{Y}) \leq 1$ gives

$$\sum_{j\neq i} |g_{ij}(\overline{\gamma})| \leq -(p_i - p_i w_i) - m_i(Y)\delta_i v_i + \overline{\gamma}p_i(1-v_i) + \overline{\gamma}v_i(1-p_0-p_i).$$

In view of the equation $p_i w_i = v_i - m_i(Y)\delta_i v_i - p_0\delta_i v_i$, this implies

$$\begin{aligned}
\sum_{j\neq i} |g_{ij}(\overline{\gamma})| &\leq -p_i + v_i - m_i(Y)\delta_i v_i - p_0\delta_i v_i - m_i(Y)\delta_i v_i \\
&\qquad + \overline{\gamma}p_i(1-v_i) + \overline{\gamma}v_i(1-p_0-p_i) \\
&= -p_i(1-\overline{\gamma}) + v_i(1 - 2m_i(Y)\delta_i - p_0\delta_i + \overline{\gamma}(1-p_0)) - 2\overline{\gamma}p_i v_i \\
&= -(p_i+v_i)(1-\overline{\gamma}) + v_i(2 - 2(m_i(Y)+p_0)\delta_i - \overline{\gamma} + \overline{\gamma}(1-p_0)) - 2\overline{\gamma}p_i v_i \\
&\leq -(p_i+v_i)(1-\overline{\gamma}) + 2(1-(m_i(Y)+1-p_i)\delta_i)v_i - 2\overline{\gamma}p_i v_i \\
&< (1-(m_i(Y)+p_0)\delta)v_i - 2\overline{\gamma}p_i v_i = g_{ii}(\overline{\gamma}).
\end{aligned}$$

(Here the penultimate inequality follows from $p_0 > 0$, and the final inequality derives from the fact that $\overline{\gamma} < 1$, which is a consequence of the definition and the fact that $p_i < 1 - \delta_i(m_i(Y) + p_0)$.) □

9 The Proof of Lemma 9

Fix $t \in [0,1]$ and $\tilde{Y} \in W$. Our goal is to show that $\tilde{Y}$ is a fixed point of H_t if $\tilde{Y} \in \mathcal{E}$ and not otherwise. The image of H_t is contained in $\mathcal{Y}^*(v)$, so we may assume that $\tilde{Y} \in \mathcal{Y}^*(v)$. Therefore $\tilde{Y}$ decomposes as $\tilde{Y} = \tilde{Y}_0 + Z$ where $\tilde{Y}_0 \in U$ and $Z \in V$.

If $\tilde{Y} \in \mathcal{E}$, then $Z = 0$ (so that $\zeta^Z = 0$) and $\nu(\mathcal{Y}) = v$, so[5]

$$H_i(\tilde{Y}, t) = \mathrm{argmin}_{\hat{y}_i \in \mathcal{Y}_i^*(v)} \hat{y}_i \Delta v,$$

and $\tilde{Y}$ is a fixed point of H_t because it is a fixed point of F. Thus $\mathcal{E}$ is contained in the set of fixed points of H_t.

Now suppose that $\tilde{Y} \in W \setminus \mathcal{E}$. Let Y be an element of $\mathcal{E}$. Then $Z \neq 0$. The definition of $\hat{\mathcal{E}}$ implies that $p^T(\tilde{Y}_0 - Y) = 0$, so Lemma 6 and $\tilde{Y} = \tilde{Y}_0 + Z$ give

$$p^T \tilde{Y} \Delta \frac{d\nu}{dY}(Y)\zeta^Z > p^T \tilde{Y}_0 \Delta \frac{d\nu}{dY}(Y)\zeta^Z = p^T Y \Delta \frac{d\nu}{dY}(Y)\zeta^Z.$$

Below we will show that

$$p^T \tilde{Y} \Delta \nu(\tilde{Y}) > p^T Y \Delta \nu(\tilde{Y}). \tag{9}$$

Multiplying the first inequality by t and the second by $1-t$, then summing, yields

$$p^T \tilde{Y} \Delta\big((1-t)\nu(\tilde{Y}) + t\frac{d\nu}{dY}(Y)\zeta^Z\big) > p^T Y \Delta\big((1-t)\nu(\tilde{Y}) + t\frac{d\nu}{dY}(Y)\zeta^Z\big).$$

But $p^T\tilde{Y} = \sum_i p_i \tilde{y}_i$ and $p^T Y = \sum_i p_i y_i$, so there must be some i such that

$$\tilde{y}_i \Delta\big((1-t)\nu(\tilde{Y}) + t\frac{d\nu}{dY}(Y)\zeta^Z\big) > y_i \Delta\big((1-t)\nu(\tilde{Y}) + t\frac{d\nu}{dY}(Y)\zeta^Z\big).$$

This implies that $\tilde{y}_i \notin H_i(\tilde{Y}, t)$ and thus $\tilde{Y} \notin H(\tilde{Y}, t)$.

The remaining task is to establish (9). We have

$$\begin{aligned} p^T \tilde{Y} \Delta \nu(\tilde{Y}) &= p^T \tilde{Y}_0 \Delta \nu(\tilde{Y}) + p^T Z \Delta \nu(\tilde{Y}_0 + Z) \\ &= p^T Y \Delta \nu(\tilde{Y}) + p^T Z \Delta \nu(\tilde{Y}_0 + Z) \\ &> p^T Y \Delta \nu(\tilde{Y}) + p^T Z \Delta \nu(\tilde{Y}_0) \\ &= p^T Y \Delta \nu(\tilde{Y}). \end{aligned}$$

[5]This is not quite correct when $t = 1$ because then $1 - t = 0$, but in this case the argument is even simpler; cf. Lemma 8.

Here the first equality is $\tilde{Y} = \tilde{Y}_0 + Z$, the second equality results from $p^T\tilde{Y}_0 = p^TY$, which is true because $Y, \tilde{Y}_0 \in \hat{\mathcal{E}}$, the inequality is from (7), and the third equality is due to the fact that directions in $\mathcal{Z}$ are parallel to $\mathcal{Y}^*(v)$.

10 Future Research

It would be desirable to extend this paper's model to allow for different different proposer-coalition pairs to generate pies of different size, since one could then investigate the hypothesis that the coalitions that form are the most productive. Whether our uniqueness result extends to such a model is an open question.

A transferable utility (TU) cooperative game specifies a payoff to each coalition of agents. The Shapley (Shapley (1953), Shapley and Shubik (1954)) value is a function that assigns a vector of payoffs to each TU game. A TU game is said to be *simple* if each coalition's payoff is either zero or one. That is, a simple game is essentially a specification of a system of winning coalitions. The power indices of Banzhaf (1965, 1968) Deegan and Packel (1972) and Johnston (1978) each assign a vector of individual "powers" to each simple game, and the Shapley power index is the restriction of the Shapley value to simple games. In our framework there is a simple game (the system of winning coalitions) and other parameters, namely the recognition probabilities and the discount factors. To obtain a power index comparable to those of Banzhaf and Shapley one may take the limit of our vector of equilibrium continuation payoffs, for the case of symmetric recognition probabilities, as the common discount factor goes to one. Another possibility studied by Montero (2005) is to search for vectors of recognition probabilities that are *self-confirmng* in the sense that they coincide with the resulting vector of continuation values. She shows that the nucleolus of a proper simple game[6] has this property.

Comparison of the properties of the various power indices seems like an interesting direction for theoretical investigation.

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[6]A simple game v is *proper* if $v(S) = 1$ implies that $v(T) = 0$ for all $T \subset N \setminus S$.

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