On Breakthroughs, Deadlines, and the Nature of Progress: Contracting for Multistage Projects

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Abstract

We study the optimal incentive schemes for multistage projects. Staged financing, pervasive in both venture capital and research grants, arises as a feature of the optimal contract. The nature of progress plays an important role. When progress is tangible, incentives are provided through a series of deadlines and a reward scheme that decreases over time. Early progress in one stage gives the agent more time and funding to complete the next one. When progress is intangible, unverifiable progress reports are used despite the fact that the information in the report has no social value. The optimal contract involves a soft deadline wherein the principal guarantees funding up to a certain date – if the agent reports progress at that date, then the principal gives him a relatively short deadline to complete the project – if not, then a probationary phase begins in which the project is terminated at a constant rate until progress is reported. We explore the implications for welfare and optimal project design.

JEL Classification: J41, L14, M55, D82

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1 Introduction

Many projects require completion of multiple stages before their benefits can be realized. This is especially true of R&D intensive industries. For example, developing a new drug requires identifying the specific chemical compounds or molecules in vitro (i.e., in test tubes), and then demonstrating its efficacy in pre-clinical (i.e., animal) trials. Once these two stages are successfully completed, clinical trials progress through three additional stages: dosing, small scale and large scale, all of which must be successfully completed before a drug can be submitted to the FDA for approval. Only after obtaining FDA approval can the drug be brought to market, generating revenue for its developer and health benefits for society.

Other examples of multistage projects are ubiquitous. In venture capital, entrepreneurs generally progress through numerous steps (e.g., prototype development, patenting, and production) before realizing profits. Basic research, almost by definition, requires numerous successive advancements before its societal benefits are realized. Naturally, these “projects” require substantial amounts of funding in order to come to fruition, and the preferences of the entrepreneur, employee, or scientist (the “agent”) regarding the timing, intensity and direction of investment are unlikely to be perfectly aligned with the preferences of the firm, institution, or funding entity (the “principal”). In this paper, we explore the optimal provision of incentives in such environments.

The model features a project that requires successful completion of two stages in order for the principal to realize any benefits. We refer to the successful completion of a stage as a breakthrough. Both principal and agent are risk-neutral — the agent, however, has limited financial resources and is protected by limited liability.\(^1\) The model is set in continuous time with breakthroughs arriving randomly according to a Poisson process. The (potential) value of the project is commonly known, but the arrival rate of breakthroughs depends on whether the agent “shirks” (e.g., diverts funds) for private benefit.\(^2\)

Our interest is in how the nature of progress influences the provision of incentives and the realized value of the project. In some settings, progress can be easily observed and verified by the principal (e.g., whether a successful prototype has been developed). In others, progress is less well defined and difficult, if not impossible, for the principal to gauge (e.g., experimental results).\(^3\) We say that progress is tangible if it is observable to both parties and can be

\(^1\)Without financial constraints and limited liability, the first-best outcome can be sustained by simply “selling” the project to the agent. However, this solution is practically infeasible for many if not most applications because of the significant capital investments required.

\(^2\)Tirole (2006, p. 92) gives several reasons why the relationship between researchers and their funding sources “is fraught with moral hazard,” which include (i) multi-tasking, (ii) incongruent objectives, and (iii) career concerns. Bearing this in mind, we adopt the overly-simplified (but standard) “work”/“shirk” nomenclature below (see Remark 1 for further discussion).

\(^3\)Recent examples of extensive data falsification and alleged falsification are the respective cases of Diederik Stapel, formerly a professor of social psychology at Tilburg University, and Anil Potti, formerly a cancer
contracted upon. On the other hand, *intangible* progress is privately observed by the agent and hence, can only be contracted upon indirectly through unverifiable progress reports from the agent.

To build intuition and a baseline of comparison for our results, we start by analyzing a single-stage project in which only one tangible breakthrough is needed to complete the project. In this case, the optimal incentive scheme can be implemented with a *simple contract*, which involves a single deadline $T^*$ and a reward that depends only on the date at which the breakthrough arrives. If the agent realizes the breakthrough at $\tau \leq T^*$, he collects the reward of $R(\tau)$. If a breakthrough is not realized by the deadline, the principal terminates the project. While the first-best policy involves no termination, the use of a deadline plays a crucial role in the provision of incentives. In setting the optimal deadline the principal faces a trade off; a longer deadline can deliver a higher probability of project completion but also requires giving the agent more rents in order to prevent shirking.

Having established the benchmark, we incorporate a second stage to the project. There is a preliminary (e.g., research) stage, which must be successfully completed before moving on to the ultimate (e.g., development) stage. A breakthrough in the preliminary stage does not generate any direct benefits and may or may not be tangible, whereas when a breakthrough in the second stage occurs the project benefits are tangibly realized. In the multistage setting, simple contracts are suboptimal. If the principal uses a simple contract then, as the deadline approaches, an agent who has not yet made a breakthrough will “run out of steam” and begin shirking, which is inefficient. Thus, a simple contract can be improved upon. However, the way in which improvements can be made depends on the nature of progress.

When progress is tangible, the principal can improve on a simple contract by incorporating a preliminary deadline for completing the first stage. Thus, the principal provides incentives to the agent through three separate channels. The first channel is the first-stage deadline; if the agent has not made a breakthrough by $T_1$, then the principal discontinues investment and the project is terminated. The second channel is a deadline for completing the second stage that weakly decreases with the amount of time it takes the agent to make his first breakthrough. That is, the earlier the agent makes the first breakthrough, the more time he gets to make the second one. However, the second clock need not run continuously. We derive precise conditions under which it is optimal for the principal to pause the second clock during the first stage in order to preserve the likelihood of ultimate success, while maintaining strong incentives in the first stage. We also show that conditional on making the first breakthrough, the expected amount of time that the agent is afforded in the second stage is strictly greater than $T_1$. Therefore, the optimal contract can be interpreted as putting the researcher at Duke University. Both Stapel’s misconduct and Potti’s alleged misconduct went undetected by their employers, funding agencies, and professions for a number of years.
agent on a “short leash” until progress is made at which point, he is given a longer leash to complete the second stage. The third channel is a monetary reward schedule (or equity stake) paid to the agent only upon the ultimate success of the project. The reward schedule is also (weakly) decreasing in both the time of the preliminary breakthrough $\tau_1$ and the time of the ultimate breakthrough $\tau_2$.

When progress is intangible, the revelation principle implies that it is without loss to focus on contracts in which the agent reports breakthroughs truthfully and immediately. An important observation is that the optimal contract with tangible progress will not induce truthful reporting when progress is intangible. To see why, suppose the agent has not yet had a breakthrough and time is nearing $T_1$. Rather than being fired at $T_1$, the agent strictly prefers to “falsely report” a breakthrough to extend his clock and, by doing so, obtain a payoff equal to the value of shirking for the rest of his tenure. Similarly, “hiding” a breakthrough and shirking until $T_1$ is the agent’s best response if the second clock is paused, because he gets the same continuation value upon reporting the breakthrough at $T_1$, while enjoying private benefits prior to making the report. To incentivize truth telling therefore requires the contract to satisfy two additional constraints, which we refer to as the no-false-progress and no-hidden-progress constraints.

The optimal contract with intangible progress requires that the principal use a first-stage deadline that is not deterministic. Instead, the principal uses random termination: funding is guaranteed up to a certain date, which we refer to as a soft deadline, after which a probationary phase ensues in which the principal randomizes over whether or not to terminate the project. Probation ends when the agent reports progress, at which point he is given a relatively short amount of time to complete the project. Random termination provide incentives for the agent not to shirk or falsely report in the first stage, while simultaneously not reducing the probability of success in the second stage conditional on making a breakthrough before the soft deadline expires.

Perhaps our most striking finding is that self-reported progress from the agent to the principal regarding the status of the project plays an essential role in implementing the optimal contract. This is despite the facts that the agent’s information has no social value and that his reports cannot be substantiated. Interestingly, the required communication is non-stationary over the duration of the project. Early in the life of the project, communication is unnecessary. If the agent makes a breakthrough, he need not report it immediately. Instead, the principal simply requires that the agent submit a report by the soft deadline as to whether he has made a breakthrough. If the agent reports “yes,” then he is given a (relatively short) hard future deadline and faces a decreasing reward scheme. If the agent reports “no,” then he is put on probation and the principal remains in constant communication with him regarding the project status. That is, after the soft deadline, the principal requires the agent
to report progress *immediately* (unlike before the soft deadline) in order to avoid suboptimal termination. In addition, the short-leash long-leash deadline structure that is optimal under tangible progress is reversed when progress is intangible.

The nature of progress also has important welfare implications. Keeping all other aspects fixed, projects with tangible progress deliver higher payoffs to the principal, are more likely to eventually succeed, and generate higher total welfare. On the other hand, the agent’s equilibrium payoff under the (principal’s) optimal contract is higher when progress is intangible. One implication is that projects with tangible progress are more likely to be to be funded than those with intangible progress even if the expected benefits are somewhat lower.

We explore three variations of the model with implications for optimal project design. We first consider the case in which progress is unobservable to both players in order to see whether the principal can benefit from suppressing the agent’s access to information about the status of the project. We find that information suppression is suboptimal; the principal does better under the optimal contract with intangible progress than in a setting where neither party can observe progress. Second, we ask whether there is scope for making communication costly. When progress is intangible, we show that the principal can indeed benefit by imposing a small cost to the agent of reporting progress. Therefore, formal channels of communication that require time and effort (e.g., paperwork) can be useful even if the same information could be communicated at no cost. Finally, we consider a project with asymmetric stages and show that *ceteris paribus* the principal’s payoff is higher when the first stage is somewhat moderately more difficult than the second stage. However, the principal does strictly worse by making the first stage too difficult relative to the second.

We believe our findings are not only of theoretical interest, but also have empirical significance. For instance, when summarizing the empirical evidence on venture capital covenants (e.g., Gompers and Lerner (1999, 2001)), Tirole (2006, p. 91) writes, “Venture capital deals usually include:

- A very detailed outline of the stages of financing (e.g., seed investment, prototype testing, early development, growth stage, etc.), At each stage the firm is given just enough cash to reach the next stage.
- The right for the venture capitalist to unilaterally stop funding at any stage...
- The right for the venture capitalist to demote or fire the managers if some key investment objective is not met...
- The right to control future financing.”

We demonstrate that these features arise naturally as elements of an optimal contract for a multistage project with tangible progress. Moreover the dynamics of the optimal reward
schedule for the agent (increasing with progress and decreasing with lack thereof) accord neatly with the findings of Kaplan and Strömberg (2003) concerning the financial stake entrepreneurs retain in the ventures they operate contingent on their performance. Also, the short-leash long-leash deadline structure mentioned above is reminiscent of the staged-financing contracts commonly observed in venture capital in which the amount of funding tends to increase conditional on meeting milestones.

Our most novel findings concern the optimal incentive scheme when progress is intangible. In particular, even in complex environments where it may be difficult or impossible to assess directly whether milestones have been achieved, staged financing nevertheless can and should be used. Specifically it is optimal to make continuation funding contingent on self-reported progress from the expert running the project. This is true even if the funding entity is not able to evaluate the veracity of the reports. While we are not aware of any extant empirical work investigating standard terms appearing in research grants and awards, making continued funding contingent on periodic progress reports appears to be a common arrangement. For example, the following text that appears on the website of the Amyotrophic Lateral Sclerosis (ALS) Association is broadly representative.

The ALS Association financial officer makes grant award payments to the Principal Investigator institution for disbursement for the project. Payments are made on a specified quarterly schedule and in the case of multi-year grants, after the first year, are contingent upon the receipt by ALS Association of satisfactory annual progress reports and documentation of research funds expended.

Regarding the use of probation, grant awarding entities do not, of course, explicitly specify random termination. As a matter of implementation, however, grant policies often are quite vague about the consequences for delays in reporting progress. For instance, the National Institute of Health’s (NIH) web site states “If your [progress] report is extremely late, you risk losing funding for the period of time between the end of the current budget period and when we finish processing your report.”4 Similarly, the National Science Foundations (NSF) Grant Policy Manual states, “NSF reserves the right, ... to withhold future payments after a specified date if the recipient fails to comply with the conditions of an NSF grant, including the reporting requirements.”5 While these policies do not specify random termination as such, it certainly seems more appropriate to view the indefinite penalties for late reporting as involving soft deadlines rather than hard ones.

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1.1 Related Literature

There is a large and growing literature studying the optimal provision of incentives in dynamic environments.\(^6\) Our benchmark single-stage model is similar to Hopenhayn and Nicolini (1997),\(^7\) who looks at providing incentives to search for employment while simultaneously providing unemployment insurance, and Mason and Välimäki (2015) who consider a dynamic moral hazard setting in which the principal must provide incentives to the agent to complete a project. In our model, it is optimal to use termination deadlines to provide incentives to the agent, whereas in their setting, the use of termination is strictly suboptimal.\(^8\) A novel aspect of our model is that we explicitly consider environments with multiple sequential stages in which progress by the agent could be private information. Thus, we “sandwich” a hidden-information problem between two stages of a hidden-action problem.

Our multistage environment with tangible progress is related to Biais et al. (2010), who analyze a model in which large (observable) losses may arrive via a Poisson process, and an agent must exert unobservable effort in order to minimize the likelihood of their arrival. They allow for investment and characterize firm dynamics as well as asymptotic properties. Our model differs in that it (i) features only a finite number of arrivals, (ii) the arrival of a breakthrough is “good news”, and (iii) we consider the case in which arrivals are unobservable to the principal. Several other recent papers that involve observable Poisson arrivals include Hoffmann and Pfeil (2010), Piskorski and Tchistyi (2011), DeMarzo et al. (2014). Given the multistage setting, a key difference in this paper is that the agent’s continuation utility is not a sufficient state variable. Toxvaerd (2006) considers a setting in which a finite number of (observable) arrivals are needed in order to complete a project. In his setting, the agent is risk averse and the optimal contract trades off optimal risk-sharing for incentive provision, but does not involve deadlines or inefficient termination.

When progress is intangible, the agent has access to private information that is persistent. Dynamic contracting with persistent private information has been studied in discrete type settings by Fernandes and Phelan (2000), Battaglini (2005), Tchistyi (2013), and with a continuum of types using a first order approach by Williams (2011) and Edmans et al. (2012). Our approach is most similar to Zhang (2009) and Guo and Hörner (2015). From a theoretical perspective, our work differs from this literature along several dimensions. One

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\(^7\)See also Lewis (2012) who considers a delegated search model in which the optimal contract includes a deadline and a bonus for early completion.

\(^8\)Though the problems are similar in spirit, the optimal contract in our one-stage benchmark looks quite different from Mason and Välimäki (2015). This difference arises due to the nature of the moral hazard problem we consider (e.g., private benefit from shirking) rather than their costly effort model. See Remark 1 for further discussion.
key difference in our environment is the presence of public (contractible) information (i.e., the ultimate success of the project), which can be used to screen the agent’s underlying private information. Another difference is that the transition probabilities across states are endogenously determined in our setting by the agent’s action.

In a contemporaneous working paper, Hu (2014) considers a setting similar to our model with intangible progress. His model takes place in discrete time where shirking is not socially inefficient, and he restricts attention to mechanisms with a deterministic deadline. Our results show that this restriction is not without loss of generality and leads to substantively different findings. For instance, we prove that the optimal mechanism (i) does not involve shirking, (ii) requires communication and (iii) uses random termination. Restricting attention to a single deterministic deadline leads Hu (2014) to precisely the opposite conclusions. In more general settings, optimal dynamic mechanisms are explored by Board (2007), Eso and Szentes (2007), Bergemann and Valimaki (2010) and Pavan et al. (2014) among others.

There is a rich existing literature exploring settings where parties learn about the value of a project over time. In these environments, lack of success typically indicates that the project is bad, and it is socially efficient to discontinue investment at some point. One common finding within this literature is that agency considerations may cause the principal to terminate the project earlier than socially optimal. By contrast, we focus on a setting in which the project is commonly known to be good from the outset (i.e., there is no learning or social value of information) in order to isolate the extent to which progress can be used to provide stronger incentives. Bonatti and Hörner (2011) study experimentation in teams for a project that requires a single breakthrough, which introduces a free-riding problem. They show that the equilibria of the game involve inefficient delays in effort provision and that deadlines, which terminate the project prior to the socially efficient time, are useful in mitigating delays despite forfeiting value when the deadline is reached. The free-riding problem also arises in Moroni (2015), who studies experimentation with multiple agents for a project that requires several (tangible) breakthroughs. The optimal contract in her setting exhibits some qualitatively similar features to our setting with tangible progress. For example, it is optimal to reward early success in the first stage with better terms in the second.

Holmstrom and Milgrom (1991) and Laux (2001) look at settings with simultaneous tasks whereas in our setting the project involves stages that must be completed sequentially. Lerner and Malmendier (2010) investigate the role of property rights and contractibility in the design of research agreements within a multi-task setting and empirically document more prevalent use of termination options in settings where effort is not contractible. Varas (2014) studies a dynamic multi-tasking model in which an agent can complete the project faster by reducing

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its quality. Like us, he finds that a random termination policy may be optimal. However, the underlying mechanisms are somewhat different. In his model, stochastic termination is used to prevent multi-tasking and requires the agent to be relatively more impatient: if the principal and agent are equally patient, the optimal contract in his model resembles our single stage benchmark. Moreover, the use of a deterministic deadline is never optimal in his model unlike in our multistage setting where the principal may use a combination of random and deterministic deadlines.

The rest of the paper proceeds as follows. In Section 2, we present a single-stage version of the model as a benchmark. We introduce the second stage and present some preliminary findings in Section 3. Our main results are located in Sections 4 through 7. We derive the optimal contracts under tangible and intangible progress in Sections 4 and 5 respectively and compare them in Section 6. Section 7 explores the possibility of restricting the agent’s access to information, the role of costly reporting, and projects with asymmetric stages. Concluding remarks appear in Section 8. Most proofs and several technical lemmas are located in the Appendix.

2 Benchmark model: a single-stage project

A principal (she) contracts with an agent (he) to undertake a project. Time is continuous and the project can be operated over a potentially infinite horizon. The project requires the successful completion of a stage, also referred to as a breakdown, in order for its benefits to be realized. Operating the project requires resources, which we model as a flow cost (or “burn rate”) $c$ per unit time that the project is in operation. The principal has unlimited resources to fund the project. The agent has no funds and is protected by limited liability, but has the skills necessary to run the project. Both parties are risk-neutral and do not discount the future. The principal can terminate the project (i.e., discontinue paying the flow cost) at any point in time. Project termination is irreversible; if the principal terminates the project prior to the breakthrough, the project delivers no benefit to the principal, and the game ends.

While the project is in operation, the agent chooses an action $a_t \in \{0, 1\}$, where $a_t = 1$ indicates that the agent appropriately invests or “works,” and $a_t = 0$ indicates that the agent diverts funds or “shirks” for private benefit. The arrival rate of a breakthrough is then given by $\lambda a_t$. Thus, if the agent works over an interval of length $dt$, then the probability of a breakthrough in the interval is $\lambda dt$. If the agent shirks, then the arrival rate of a breakthrough is zero but he receives a private flow benefit of $\phi dt$, where $\phi > 0$ measures the severity of the
agency problem. The agent receives no intrinsic benefit from project success; he benefits solely from the compensation delivered by the principal and any private benefits from shirking.

**Remark 1.** For many relevant applications, the most natural interpretation of the moral hazard problem is that the agent can secretly divert the principal’s investment for private benefit. For example, an entrepreneur can use venture capital funding for private consumption, or a scientist may fund a pet project not authorized under his current grant. Nevertheless, we will adopt the standard “shirk”/“work” terminology (e.g., Tirole, 2006).\(^\text{11}\)

The success of the project is publicly observed and contractible. Upon the arrival of a breakthrough the principal realizes a payoff \(\Pi > 0\), makes any outstanding contractual payments to the agent and the game ends.\(^\text{12}\) Let \(\tau\) denote the random variable representing the date of project success. Throughout our analysis, we employ the following assumptions.

**Assumption 1.** The expected value of the project (absent agency costs) is strictly positive

\[
\Pi - \frac{c}{\lambda} > 0.
\]

**Assumption 2.** Shirking is non-trivial and inefficient

\[
0 < \phi \leq c.
\]

**Remark 2.** Assumption 1 and 2 imply that under the first-best policy, the agent never shirks and the project is never terminated prior to success.

At \(t = 0\), the principal offers the agent a contract. We assume that the principal can fully commit to all terms. If the agent rejects the offer, then both parties receive their outside options normalized to zero. A contract is a triple, denoted by \(\Gamma = \{a, Y, T\}\), where \(a_t\) is the recommended action to the agent at time \(t\), \(dY_t\) is a monetary payment made to the agent at \(t\) which can be conditioned on project success, and \(T\) is the date at which the project is terminated.\(^\text{13}\) An action process, \(a\), induces a probability distribution \(\mathbb{P}^a\) over \(\tau\). Let \(\mathbb{E}^a\) denote the corresponding expectation operator. Given any contract, the principal’s (ex-ante)

\(^{11}\)In the analog of our model where the agent incurs a cost of effort rather than a benefit from diversion (i.e., arrival rate of breakthroughs is zero without effort), the limited liability constraint has no bite and the first-best outcome is attainable. In a costly-effort model with discounting and a strictly positive arrival rate even when the agent shirks, it is possible to obtain results similar to those presented in this paper under certain parametric restrictions.

\(^{12}\)It is without loss to assume that any remaining payments are made upon the arrival of success.

\(^{13}\)In the multistage setting, the solution to the principal’s problem may require randomization over termination dates. However, randomization is unnecessary for a single-stage project so we delay introducing this possibility and additional notation until it is needed.
expected utility is given by
\[
\mathcal{P}_0(\Gamma) = \mathbb{E}^a \left[ \Pi \cdot \mathbb{1}_{\{\tau \leq T\}} - \int_0^{T \wedge \tau} \{cdt + dY_t\} \right],
\]  
\hspace{1cm} (1)

and the agent’s expected utility is given by
\[
U_0(\Gamma) = \mathbb{E}^a \left[ \int_0^{T \wedge \tau} \{g(a_t)dt + dY_t\} \right],
\]  
\hspace{1cm} (2)

where \(g(a) \equiv \phi(1 - a)\). The contract \(\Gamma\) is said to be incentive compatible if \(a\) maximizes the agent’s expected utility \((2)\) given \((Y,T)\). The principal’s problem is to find an incentive compatible contract that maximizes \((1)\) subject to delivering an expected utility to the agent of at least his outside option.

Note that because the principal and the agent have linear utility and are equally patient, it is without loss of generality to backload all monetary payments to the agent (e.g., Ray, 2002). Therefore, let \(R_t = dY_t \mathbb{1}_{\{t = \tau\}}\) denote the reward to the agent for success at time \(t\). As is typical of most principal-agent models, making a payment to the agent upon “failure” (i.e., termination prior to project success) is suboptimal.

Given any contract \(\Gamma\), let \(U_t(\Gamma) \in \mathbb{R}_+\), denote the agent’s continuation value after any non-terminal history, i.e.,
\[
U_t(\Gamma) = \sup_a \mathbb{E}^a \left[ \int_t^{T \wedge \tau} \{g(a_s)ds + R_s \mathbb{1}_{\{s = \tau\}}\} \mid t \leq \tau \right].
\]  
\hspace{1cm} (3)

Assuming that \(U_t \in C^1\) for \(t \in [0, T]\) (which will be verified later), the Hamilton-Jacobi-Bellman (HJB) equation for the agent’s problem is given by
\[
0 = U'_t + \sup_{a_t} \{g(a_t) + \lambda a_t (R_t - U_t)\}. 
\]  
\hspace{1cm} (4)

If the agent works over an interval \(dt\), then he makes a breakthrough with probability \(\lambda dt\) and gets the reward \(R_t\). However, by doing so, he forgoes the private benefit of shirking \(g(0) = \phi dt\) as well as the continuation utility he would get if the breakthrough did not arrive. Thus, in order to give the agent incentives to work the principal must reward the agent with additional utility of at least \(\phi / \lambda\) upon arrival of success. The incentive compatibility condition can therefore be summarized by the following lemma.

Lemma 2.1. Given any contract \(\Gamma\), the optimal action for the agent at time \(t\) is
\[
a_t = 1 \iff R_t \geq U_t(\Gamma) + \frac{\phi}{\lambda}. 
\]  
\hspace{1cm} (5)
Let $I$ denote the set of all contracts such that (5) holds for all $t \in [0, T]$. The principal’s objective is to maximize $P_0(\Gamma)$ subject to $\Gamma \in I$. Following standard arguments (e.g., Spear and Srivastava, 1987), the principal’s problem can be formulated recursively, where the state variable is the promised utility to the agent, denoted by $u \in \mathbb{R}_+$. Let $V(u)$ denote the principal’s value function, which solves

$$V(u) = \sup_{\Gamma \in I} P_0(\Gamma),$$

subject to the additional “promise keeping” condition

$$U_0(\Gamma) = u. \quad (7)$$

Notice that $U_t(\Gamma)$ is strictly positive for any $t < T$, therefore if $u = 0$, then the only possible solution is for the project to be terminated immediately and hence $V(0) = 0$. For $u > 0$, the HJB equation for the principal’s problem is

$$0 = \max_{R,a} \left\{ \lambda a (\Pi - R - V(u)) - c + V'(u) \frac{du}{dt} \right\}$$

s.t. $$\frac{du}{dt} = -\max_{a} \{g(a) + \lambda a (R - u)\}.$$ 

Clearly, the solution to the principal’s problem must involve $V'(u) \geq -1$ since the principal has the option to make direct payments. Hence, it is without loss to focus on contracts such that $a = 1.\textsuperscript{14}$ The first-order condition then requires that any solution to the HJB involves $R = u + \frac{\phi}{\lambda}$ (i.e., the incentive compatibility condition binds), in which case the agent’s continuation value decreases at a constant rate $\phi$ prior to termination or success and the principal’s value function satisfies the ordinary differential equation

$$\lambda V(u) = \lambda \left( \Pi - u - \frac{\phi}{\lambda} \right) - c - \phi V'(u).$$

Using the boundary condition at $u = 0$ to pin down the constant, we arrive at the following candidate for the principals’ value function

$$\bar{V}(u) \equiv \left( \Pi - \frac{c}{\lambda} - u \right) - \underbrace{\left( \Pi - \frac{c}{\lambda} \right)}_{\text{Agency cost}} e^{-\frac{\lambda u}{\phi}}. \quad (8)$$

The first term on the right side represents the first best value of the project net of delivering

\textsuperscript{14}If $a = 0$, the HJB is satisfied if and only if $V'(u) = -c/\phi \leq -1$ in which case a direct payment is a more efficient form of compensation than letting the agent shirk.
Proposition 2.2. The principal’s value function solving (6)-(7) is given by \( \bar{V}(u) \). Given any level of agent utility \( u \in \mathbb{R}_+ \), this payoff can be attained under the contract with a deadline

\[
T^*(u) = \frac{u}{\phi},
\]

and a reward payment for success that is decreasing over time according to

\[
R^*(\tau) = \phi \left( \frac{1}{\lambda} + T^*(u) - \tau \right), \quad \forall \tau \in [0, T^*(u)].
\]

The only thing left to pin down is the initial utility level for the agent, which is equivalent to the optimal termination date of the contract. Naturally, the division of surplus will depend on the relative bargaining power and outside options of each player. To fix ideas, throughout the paper we endow the principal with all of the bargaining power and set the agent’s outside option to zero. We refer to the \textit{optimal contract}, as the contract that maximizes the principal’s payoff over all \( u \in \mathbb{R}_+ \).\footnote{Henceforth, the statement “for all (any) \( u \)” refers to all (any) \( u \in \mathbb{R}_+ \).} Therefore, let

\[
u^* \equiv \arg \max_{u \in \mathbb{R}_+} \bar{V}(u).
\]

We call a project \textit{feasible} if the optimal contract does not involve immediate termination.

Corollary 2.3. A single-stage project is feasible if and only if

\[
\lambda \Pi - c > \phi. \tag{C.1}
\]

If (C.1) holds, then the optimal contract has a deadline \( T^* \equiv \frac{1}{\lambda} \ln \left( \frac{\lambda \Pi - c}{\phi} \right) \).

The parametric restriction in (C.1) ensures that \( T^* > 0 \). Intuitively, the principal must give the agent \( \phi \) per unit time in continuation value in order to prevent shirking. The condition (C.1) therefore ensures that the principal’s expected flow benefit, \( \lambda \Pi \), outweighs her total flow cost of operating the project and inducing effort \((c + \phi)\). Notice that this inequality is stronger than would be needed in a first-best situation where the principal ran the project herself, namely \( \lambda \Pi - c > 0 \), in which case, as noted in Remark 2, the first-best policy is to invest indefinitely until the innovation arrives. Also notice that the optimal deadline increases as the agency problem becomes less severe, with \( \lim_{\phi \to 0} T^* = \infty \). That is,

\footnote{Notice that \( \bar{V}(u) \) is strictly concave, which confirms that randomization over the termination deadline is suboptimal and \( \bar{V}'(u) \geq -1 \), which confirms incentive compatibility binds.}
the (second-best) outcome and principal payoffs converge to first-best as the agency conflict goes to zero. Hence, the deadline $T^*$ exists only to mitigate agency costs.

3 Two-stage projects

Having derived the optimal contract for a single-stage project in the previous section, we now introduce an additional stage. Henceforth, the agent must make two breakthroughs in order for the principal to realize the project benefits. To simplify exposition, we assume that the parameters $(\phi, c, \lambda)$ are the same for both stages.\footnote{We analyze projects with asymmetric stages in Subsection 7.3.} Analogous to Assumption 1, We assume that the expected value of the two-stage project is strictly positive; i.e., $\Pi - 2c/\lambda > 0$. Note that this assumption does not imply that the (two-stage) project is feasible. Indeed part of our interest is in precisely characterizing the conditions under which the principal can profitably undertake multistage projects.

We let $\tau_1$ and $\tau_2$ denote the (random) times at which the first and second breakthrough occur. We distinguish between the first stage of the project, $t \in [0, \tau_1)$, and the second stage, $t \in [\tau_1, \tau_2)$. As before, we assume that the date at which the project ultimately succeeds, now denoted $\tau_2$, is publicly observed and can be contracted upon. In most of our analysis, we assume that intermediate progress is observed by the agent and focus our attention on the following environments.

**Definition 3.1 (Tangibility of Progress).** We say that progress is **tangible** if $\tau_1$ is publicly observed and can be directly contracted upon at no cost to either party. We say that progress is **intangible** if $\tau_1$ is privately observed by the agent and can only be contracted upon indirectly via unverifiable “progress reports” from the agent.

A priori, it is not clear that being able to condition on unverifiable reports from the agent has any value to the principal. Because the information itself has no social value, it can only be beneficial to the principal if it can be used to mitigate the agent’s rents. Of course, in order to induce truthful reporting the agent must be given appropriate incentives. So, the question becomes whether the principal can elicit information from the agent in order to reduce his rents. When posed this way, two of our main results may seem surprising. In particular, we will show that conditioning on unverifiable progress reports by the agent is in fact a crucial part of the optimal contract when progress is intangible.

In Section 3.1, we illustrate how the simple class of contracts used in the single-stage project can be improved upon in a multistage setting. We then derive the optimal contract under tangible progress in Section 4 and under intangible progress in Section 5.
3.1 Simple contracts

In a single-stage project, the principal can implement the optimal contract with a single (deterministic) deadline, $T$, and reward scheme that depends only on the project completion date. We refer to such contracts as *simple contracts*. Notice that simple contracts preclude contracting on progress, either directly (when it is tangible) or indirectly through communication with the agent (when it is intangible). This leads to the undesirable feature that the agent will begin shirking as the deadline approaches if he has not made progress.

**Proposition 3.2.** For a two-stage project and any simple contract with a bounded reward scheme, there exists a $\bar{t} < T$ such that if the agent has not made the first breakthrough by $\bar{t}$, he will shirk for all $t > \bar{t}$.

*Proof.* The probability of completing the project by $T$ given that the first breakthrough has not been made by $\bar{t}$ is

$$\Pr(\tau_2 \leq T | \tau_1 > \bar{t}) = 1 - e^{\lambda(T-\bar{t})} (1 + \lambda(T - \bar{t})),$$

and as $(T - \bar{t}) \to 0$, the above expression, and hence the benefit to the agent of working, converges to zero at a rate proportional to $(T - \bar{t})^2$, whereas the benefit of shirking is proportional to $(T - \bar{t})$, implying that for $t$ close enough to $T$, the agent will prefer to shirk unless the reward for success is arbitrarily large. $\square$

A corollary of Proposition 3.2 is that simple contracts are not the most efficient way to provide incentives in a multistage setting regardless of the nature of progress. Yet, the way in which a simple contract can be improved upon depends on the nature of progress. When progress is tangible, the principal can improve upon a simple contract by using a first-stage deadline, whereby the project is terminated if the agent has not made progress by $\bar{t}$ (rather than permit him to shirk until $T$). Firing an agent who has not had a breakthrough by $\bar{t}$ saves the principal $c(T - \bar{t})$ in resources and provides stronger incentives since the return to working is unchanged and the return to shirking is diminished.

When progress is intangible, the principal cannot simply terminate the project if the agent has not reported a breakthrough by $\bar{t}$ because doing so would induce him to make false claims of progress and then shirk. Nevertheless, simple contracts can still be improved upon with communication. To see how, suppose that upon reaching $\bar{t}$, the principal asks the agent “have you made a breakthrough yet?” To induce the agent to report truthfully, the principal promises to make a “severance” payment to the agent of $P = \phi(T - \bar{t})$ and terminate the project if he answers “no” and continue funding the project until $T$ with the same reward scheme if he answers “yes.” This arrangement clearly induces truthful reporting for an agent who has not made a breakthrough. It also induces truthful reporting for an agent.
who has made a breakthrough provided that following a breakthrough working is incentive compatible under the original contract. Moreover, adding a severance payment to the original contract does not change the probability of ultimate success because the agent who reports “no” and takes severance would have otherwise shirked from $\tilde{t}$ on. This scheme saves the principal $(c - \phi)(T - \tilde{t})$ conditional on the agent reporting “no.” Thus, this simple form of communication with the agent weakly improves the principal’s expected payoff (and strictly if $\phi < c$). Of course, it need not be (and in general will not be) the case that making a severance payment is optimal as doing so is costly to the principal, and as we will see in Section 5, there are more effective ways of providing incentives.

4 The optimal contract with tangible progress

In this section we assume that progress is tangible. That is, the times of both the first and second breakthroughs are publicly observable and can be contracted upon. As before, a contract consists of $\Gamma = \{a, Y, T\}$, where each of the elements can depend on the history as well as a public randomization device. Formally, we let $\mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0}$ denote the filtration generated by $(\tau_1, \tau_2)$. Under any contract that does not employ randomization, $a$ is an $\mathcal{F}$-predictable process, $Y$ is a non-decreasing, $\mathcal{F}$-adapted process and $T$ is an $\mathcal{F}$-adapted stopping rule.

Since the project stage is publicly observable, the principal’s problem can be formulated recursively using the state variable, $(s, u) \in \{1, 2\} \times \mathbb{R}_+$, which can be interpreted as the set of histories such that the current stage is $s$ and the agents promised utility is $u$. Conditional on the state variable, we let $V_s(u)$ denote the principal’s value function in state $(s, u)$. That is, $V_s(u)$ is the maximal expected payoff that the principal can achieve starting in stage $s$ subject to incentive compatibility and delivering $u$ in promised utility to the agent.

We solve the principal’s problem by backward induction on $s$. Given any level of promised utility to the agent, the maximal payoff the principal can achieve in the second stage is the same as that from the single-stage problem. Specifically, it involves (i) $R(u) = u + \phi/\lambda$ and (ii) $\frac{du}{dt} = -\phi$, until $u = 0$ at which point the project is terminated. Therefore, $V_2(u) = \bar{V}(u)$ (from equation (8)) and conditional on reaching the state $(2, u)$, the principal’s continuation payoff under the optimal contract is $V_2(u)$.

Moving back to the first stage, the key question facing the principal is how much to reward the agent (in continuation value) for making a breakthrough at each $t < T$, which we

\footnote{It cannot be larger than $V_2(u)$ and satisfy promise keeping. Moreover, any contract under which it is strictly less can be improved upon without affecting the agent’s incentives.}
denote by $W_t$. The principal’s problem can be written as

$$\sup_{\Gamma} \mathbb{E}^{a} \left[ \int_{0}^{T \wedge \tau_1} \{ V_2(W_t) \mathbb{1}_{\{ t = \tau_1 \}} - (cdt + dY_t) \} \bigg| s = 1 \right],$$

subject to incentive compatibility

$$a \in \arg\max_{\tilde{a}} \mathbb{E}^{\tilde{a}} \left[ W_{\tau_1} \mathbb{1}_{\{ \tau_1 \leq T \}} + \int_{0}^{T \wedge \tau_1} \{ g(a_t) dt + dY_t \} \bigg| s = 1 \right].$$

Let $V_1(u)$ denote the principal’s first-stage value function. Using arguments analogous to those in the single-stage case, it can easily be shown that the solution to the principal’s first-stage problem involves no shirking ($a_t = 1$ for all $t < T$) and that providing incentives not to shirk in the first stage requires that

$$W_t \geq U_t + \phi/\lambda.$$ 

We will proceed by assuming the principal uses a deterministic contract and then verify the resulting value function is concave. Under a deterministic contract in which the agent never shirks, the promise-keeping condition is given by $0 = U_t' + \lambda(W_t - U_t)$. Given the time-invariance of the problem, the principal’s HJB in the first stage can be written as

$$0 = \sup_w \left\{ \lambda(V_2(w) - V_1(u)) - c + V_1'(u) \frac{du}{dt} \right\}$$

subject to

$$w \geq u + \phi/\lambda,$$  \hspace{1cm} (IC1)

$$\frac{du}{dt} = -\lambda(w - u),$$ \hspace{1cm} (PK1)

$$V_1(0) = 0.$$ \hspace{1cm} (BC1)

The principal’s tradeoff can be summarized as follows. Higher promised utility for making a breakthrough (higher $w$), corresponds to giving the agent more time in the second stage and hence a higher probability of ultimate success conditional on reaching the second stage. However, a higher $w$ also causes the agent’s continuation utility to decrease faster, which implies less time to make a breakthrough in the first stage and hence a lower probability of reaching the second stage.

Based on the results for the single-stage project, one might expect that the incentive compatibility condition also binds everywhere in the first stage of a multistage project. As we will see, this is not necessarily true. Nevertheless, it is a useful starting point. Therefore,
consider any interval over which (IC1) binds. In this case, $V_1$ satisfies

$$\lambda V_1(u) = \lambda V_2(u + \phi/\lambda) - c - \phi V'_1(u),$$

which has a solution of the form

$$ \left( \Pi - \frac{2c}{\lambda} - u \right) - \frac{\lambda u}{\phi} \left( \Pi - \frac{c}{\lambda} \right) e^{-(1+\lambda u/\phi)} + K_1 e^{-\lambda u/\phi}. $$

Let $V_{1c}$ denote the candidate value function in which (IC1) binds for all $u$. The terminal boundary condition (BC1) implies that $K_1 = 2c/\lambda - b$. And thus we have that

$$ V_{1c}(u) \equiv \left( \Pi - \frac{2c}{\lambda} - u \right) - \left( \Pi - \frac{2c}{\lambda} \right) e^{-\lambda u/\phi} - \frac{\lambda u}{\phi} \left( \Pi - \frac{c}{\lambda} \right) e^{-(1+\lambda u/\phi)}. $$

Similar to the value function in equation (8), the first term on the right-hand side is the first-best value of the project net of delivering $u$ to the agent, while the remaining two terms represent the (expected) agency cost associated with project termination in the first and second stage respectively. Given the closed form expression for the candidate value function, it is easy to check for its concavity.

**Lemma 4.1.** $V_{1c}$ is concave for all $u \in \mathbb{R}^+$ if and only if

$$ \Pi \geq \frac{2c}{\lambda} \left( \frac{e - 1}{e - 2} \right). $$

Note that this condition does not depend on the degree of the agency conflict ($\phi$); it simply says that the ratio of the project’s benefit to the expected costs is sufficiently large. The next result shows that when (C.2) holds, the candidate value function indeed characterizes the solution to the principal’s problem.

**Proposition 4.2.** If (C.2) holds then the principal’s value function in the first stage is given by $V_1 = V_{1c}$, and under the optimal policy (IC1) binds for all $u \in \mathbb{R}^+$.

Figure 1 illustrates the value function and optimal dynamics. The agent’s initial promised utility is $u_c \equiv \arg \max_{u \geq 0} V_{1c}(u)$ (i.e., the red asterisk in Figure 1).\(^{19}\) Prior to the first breakthrough, the agent’s continuation utility gradually decreases at rate $\phi$. If a breakthrough arrives prior to $u$ reaching zero, then the agent’s continuation utility jumps up by $\phi/\lambda$ and then gradually decreases over time until either the ultimate success is realized (and he is paid $u + \phi/\lambda$) or continuation utility reaches zero in the second stage.

\(^{19}\)That $V_{1c}$ is concave and $\lim_{u \to \infty} V'_{1c}(u) = -1$ ensures that $u_c \in \mathbb{R}^+$ exists. Also, it can easily be verified that (C.2) implies $u_c > 0$. 

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Figure 1: This figure illustrates the value function and optimal dynamics for the case when (C.2) holds. Starting from the red asterisk, the continuation values follow the left pointing arrows on the red line until a breakthrough occurs (e.g., at the black diamond) at which point they jump upward to the blue line as indicated by the solid arrow, or the origin is reached at which point the project is terminated.

If (C.2) does not hold, then \( V'_2(u + \phi/\lambda) > V_{1c}(u) \) for \( u \) sufficiently close to zero. Therefore, (IC1) does not bind; the principal can benefit by increasing the promised utility for success above the level required to induce effort. Doing so increases the chances of success conditional on reaching the second stage (recall that promised utility in the second stage is proportional to the amount of time the agent is given to complete the project). In this case, the principal’s value function is characterized by the solution to a free-boundary problem, where the boundary is the level of agent utility at which (IC1) becomes slack.

Proposition 4.3. If (C.2) does not hold then there exists \( u > 0 \) such that the principal’s value function in the first stage is given by

\[
V_1(u) = \begin{cases} 
\Pi - \frac{2c}{\lambda} - u - \left( \Pi - \frac{c}{\lambda} \right) \left( 2 + \frac{\lambda(u - u)}{\phi} \right) e^{-(1 + \lambda u/\phi)} & \text{if } u \in [0, u) \\
\Pi - \frac{2c}{\lambda} - u - \left( \Pi - \frac{c}{\lambda} \right) \left( 2 + \frac{\lambda(u - u)}{\phi} \right) e^{-(1 + \lambda u/\phi)} & \text{if } u \geq u.
\end{cases}
\]

The optimal first-stage policy is \( w = u + \phi/\lambda \) for \( u \in [0, u) \), and \( w = u + \phi/\lambda \) for \( u \geq u \).

Figure 2 illustrates the value function and optimal dynamics in the case that (C.2) does not hold. The agent’s promised utility starts at the blue asterisk and decreases gradually at rate \( \phi \) until reaching \( u \), at which point the continuation utility falls at a rate faster than \( \phi \). Conditional on realizing a breakthrough at some \( u \in (0, u] \), continuation utility jumps up to the same point, \( w(u) = w(u) = u + \phi/\lambda \), regardless of \( u \).
Figure 2: These figures illustrate the optimal dynamics when (C.2) does not hold. Starting from the asterisk in the left panel, the continuation values follow the arrows until either a breakthrough occurs at which point they jump upward as illustrated in the right panel, or the blue dot is reached. Following a breakthrough at any \( u \in (0, \bar{u}] \), the agent’s continuation value jump to the same point (i.e., \( \bar{w} \)) regardless of the time at which the breakthrough occurs.

4.1 Implementation with deadlines under tangible progress

First suppose \((C.2)\) is satisfied. To implement the optimal contract, the principal sets two deadlines, a short one and a long one. If the agent has not made a breakthrough by the short deadline, then the project is terminated. On the other hand, if he makes a breakthrough before the short deadline, then he gets until the long deadline to make the second breakthrough. Thus he is rewarded for an early first-stage breakthrough with more time to make a second-stage one, and he is rewarded by making an early second-stage breakthrough with a higher payment. The details are summarized in the following result.

**Proposition 4.4.** If \((C.2)\) holds then the project is feasible and the optimal contract can be implemented with a deadline to complete the first stage \( T^c_1 = u_c/\phi \), a deadline to complete the project \( T^c = \lambda + T^c_1 \) and a reward schedule \( R^c(\tau_2) = \phi \left( \frac{1}{\lambda} + T^c - \tau_2 \right) \) such that:

(i) If the agent is not successful in the first stage prior to \( T^c_1 \), then the project is terminated.

(ii) If the agent has a breakthrough in the first stage at \( \tau_1 < T^c_1 \), he has until time \( T^c \) to complete the project. If the agent does not complete the project prior to \( T^c \), then the project is terminated.

(iii) The agent is rewarded only if the project ultimately succeeds and in the amount \( R^c(\tau_2) \).\(^{20}\)

Note that \( T^c \) is the total amount of time to complete the project (not the additional amount of time to make the second breakthrough). Thus, the termination rule in Proposition 4.4 is

\(^{20}\)That compensation for success depends only on \( \tau_2 \) relies on the fact that the private benefit from shirking is identical across stages. More generally, if \( \phi \) differs across stages, then the reward for success depends on both \( \tau_1 \) and \( \tau_2 \).
equivalent to giving the agent a deadline for the project of $T^c_1$ that is extended by an amount of time $\frac{1}{\lambda}$ if the first breakthrough occurs prior to $T^c_1$.

Implementing the optimal contract when (C.2) does not hold is more delicate. One way is for the principal to set three clocks: (i) a short clock with deadline $T_S$, (ii) a medium clock with deadline $T_M > T_S$, and (iii) a long clock with deadline $T_L > T_M$. If the first breakthrough occurs at $\tau_1 < T_S$, then the agent receives the rest of the time on the long clock to make the second breakthrough. If the short clock expires before the agent makes the first breakthrough, then the long clock is stopped with $T_L - T_S$ remaining and restarted if the agent makes a breakthrough before the medium clock expires. If the agent does not make the first breakthrough before the medium clock expires, then the project is terminated.\footnote{It is also possible to implement the optimum by randomly terminating the project at a constant rate once the agent’s continuation value reaches $u$. Because randomization is not necessary, we focus on the deterministic implementation here and use the randomization device only when it is necessary (see Section 5).}

The crucial part of any implementation when (C.2) is violated is that the agent’s continuation payoff following a breakthrough is constant over $\tau_1 \in [T_S, T_M]$. By not reducing the agent’s continuation value over this interval, the principal preserves the amount of time the agent has to complete the second stage at the expense of giving the agent less time to complete the first stage. Intuitively, shifting resources away from the first stage and toward the second stage is beneficial because a first-stage breakthrough has no value to the principal without its counterpart.\footnote{In a model where each breakthrough is worth $\Pi$ to the principal, this resource shifting is never optimal (i.e., incentive compatibility always binds and the long clock is never paused).}

What is surprising is that shifting resources to the second stage is optimal only when the ultimate benefit from success is relatively small (i.e., when (C.2) fails).

We formalize the three-clock implementation in the following result.

**Proposition 4.5.** Suppose (C.2) does not hold. Then the project is feasible if and only if

$$\Pi - \frac{2c}{\lambda} > \phi \left( \frac{1}{\lambda} + T^* \right).$$

If (C.3) holds, then there exists $T_S < T_M < T_L$ such that the optimal contract can be implemented with a deadline in the first stage of $T^n_1 = T_M$, a deadline to complete the project that is conditional on the time of the first breakthrough $T^n(\tau_1) = T_L + \max\{0, \tau_1 - T_S\}$, and the reward schedule $R^n(\tau_1, \tau_2) = \phi \left( \frac{1}{\lambda} + T^n(\tau_1) - \tau_2 \right)$ as in (i)-(iii) of Proposition 4.4.

Thus, a fixed first-stage deadline remains part of the optimal contract, but the deadline for project completion and the agent’s compensation depend on the date at which the first breakthrough is made. Of course, the implementation in both Proposition 4.4 and 4.5 depends critically on the principal’s ability to condition on $\tau_1$. Next, we turn to projects with intangible progress in which conditioning directly on $\tau_1$ is not possible.
5 The optimal contract with intangible progress

Recall that intangible progress means the agent privately observes $\tau_1$ and therefore has (persistent) private information about the history of progress, while the date of the second breakthrough, $\tau_2$, remains publicly observable and contractible. Clearly, the principal cannot condition elements of the contract directly on $\tau_1$, rather, she can only condition on information communicated by the agent (and $\tau_2$). Given an arbitrary contract, the agent’s continuation value can depend on any information communicated and the actual stage of the project. However, by the revelation principle, in searching for the principal’s optimal mechanism it is without loss to focus on direct mechanisms that induce truthful reporting (see Myerson (1986) or Pavan et al. (2014) for further discussion). We can therefore restrict attention to direct mechanisms that induce the agent to report truthfully and immediately. We use $\hat{\tau}_1$ to denote the time at which the agent reports the first breakthrough in order to distinguish it from the time at which the first breakthrough is made.

As in other settings with persistent private information (e.g., Fernandes and Phelan (2000), Zhang (2009) and Guo and Hörner (2015)), it is convenient to formulate the principal’s problem as a dynamic program using the vector of promised utilities as the state variable. Thus, we will now make use of three state variables: the project stage $s \in \{1, 2\}$ (as reported by the agent) and the pair of promised continuation values to each type, $(u_1, u_2) \in \mathbb{R}_+^2$. Henceforth, we refer to the agent who has made a breakthrough as the “high” type and agent who has not yet made a breakthrough as the “low” type. For states in which $s = 2$, $u_1$ can be interpreted as the maximal payoff that a low type could obtain by falsely reporting a breakthrough. For states in which $s = 1$, $u_2$ can be interpreted either as the maximal payoff that a high type could obtain by not reporting progress or as the promised reward to the low type for making a breakthrough in that state.

5.1 Implementable utility levels

It will be useful to characterize the set of utility pairs that are implementable. A utility pair $\vec{u} = (u_1, u_2) \in \mathbb{R}_+^2$ is implementable if there exists a mapping from the agent’s type, $s$, to a contract $\Gamma_s$ such that (i) each agent prefers to report his type truthfully, (ii) the recommended action is incentive compatible, and (iii) the contract delivers expected payoff of $u_s$ to an agent who truthfully reports $s \in \{1, 2\}$. We denote the set of implementable utility levels by $U \subset \mathbb{R}_+^2$. Of course, not all utility pairs are implementable. For example, the high type can always “mimic” the low type, and thus $u_1 > u_2$ is not implementable. The following lemma says that this is essentially the only restriction on $U$.

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23Note that the actual time at which the breakthrough was made is irrelevant for the agent’s continuation value. Only whether a breakthrough was made and if and when it was reported matters.
Lemma 5.1. For a two-stage project with intangible progress, the set of implementable utility pairs is given by \( U = \{ (u_1, u_2) \in \mathbb{R}^2_{++} : u_2 \geq u_1 \} \cup (0, 0) \).

Let \( L_H = U \cap \{ u_2 = u_1 + \phi/\lambda \} \) denote the line along which the incentive compatibility condition for the low-type agent holds with equality and define the two complimentary subregions \( U_L = U \cap \{ u_2 < u_1 + \phi/\lambda \} \) and \( U_H = U \cap \{ u_2 \geq u_1 + \phi/\lambda \} \). Notice that the low-type agent will strictly prefer to shirk for all \( \vec{u} \in U_L \) and will (weakly) prefer not to shirk for all \( \vec{u} \in U_H \). Finally, let \( L_L = U \cap \{ u_1 = u_2 \} \) denote the lower boundary of \( U \). These regions are illustrated in Figure 3.

5.2 Second stage problem

With the additional state variable, we can again solve for the optimal contract using backward induction. That is, given an arbitrary implementable utility pair, \( (u_1, u_2) \in U \), we first solve for the principal’s value function and optimal policy in the second stage and then use these payoffs to find the optimal first stage policy. We let \( F_2 : U \to \mathbb{R} \) denote the principal’s second stage value function, which maximizes her payoff subject to incentive compatibility, delivering the required promised utility \( u_2 \) to a high type and delivering no more than \( u_1 \) to the low type. Formally, \( F_2 \) solves

\[
F_2(u_1, u_2) = \sup_{\Gamma} \mathbb{E}^a \left[ \Pi \cdot \mathbb{1}_{\{\tau_2 \leq T\}} - \int_0^{T \wedge \tau_2} \{c dt + dY_t\} \mid s = 2 \right]
\]

s.t. \( a \in \arg \max_{\tilde{a}} \mathbb{E}^{\tilde{a}} \left[ \int_0^{T \wedge \tau_2} \{g(\tilde{a}_t) dt + dY_t\} \mid s = 2 \right] \)

\[
u_2 = \mathbb{E}^a \left[ \int_0^{T \wedge \tau_2} \{g(a_t) dt + dY_t\} \mid s = 2 \right]
\]

\[
u_1 \geq \max_{\tilde{a}} \mathbb{E}^{\tilde{a}} \left[ \int_0^{T \wedge \tau_2} \{g(\tilde{a}_t) dt + dY_t\} \mid s = 1 \right].
\]
Proposition 5.2. For a two-stage project with intangible progress and any \( u \in U \), the principal’s value function in the second stage is given by

\[
F_2(u_1, u_2) = (1 - e^{-\lambda u_1/\phi}) \left( \Pi - \frac{c}{\lambda} \right) - u_1.
\]  

The expression for the principal’s value function is intuitive in light of the implementation used for the single-stage benchmark (see equation (8)). The larger is \( u_1 \), the longer is the deadline the principal can give the agent without violating (19), and therefore the higher is the probability of making the last breakthrough. Importantly, (19) decouples the link between the promised utility to the agent (\( u_2 \)) and the probability of making the second breakthrough \( (1 - e^{-\lambda u_1/\phi}) \).

5.3 First-stage problem

Having derived the principal’s value function after the first breakthrough for any given pair of implementable utilities, we can now turn to the problem in the first stage. Prior to a reported breakthrough, the principal chooses a termination rule, reward scheme and recommended action, as well as how much utility to deliver to each type of agent upon reporting a breakthrough, denoted by \( W_1, W_2 \). To induce truth telling, two additional constraints are required. We incorporate random termination (a necessary feature of the optimal contract), by letting the principal choose a distribution over termination dates denoted by \( S \), which combined with \( a \) induces a distribution over \( (\tau_1, \tau_2, T) \).\(^{24}\) Denote the corresponding expectation operator by \( E^{(a, S)} \). The principal’s problem in the first stage can be written as follows.

\[
\sup_{\Gamma} \mathbb{E}^{(a, S)} \left[ F_2(W_1(\tau_1), W_2(\tau_1)) \mathbb{I}_{\{\tau_1 \leq T\}} - \int_{0}^{T \wedge \tau_1} \{cdt + dY_t\} \bigg| s = 0 \right] \quad (OBJ1')
\]

subject to

\[
a \in \arg \max_{a} \mathbb{E}^{(\bar{a}, S)} \left[ W_2(\tau_1) \mathbb{I}_{\{\tau_1 \leq T\}} + \int_{0}^{T \wedge \tau_1} \{g(a_t) dt + dY_t\} \bigg| s = 1 \right] \quad (21)
\]

\(^{24}\)Naturally, we require \( S \) to be a right-continuous process and \( S_t \) to be measurable with respect to the principal’s information set, including whether a breakthrough has been reported at (or prior to) time \( t \).
and for all \( t < T \), the truth-telling constraints are given by

\[
W_1(t) \leq U_1(t) \equiv \mathbb{E}_t^{(a,S)} \left[ W_2(\tau_1) \mathbb{I}_{(\tau_1 \leq T)} + \int_t^{T \wedge \tau_1} \{ g(a_t) dt + dY_t \} \right | s = 1 \]  \quad (22)
\]

\[
W_2(t) \geq \max_{\tilde{a}, \tilde{\tau}_1 \geq t} \mathbb{E}_t^{(\tilde{a},S)} \left[ W_2(\tilde{\tau}_1) \mathbb{I}_{(\tilde{\tau}_1 \leq T)} + \int_t^{T \wedge \tilde{\tau}_1} \{ g(\tilde{a}_t) dt + dY_t \} \right | s = 2 \].  \quad (23)
\]

The first truth-telling constraint ensures that the low-type agent does not want to falsely report a breakthrough and the second ensures that the high-type agent cannot benefit from “hiding” a breakthrough from the principal. To solve the principal’s first stage problem, we show that it is without loss to focus on contracts in which the agent does not shirk along the equilibrium path (Lemma A.1). We then formulate a recursive version of the problem that relaxes (23) and hence only requires keeping track of the low-type agent’s continuation value. The solution to the relaxed program is characterized in Proposition 5.3. Finally, we show that there exists a contract that satisfies the neglected constraint under which the principal obtains the same value as in the solution to the relaxed program (Proposition 5.4).

We incorporate the principal’s ability to randomly terminate the project by letting \( \sigma \) denote the hazard rate of termination. The HJB for the relaxed problem is as follows.

\[
0 = \max \left\{ \sup_{w_1, w_2, \sigma} \left\{ \lambda F_2(w_1, w_2) - (\lambda + \sigma) F_1(u_1) - c + F_1'(u_1) \frac{du_1}{dt} \right\}, \right. \\
\left. u_1 F_1'(u_1) - F_1(u_1) \right\} \quad \text{subject to} \\
u_1 \geq w_1 \quad \text{(NFP)}
\]

\[
\lambda(w_2 - u_1) \geq \phi \quad \text{(IC1')} \\
\frac{du_1}{dt} = -\lambda(w_2 - u_1) + \sigma u_1 \quad \text{(PK1')} \\
F_1(0) = 0. \quad \text{(BC1')}
\]

The key difference between the problem above and the first-stage HJB with tangible progress is the no-false-progress constraint (NFP), which decouples the link between promised utility and probability of success following a breakthrough. To see the importance of this decoupling, consider the principal’s problem of optimally choosing \( w_1 \) and \( w_2 \). Because \( F_2 \) is increasing in \( w_1 \), (NFP) binds; the principal would like to increase \( w_1 \) following a reported breakthrough in order to give the agent more time to make the second breakthrough, but cannot (given the promised utility to the low-type agent) without inducing a false report. Because \( F_2 \) is decreasing in \( w_2 \), the incentive compatibility condition (IC1') also binds. Unlike when
progress is tangible, the principal is not tempted to choose \( w_2 \) above the level necessary to induce effort; doing so would simply mean giving more rents to an agent who makes a breakthrough without increasing the likelihood of ultimate success.

The solution to the relaxed problem requires stochastic termination. To understand why, recall that we have endowed the principal with a public randomization device and therefore her value function must be weakly concave. Now, suppose that the optimal contract is deterministic. Then, \((\text{HJB}1')\) becomes

\[
\lambda F_1(u_1) = \lambda F_2(u_1, u_1 + \frac{\phi}{\lambda}) - c - \phi F_1'(u_1),
\]

which has a solution of the form

\[
F_{1c}(u_1) = b - \frac{2c}{\lambda} - u_1 - u_1 e^{-\frac{\lambda u_1}{\phi}} \left( \frac{\lambda b - c}{\phi} \right) + H_1 e^{-\frac{\lambda u_1}{\phi}},
\]

where \( H_1 \) is an arbitrary constant. Imposing the terminal boundary condition, \( F_{1c}(0) = 0 \), we get that \( H_1^c = \frac{2c}{\lambda} - \Pi \). Evaluating the second derivative at \( u_1 = 0 \) gives

\[
F_{1c}''(0) = \frac{\lambda^2 \Pi}{\phi^2} > 0.
\]

Hence, the principal can improve her payoff (above \( F_{1c} \)) by randomizing over the termination date, violating our supposition. Importantly, the principal cannot achieve the same payoff by setting \( w_1 \) to be constant for \( u_1 \) below some threshold (as in the case of tangible progress when \((\text{C.2})\) fails) due to the decoupling. Thus, random termination is an essential feature of the optimal contract when progress is intangible.

The second term in \((\text{HJB}1')\) is the net effect of increasing \( \sigma \). If this term is strictly positive, then the optimal rate of termination would be infinity (i.e., it would be optimal to terminate with an atom). Therefore, in order for a flow rate of termination to be optimal at some \( u_1 = u_s \), it must be that

\[
\lim_{u_1 \downarrow u_s} F_1'(u_1) = \lim_{u_1 \downarrow u_s} \frac{F_1(u_1)}{u_1}.
\]

Given that \((26)\) must hold, choosing the optimal \( u_s \) is equivalent to the super contact condition (Dumas, 1991),

\[
\lim_{u_1 \downarrow u_s} F_1''(u_1) = 0.
\]

The solution to the free-boundary problem implied by \((25)-(27)\) characterizes the principal’s value function.
Proposition 5.3. There exists $u_s \geq 0$ such that the solution to \((HJB1')\) is given by

$$F_1(u_1) = \begin{cases} \Pi - \frac{2c}{\lambda} - u_1 - \left(\Pi - \frac{c}{\lambda}\right) \left(2 + \frac{\lambda(u_s - u_1)}{\phi}\right) e^{-\lambda u_s/\phi} & \text{if } u_1 \in [0, u_s), \\ \Pi - \frac{2c}{\lambda} - u_1 - \left(\Pi - \frac{c}{\lambda}\right) \left(2 + \frac{\lambda(u_1 - u_s)}{\phi}\right) e^{-\lambda u_1/\phi} & \text{if } u_1 \geq u_s. \end{cases}$$

(28)

(29)

For all $u_1 \geq u_s$, the optimal policy involves $w_2 = u_1 + \phi/\lambda$, $w_1 = u_1$, $\sigma = \frac{\phi}{u_s} \mathbb{1}_{\{u_1 = u_s\}}$. For $u_1 < u_s$, the optimal policy involves terminating the project with probability $(u_s - u_1)/u_s$. The principal’s ex-ante payoff under the optimal contract with intangible progress is $\max_u F_1(u)$.

The similarity between the value functions presented in Proposition 4.3 and Proposition 5.3 is not a coincidence. Both value functions are linear for promised utility sufficiently close to zero, stemming from the importance of preserving time to complete the project following the first breakthrough. When progress is tangible, the first-stage value function exhibits the linear segment only when the project fundamentals are relatively weak (i.e., when \((C.2)\) does not hold), whereas the value function always has a linear segment with intangible progress. Intuitively, the no-false-progress constraint effectively prohibits the principal from granting more time following a reported breakthrough, meaning that preserving time in the second stage (at the expense of less time in the first) is even more important.

5.4 Optimal dynamics

The optimal dynamics are illustrated in Figure 4. At $t = 0$, the agent’s continuation utility starts at $u_I \equiv \arg \max_u F_1(u)$, the blue asterisk in the upper right corner of the figure. Prior to reporting a breakthrough, the continuation utilities evolve along $L_H$ toward $\kappa^*$. If a breakthrough is reported prior to reaching $\kappa^*$, the optimal dynamics are not uniquely pinned down. That is, continuation values can continue evolving downward along $L_H$ or they can travel to the interior of $U_L$. Eventually, however, the state must travel toward the origin and if $\vec{u}$ reaches the origin prior to the second stage breakthrough, the project is terminated.

If a breakthrough is not reported prior to reaching $\kappa^*$, then the principal initiates a probationary phase in which she randomizes over terminating the project (the state jumps to the origin) and maintaining promised utilities at $\kappa^*$. Hence, $\kappa^*$, serves as a partially absorbing state until the probationary phase ends with either project termination or a reported breakthrough, at which point, the state variable again evolves toward the origin. Conditional on reaching $\kappa^*$ and not being terminated, the promised utility to the agent once he (truthfully) reports a breakthrough is independent of when it is reported.

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Figure 4: This figure illustrates the optimal dynamics of continuation values under the MOCC. Prior to the soft deadline ($\kappa^*$), continuation values drift down along $\mathcal{L}_H$. Upon reaching $\kappa^*$, the evolution of continuation values depends on whether a breakthrough has been reported. If it has not, they stop drifting and either jump to the origin (if terminated) or remain at $\kappa^*$. If a breakthrough is reported at the soft deadline, then the continuation values drift down along the dotted line toward the origin under the implementation in Proposition 5.4.

5.5 Implementation with deadlines under intangible progress

Recall that with tangible progress when (C.2) does not hold, the optimal contract can be implemented by using three clocks (short, medium and long). With intangible progress, the three clock implementation no longer works. To see why, note that stopping the long clock and allowing the medium clock to run would induce the low-type agent to falsely report progress just before the medium clock expired and it would also induce the high-type agent to hide progress and shirk until just before the medium clock expired.

Instead of using three clocks, the principal uses a soft deadline, $T_s$, a long clock, and a termination rate $\sigma = \phi / u_s$. The principal guarantees funding for the project up to $T_s$ – if the agent does not report a breakthrough by $T_s$, the long clocked is stopped and the project is terminated at rate $\sigma$. If the probationary phase ends with the agent reporting a breakthrough then the agent gets the remaining time on the long clock to complete the project.

To implement the optimal contract, communication between the principal and agent is critical. However, this communication is not strictly required until the state reaches $\kappa^*$. That is, early on in the life of the project the principal need not be in communication with the agent regarding the status of the project. Instead, at $t = 0$, the principal simply gives the agent a future date (i.e., the soft deadline $T_s$), at which a progress report is required. If the ultimate success of the project is realized prior to $T_s$ then there is no need for the agent to make any report at all.\footnote{Even though we have employed the revelation principle and solved for the optimal mechanism assuming that the agent reports progress immediately, the optimal contract does not require conditioning on the time at which the first breakthrough occurs if it is less than $T_s$. Therefore, the principal can achieve the same}
hand, once the state reaches $\kappa^*$ (i.e., at all $t \geq T_s$), the agent must report a breakthrough as soon as it arrives in order to avoid suboptimal termination.

We refer to the implementation described above as the Minimally Optimal Communication Contract (MOCC). It is the mechanism that minimizes the expected number of reports that the agent will be forced to make over the life of the project subject to delivering the maximal payoff to the principal.\footnote{Notice that whenever $\tau_2 < T_s$, the agent does not make a report.} Because of this property, the MOCC is uniquely optimal when the agent must incur (or the principal can impose) a small cost in order to report progress (see Section 7.2). Formally, it can be implemented as follows.

**Proposition 5.4.** A two-stage project with intangible progress is feasible if and only if

$$\Pi - \frac{2c}{\lambda} > \frac{2\phi}{\lambda} + \phi T^*.$$  \hspace{1cm} (C.4)

If \((C.4)\) holds, the optimal contract can be implemented by use of a soft deadline $T_s$, a long clock $T_s + us/\phi$, a termination rate $\sigma = \phi/\lambda$, and a reward function

$$R_s(\hat{\tau}_1, \tau_2) = \begin{cases} 
\phi \left( \frac{2}{\lambda} + \frac{1}{\sigma} + T_s - \tau_2 \right), & \text{if } \tau_2 \leq T_s \\
\phi \left( 1 + \frac{\phi}{\lambda} \right) \left( \frac{1}{\lambda} + \frac{1}{\sigma} + \max\{T_s, \hat{\tau}_1\} - \tau_2 \right), & \text{if } 0 < \tau_2 - \max\{T_s, \hat{\tau}_1\} \leq \frac{1}{\sigma} \\
0, & \text{otherwise.}
\end{cases}$$

such that:

- If the project is not completed prior to the soft deadline $T_s$, the principal asks the agent for a progress report at $t = T_s$.
  - If the agent reports that he has made the first breakthrough $(\hat{\tau}_1 \leq T_s)$, then he is given the remaining time on the long clock, $u_s/\phi$, to complete the project.
  - If the agent reports that he has not yet made the first breakthrough, then the principal stops the long clock and initiates a probationary phase in which the project is terminated at constant rate $\sigma$.
  - If the agent reports a breakthrough during the probationary phase, then he is given the remaining time on the long clock, $u_s/\phi$, to complete the project.

- The agent gets the reward $R_s(\hat{\tau}_1, \tau_2)$ only if the project succeeds prior to being terminated.

Even though the principal cannot substantiate the agent’s progress reports, such reports are an essential aspect of the optimal mechanism because they are used to govern the ex-ante payoff with no communication prior to $T_s$. Under this implementation, the principal’s expected payoff at $t \in (0, T_s)$ will depend on her belief about the project stage and will therefore be a weighted combination of $F_1$ and $F_2$.\footnote{Notice that whenever $\tau_2 < T_s$, the agent does not make a report.}
continuation contract: a report of “no” resulting in probation and one of “yes” resulting in a relatively short time to complete the project.\footnote{Lemma A.2 in the appendix shows that $u_s/\phi \leq T^*$.} Indeed, permitting the agent to “remain silent” regarding his progress would make the principal worse off as was demonstrated in Proposition 3.2.

Several other aspects of Proposition 5.4 also warrant discussion. First, instead of using a hard deadline and a severance payment to induce truthful reporting by the low-type agent (as was suggested in Section 3.1), the principal finds it optimal to screen types with a soft deadline and no severance. Specifically, at every instant of the probationary phase the low-type agent is indifferent between honestly reporting his lack of progress (and facing continued probation while he works) and falsely reporting progress (and optimally shirking over the time left on the long clock). The high-type agent, however, strictly prefers reporting his true status at this point – that is, he strictly prefers having a short period of time to complete the project to facing probation.

Second, even under the MOCC, the reward schedule is not uniquely pinned down for $\tau_2 > T_s$. Any reward function satisfying incentive compatibility, promise keeping, and the boundary conditions $u_2 = \phi(1/\lambda + 1/\sigma)$ at $t = T_s$ and $u_2 = 0$ at $t = T_s + 1/\sigma$ will suffice. The reward function given in Proposition 5.4 is the unique one meeting these criteria that induces piecewise linear promised utility for the high-type agent.

6 Implications for Deadlines and Welfare

In this section we explore how the nature of progress affects the amount of time the agent is given to complete the project, ex-ante payoffs, and the overall likelihood of project success.

6.1 Deadlines

One natural question is whether the agent should be given more time to make a breakthrough in the first stage or in the second. When progress is tangible, the second stage deadline depends on the realization of $\tau_1$ and may be longer or shorter than the first stage deadline. However, conditional on making a breakthrough prior to being terminated, the agent is on average granted more time to complete the second stage.

**Proposition 6.1.** For any feasible two-stage project with tangible progress, conditional on making a breakthrough in the first stage, the expected amount of time the agent has to complete the second stage is strictly greater than the first-stage deadline.

In other words, the principal keeps the agent on a relatively “short leash” until he makes the initial breakthrough. After that, he expects to have a longer horizon to bring the project
to fruition. While this result is intuitive, it depends critically on the tangible nature of progress as illustrated by the following result.

**Proposition 6.2.** For any feasible two-stage project with intangible progress, conditional on making a breakthrough in the first stage, the expected amount of time the agent has to complete the second stage is strictly less than the expected amount of time the agent has to complete the first stage.

Under either information regime, the agent’s continuation utility increases by $\phi/\lambda$ following the first breakthrough. The difference is that when progress is tangible, this is achieved through granting the agent more time to complete the project, whereas when progress is intangible it is done purely via the reward function. Therefore, while a short leash–long leash structure is optimal under tangible progress, the opposite holds when progress is intangible.

### 6.2 (Ex-ante) Payoffs and the likelihood of project success

Because the agent never diverts investment under either regime, the only source of inefficiency is terminating the project prior to success. Hence total welfare is proportional to the probability that the project succeeds and the principal’s value function is equal to total welfare minus the agent’s utility. Moreover, from the principal’s prospective, the only difference between tangible and intangible progress is that additional truth telling constraints must be satisfied. These observations underpin the following result.

**Proposition 6.3.** For any level of promised utility $u_1 > 0$ to the agent in the first stage, the principal is strictly better off and the probability of project completion is strictly higher if progress is tangible: $V_1(u_1) > F_1(u_1)$.

This results says that the principal prefers progress to be tangible for any given level of promised utility to the agent and therefore is also better off from an ex-ante perspective under an optimal contract. In contrast, the next result shows that under an optimal contract, the agent prefers intangible progress. Let $u_J = \arg \max_u V_1(u)$ (i.e., $u_J = u_c$ if (C.2) holds and $u_n$ otherwise) and recall that $u_I = \arg \max_u F_1(u)$. Thus, $u_J$ and $u_I$ denote the agent’s ex-ante payoff under the optimal contract with tangible and intangible progress respectively.

**Proposition 6.4.** For any two-stage project that is feasible under both tangible and intangible progress, the agent is strictly better off ex-ante under an optimal contract if progress is intangible than if it is tangible: $u_I > u_J$.

When progress is privately observed by the agent, he earns informational rents because he must be given incentives not to falsely report a breakthrough. Figure 5 illustrates an
example with the solutions under tangible and intangible progress in the case where (C.2) holds. Notice that $V_1$ is everywhere higher than $F_1$ (consistent with Proposition 6.3) and the peak of $F_1$ lies to the right of the peak of $V_1$ (consistent with Proposition 6.4).

Because the agent never diverts investment under either regime, the only source of inefficiency is terminating the project prior to success. Hence total welfare is proportional to the probability that the project succeeds. We have just shown that the principal prefers progress to be tangible and the agent prefers it to be intangible. *A priori* it is not clear which informational regime yields higher total ex-ante welfare and hence which type of projects that are more likely to succeed. On the one hand, the agent earns higher rents and hence has a longer expected first-stage deadline when progress is intangible. On the other hand, the principal can give the agent additional time following a breakthrough in the first stage when it is tangible. The next result shows that the latter effect dominates the former. Thus when moving from a setting with intangible to tangible progress, the principal gains more than the agent loses.

**Proposition 6.5.** Fix any two-stage project that is feasible. Under the optimal contract, the total ex-ante expected welfare and the probability that the project ultimately succeeds is strictly higher when progress is tangible than when it is intangible.

## 7 Project Design

In this section we investigate three variations of the model that have implications for certain aspects of project design. We first consider the case in which progress is unobservable to both players in order to see whether the principal can benefit from suppressing the agent’s parameter values for which she is willing to contract with the agent is smaller. In other words, there are parameter values for which the agent prefers progress to be tangible because the principal would not initiate the project otherwise (e.g., (C.2) holds but (C.4) does not.).
access to information about the status of the project. We then consider costly reporting and ask whether there is scope for imposing a cost on the agent in order to submit a progress report. Finally, we explore the case of asymmetric stages and ask whether it can be beneficial for the principal to design one of the stages to be more difficult than the other.

7.1 Unobservable Progress

When progress is intangible, the principal must give the agent information rents in order to induce truthful reporting. However, the principal also uses this information to more efficiently provide incentives. In considering how to design projects, it is then natural to ask whether these benefits outweigh the costs. That is, if the principal cannot observe progress, is it better to also restrict the agent’s ability to do so?

Our results suggest that the answer to the question posed above is “no.” That is, the principal benefits more from effectively using this information than the rents she gives up to acquire it. One implication is that, in designing projects and their incentives schemes, the principal should focus on how best to illicit and effectively use the agent’s private information (as was done in Section 5) rather than trying to suppress his access to it.

By way of terminology, we refer to progress as being unobservable if $\tau_1$ is not observed by either party and hence cannot be contracted upon either directly or indirectly.

Proposition 7.1. When progress is unobservable, the principal's ex-ante payoff under the optimal contract is given by

$$\max_u \left( \Pi - \frac{2c}{\lambda} \right) \left( 1 - e^{-\lambda u/\phi} \right) - \left( b - \frac{c}{\lambda} \right) \frac{\lambda u}{\phi} e^{-\lambda u/\phi} - u,$$

which can be implemented using a simple contract.

When progress is intangible, the principal can do strictly better than a simple contract by using a more sophisticated termination policy in which the project is terminated at a constant rate if the agent has not reported progress by the soft deadline. This allows the principal to give a low-type agent more time to complete the project conditional on having a breakthrough prior to being terminated. Why is it that the principal cannot benefit from a similar strategy with unobservable progress? The reason, of course, is that the principal has no way of knowing whether the project is still in the first stage or has progressed to the second. Any policy which uses random termination must do so indiscriminately, which means terminating projects that have already progressed to the next stage. This highlights the cost of unobservable progress relative to intangible progress.

The benefit of unobservable progress is that the agent also does not know whether progress has been made, and it is therefore easier to induce effort. For example, it is no longer the case
that the agent necessarily begins shirking as the deadline approaches (i.e., Proposition 3.2 no longer holds), which is why a simple contract remains optimal. Having derived the principal’s maximal payoff in both scenarios, we can easily compare them.

**Corollary 7.2.** For any two-stage project that is feasible with intangible progress, the principal is strictly better off under the optimal contract when progress is intangible than when it is unobservable.

### 7.2 Costly Reporting

Next we consider the case of intangible progress, but permit two channels through which the agent may report progress: an informal one (e.g., verbal) and a formal one (e.g., written). Both types of reports can be contracted upon and both are falsifiable. The only operational difference between the two channels is that the formal channel requires incurring a cost \( \rho > 0 \) (e.g., the time and effort of filling out documentation), whereas the informal channel remains costless for the agent to use (as it was in Section 5).\(^{29}\) We will assume that using the formal channel is equally costly for the agent whether he submits a true or false report though our results hold *a fortiori* if the cost is higher for a false report.

The presence of the informal channel allows us to again invoke the revelation principle and focus only on direct mechanisms in which the agent truthfully reports progress as soon as it arrives. If the principal elects to use only the informal channel, then the setting is identical to the one studied in Section 5. The question is whether it is ever advantageous for her to require the agent to use the costly channel for reporting progress.

**Proposition 7.3.** If the project is feasible using only the informal channel and if \( \rho \) is sufficiently small, then:

(i) There exists \( u_s(\rho) > 0 \) such that for \( u_1 \geq u_s(\rho) \) the principal’s value function in the first stage is

\[
F_1(u_1; \rho) = \left( b - \frac{2c}{\lambda} - u_1 \right) - \left( \rho + \left( b - \frac{c}{\lambda} \right)^{-\lambda \rho / \phi} \right) \left( 2 + \frac{\lambda(u_1 - u_s(\rho))}{\phi} \right) e^{-\lambda u_1 / \phi}. \tag{30}
\]

(ii) There exists a \( T_s(\rho) > 0 \) such that it is optimal for the principal to require the agent to report progress through the formal channel if the project has not succeeded by \( T_s(\rho) \).

Thus, the optimal contract requires the high type to substantiate his informal claim of progress with a formal report if he does not complete the project by \( T_s(\rho) \), which serves as the soft deadline. Additionally, any claims of progress during the probationary phase must be made through the formal channel.

\(^{29}\)To respect limited liability, \( \rho \) should be interpreted as a direct loss of utility rather than a monetary cost.
To understand this result, notice that the principal’s value function converges to the case of intangible progress investigated in Section 5 as $\rho$ goes to zero; i.e.,

$$\lim_{\rho \to 0} F_1(u_1; \rho) = F_1(u_0).$$

For $\rho > 0$, however, the principal’s payoff is impacted in two ways, one positive and one negative. On the up side, requiring formal reports at the soft deadline relaxes the no-false-progress constraint and allows the principal to add extra time of $\rho/\phi$ to the clock. On the down side, with probability $e^{-\lambda(u_1-u_s(\rho))/\phi}$ the high-type agent will have to make a costly formal report, and promise keeping necessitates that the principal compensate him for this event. In the proof of the proposition, we show that for relatively small values of $\rho$, the positive effect dominates the negative effect. In other words, the principal is better off from an ex-ante perspective if she requires the agent to use the costly channel if the project is not completed before the soft deadline is reached. On the other hand, if $\rho$ is sufficiently large, then the second effect dominates and it is optimal for the principal to use only the informal channel.

Essentially, imposing a small cost of reporting progress on the agent after the soft deadline helps the principal to more effectively screen types. Note that this holds even though we have assumed that the reporting cost is the same whether or not the agent submits an honest report. The intuition is that reporting costs relax the no-false-progress constraint for the low type with probability one, while the cost of additional compensation to the high type is incurred with probability less than one (i.e., an agent who has made the first breakthrough will avoid paying the reporting cost if he completes the project prior to the soft deadline). Reporting costs, therefore, have greater impact on an agent who has not yet made progress, and it is this differential impact that allows the principal to benefit from making communication costly. To illustrate this point further, it is worth noting that the principal is strictly worse off under a direct mechanism that uses only the formal channel than under a direct mechanism using only the informal one.

Finally, recall that the optimal contract with intangible progress can be implemented via the MOCC (see Section 5), where no communication takes place prior to the soft deadline. With costly reporting, communication prior to the soft deadline is also not required. That is, the optimal contract can be implemented with a termination policy and wage scheme that do not depend on reports made using the informal channel. Therefore, provided that $\rho$ is not too large, the informal channel is unnecessary; the MOCC using only the formal channel can be used to implement the optimum. Further, if the informal channel is unavailable (i.e., any communication requires the agent to incur a cost $\rho$), then the MOCC is uniquely optimal.

\[30\text{See footnote 25 for further discussion.}\]
since it minimizes the probability of incurring the cost.

### 7.3 Asymmetric Stages

In many (probably most) relevant applications, each stage of the project is different. For example, one stage may be expected to take more time (have a smaller $\lambda$), require more working capital (higher $c$), and/or yield greater private benefits to the agent from shirking (higher $\phi$). In this subsection, we extend our analysis to a setting with stages that are not identical.

In general, a stage $k \in \{1, 2\}$ can be described by the pair $(\phi_k/\lambda_k, c_k/\lambda_k)$. To fix ideas, we set $\phi_1 = \phi_2 = \phi$ and $c_1 = c_2 = c$ and parameterize the asymmetry of stages by $\alpha \in [-1, 1]$, where

$$\frac{1}{\lambda_1} = \frac{1 + \alpha}{\lambda} \quad \text{and} \quad \frac{1}{\lambda_2} = \frac{1 - \alpha}{\lambda}$$

for some fixed $\lambda$. This parametrization maintains a fixed project value as we vary $\alpha$ (i.e., $\Pi - 2c/\lambda$) therefore allowing us to isolate the effect of the asymmetry. Also, note that the probability distribution of $\tau_2$ is symmetric in $\alpha$. For $\alpha = 0$, the two stages are identical. For $\alpha > 0$, the first stage is expected to take more time and require a larger fraction of the total resources than the second stage. We therefore refer to the first stage as being harder if $\alpha > 0$, and easier if $\alpha < 0$.

The analysis from Sections 4 and 5 can easily be extended to this setting. Recall that $V_1$ and $F_1$ were used to denote the principal’s value function in the first stage with symmetric stages under tangible and intangible progress. We use a superscript on the value function (i.e., $V_1^\alpha$ and $F_1^\alpha$) to denote the principal’s value function as it depends on $\alpha$ under tangible and intangible progress respectively.

**Proposition 7.4.** Suppose that (C.2) holds strictly. Then for a two-stage project with tangible progress and asymmetric stages, the following statements hold.

1. There exists an $\tilde{\alpha}^{\tan} \in (0, 1)$ such that $V_1^\alpha(u)$ is strictly increasing in $\alpha$ for all $\alpha \in (-\tilde{\alpha}^{\tan}, \tilde{\alpha}^{\tan})$ and $u > 0$.

2. There exists $\bar{\alpha}^{\tan} \in (\alpha^{\tan}, 1)$ such that $V_1^\alpha(u)$ is strictly decreasing in $\alpha$ for all $\alpha \in (\bar{\alpha}^{\tan}, 1)$ and $u > 0$.

3. As $\alpha \to 1$, $V_1^\alpha$ converges uniformly to $\bar{V}^{\frac{\lambda}{2}}$, which is the principal’s value function in a single-stage project where the arrival rate is $\lambda/2$.

4. $\overline{V}^{\frac{\lambda}{2}}(u) < V_1^\alpha(u)$ for any $\alpha \in [0, 1)$ and $u > 0$.

The first result shows that with tangible progress, the principal is better off (worse off) if the first stage is moderately more difficult (easier) than the second. Intuition for this result is
as follows. First, recall that when (C.2) holds, (IC1) binds everywhere, meaning the principal does not want to reward the agent more than necessary to induce effort.\footnote{If (C.2) does not hold then, based on numerical examples, (i)-(iii) still appear to hold. However, (iv) does not hold: there exist $u$ small enough such that $\bar{V}^{\lambda/2}(u) > V^\alpha_1(u)$.} Due to incentive compatibility, a positive $\alpha$ requires that the agent get a larger reward following a breakthrough in the first stage, $w(u) = u + \frac{(1+\alpha)\phi}{\lambda}$, and a smaller reward following a breakthrough in the second, $R(u) = u + \frac{(1-\alpha)\phi}{\lambda}$. On the margin it is cheaper to compensate the agent with continuation utility for a breakthrough in the first stage rather than a monetary payment for a breakthrough in the second, since the former can be achieved with an increase in the probability of success, while the latter is a pure transfer. This is why the principal can benefit from making the first stage marginally more difficult.

Parts (iii) and (iv) of Proposition 7.4 show that the principal is strictly worse off by making the first stage too difficult. The intuition is that as $\alpha \to 1$, the second stage becomes irrelevant and the project effectively has only one-stage with an arrival rate of $\lambda/2$. In a one-stage project, the principal has an inferior monitoring technology and therefore does worse than in any two-stage project with tangible progress, regardless of the degree of asymmetry.

The next proposition shows that with intangible progress, $\alpha$ has a similar effect on the principal’s ex-ante payoff under the optimal contract.

**Proposition 7.5.** For any feasible two-stage project with intangible progress and asymmetric stages, the following statements hold.

(i) There exists an $\alpha^{\text{int}} \in (0,1)$ such that $F_1^\alpha(u_1)$ is strictly increasing in $\alpha$ for all $\alpha \in (-\alpha^{\text{int}}, \alpha^{\text{int}})$ and $u_1 > 0$.

(ii) There exists an $\bar{\alpha}^{\text{int}} \in (\alpha^{\text{int}},1)$ such that $\max_u F_1^\alpha(u_1)$ is strictly decreasing in $\alpha$ for all $\alpha \in (\bar{\alpha}^{\text{int}},1)$.

(iii) As $\alpha \to 1$, $F_1^\alpha$ converges uniformly to $\bar{V}^{\lambda/2}$.

This result has a similar flavor to Proposition 7.4. Starting from a situation with symmetric stages, the principal does better (worse) by making the first stage slightly harder (easier). However, making the first stage too difficult is not advantageous for the principal as it reduces her ability to allocate the remaining (expected) time until termination in the most efficient manner.

Taken together, these two propositions have several novel implications for the optimal design of projects regardless of the nature of progress. First, it is better to put more difficult stages first. Second, to the extent that it is possible to break up a project into several stages, the principal can always improve her payoff by doing so. Finally, to the extent that the project stages can be designed, an interior level of $\alpha \in (0,1)$ maximizes the principal’s payoff.
8 Conclusion

In this paper we study the optimal provision of incentives for multistage projects. We characterize optimal contracts under both tangible and intangible progress and explore the implications for welfare and optimal project design.

The optimal contract under tangible progress resembles the structure of venture capital financing arrangements that typically tie future funding to attainment of certain observable milestones. When progress is observed, the principal rewards the agent with additional time and the necessary funding to complete the next phase of the project. We identify the precise condition under which the incentive compatibility condition does not bind near the end of the first stage and the principal finds it optimal to “pause the clock” in order to maximize the probability of ultimate success.

The optimal contract when progress is intangible shares features in common with multi-year research grants and awards: funding is partially contingent on periodic self-reported progress by the researcher, although penalties for late reports appear often to be vague and imprecise. Self-reported progress plays an important, and yet simple, role in implementing the optimal contract. The optimal contract with the minimal amount of communication involves the use of a soft deadline, prior to which the agent is not required to communicate with the principal and after which breakthroughs must be reported as soon as they arrive in order to avoid suboptimal termination.

Regarding the optimal design of projects, we demonstrate several results. First, the principal does better when progress is intangible than when it is unobservable. Second, when progress is intangible, the principal achieves a higher payoff by imposing a small cost to the agent for reporting a breakthrough. Third, the principal can benefit from making the first stage somewhat more difficult than the second.

There are numerous avenues for future work in this vein. For instance, progress need not be completely tangible or intangible, but might be imperfectly observed by the principal, perhaps as the result of a costly audit. Also, we have focused on a setting with two discrete stages. Natural extensions would be to consider a setting with more stages (or model progress as a continuous process) and allow for the possibility of setbacks along the path to project completion. Additionally, in order to isolate progress as an instrument for providing incentives, we have suppressed uncertainty about the underlying value of the project. It would be edifying to study the role of both tangible and intangible progress in an environment where parties learn about the value of the project so long as it is funded. Finally, our analysis is couched in a setting with full commitment by the principal, and relaxing this assumption may shed important light on other aspects of the subject.
References


A Appendix

Proof of Lemma 2.1. The HJB equation for the agent’s problem can be derived in the usual way.
\[ U_t = \sup_{a_t} (\lambda a_t R_t + g(a_t)) dt + (1 - \lambda a_t dt) U_{t+dt} \]

Using a Taylor expansion \( U_{t+dt} = U_t + U'_t dt + o(dt) \), canceling \( U_t \) on both sides, dividing by \( dt \) and taking the limit as \( dt \to 0 \), we obtain (4). The lemma follows because (i) the HJB is a necessary condition for \( a \) to solve (3) and (ii) it is satisfied if and only if (5) holds.

Proof of Proposition 2.2. Fix an arbitrary \( u \in \mathbb{R}_+ \), we first show that \( V(u) \leq \bar{V}(u) \). Since \( \max_u U_0(\Gamma) \geq \phi T \), incentive compatibility and promise keeping requires that \( T \leq u/\phi \). Because \( \lambda b > c \geq \phi \), given any finite deadline \( T \) satisfying \( T \leq u/\phi \), total surplus is maximized by \( T = u/\phi \) and \( a_t = 1 \) for all \( t \in [0,u/\phi] \). That is,
\[ V(u) + u \leq \max_{a,T \leq u/\phi} \mathbb{E}^a \left[ \Pi \cdot 1_{\{T \leq T\}} - \int_0^{T \land \tau} cdt \right] = \left( 1 - e^{-\lambda \phi u} \right) \left( \Pi - \frac{c}{\lambda} \right). \]

Hence,
\[ V(u) \leq \left( 1 - e^{-\lambda \phi u} \right) \left( \Pi - \frac{c}{\lambda} \right) - u = \bar{V}(u). \] (A.1)

Next, let \( \Gamma(u) \) denote the contract with deadline \( T_u = u/\phi \), reward scheme \( R_u(t) = \phi \left( \frac{1}{\lambda} + u/\phi - t \right) \) and \( a_t = 1 \) for all \( t \in [0,T(u)] \). Notice that \( \mathcal{P}_0(\Gamma(u)) = \bar{V}(u), U_0(\Gamma(u)) = u \) and \( \Gamma(u) \in \mathcal{I} \). Hence \( V(u) \geq \bar{V}(u) \), which combined with (A.1) implies \( V(u) = \bar{V}(u) \).

Proof of Lemma 4.1. Twice differentiating \( V_{1c}(u) \) gives
\[ \frac{d^2}{du^2} V_{1c}(u) = \frac{\lambda e^{-\lambda \phi} - 1}{\phi^3} \lambda u (e - \lambda \Pi) + \phi (2e(e - 1) - \lambda \Pi(e - 2)). \] (A.2)

The term inside the parentheses is strictly decreasing in \( u \) given (C.1). Therefore, the function is concave for all \( u \in \mathbb{R}_+ \) if and only if \( \frac{d}{du} V_{1c}(0) \leq 0 \). Evaluating (A.2) at \( u = 0 \), we have that
\[ \text{sign} \left( \frac{d}{du} V_{1c}(0) \right) \leq 0 \iff \text{sign} \left( 2e(e - 1) - \lambda \Pi(e - 2) \right) \leq 0. \]

The second inequality is equivalent to (C.2).

Proof of Proposition 4.2. By construction, \( V_{1c}(u) \) is the principal’s payoff under the first-stage policy \( w_{1c}(u) = u + \phi/\lambda, T(u) = u/\phi \) and \( a_t = 1 \) for all \( t \leq T(u) \). Since this policy is clearly incentive compatible and satisfies promise keeping, \( V_{1c}(u) \leq V_1(u) \). Therefore, it is sufficient to show that if (C.2) holds then \( V_1 \leq V_{1c} \).

To do so, we first show that when (C.2) holds, \( V_{1c} \) solves (HJB1) and then prove that any solution to (HJB1) implies the desired inequality. To verify that \( V_{1c} \) solves (HJB1), define
\[ L_c(w,u) \equiv \lambda (V_2(w) - V_{1c}(u)) - c - \lambda (w - u) V_{1c}'(u). \]
First, we show that \( L_c \) is decreasing in \( w \) for all \( w \geq u + \phi/\lambda \) and \( u \in \mathbb{R}_+ \). Because \( V_2 \) is strictly concave, showing that \( L_c \) is decreasing at \( w = u + \phi/\lambda \) is sufficient. Using the closed-form expressions for \( V_2 \) and \( V_{1c} \), we have that
\[
\frac{d}{dw} L_c(w, u) \big|_{w = u + \phi/\lambda} = V'_c(u + \phi c/\lambda) - V'_{1c}(u) = e^{-\lambda w} - \frac{1}{\phi^2} \left[ c(\lambda u + 2(\varepsilon - 1)\phi) - \Pi \lambda (\lambda u + (\varepsilon - 2)\phi) \right].
\]

The term in the brackets on the right-hand side, which determines the sign of the expression, is decreasing in \(u\) and hence we need only verify that it is weakly negative when evaluated at \(u = 0\). Doing so gives \(V'_c(\phi c/\lambda) - V'_{1c}(0) \leq 0 \iff \frac{2(\varepsilon - 1)c - (\varepsilon - 2)\Pi \lambda}{\phi} \leq 0\). The latter inequality holds if and only if (C.2) does. Therefore, if (C.2) holds then
\[
\max_{w \geq u + \phi/\lambda} L_c(w, u) = \lambda(V_2(u + \phi/\lambda) - V_{1c}(u)) - c - \phi V'_{1c}(u) = 0,
\]
where the second equality follows from the fact that \(V_{1c}\) satisfies (11).

To verify that a solution to the (HJB1) is sufficient, write the payoff to the principal under an arbitrary (deterministic) contract that induces effort and delivers a payoff of \(U\) or \(\hat{V}\) to the agent, we conclude that \(V_{1c}\) satisfies (11).

Suppose that \(\hat{V}\) is a \(C^1\) function that solves (HJB1) subject to the constraints. Consider an arbitrary feasible contract \(\Gamma\), which induces some \((W_s, U_s)\) for \(s \in [0, T]\). Using (HJB1) we have that
\[
0 \geq \lambda e^{-\lambda t} \left[ \lambda(V_2(W_t) - \hat{V}(U_t)) - c - \lambda(W_t - U_t)\hat{V}'(U_t) \right]
\]
or
\[
\frac{d}{dt} \left( e^{-\lambda t} \hat{V}(U_t) \right) \geq e^{-\lambda t} (\lambda V_2(W_t) - c).
\]
Integrating over \([0, T]\) and noting that \(U_T = 0\), we get that
\[
\hat{V}(U_0) - e^{-\lambda T} \hat{V}(U_T) = \hat{V}(U_0) \geq \int_0^T e^{-\lambda t} (\lambda V_2(W_t) - c) dt = \mathcal{P}_0(\Gamma).
\]
Since the inequality holds for all \(\Gamma\) delivering an arbitrary \(U_0 \in \mathbb{R}_+\) to the agent, we conclude that \(V_1 \leq \hat{V}\) as desired. In summary, when (C.2) holds, \(V_{1c}\) is a \(C^1\) function that solves the principal’s program and is attained by the policy \(w_{1c}\), which is sufficient for optimality within the class of deterministic contracts. By Lemma 4.1, \(V_{1c}\) is concave and therefore the policy is also optimal among contracts employing randomization.

**Proof of Proposition 4.3.** We prove the proposition in three steps. First, we construct the value function under the stated policy and posit a system of boundary conditions. Second, we show that the system of boundary conditions has a unique solution, which characterizes \(u\). Third, we verify that, given this \(u\), the value function in (14)-(15) solves the principal’s first stage problem. The final (and omitted) step is to apply the same verification argument given in the proof of Proposition 4.2.

**Step 1:** Construct the value function under the stated policy. For \(u > u_\ast\), \(V_1\) evolves according to
(11) and therefore has a solution of the form (12). For \( u < u \), \( w(u) = w \) and therefore \( V_1 \) satisfies

\[
\lambda V_1(u) = \lambda V_2(w) - c - \lambda (w - u) V_1'(u),
\]

which is linear in \( u \) and has a solution of the form

\[
V_1(u) = V_2(w) - \frac{c}{\lambda} + K_2(u - w).
\]

In total, there are four unknowns to pin down: \((K_1, K_2, u, w)\). There are four boundary conditions, which are necessary for \( V_1 \) to be a valid solution to the principal’s problem. They are given by the terminal boundary condition at \( u = 0 \), as well as value-matching, smooth-pasting and twice differentiability at \( u \) (which is required to prove Step 3):

\[
\begin{align*}
V_1(0) &= 0 \quad \text{(A.5)} \\
V_1(u^-) &= V_1(u^+) \quad \text{(A.6)} \\
V_1'(u^-) &= V_1'(u^+) \quad \text{(A.7)} \\
V_1''(u^-) &= V_1''(u^+) \quad \text{(A.8)}
\end{align*}
\]

where \( f(x^-), f(x^+) \) denote left and right limits. Given (A.5)-(A.6), (A.7)-(A.8) can be interpreted as optimality conditions, which maximize the principal’s value function over all possible \((u, w)\).\(^{32}\)

**Step 2: Solution to the system of boundary conditions.** Relatively straightforward (though somewhat involved) algebra can be used to show (see footnote 32) that the solution to the system of boundary conditions requires that \( u \) satisfy

\[
\Pi - \frac{2c}{\lambda} - \left( \frac{\Pi - c}{\lambda} \right) \left( 2 + \frac{\lambda u}{\phi} \right) e^{-(1+\lambda u/\phi)} = 0, \quad \text{ (A.9)}
\]

which has a unique solution in \( \mathbb{R}_+ \) provided that (C.2) holds, and \((w, K_1, K_2)\) are given by

\[
\begin{align*}
w &= u + \phi/\lambda \quad \text{(A.10)} \\
K_1 &= \left( \frac{\Pi - c}{\lambda} \right) \left( \frac{\lambda u}{\phi} - 2 \right) e^{-1} \quad \text{(A.11)} \\
K_2 &= \frac{V_2(w) - c/\lambda}{w} = \frac{\lambda}{\phi} \left( \frac{\Pi - c}{\lambda} \right) e^{-(1+\lambda u/\phi)} - 1. \quad \text{(A.12)}
\end{align*}
\]

Using (A.12) in (A.4) gives the expression in (14) for \( V_1(u) \) when \( u \leq u \). Using (A.11) in (12) gives the expression in (15) for \( V_1(u) \) when \( u \geq u \).

**Step 3: Demonstrate that the solution solves (HJB1).** Define

\[
L(w, u) \equiv \lambda (V_2(w) - V_1(u) - (w - u) V_1'(u)) - c,
\]

where \( V_1 \) is the value function derived above. It suffices to check that \( \max_{w \geq u + \phi/\lambda} L(w, u) = 0 \) for all \( u \in \mathbb{R}_+ \). Note that \( L \) is concave in \( w \). Thus, the derivative of \( L \) w.r.t. \( w \) is everywhere decreasing and the maximum is attained either at the boundary (when \( V_2(u + \phi/\lambda) - V_1'(u) \leq 0 \)) or at an interior point (when \( V_2(u + \phi/\lambda) - V_1'(u) > 0 \)). The rest of the proof is broken into two cases.

\(^{32}\)See working paper version dated July, 2015 for a derivation of the boundary conditions as optimality conditions as well as a step-by-step derivation of the solution.
Case 1: For \( u < u_c \), \( V_1'(u) = V_2'(w) < V_1'(u + \phi/\lambda) \). Therefore,
\[
w^\ast \in \arg \max_w L(w, u) = \arg \max_w V_2(w) - wV_2'(w) \implies w^\ast = w
\]
and \( \max_w L(w, u) = L(w, u) = \lambda V_2(w) - V_1(u) - (w - u)V_1'(u) - c = 0 \), where the last equality follows from (A.3).

Case 2: For \( u \geq u_c \), we follow a similar approach as in the proof of Proposition 4.2 by showing that \( V_2(w) - wV_1'(u) \) is decreasing in \( w \) for all \( w \geq u + \phi/\lambda \). Using the closed-form expression in (15), we have that
\[
V_2'(u + \phi/c) - V_1'(u) = -\frac{\lambda}{\phi^2} (u - u)(\lambda \Pi - c) e^{-(1 + \lambda u)} \leq 0.
\]
Hence, \( \max_{u \geq u + \phi/\lambda} L(w, u) = L(u + \phi/\lambda, u) = 0 \), where the last equality follows from the fact that \( V_1 \) satisfies (11) for \( u \geq u_c \).

Proof of Proposition 4.4. To establish feasibility, note that \( V_1'(0) > 0 \) if and only if \( \Pi - \frac{2c}{\lambda} > \frac{\phi}{\lambda} + (\Pi - \frac{\phi}{\lambda}) e^{-1} \), which is implied by (C.2) and \( \phi < c \). Therefore, (C.2) is sufficient to ensure that \( u_c > 0 \) and hence the project is feasible. Also \( \lim_{u \to \infty} V_1'(u) = -1 \) and global concavity ensure that \( u_c \) is unique and finite.

Next we show that the contract in the proposition (denoted by \( \Gamma_c \)) is incentive compatible. Let \( W_t \) and \( U_t \) be the continuation value to the agent who has and has not made a breakthrough respectively and who follows the prescribed action under \( \Gamma_c \). For \( t \leq T^c \), we have that
\[
W_t = \int_t^{T^c} \lambda e^{-\lambda(T^c - t)} R^c(\tau) d\tau = \phi (T^c - t).
\]
Similarly, for \( t < T^c_1 \), we have that
\[
U_t = \int_t^{T^c_1} \lambda e^{-\lambda(T^c_1 - t)} W_{\tau} d\tau = \phi (T^c_1 - t).
\]
To verify that the contract is incentive compatible, notice that for all \( t < T^c_1 \), \( \lambda(W_t - U_t) = \lambda\phi(T^c - T^c_1) = \phi \) and similarly \( \lambda(R^c(t) - W_t) = \lambda (\phi (\frac{1}{\lambda} + T^c - t) - \phi) = \phi \) for all \( t < T^c \). Therefore, it is weakly optimal for the agent to work prior to termination and hence the contract is incentive compatible.

Finally, we show that the contract yields the value to the principal as given in Proposition 4.2. Conditional on a breakthrough at time \( t \), the principal’s expected continuation value is given by
\[
\int_t^{T^c} \lambda e^{-\lambda(T^c - t)} \left( b - \frac{c}{\lambda} - R^c(\tau) \right) d\tau = \left( b - \frac{C}{\lambda} \right) \left( 1 - e^{-\lambda(T^c - t)} \right) - \phi(T^c - t)
\]
\[
= V_2(W_t).
\]
Prior to the first breakthrough, the agent works for all \( t \in [0, T^c_1] \). Therefore, at any time \( t < \min\{T^c_1, \tau_1\} \), the principal’s expected continuation value is
\[
\int_t^{T^c_1} \lambda e^{-\lambda(T^c_1 - t)} \left( V_2(W_{\tau} - \frac{c}{\lambda}) \right) d\tau_1.
\]
Substituting in for \( V_2(W_{\tau_1}) \) and integrating yields \( V_1(U_t) \) as desired. Moreover, \( U_0 = \phi T^c_1 = u_c \) so that \( \mathcal{P}_0(\Gamma_c) = V_1(U_0) = V_1(u_c) \), which completes the proof. \( \square \)
Proof of Proposition 4.5. To establish feasibility, first note that for \( u \geq u \)

\[
V'_1(u) = \left( \frac{\lambda T - e}{\phi} \right) \left( \frac{\lambda (u - u)}{\phi} + 1 \right) e^{-(1+\lambda u/\phi)} - 1.
\]  

(A.13)

From this we obtain \( V'_1(u) = K_2 = F'_1(0) \). Because the value function is weakly concave, the principal prefers her outside option if \( K_2 < 0 \) (\( V'_1(u) < 0 \) for \( u > 0 \)) and prefers contracting with the agent if \( K_2 > 0 \) (\( V'_1(0) > 0 \)). Thus we wish to show that \( K_2 > 0 \) is equivalent to \((C.3)\). Algebraically manipulating \((A.12)\) shows that \( K_2 > 0 \) is equivalent to the condition \( w < \phi T^* \) which holds if and only if \( G(\phi T^*) > 0 \), where \( G(\cdot) \) is defined by

\[
G(w) \equiv \Pi - \frac{2c}{\lambda} - \left( \Pi - \frac{c}{\lambda} \right) \left( \frac{\lambda w}{\phi} + 1 \right) e^{-\lambda w/\phi}.
\]  

(A.14)

Using \((9)\) to simplify this condition and rearranging yields \((C.3)\).

For the rest of the proof assume \((C.3)\) holds (i.e., \( V'_1(0) > 0 \)). Note that \( \lim_{u \to \infty} V'_1(u) = -1 \) and \( V'_1(u) \) is strictly concave for \( u > u \), which establishes both the existence of the unique value \( u_n \equiv \arg \max_{u \geq 0} V'_1(u) \) and \( u_n > u \). Next, we define \( T_S, T_M \) and \( T_L \) by

\[
T_S \equiv \frac{u_n - u}{\phi},
\]  

(A.15)

\[
w \left( 1 - e^{-\lambda (T_M - T_S)} \right) \equiv u,
\]  

(A.16)

\[
T_L = T_S + \frac{w}{\phi}.
\]  

(A.17)

That \( T_S < T_M \) follows from \((A.16)\) and noting that \( w > u \). To show \( T_M < T_L \) note that \((A.16)\) can be rewritten in the form

\[
T_M - T_S = \frac{1}{\lambda} \ln \left( \frac{\lambda u}{\phi} \right).
\]  

(A.18)

Thus, from \((A.17)\), \( T_L > T_M \) iff \( \frac{\lambda u}{\phi} > \ln \left( \frac{\lambda u}{\phi} \right) \), which holds because \( \arg \min_{x \in \mathbb{R}} x - \ln(x) = 1 \).

Next, we show that the contract stated in the proposition (denoted by \( \Gamma_n \)) for \( T_S, T_M, T_L \) as defined above is incentive compatible. Let \( \mathcal{H}_1 \) denote the set of non-terminal histories after a breakthrough has been made, i.e., \( \mathcal{H}_1 \equiv \{(\tau_1, t) : \tau_1 \in (0, T_M], t \in [\tau_1, T^n(\tau_1))\} \). Let \( W(\tau_1, t) \) denote the continuation value at time \( t \geq \tau_1 \) to an agent who made a breakthrough at \( \tau_1 \) and follows the recommended action prescribed by \( \Gamma_n \). Using the functional form for the reward function of \( \Gamma_n \), we have that

\[
W(\tau_1, t) = \int_{t}^{T^n(\tau_1)} e^{-\lambda (t-t)} R^n(\tau_1, \tau_2) d\tau_2 = \phi (T^n(\tau_1) - t), \quad \forall (\tau_1, t) \in \mathcal{H}_1.
\]  

(A.19)

Thus, \( \lambda (R^n(\tau_1, t) - W(\tau_1, t)) = \phi \), for all \( (\tau_1, t) \in \mathcal{H}_1 \) and hence the contract is incentive compatible for all \( t \geq \tau_1 \).

For \( t < \tau_1 \), let \( U_t \) denote the continuation value (under \( \Gamma_n \)) to an agent who has not made a breakthrough by time \( t \). Conditional on making a breakthrough at time \( \tau_1 \), the agent’s continuation value under \( \Gamma_n \) is \( W(\tau_1, \tau_1) \). Thus,

\[
U_t = \int_{t}^{T_M} e^{-\lambda (t-t)} W(\tau_1, \tau_1) d\tau_1.
\]

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where the last equality follows from (A.10) and (A.16). Similarly, for \( t \) weakly higher payoff than the low type given any \( \Gamma \). Hence, any implementable pair must satisfy with probability one, otherwise the low-type agent can shirk and obtain a strictly positive expected payoff. Hence, if

\[
\begin{align*}
\text{Proof of Lemma 5.1.} \quad \text{Any implementable } \vec{u} \text{ such that } u_1 = 0 \text{ must involve immediate termination with probability one, otherwise the low-type agent can shirk and obtain a strictly positive expected payoff. Hence, if } u_1 = 0 \text{ then it must also be that } u_2 = 0. \text{ Also, clearly the high type can achieve a weakly higher payoff then the low type given any } \Gamma. \text{ Hence, any implementable pair must satisfy } u_2 \geq u_1. \text{ Therefore, } \mathcal{U} \subseteq \{(u_1, u_2) \in \mathbb{R}^2_+ : u_2 \geq u_1\} \cup (0, 0). \text{ The rest of the proof is by construction.}
\end{align*}
\]
Any \( \tilde{u} \in U_L \) can be implemented by providing the agent a choice between two contracts. The first contract terminates the project immediately with a severance payment, \( P = u_1 \). The second contract has a deadline \( T = u_1/\phi \) and a reward function \( R(t) = \phi \left( \frac{1}{\lambda} + T - t \right) + q \) where \( q = \frac{u_2 - u_1}{1 - e^{-\lambda u_1/\phi}} \). Clearly, the first contract delivers \( u_1 \) to both types. Under the second contract, the low type will strictly prefer to shirk (yielding a payoff of \( u_1 \)) and the high type will strictly prefer to work (yielding an expected payoff of \( u_2 \geq u_1 \)). Hence, each type weakly prefers to report truthfully and any \( \tilde{u} \in U_L \) is implementable.

To prove that any \( \tilde{u} \in U_H \) can be implemented, we use a contract that is independent of the agent’s report for a period of length \( \Delta \) at which point the continuation utilities are \( (\hat{u}_1, \hat{u}_2) \in U_L \) and the contract in the above paragraph is implemented for \( t > \Delta \). To do so, define \( S = \frac{u_2}{u_1} > 1 \) (since \( (u_1, u_2) \in U_H \)). Let \( (\hat{u}_1, \hat{u}_2) \) be the point in \( L_H \) that intersects the ray with slope \( S \) that goes through the origin and \( (u_1, u_2) \):

\[
S \hat{u}_1 = \hat{u}_1 + \frac{\phi}{\lambda} \iff \hat{u}_1 = \frac{\phi}{\lambda} \left( \frac{u_1}{u_2 - u_1} \right), \quad \text{and} \quad \hat{u}_2 = \frac{\phi}{\lambda} \left( \frac{u_2}{u_2 - u_1} \right).
\]

Let \( U_2(t), U_1(t) \) be the functions jointly satisfying

\[
\frac{dU_2}{dt} = \lambda S(U_1(t) - U_2(t)), \quad U_2(0) = u_2
\]

\[
\frac{dU_1}{dt} = \lambda(U_1(t) - U_2(t)), \quad U_1(0) = u_1,
\]

which have unique solutions given by

\[
U_2(t) = \frac{1}{S - 1} \left( S \left( u_1 + (u_2 - u_1)e^{-\lambda t(S-1)} \right) - u_2 \right) \tag{A.20}
\]

\[
U_1(t) = \frac{1}{S - 1} \left( Su_1 - u_2 + (u_2 - u_1)e^{-\lambda t(S-1)} \right). \tag{A.21}
\]

Note that \( U_2(t) - U_1(t) = (u_2 - u_1)e^{-\lambda t(S-1)} \), so define \( \Delta \) such that \( U_2(\Delta) - U_1(\Delta) = \frac{\phi}{\lambda} \). That is

\[
\Delta = \frac{1}{\lambda(S - 1)} \ln \left( \frac{u_2 - u_1}{\phi/\lambda} \right).
\]

Finally, let \( \hat{R}(t) = U_2(t) + S(U_2(t) - U_1(t)) \) for all \( t < \Delta \). Note that for \( t < \Delta \), neither type shirks since \( \hat{R}(t) \geq U_2(t) + \frac{\phi}{\lambda} \) and \( U_2(t) \geq U_1(t) + \frac{\phi}{\lambda} \). By construction, the contract that (i) pays \( \hat{R}(t) \) for ultimate success prior to \( \Delta \) and (ii) implements the contract from the first paragraph at \( t = \Delta \) (at which point \( (U_1(\Delta), U_2(\Delta)) = (\hat{u}_1, \hat{u}_2) \in U_L \)) delivers the desired expected utilities and induces truth telling, which completes the proof.

\[\square\]

**Proof of Proposition 5.2.** We first show that the expression on the right side of (20) is an upper bound. For \( \tilde{u} \in U_L \cup L_H \), we then show there exists a simple contract that achieves this bound. For \( \tilde{u} \in U_H \setminus L_H \), we construct a sequence of contracts under which the principal’s value converges to
Therefore, to satisfy \( (19) \), the termination policy must be such that \( \mathbb{E}[T] \leq u_1/\phi \). The promise keeping constraint requires

\[
\begin{align*}
    u_2 - \mathbb{E}^a &\left[ \int_0^{T \wedge \tau_2} g(a_t) dt \mid s = 1 \right] = \mathbb{E}^a \left[ \int_0^{T \wedge \tau_2} dY_t \mid s = 1 \right],
\end{align*}
\]

and substituting into \( (16) \), we get that

\[
F_2(u_1, u_2) \leq \sup_{a,T} \mathbb{E}^a \left[ b \cdot 1_{(\tau_2 \leq T)} - \int_0^{T \wedge \tau_2} (c - g(a_t)) dt \mid s = 1 \right] - u_2, \quad \text{s.t.} \; \mathbb{E}[T] \leq u_1/\phi.
\]

The right-hand side is increasing in \( a_t \), hence

\[
F_2(u_1, u_2) \leq \sup_T \left( \Pi - \frac{c}{\lambda} \right) \mathbb{E} \left[ 1 - e^{-\lambda T} \right] - u_2, \quad \text{s.t.} \; \mathbb{E}[T] \leq u_1/\phi.
\]

By Jensen’s inequality we have that \( \mathbb{E} \left[ 1 - e^{-\lambda T} \right] \leq 1 - e^{-\lambda \mathbb{E}[T]} \), which is strictly increasing in \( \mathbb{E}[T] \), and therefore the constraint binds. Inserting \( \mathbb{E}[T] = u_1/\phi \) completes the proof of the bound.

For \( \tilde{u} \in U_L \cup L_H \), the bound can be achieved by a simple contract with a deadline of \( T = u_1/\phi \) and \( R(t) = \phi \left( \frac{1}{\lambda} + T - t \right) + q \) where \( q = \frac{u_2 - u_1}{1 - e^{-\phi u_1/\phi}} \). For \( \tilde{u} \in U_H \setminus L_H \), consider the sequence of simple contracts indexed by \( S \) and defined as follows:

\[
R_S(t) = \begin{cases} 
    U_2(t) + S(U_2(t) - U_1(t)), & t \in [0, \Delta_S] \\
    U_2(t) + \frac{\phi}{\lambda}, & t \in (\Delta_S, T_S]
\end{cases}
\]

\[
\Delta_S = \frac{1}{\lambda(S - 1)} \ln \left( \frac{u_2 - u_1}{\phi/\lambda} \right), \quad T_S = \frac{U_1(\Delta_S)}{\phi},
\]

where \( U_2(t) \) and \( U_1(t) \) are given by \( (A.20) \) and \( (A.21) \) respectively. By construction, we have that

\[
\begin{align*}
    u_2 &= \int_0^{T_S} \lambda e^{-\lambda t} R_S(t) dt \\
    u_1 &= \int_0^{\Delta_S} \lambda^2 t e^{-\lambda t} R_S(t) dt + e^{-\lambda \Delta_S} (1 + \lambda \Delta S) \phi(T_S - \Delta_S)
\end{align*}
\]

\[
R_S(t) \geq U_2(t) + \frac{\phi}{\lambda}.
\]

Therefore, it is incentive compatible for the high-type agent to work for all \( t \in [0, T_S] \) and his expected payoff from doing so is exactly \( u_2 \). Furthermore, given that \( U_2(t) = U_1(t) + \frac{\phi}{\lambda} \) for all \( t \geq \Delta_S \), the maximal payoff to the low type is exactly \( u_1 \). Finally, the expected payoff to the principal under this contract is given by

\[
P_S = (1 - e^{-\lambda T_S}) \left( b - \frac{\phi}{\lambda} \right) - u_2 \quad \text{and} \quad \lim_{S \to \infty} T_S = u_1/\phi.
\]
implies that \( \lim_{S \to \infty} P_S = F_2(u_1, u_2) \).

\[ \text{Lemma A.1. For any } \Gamma \text{ satisfying (21)-(23) that involves the agent shirking over some interval of time, there exists a } \hat{\Gamma} \text{ that also satisfies (21)-(23) such that the agent does not shirk and } \mathcal{P}_0(\hat{\Gamma}) \geq \mathcal{P}_0(\Gamma). \]

\[ \text{Proof. Let } (t_1, t_2) \text{ denote an arbitrary interval of time over which the agent shirks in the first stage under } \Gamma, \text{ where } 0 \leq t_1 < t_2 \leq T. \text{ To conserve notation, we assume that } \Gamma \text{ involves a deterministic termination rule (the arguments in the proof can easily be extended). Define } \Delta = t_2 - t_1. \text{ Let } \hat{\Gamma} \text{ be such that prior to a reported breakthrough:} \]

\[ \text{(i) } \hat{T} = T - \Delta, \]
\( \text{(ii) For all } t < t_1, \text{ (} \hat{a}_t, \hat{Y}_t, \hat{W}_1(t), \hat{W}_2(t) \text{) is identical to } (a_t, Y_t, W_1(t), W_2(t)), \)
\[ \text{(iii) For } t \in (t_1, \hat{T}), \text{ let } (\hat{a}_t, \hat{Y}_t, \hat{W}_1(t), \hat{W}_2(t)) = (a_{t+\Delta}, Y_{t+\Delta}, W_1(t + \Delta), W_2(t + \Delta)), \]
\[ \text{(iv) At time } t_1, \text{ the principal makes an unconditional payment to the agent in the amount of} \]
\[ d\hat{Y}_{t_1} = \phi \Delta + \mathbb{E}^{a=0} \left[ \int_{t_1}^{t_2} dY_t | s = 0 \right]. \]

It is straightforward to check that if \( \Gamma \) satisfies (21)-(23) for all \( t < T \) then \( \hat{\Gamma} \) satisfies (21)-(23) for all \( t < \hat{T} \). Prior to \( t_1 \), both the agent’s action and the principal’s payoff conditional on a breakthrough is the same under both contracts. If a breakthrough does not happen prior to \( t_1 \), then \( \mathcal{P}_{t_1}(\hat{\Gamma}) = \mathcal{P}_{t_1}(\Gamma) + (c - \phi) \Delta \geq \mathcal{P}_{t_1}(\Gamma) \). Hence, \( \mathcal{P}_0(\hat{\Gamma}) \geq \mathcal{P}_0(\Gamma) \).

\[ \text{Proof of Proposition 5.3. First, we construct the value function under the stated policy. Using the boundary conditions (26) and (27), we pin down } u_s \text{ and show that the value function under the stated policy indeed has the form given by (28)-(29). We then verify that, given the } u_s \text{ implied by the boundary conditions, the value function in (28)-(29) solves (HJB1’). That max}_u F_1(u) \text{ is an upper bound on the solution to (OBJ1’) is immediate (since it relaxes (23)). In the Proof of Proposition (5.4), we show there exists a contract satisfying this additional constraint that achieves the bound.} \]

For \( u > u_s \), \( F_1 \) evolves according to (24) and therefore has a solution of the form (25). For \( u < u_s \), the principal’s value under the stated policy is given by
\[ F_1(u_1) = \frac{u_1}{u_s} F_1(u_s) \quad \text{(A.22)} \]

There are two unknowns to pin down: \((u_s, H_1)\). Solving (26) for \( H_1 \) gives
\[ H_1 = \frac{\lambda u_s}{c} \left( -c \lambda \left( \Pi \phi \left( e^{\frac{\lambda u_s}{\phi}} - 2 \right) + u_s \right) + b \lambda^2 u_s + 2 c^2 \phi \left( e^{\frac{\lambda u_s}{\phi}} - 1 \right) \right) - \frac{c \phi (c \phi + \lambda u_s)^2}{c \phi (c \phi + \lambda u_s)^2} \]

Plugging this expression into (27), (or, equivalently, maximizing \( H_1 \) over all possible \( u_s \)), we get that \( u_s \) is defined implicitly by
\[ b - \frac{2c}{\lambda} - \left( \Pi - \frac{c}{\lambda} \right) \left( 2 + \frac{\lambda u_s}{\phi} \right) e^{-\lambda u_s/\phi} = 0, \quad \text{(A.23)} \]
and hence
\[ H_1 = \left( \frac{\lambda u_s}{\phi} - 2 \right) \left( \Pi - \frac{c}{\lambda} \right). \quad \text{(A.24)} \]
Substituting (A.24) into (25), we get (29). Then (28) follows from (A.22), which verifies that the value function under the stated policy for $u_s$ given by (A.23) indeed has the stated form.

Finally, we verify that $F_1$ solves (HJB1') subject to the four constraints. Given $u_s$ as defined implicitly by (A.23), one can easily check that $F_1(0) = 0$ and thus (BC1') is satisfied. Using arguments already given in the text, subject to (IC1'), (PK1'), and (NFP), we have that

$$
\sup_{w_1,w_2,\sigma} \left\{ \lambda F_2(w_1, w_2) - (\lambda + \sigma) F_1(u_1) - c + F_1^*(u_1) \frac{du_1}{dt} \right\}
= \lambda \left( F_2(u_1, u_1 + \phi/\lambda) - F_1(u_1) \right) - c - \phi F_1^*(u_1) + \sup_{\sigma} \left\{ \sigma \left( u_1 F_1^*(u_1) - F_1(u_1) \right) \right\},
$$

and by construction, $L_s(u_1) = 0$ for $u_1 \geq u_s$ and $u_1 F_1^*(u_1) - F_1(u_1) = 0$ for $u_1 \leq u_s$. Hence, it is sufficient to show that (i) $L_s(u_1) \leq 0$ for $u_1 < u_s$, and (ii) $u_1 F_1^*(u_1) - F_1(u_1) \leq 0$ for $u_1 > u_s$.

For (i), notice from (29) that $L_s$ is concave and hence $L_s'$ is decreasing for all $u_1 \in [0, u_s)$. That $L_s'(u_s) = 0$ implies $L_s$ is increasing below $u_s$, and that $L_s(u_s) = 0$ then gives the result. For (ii), notice from (29) that $F_1''(u_1) < 0$ and hence $u_1 F_1^*(u_1) - F_1(u_1)$ is decreasing for all $u_1 > u_s$. The result then follows since $u_s F_1^*(u_s) = F_1(u_s)$. \hfill \Box

The following lemma will be used in the proof of Proposition 5.4.

**Lemma A.2.** Define $u_I \equiv \arg \max_{u \geq 0} F_1(u)$.

1. If (C.4) holds strictly, then $u_s < \phi T^* < u_I$ and $F_1(u_I) > 0$.
2. If (C.4) holds with equality, then $u_s = \phi T^* = u_I$ and $F_1(u_I) = 0$.
3. If (C.4) is violated, then $F_1(u_I) < 0$.

**Proof.** A non-trivial solution to the contract-design problem exists under intangible progress if and only if $u_I > u_s$ satisfies the first-order condition

$$
\left( \frac{\lambda I - c}{\phi} \right) \left( \frac{\lambda (u_I - u_s)}{\phi} + 1 \right) e^{-\lambda u_I/\phi} - 1 = 0,
$$

(A.25)

or

$$
u_s = u_I + \frac{\phi}{\lambda} \left( 1 - e^{\lambda (u_I/\phi - T^*)} \right).
$$

The right hand side is maximized when $u_I = \phi T^*$, in which case $u_s = \phi T^*$. Therefore $u_I > u_s$ iff $\phi T^* > u_s$. To determine when $\phi T^* > u_s$, consider the function

$$
J(u) \equiv b - \frac{2c}{\lambda} \left( \Pi - \frac{c}{\lambda} \right) \left( \frac{\lambda u}{\phi} + 2 \right) e^{-\lambda u/\phi}.
$$

Observe that $J(0) = -b$, $\lim_{u \to \infty} J(u) = b - 2c/\lambda$, and $J'(u) > 0$ for all $u < \infty$. Moreover, from (A.23), $J(u_s) = 0$. Therefore, $\phi T^* > u_s \iff J(\phi T^*) > 0$. The latter is equivalent to (C.4). \hfill \Box

**Proof of Proposition 5.4.** Let $\Gamma_s$ denote the contract stated in the proposition with

$$
T_s \equiv \max \left\{ \frac{u_I - u_s}{\phi}, 0 \right\}.
$$

We first show that $\Gamma_s$ induces the prescribed behavior by the agent (i.e., truth telling and no shirking). Start from any $t \geq \tau_1$ (i.e., after a breakthrough has been made), and let $U_2(\tau_1, t)$ denote
the high-type agent’s equilibrium continuation value at time $t$ from following the prescribed behavior in $\Gamma_s$.

- If $\tau_1 \geq T_s$, then
  \[ U_2(\tau_1, t) = \int_t^{\tau_1 + 1/\sigma} \lambda e^{-\lambda(\tau_2 - t)} R_s(\tau_1, \tau_2) d\tau_2 = \phi \left( 1 + \frac{\sigma}{\lambda} \right) \left( \frac{1}{\sigma} + \tau_1 - t \right). \] (A.26)

To verify it is optimal for the agent to work (conditional on reporting truthfully) until making the second breakthrough or until running out of time, notice that

\[ \lambda(R_s(\tau_1, t) - U_2(\tau_1, t)) = \phi \left( 1 + \frac{\sigma}{\lambda} \right) > \phi. \]

Hence, working is strictly optimal. To verify that it is optimal for the agent to report progress immediately (i.e., that (23) holds), note that for any $t \geq T_s$

\[ W_2(t) = U_2(t, t) = \phi \left( \frac{1}{\lambda} + \frac{1}{\sigma} \right). \]

Due to the stationarity of the continuation contract, if the agent prefers to delay reporting progress at $t = \tau_1$, then he prefers to delay reporting progress indefinitely (i.e., to never report progress). If he delays a report then it is strictly optimal to shirk (since the reward is zero if $\tau_2 \in (T_s, \hat{T})$) and his expected payoff is

\[ \int_t^\infty \sigma e^{-\sigma(s-t)} \phi(s-t) ds = \frac{\phi}{\sigma} < U_2(t, t). \]

Hence, the agent strictly prefers to report progress as soon as it arrives for all $\tau_1 \geq T_s$.

- If $\tau_1 < T_s$, then
  \[ U_2(\tau_1, t) = \int_t^{T_s} \lambda e^{-\lambda(\tau_2 - t)} R_s(\tau_1, \tau_2) d\tau_2 + \int_{T_s}^{\tau_1 + 1/\sigma} \lambda e^{-\lambda(\tau_2 - t)} R_s(\tau_1, \tau_2) d\tau_2 = \phi \left( \frac{1}{\lambda} + \frac{1}{\sigma} + T_s - t \right). \]

Hence, $\lambda(R_s(\tau_1, t) - U_2(\tau_1, t)) = \phi$ and $a_t = 1$ is weakly optimal, regardless of whether progress is reported (since $R_s$ is independent of $\hat{T}$ for $\tau_2 \leq T_s$). Further, the same argument as in the bullet above shows that the agent prefers to report progress at $t = T_s$ and work from that point forward.

Now consider any $t \leq \tau_1$ (i.e., prior to a breakthrough being made). Let $U_1(t)$ denote the low-type agent’s equilibrium continuation value at time $t$ from following $\Gamma_s$.

- If $t \geq T_s$, then $W_2(t) = \phi \left( \frac{1}{\lambda} + \frac{1}{\sigma} \right)$ and thus
  \[ U_1(t) = \int_t^\infty \lambda e^{-(\lambda + \sigma)(\tau_1 - t)} W_2(\tau_1) d\tau_1 = \frac{\phi}{\sigma}. \] (A.27)

Since $\lambda(W_2(t) - U_1(t)) = \phi$, working is (weakly) optimal for the agent. Next, we verify that the agent does not want to falsely report progress at any $t \geq T_s$. Due to the stationarity of the continuation contract, if the agent prefers to falsely report progress at $t > T_s$, then he prefers to do so at $t = T_s$. Thus, suppose he falsely reports progress at the soft deadline. Let $\hat{U}(t)$ be his expected payoff at $t \geq T_s$ from acting optimally henceforth. Consider the following chain
\[ T_s < t \iff \phi \left( \frac{\sigma}{\lambda} \right) \left( \frac{1}{\sigma} + T_s - t \right) < \frac{\phi}{\lambda} \]
\[ \iff \phi \left( 1 + \frac{\sigma}{\lambda} \right) \left( \frac{1}{\sigma} + T_s - t \right) - \phi \left( \frac{1}{\sigma} + T_s - t \right) < \frac{\phi}{\lambda} \]
\[ \iff U_2(\tau_1, t) - \phi \left( \frac{1}{\sigma} + T_s - t \right) < \frac{\phi}{\lambda} \]

where the last line follows from (A.26). Next note that

\[ \phi \left( \frac{1}{\sigma} + T_s - t \right) \leq \tilde{U}(t), \]

because the left side is the payoff to the agent from falsely reporting progress at \( T_s \), having no breakthrough, and shirking from date \( t \) to \( T_s + \frac{1}{\sigma} \) and the right side is his expected payoff from falsely reporting progress at \( T_s \), having no breakthrough, and acting optimally from date \( t \) to \( T_s + \frac{1}{\sigma} \). Thus we have

\[ U_2(t) - \tilde{U}(t) < \frac{\phi}{\lambda}, \]

which says that it is suboptimal for the agent to work at any \( t > T_s \) after falsely reporting progress at \( T_s \). Thus, if the agent falsely reports progress at \( T_s \), then it is optimal for him to shirk until time runs out; i.e., \( \tilde{U}(t) = \phi \left( \frac{1}{\lambda} + T_s - t \right) \). Finally, recall that \( \phi/\sigma \) is the agent’s expected payoff from honestly reporting no progress at \( T_s \) and then working while on probation. Thus, at \( T_s \) the agent is indifferent between honestly reporting no progress and then (optimally) working while on probation and falsely reporting progress and then (optimally) shirking until time runs out. Therefore, honestly reporting no progress at \( T_s \) is weakly optimal.

- Next, consider \( t < T_s \). Noting that

\[ W_2(t) = \begin{cases} 
\phi \left( \frac{1}{\lambda} + \frac{1}{\sigma} + T_s - t \right), & \text{if } t < T_s \\
\phi \left( \frac{1}{\lambda} + \frac{1}{\sigma} \right), & \text{if } t \geq T_s.
\end{cases} \]

and

\[ U_1(t) = \int_t^{T_s} \lambda e^{-\lambda(\tau_1 - t)} W_2(\tau_1) d\tau_1 + \int_{T_s}^{\infty} \lambda e^{-(\lambda + \sigma)(\tau_1 - t)} W_2(\tau_1) d\tau_1 = \phi \left( \frac{1}{\sigma} + T_s - t \right), \]

we get that \( \lambda(W_2(t) - U_1(t)) = \phi \) for all \( t < T_s \), which shows that working is weakly optimal. Finally note that working until \( T_s \) is also optimal for the agent if he plans to falsely report progress at \( T_s \), that is \( \tilde{U}(t) = U_1(t) \) for \( t \leq T_s \).

Thus we have shown that \( \Gamma_s \) induces truth-telling and no shirking. Following similar steps to those in the proof of Proposition 4.5, it is straightforward to verify that the principal’s expected payoff under \( \Gamma_s \) corresponds to \( F_2(\tilde{U}(t), U_2(t)) \) in the second stage and \( F_1(U_1(t)) \) in the first stage. Further, \( U_1(0) = \phi(T_s + 1/\sigma) = u_1 \), so that \( F_1(U_1(0)) = F_1(u_1) \), which we have already argued is an upper bound on the solution to the principal’s problem (see Proof of Proposition 5.3). Thus, we can conclude that \( \Gamma_s \) is an optimal contract. That (C.4) is necessary and sufficient for a feasible contract to exist follows immediately from Lemma A.2.
B Supplemental Appendix (for online publication only)

Proof of Proposition 6.1. First, suppose that (C.2) holds. From Proposition 4.4, we have that

\[ \mathbb{E}[T^c | \tau_1 \leq T^c_1] = 1/\lambda + T^c_1 - \mathbb{E}[\tau_1 | \tau_1 \leq T^c_1] \]

Noting that \( \mathbb{E}[\tau_1 | \tau_1 \leq T^c_1] < \mathbb{E}[\tau_1] = 1/\lambda \) yields \( \mathbb{E}[T^c | \tau_1 \leq T^c_1] > T^c_1 \), completing the argument for this case. Next, suppose (C.2) does not hold. Observe that \( \frac{1}{\lambda} = \mathbb{E}[\tau_1] > \mathbb{E}[\min\{\tau_1, T_S\}] | \tau_1 \leq T_M \) or \( 1 > \lambda \mathbb{E}[\min\{\tau_1, T_S\}] | \tau_1 \leq T_M \). Next, \( \min x - \ln(x) = 1 \) implies

\[ \frac{\lambda w}{\phi} - \ln \left( \frac{\lambda w}{\phi} \right) > \lambda \mathbb{E}[\min\{\tau_1, T_S\}] | \tau_1 \leq T_M]. \]

Substituting from (A.18) gives \( \frac{w}{\phi} - (T_M - T_S) > \mathbb{E}[\min\{\tau_1, T_S\}] | \tau_1 \leq T_M]. \) Substituting from (A.17) gives \( \mathbb{E}[T^c(\tau_1)|\tau_1 \leq T^c_1] > T^c_1 \), which establishes the claim.

\( \square \)

Proof of Proposition 6.2. This result follows immediately from the fact that the total expected amount of time the agent is given to complete the project is fixed at \( t = 0 \) to be \( T_s + u_s/\phi \) and does not increase following a (reported) breakthrough.

\( \square \)

Proof of Proposition 6.3. Suppose that the agent has the first breakthrough when the state is \( u_1 > 0 \). By (8) and (20), \( V_2(w_2) > F_2(w_1, w_2) \) iff \( w_2 > w_1 \), which holds because \( w_2 = w_1 + \phi/\lambda \). Now, consider (HJB1) and (HJB1'). Given that (NFP) binds, the only difference between the two programs is the second-stage value functions \( V_2(w_2) \) and \( F_2(w_1, w_2) \). But \( V_2(w_2) > F_2(w_1, w_2) \) then implies directly that \( V_1(u_1) > F_1(u_1) \). Finally, the probability of project completion is higher when progress is tangible because

\[ \frac{V_1(u_1) + u_1}{b - 2c\lambda} > \frac{F_1(u_1) + u_1}{b - 2c/\lambda}. \]

The expressions on the left and right are the probabilities of project completion under tangible and intangible progress respectively.

\( \square \)

Proof of Proposition 6.4. We decompose the proof into two cases.

Case 1. Suppose (C.2) holds. We wish to show \( u_I > u_c \). Given concavity of the value functions, this is true iff \( F'_1(u_I) > V'_1(u_I) \). Straightforward algebra shows that this is equivalent to

\[ (\lambda \Pi - c) \left( \frac{\lambda (u_I - u_c)}{\phi} + 1 \right) > (\lambda \Pi - c) \left( \frac{\lambda u_I}{\phi} - 1 \right) e^{-1} + (\lambda \Pi - 2c) \]. \tag{A.28} \]

Now, choose \( \lambda, \Pi, \) and \( c \) to be any values satisfying (C.2), and define \( z \equiv \frac{\lambda \Pi - c}{c} \) and \( \theta \equiv \phi/c \). Let \( \bar{\theta}(z) \) be the value of \( \theta \) satisfying (C.4) with equality. That is,

\[ \frac{z}{\bar{\theta}(z)} - \ln \left( \frac{z}{\bar{\theta}(z)} \right) - 1 - \frac{1}{\bar{\theta}(z)} = 2 \equiv 0. \tag{A.29} \]

Using (C.2), (A.29), and Lemma A.2 it can be shown that a sufficient condition for (A.28) is \( z/\bar{\theta}(z) > 1.42 \). We now demonstrate that this holds for all relevant values of \( z \). Implicitly differentiating (A.29) yields

\[ \left( \frac{\bar{\theta} - \bar{\theta}' z}{\bar{\theta}^2} \right) \left( 1 - \frac{\bar{\theta}'}{z} \right) + \frac{\bar{\theta}'}{\bar{\theta}^2} = 0. \tag{A.30} \]
From this we obtain
\[ \bar{\theta}' = \frac{\bar{\theta}^2/z - \bar{\theta}}{1 + \bar{\theta} - z}. \]

The numerator on the right is negative by (C.1) and the denominator is negative by (A.29). Thus \( \bar{\theta}(z) \) is increasing. Also from (A.30) we have
\[ \frac{d}{dz} \left( \frac{z}{\bar{\theta}} \right) = \frac{\bar{\theta} - \bar{\theta}' z}{\bar{\theta}^2} = -\frac{\bar{\theta}' / \bar{\theta}^2}{1 - \bar{\theta}/z}. \]

The denominator is positive by (C.1). Therefore, \( z/\bar{\theta}(z) \) is decreasing. Define \( z_0 \) implicitly by
\[ \bar{\theta}(z_0) = 1, \]
\( \text{corresponding to the highest possible value of } \theta \) (i.e., \( \phi = c \)). Because \( \bar{\theta}(z) \) is increasing, we need only consider values \( z \in [0, z_0] \). Moreover because \( z \bar{\theta}(z) \) is decreasing, we have
\[ z \bar{\theta}(z) > 1 \quad \forall z \in [0, z_0] \iff \frac{z_0}{\bar{\theta}(z_0)} > 1.42. \]

Evaluating (A.29) at \( z_0 \) yields
\[ \frac{z_0}{\bar{\theta}(z_0)} - \ln \left( \frac{z_0}{\bar{\theta}(z_0)} \right) - 3 = 0 \implies \frac{z_0}{\bar{\theta}(z_0)} > 1.42. \]

**Case 2.** Suppose (C.2) does not hold. We wish to show \( u_I > u_n \). Given that \( V_1(u) \) is strictly concave for \( u > u_c \), the inequality holds if \( V_1'(u_I) < 0 \). Using Lemma A.2 it can be shown that a sufficient condition for this is
\[ \frac{\lambda (u_s - u)}{\phi} < e - 1. \]

It can also be shown that \( u_s - u \) is greatest when \( u = 0 \) (i.e., when (C.2) holds with equality). Let \( u_s^0 \) be the value of \( u_s \) when (C.2) holds with equality. From (A.9) and (A.23) we get
\[ \frac{\lambda (u_s - u)}{\phi} = 1 + \ln \left( \frac{2 + \lambda u_s / \phi}{2 + \phi / \phi} \right). \quad (A.31) \]

Thus, \( \frac{\lambda u_s^0}{\phi} \equiv 1 + \ln 1 + 0.5\lambda u_s^0 / \phi \implies \frac{\lambda u_s^0}{\phi} \approx 1.583 < e - 1. \)

**Lemma B.1.** For any feasible two-stage project:

1. If progress is tangible, the ex-ante total expected welfare under an optimal contract is
\[ V_1(u_J) + u_J = \Pi - \frac{2c}{\lambda} - \left[ \frac{\phi}{\lambda} \left( 1 + e^{\lambda T^* - \lambda u_J / \phi} \right) \right]. \]

2. If progress is intangible, then ex-ante expected welfare under an optimal contract is
\[ F_1(u_I) + u_I = \Pi - \frac{2c}{\lambda} - \left[ \frac{\phi}{\lambda} \left( 1 + e^{\lambda T^* - \lambda u_I / \phi} \right) \right]. \]

**Proof.** We prove the claim for the case when (C.2) holds. The proof for the case when it is violated follows exactly the same lines. By (13) and (29) we have
\[ V_{1c}(u_c) + u_c > F_1(u_I) + u_I. \]
iff 
\[
\left( \Pi - \frac{2c}{\lambda} \right) \left( 1 - e^{-(\lambda u_c/\phi)} \right) - \left( \Pi - \frac{c}{\lambda} \right) \frac{\lambda u_c}{\phi} e^{-(1+\lambda u_c/\phi)} > b - \frac{2c}{\lambda} - \left( \Pi - \frac{c}{\lambda} \right) \left( 2 + \frac{\lambda(u_I - u_s)}{\phi} \right) e^{-\lambda u_I/\phi}
\]

iff 
\[
\left( \Pi - \frac{2c}{\lambda} + \left( \Pi - \frac{c}{\lambda} \right) \frac{\lambda u_c}{\phi} e^{-1} \right) e^{-u_c/\phi} < \left( \Pi - \frac{c}{\lambda} \right) \left( 2 + \frac{\lambda(u_I - u_s)}{\phi} \right) e^{\lambda u_I/\phi}
\]

iff 
\[
\left( \frac{\lambda \Pi - c}{\phi} \right) e^{-(1+\lambda u_c/\phi)} + \left[ \frac{\lambda}{\phi} \left( \Pi - \frac{2c}{\lambda} + \left( \Pi - \frac{c}{\lambda} \right) \left( \frac{\lambda u_c}{\phi} - 1 \right) e^{-1} \right) e^{-u_c/\phi} - 1 \right] < \left( \frac{\lambda \Pi - c}{\phi} \right) e^{-\lambda u_I/\phi} + \left[ \frac{\lambda}{\phi} \left( \Pi - \frac{c}{\lambda} \right) \left( 1 + \frac{\lambda(u_I - u_s)}{\phi} \right) e^{\lambda u_I/\phi} - 1 \right].
\]

The first expression in square brackets is zero by definition of \( u_c \) (see (A.13)) and the second expression in square brackets is zero by definition of \( u_I \) (see (A.25)). The claim then follows.  

**Proof of Proposition 6.5.** We decompose the proof into two cases.

**Case 1.** Suppose (C.2) holds (strictly). By Lemma B.1 the claim holds iff \( u_c + \phi/\lambda > u_I \). First we show by contradiction that there are no parameter values for which \( u_c + \phi/\lambda = u_I \). If \( u_c + \phi/\lambda = u_I \) then by Lemma B.1 we have
\[
V_{1c}(u_c) + u_c = F_1(u_I) + u_I,
\]
which is equivalent to
\[
\frac{\lambda u_s}{\phi} - 1 = 2 - \frac{\lambda \Pi - 2c}{\lambda \Pi - c}.
\]
From (A.23) we have
\[
\left( \frac{\lambda u_s}{\phi} + 2 \right) e^{-\lambda u_s/\phi} = \frac{\lambda \Pi - 2c}{\lambda \Pi - c},
\]
if
\[
2 - \left( \frac{\lambda u_s}{\phi} + 2 \right) e^{-(\lambda u_s/\phi-1)} = 2 - \frac{\lambda \Pi - 2c}{\lambda \Pi - c} e^{-x}.
\]
Combining this with the previous equation gives
\[
2 - (x + 3)e^{-x} = x,
\]
where \( x \equiv \lambda u_s/\phi - 1 \). The solution to this equation necessitates \( u_s < 0 \), which violates limited liability. Therefore, continuity (i.e., Berge’s Theorem) implies that either \( u_c + \phi/\lambda > u_I \) or \( u_c + \phi/\lambda < u_I \) for all parameter values satisfying (C.4). We will show that if (C.4) holds with equality, then \( u_c + \phi/\lambda > u_I \), which from Lemma B.1 will establish the claim. If (C.4) holds with equality, then by Lemma A.2, \( u_I = \phi T^* \). Now setting (A.13) equal to zero and rearranging terms yields
\[
\frac{\lambda u_c}{\phi} + 1 = \lambda T^* + \ln \left( \frac{\lambda u_c}{\phi} + 1 + \gamma \right),
\]
where
\[
\gamma \equiv \frac{\lambda \Pi - 2c}{\lambda \Pi - c} e^{-2} - 2.
\]

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From (C.2), $\gamma > 0$ and so for all parameter values satisfying (C.4), $u_c + \phi/\lambda > \phi T^*$. 

**Case 2.** Suppose (C.2) is violated (weakly). By Lemma B.1 the claim holds iff $u_n + \phi/\lambda > u_I$. To see this, first observe that there are no parameter values for which $u_n + \phi/\lambda = u_I$. If this were ever true, then we would have

$$V_1(u_n) + u_n = F_1(u_I) + u_I$$

or

$$b - \frac{2c}{\lambda} - \left(\Pi - \frac{c}{X}\right)\left(2 + \frac{\lambda(u_n - u)}{\phi}\right)e^{-(1 + \lambda u_n/\phi)} = b - \frac{2c}{\lambda} - \left(\Pi - \frac{c}{X}\right)\left(2 + \frac{\lambda(u_I - u)}{\phi}\right)e^{-\lambda u_I/\phi}$$

which reduces to

$$u + \frac{\phi}{\lambda} = u_s,$$

contradicting (A.31). Therefore, continuity (i.e., Berge’s Theorem) implies that either $V_1(u_n) + u_n > F_1(u_I) + u_I$ or $V_1(u_n) + u_n < F_1(u_I) + u_I$ for all parameter values satisfying (C.4). Suppose (C.4) holds with equality. Then Lemma A.2 yields

$$F_1(u_I) + u_I = \phi T^*.$$

Because (C.3) is satisfied strictly when (C.4) holds with equality, we have $V_1(u_n) > 0$ and $u_n > \phi T^*$, which establishes the claim.

**Proof of Proposition 7.1.** Consider first any simple contract with deadline $T$. If the agent does not shirk, then the probability that the project succeeds at $t \in [0, T]$ is given by $\lambda t e^{-\lambda t}$ and the probability that the project does not succeed prior to $T$ is given by $e^{-\lambda T}(1 + \lambda T)$. Therefore, the total surplus is given by

$$S(T) \equiv \int_0^T \lambda^2 t e^{-\lambda t}(\Pi - ct)dt + e^{-\lambda T}(1 + \lambda T)(-cT).$$

In order to induce the agent to work, he must be given rents in the amount of at least $u = \phi T$, otherwise he can do better by shirking. Making the change of variables from $T$ to $u$, we have that the principal’s ex-ante expected payoff under a simple contract with deadline $T = u/\phi$ is bounded above by

$$G(u) \equiv S(u/\phi) - u = \left(1 - e^{-\lambda u/\phi}(1 + \lambda u/\phi)\right)\Pi - \frac{2c}{\lambda}\left(1 - e^{-\lambda u/\phi}(1 + \lambda u/2\phi)\right) - u.$$

Note that $G(u)$ is not the principal’s value function under the optimal contract, since her belief about the project stage changes over time. We will construct the value function shortly. To prove the proposition, it suffices to show that

(i) The principal’s ex-ante payoff for a project with unobservable progress under any contract is bounded above by $\max_u G(u)$.

(ii) There exists an incentive-compatible simple contract under which the principal’s ex-ante expected payoff is $\max_u G(u)$.

(iii) For all $u > 0$, $G(u) < F_1(u)$. Therefore, the principal does strictly better with intangible progress than she does with unobservable progress.

33Arguments similar to those made for a single-stage project can be used to confirm shirking is suboptimal.
For (i), let \( w^* = \arg \max_u G(u) \), which is generically unique. We have already argued that the principal’s maximal payoff under a simple contract is bounded above by \( G(w^*) \). Given that neither player has any information about the status of the project, the only possibility is that the principal randomizes over the termination date. It therefore suffices to show that the principal cannot benefit from such randomization, or equivalently, that the principal’s value function (under this simple contract with the optimally chosen deadline) is globally concave in the agent’s continuation value.

Suppose that the principal can implement a simple contract in which the incentive compatibility condition holds with equality for all \( t \) (this is the best possible case for the principal). It is most intuitive to construct this value function from the pair of value functions that are conditional on \( s \) (i.e., whether a breakthrough has been made) and weight them appropriately by the probability that the principal assigns to each. Given \( u \), the principal’s payoff conditional on being in the first stage \((s = 0)\) is \( g(u) \); i.e.,

\[
F^{\text{unobs}}(u|s = 0) = \int_0^{u/\phi} \lambda^2 t e^{-\lambda t} (b - ct) dt - e^{-\lambda u/\phi} (1 + \lambda u/\phi) cu/\phi - u
= (1 - e^{-\lambda u/\phi})(b - 2c/\lambda) - \frac{\lambda u}{\phi} e^{-\lambda u/\phi}(b - c/\lambda) - u.
\]

Conditional on being in the second stage, the principal’s value function is the benchmark payoff \( V(u) \):

\[
F^{\text{unobs}}(u|s = 1) = \int_0^{u/\phi} \lambda e^{-\lambda t} (b - ct) + e^{-\lambda u/\phi} (-cu/\phi) - u = (1 - e^{-\lambda u/\phi})(b - c/\lambda) - u.
\]

Over time, the principal’s beliefs will evolve about the state of the project. Conditional on reaching state \( u < w^* \) prior to project success, a period of time of length \( t(u; w^*) = \frac{w^*-u}{\phi} \) has elapsed. Therefore, the principal’s beliefs are given by

\[
\mu(u; w^*) = \Pr(\tau_1 \leq t(u; w^*)|\tau_2 > t(u; w^*)) = \frac{\lambda \left( \frac{w^*-u}{\phi} \right)}{1 + \lambda \left( \frac{w^*-u}{\phi} \right)}.
\]

The principal’s value function for \( u \leq w^* \) is therefore given by

\[
F^{\text{unobs}}(u; w^*) = \mu(u; w^*) F^{\text{unobs}}(u|s = 1) + (1 - \mu(u; w^*)) F^{\text{unobs}}(u|s = 0).
\]

We will now verify this value function is concave for all \( u \leq w^* \). Using the functional forms given above and twice differentiating \( F^{\text{unobs}}(u; w^*) \), we get that

\[
\frac{d^2}{du^2} F^{\text{unobs}}(u; w^*) = \frac{-\lambda e^{-\lambda u/\phi}}{\phi^2 (\lambda (w^* - u) + \phi)^2} \left[ (\lambda^3 w^* (w^* - u)^2 + \lambda w^* \phi^2) (\lambda b - c) + \lambda^2 \phi (w^* - u)^2 (\lambda b - 2c) + \phi^3 \left( \lambda b + 2 (e^{\lambda u} - 1) \right) \right].
\]

All three terms inside the brackets are clearly positive, implying the value function is concave in \( u \) for all \( u \leq w^* \), which completes the proof of (i).

We prove (ii) by showing that for any \( T \), there exists a \( w : [0, T] \rightarrow \mathbb{R}_+ \) such that (1) it is incentive compatible for the agent to work for all \( t \in [0, T] \), and (2) the agent’s continuation utility at date \( t \) is \( u(t) = \phi(T - t) \). Let \( u_2(t) \) be the promised continuation value conditional on being in
We want to find \( w \). This is established using the same method of proof as for Proposition 5.2 with two straightforward alterations. First, to satisfy (19), the termination policy must be such that \( \mathbb{E}[T] \leq (u_1 + \rho)/\phi \). Second, for \( u_1 \leq y \), there is a chance that the high type will have to pay the reporting cost. Therefore,

Conditional on progress, the evolution of \( u_2 \) is given by

\[
\lambda u_2(t) = \lambda w(t) + u_2'(t).
\]  

(A.32)

We want to find \( w(t) \) such that \( u(t) = \phi(T - t) \) for all \( t \in [0, T] \). Note that this implies that \( u'(t) = -\phi \). Using the promise keeping condition,

\[
\phi(T - t + 1/\lambda) = (1 - \mu(t))u_2(t) + \mu(t)w(t).
\]

Substituting for \( w(t) \) from (A.32), we get that

\[
\phi \left( T - t + \frac{1}{\lambda} \right) = (1 - \mu(t))u_2(t) + \mu(t) \left( u_2(t) - \frac{u_2'(t)}{\lambda} \right)
\]

\[
= u_2(t) - \frac{t}{1 + \lambda t} u_2'(t).
\]

Imposing the boundary condition \( u_2(T) = 0 \), we arrive at a unique solution for \( u_2(t) \), which we can then substitute back into (A.32), to arrive at

\[
w(t) = \phi \left( T - t + \frac{1}{\lambda} + \frac{e^{-\lambda(T-t)}}{\lambda^2 T} + \frac{e^{\lambda t}}{\lambda} (q(-\lambda T) - q(-\lambda t)) \right),
\]

where \( q(z) = -\int_{-\infty}^z e^{-x}/x \, dx \). It is straightforward to check that \( w(t) > 0 \) for all \( t \in [0, T] \), which completes the proof of (ii). For (iii), note that \( G(u) \) is identical to \( F_{1c}(u) \) with \( H_1^c = \frac{2c}{\lambda} - b \) (see (25)). The result then follows from the fact that \( F_1 \) has the same form as \( F_{1c} \), but with a constant strictly larger than \( \frac{2c}{\lambda} - b \).

\[\square\]

**Proof of Proposition 7.3.** The proof takes several steps. In Steps 1 through 3 we assume that the principal must use the formal channel, but that she does so optimally. In Step 4 we prove that for \( \rho \) sufficiently small it is, in fact, optimal for the principal to use the formal channel.

**Step 1.** First observe that the only possible reason for requiring a costly report is to relax the no-false-progress constraint so as to add more time to the clock following the first reported breakthrough. This implies that the formal communication channel should only be used to report progress (not lack of progress). Next, observe that it cannot be optimal for the principal to require the costly formal report at date \( t \) and take no action until date \( t' > t \), because this is dominated by waiting until \( t' \) to require the report. (The project might be completed between \( t \) and \( t' \).) Thus, it is optimal to put off requiring a formal report as long as possible. Let \( y \) be the highest value of the low type’s continuation utility at which the principal requires a formal report, and let \( F_2(u_1, u_2; \rho) \) denote the principal’s value function in the second stage. This value function is given by

\[
F_2(u_1, u_2; \rho) = \left( b - \frac{c}{\lambda} \right) \left( 1 - e^{-\lambda(u_1 + \rho)/\phi} \right) - u_2 - \mathbb{1}_{\{u_1 \leq y\}} \rho e^{-\lambda(u_1 - y)/\phi}.
\]  

(A.33)

This is established using the same method of proof as for Proposition 5.2 with two straightforward alterations. First, to satisfy (19), the termination policy must be such that \( \mathbb{E}[T] \leq (u_1 + \rho)/\phi \). Second, for \( u_1 \leq y \), there is a chance that the high type will have to pay the reporting cost. Therefore,
the promise keeping constraint necessitates

\[
u_2 + \rho e^{-\lambda(u_1-y)/\phi} = \mathbb{E}_{t=1} \left[ \int_0^{T_{\lambda \tau_2}} dY_t \right],
\]

Because \( F_2(u_1, u_2; \rho) \) is concave in \( u_1 \) and linear in \( u_2 \), \( \sigma = 0 \) is optimal.

**Step 4.** Define \( u_I(\rho) \equiv \arg \max_{u \geq 0} F_1(u; \rho) \). We wish to show that for \( \rho \) sufficiently small, \( F_1(u_I(\rho); \rho) > F_1(u_I) \). Because \( \lim_{\rho \to 0} u_I(\rho) = u_I \), the claim will follow if we show \( \frac{\partial F_1(u_I; 0)}{\partial \rho} > 0 \).

Applying the Envelope Theorem

\[
\frac{\partial F_1(u_I; 0)}{\partial \rho} = u_s'(0) \frac{\lambda}{\phi} \left( b - \frac{c}{\lambda} \right) e^{-\lambda u_I/\phi} + \frac{\lambda}{\phi} \left( b - \frac{c}{\lambda} \right) \left( 2 + \frac{\lambda(u_I - u_s)}{\phi} \right) e^{-\lambda u_I/\phi}.
\]

Moreover

\[
u_s'(0) = -\left( b - \frac{c}{\lambda} \right) \left( 2 + \frac{\lambda u_s}{\phi} \right).
\]

Therefore

\[
\frac{\partial F_1(u_I; 0)}{\partial \rho} = \left[ \frac{\lambda}{\phi} e^{-\lambda u_I/\phi} \left( b - \frac{c}{\lambda} - \frac{\phi}{\lambda} \right) \right] \left[ 2 + \frac{\lambda(u_I - u_s)}{\phi} - \frac{2 + \lambda u_s}{1 + \frac{\lambda u_s}{\phi}} \right].
\]

The first term in brackets is evidently strictly positive. Moreover, it follows from \( u_I > u_s > 0 \) and straightforward algebra that the second term in brackets is also strictly positive. \( \square \)
Proof of Proposition 7.4. We first construct the principal’s value function. The method for doing so follows closely that in Section 4 so we provide only an overview here. The principal’s value function in the second stage is immediately given by replacing $\lambda$ with $\lambda_2$ in equation (8). Next, suppose that the incentive compatibility condition binds for all $u$ in the first stage. Using the terminal boundary condition, we then arrive at the candidate value function

$$V_{1c}^\alpha(u) = \left( b - \frac{2c}{\lambda} \right) \left( 1 - e^{-\frac{\lambda u}{(1+\alpha)\phi}} \right) - u + \left( \frac{1-\alpha}{2\alpha} \right) e^{-\frac{1+\alpha}{1-\alpha}} \left( e^{-\frac{\lambda u}{(1+\alpha)\phi}} - e^{-\frac{\lambda u}{(1+\alpha)\phi}} \right) \left( \Pi - \frac{c(1-\alpha)}{\lambda} \right).$$

It is then easy to verify that $V_{1c}^\alpha$ is globally concave provided that

$$\frac{\lambda \Pi}{2c} \geq l(\alpha) \equiv \frac{(1-\alpha)\left( 1 - e^{-\frac{1+\alpha}{1-\alpha}} \right)}{1-\alpha - 2e^{-\frac{1+\alpha}{1-\alpha}}}.$$  \hspace{1cm} (A.35)

Therefore, if (A.35) holds then $V_{1}^\alpha = V_{1c}^\alpha$. We note that (A.35) reduces to (C.2) at $\alpha = 0$ and further that $l(\alpha)$ is decreasing. Hence, if (C.2) holds strictly, then there exists an $\epsilon > 0$ such that (A.35) holds for all $\alpha \in (-\epsilon, 1)$. For all such $\alpha$, $V_{1}^\alpha = V_{1c}^\alpha$.

Noting that $V_{1c}^\alpha$ is continuously differentiable in $\alpha$ on $(-1, 1)$, we have that

$$\left. \frac{d}{d\alpha} V_{1c}^\alpha(u) \right|_{\alpha=0} = \frac{\mu e^{-(1+\frac{\lambda u}{\phi})}}{\phi} (\lambda \Pi (3 - e) - 2c( e - 2) ) > 0 \iff \lambda \Pi > 2c \left( \frac{e - 2}{3 - e} \right).$$

The second inequality is implied by (C.2). Thus, we have shown that the principal’s value function is strictly increasing in $\alpha$ in a neighborhood around $\alpha = 0$ implying (i).

We can use a similar approach to demonstrate (ii),

$$\lim_{\alpha \uparrow 1} \frac{d}{d\alpha} V_{1c}^\alpha(u) = e^{-\frac{\lambda u}{2\phi}} \frac{2c - \lambda \Pi}{4\phi} < 0.$$ 

Hence, $V_{1}^\alpha$ is strictly decreasing in $\alpha$ at (and in a neighborhood just below) $\alpha = 1$, implying the existence of an $\alpha^{tan}$.

For (iii), let $V_{1c}^0(u) \equiv \lim_{\alpha \uparrow 1} V_{1c}^\alpha(u) = \lim_{\alpha \uparrow 1} V_{1}^\alpha(u)$, where the second inequality follows because our construction of the principal’s value function is valid for all $\alpha \in [0, 1)$. Also, let $\tilde{V}_{1}^{\lambda/2}(u) \equiv \left( 1 - e^{-\frac{\lambda u}{2\phi}} \right) \left( \Pi - \frac{2c}{\lambda} \right) - u$ denote the principal’s value function in a single stage project where the arrival rate is $\lambda/2$. Notice that $V_{1c}^0(u) = \tilde{V}_{1}^{\lambda/2}(u)$ (i.e., pointwise convergence is immediate). To prove (iii), it suffices to show that $0 < V_{1c}^\alpha(u) - \tilde{V}_{1}^{\lambda/2}(u) \leq m(\alpha)$ for any $u > 0$ and some $m(\alpha)$ such that $\lim_{\alpha \uparrow 1} m(\alpha) = 0$. For the first inequality, using the expression derived above, we have that

$$V_{1c}^\alpha(u) - \tilde{V}_{1}^{\lambda/2}(u) > 0 \iff \left( \Pi - \frac{2c}{\lambda} \right) r^\alpha(\lambda u/\phi) > \left( \frac{1-\alpha}{2\alpha} \right) e^{-\frac{1+\alpha}{1-\alpha}} \left( \Pi - \frac{c(1-\alpha)}{\lambda} \right) \hspace{1cm} (A.36)$$

where

$$r^\alpha(x) \equiv \frac{e^{-\frac{x}{\phi}} - e^{-\frac{x}{1+\alpha}}}{e^{-\frac{x}{1+\alpha}} - e^{-\frac{x}{1-\alpha}}}.$$ 

Since $r^\alpha(x)$ is increasing in $x$, it suffices to check that the inequality holds at $u = 0$. Taking the
limit gives \( r^\alpha(0) = \frac{(1-\alpha)^2}{\alpha} \). Plugging this into (A.36) and performing some algebra, can be used to show that (A.36) is equivalent to (A.35) at \( u = 0 \).

For the second inequality, we can use the fact that

\[
V^\alpha_{lc}(u) - \bar{V}^2(u) \leq \left( \Pi - \frac{2c}{\lambda} \right) \left( e^{-\frac{\lambda u}{2\alpha}} - e^{-\frac{\lambda u}{(1-\alpha)\beta}} \right),
\]

where \( e^{-\frac{x}{2}} - e^{-\frac{x}{(1-\alpha)\beta}} \) is a hump-shaped function in \( x \) that achieves its maximum at \( x^*(\alpha) = 2(1+\alpha)\ln(\frac{1+\alpha}{2})/(\alpha-1) \). Therefore, \( m(\alpha) = \left( \Pi - \frac{2c}{\lambda} \right) \left( e^{-\frac{\lambda u}{2}} - e^{-\frac{\lambda u}{(1+\alpha)}} \right) \) serves as a uniform bound that converges to zero as \( \alpha \to 1 \), which completes the proof of (iii).

Proof of Proposition 7.5. First, we extend the analysis from Section 5.3 to characterize \( F^\alpha_1 \) for an arbitrary \( \alpha \). Using the same arguments as in Proposition 5.2, we have that

\[
F^\alpha_2(u_1, u_2) = \left( 1 - e^{-\frac{\lambda u_1}{(1+\alpha)\beta}} \right) \left( \Pi - \frac{c(1-\alpha)}{\lambda} \right) - u_1.
\]

Replacing \( F_2 \) with \( F^\alpha_2 \) in (HJB1') and the appropriately modified (binding) constraints, we find that \( F^\alpha_1 \) has the form

\[
F^\alpha_{lc}(u_1) = \left( \Pi - \frac{2c}{\lambda} - u_1 \right) + \left( 1 - \frac{\alpha}{2\alpha} \right) \left( \Pi - \frac{(1-\alpha)c}{\lambda} \right) e^{-\frac{\lambda u_1}{(1-\alpha)\beta}} + C^\alpha_1 e^{-\frac{\lambda u_1}{(1+\alpha)\beta}}. \tag{A.37}
\]

Notice that the first term on the right hand side is the first-best value. The second term is strictly positive. Therefore, any solution must involve \( C^\alpha_1 < 0 \). If the terminal boundary condition is imposed \( (F^\alpha_{lc}(0) = 0) \) then \( C^\alpha_1 = -\frac{(1+\alpha)(\Pi - c(1+\alpha))}{2\alpha \lambda} \) and \( F^\alpha_{lc}(0) = \frac{\lambda^2 \Pi}{(1-\alpha)\beta} > 0 \). Hence, there exists some \( u_s(\alpha) \) such that random termination is optimal for \( u \in (0, u_s(\alpha)) \). Let \( c^\alpha(u) \) denote the constant in the principal’s value function that satisfies the smooth-pasting condition (i.e., (26)) at an arbitrary \( u > 0 \). That is,

\[
c^\alpha(u) \equiv \frac{2\alpha \lambda u}{(\alpha^2 - 1)\beta} \left( \phi \left( a^2 - 2a + 4ae^{\frac{\lambda u}{\alpha(\phi + \lambda u + \phi)}} + 1 \right) + (\lambda u - \alpha \lambda) - \lambda \Pi \left( -\alpha \phi + 2\alpha e^{\frac{\lambda u}{\phi}} + \lambda u + \phi \right) \right). \tag{A.38}
\]

Twice differentiability at \( u_s(\alpha) \) (i.e., (27)) is equivalent to \( u_s(\alpha) = \max_u c^\alpha(u) \), which requires the first-order condition

\[
\frac{e^{\lambda u_s(\alpha)}/\phi(1-\alpha)}{\lambda u_s(\alpha) + 2\phi} = \frac{\lambda \Pi - c(1-\alpha)}{\phi(1-\alpha)(\lambda \Pi - 2c)}. \tag{A.39}
\]

The right-hand side of the above expression is strictly greater than \( 1/2\phi \) for all \( \alpha \in (0, 1) \). The left-hand side is equal to \( 1/2\phi \) at \( u_s(\alpha) = 0 \), strictly increasing in \( u_s(\alpha) \) and unbounded. This guarantees the existence of a unique \( u_s(\alpha) \) satisfying (A.39), which completes the characterization of \( F^\alpha_1 \). To summarize,

- For \( u \geq u_s(\alpha) \), \( F^\alpha_1 \) is of the form given in (A.37) with \( C^\alpha_1 = c^\alpha(u_s(\alpha)) \), where \( u_s^\alpha \) is the unique solution to (A.39).
- For \( u \in (0, u_s(\alpha)) \), \( F^\alpha_1(u) = \frac{u}{u_s(\alpha)} F^\alpha_{lc}(u_s(\alpha)). \)

To prove (i), first note that by the envelope theorem \( \frac{d}{d\alpha} c^\alpha(u_s(\alpha)) = \frac{\partial}{\partial \alpha} c^\alpha(u_s(\alpha)). \) Using this
fact, evaluating the derivative and taking the limit as \( \alpha \to 0 \), we get that for \( u \geq u_s(0) = u_s \),

\[
\lim_{\alpha \to 0} \left( \frac{d}{d\alpha} F_1^\alpha(u) \right) = \left( \frac{e^{-\frac{\lambda u}{\phi}}(u(\lambda u_s + \phi) - \lambda u_s^2)}{\phi^2(\lambda u_s + \phi)^2} \right) \times \\
\left( \Pi \lambda \left( \lambda^2 u_s^2 - \phi^2 \left( \frac{\lambda u_s}{\phi} - 1 \right) + \lambda u_s \phi \right) - \left( \lambda^2 u_s^2 - 2\phi^2 \left( \frac{\lambda u_s}{\phi} - 1 \right) + 2\lambda u_s \phi \right) \right).
\]

The first-term on the right hand side is clearly positive for \( u \geq u_s \). Using (A.39), the second term reduces to \( \lambda(\lambda u_s + \phi)((\lambda^2 c)u_s - \Pi \phi) \), which is also clearly positive if \( \lambda u_s / \phi > \frac{\lambda^2 c}{\lambda^2 c - 1} \). We now claim that if \( u_s \) solves (A.39) for \( \alpha = 0 \), then this latter inequality must hold. Let \( x \equiv \lambda u_s / \phi \geq 0 \), \( y \equiv \lambda^2 c / \phi - 2 > 0 \), and \( \alpha = 0 \). The claim is that

\[
\frac{e^x}{x+2} = \frac{y+1}{y} \implies x > \frac{y+2}{y+1}.
\]

To see that this is true, suppose that \( \frac{e^x}{x+2} = \frac{y+1}{y} \) and \( x \leq \frac{y+2}{y+1} \). Note that \( \frac{e^x}{x+2} \) is strictly increasing. Therefore,

\[
\frac{e^x}{x+2} \leq \frac{e^{y/2}}{y+1} < \frac{y+1}{y},
\]

which gives the contradiction. We have thus shown that at \( \alpha = 0 \), the derivative of \( F_1^\alpha(u) \) w.r.t. \( \alpha \) is strictly positive for all \( u \geq u_s(\alpha) \). That the same statement is true for \( u \in (0, u_s) \) is immediate by the linearity of the value function below \( u_s \). Since \( F_1^\alpha \) is also continuously differentiable in both of its arguments, it must be strictly increasing in a neighborhood around \( \alpha = 0 \) for all \( u > 0 \), which completes the proof of (i).

For (ii) and (iii), we first show that \( \lim_{\alpha \to 1} u_s(\alpha) = 0 \). To do so, rewrite (A.39) as

\[
\phi(1-\alpha)e^{\lambda u_s(\alpha)/\phi(1-\alpha)} = \frac{(\lambda^2 c - (1-\alpha))(\lambda u_s(\alpha) + 2\phi)}{\lambda^2 c - 2\phi}.
\]

Suppose \( u_s(1) \equiv \lim_{\alpha \to 1} u_s(\alpha) \in (0, \infty) \). Then \( \lim_{\alpha \to 1} \phi(1-\alpha)e^{\lambda u_s(\alpha)/\phi(1-\alpha)} = \infty > \frac{(\lambda^2 c - (1-\alpha))(\lambda u_s(1) + 2\phi)}{\lambda^2 c - 2\phi} \), a contradiction. Also, clearly \( u_s(1) < \infty \) otherwise the principal’s value function would be arbitrarily negative. The only remaining possibility is \( u_s(1) = 0 \).

From (A.37), we have that for \( u \geq u_s^\alpha \),

\[
\lim_{\alpha \to 1} \left( \frac{d}{d\alpha} F_1^\alpha(u) \right) = e^{-\frac{\lambda u}{\phi}} \lim_{\alpha \to 1} \left( \frac{\lambda u}{\phi} e^\alpha(u_s(\alpha)) + \frac{\partial}{\partial \alpha} e^\alpha(u_s(\alpha)) \right).
\]

Notice from (A.38) that \( \lim_{\alpha \to 0} \left( \lim_{\alpha \to 0} e^\alpha(u) \right) = \lim_{\alpha \to 0} \left( \lim_{\alpha \to 1} e^\alpha(u) \right) = -(\Pi - 2c/\lambda) \). Therefore, we can conclude that \( \lim_{\alpha \to 1} \phi^\alpha(u_s(\alpha)) = -(\Pi - 2c/\lambda) \). Hence, to prove (ii), it is sufficient to show that \( \lim_{\alpha \to 1} \phi^\alpha(u_s(\alpha)) = 0 \), for this implies \( \frac{d}{d\alpha} F_1^\alpha(u) \approx e^{-\frac{\lambda u}{\phi}} (\Pi - 2c/\lambda) \frac{\lambda u_s}{\phi} < 0 \) for \( u \geq u_s(\alpha) \) and \( \alpha \) sufficiently close to 1. To see that \( \lim_{\alpha \to 1} \phi^\alpha(u_s(\alpha)) = 0 \), first notice from (A.39) that

\[
\lim_{\alpha \to 1} e^{-\frac{\lambda u_s(\alpha)}{\phi(1-\alpha)}} \equiv 0 \implies (1-\alpha) \in (0, \infty),
\]

implying that \( u_s(\alpha) = O((1-\alpha) \ln(1-\alpha)) \) as \( \alpha \to 1 \). Differentiating (A.38) with respect to \( \alpha \) and
omitting the argument of \( u_s(\alpha) \), we get that

\[
\frac{\partial}{\partial \alpha} e^\alpha (u_s(\alpha)) = \\
\frac{2 \lambda u_s}{\phi (\alpha + 1)} e^{2 \lambda u_s (\alpha)} + 2 \lambda u_s (\alpha) (\alpha^2 + 1) \\
+ \lambda^2 u_s^2 \phi \left[ 3 \alpha + 2 \alpha^2 (\alpha + 1) + \alpha^3 \left( 5 - 4 e^{-\lambda u_s (\alpha)} \right) \right] + 2 (1 - \alpha) (\alpha + 1) \lambda^2 u_s^2 \phi \left( 4 \alpha^4 - 4 \alpha + 4 \alpha^3 \right)
\]

Using (A.40), we know that \( \phi (\alpha + 1) \phi (1 - \alpha) \) is \( O(1 - \alpha) \) as \( \alpha \to 1 \). Therefore, any term inside the outermost brackets that goes to zero faster than \( O(1 - \alpha) \) will converge to zero when scaled by the fraction outside the brackets. By inspection, the only terms that do not clearly go to zero faster than \( O(1 - \alpha) \) are

\[
\lambda^2 u_s^2 (\alpha) \phi \left[ 4 \alpha^4 - 4 \alpha + 4 \alpha^3 \right] + 2 \alpha \phi \left( \phi (\alpha + 1) \phi (1 - \alpha) \right)
\]

Thus, we get have that

\[
\lim_{\alpha \to 1} \left( \frac{\partial}{\partial \alpha} e^\alpha (u_s(\alpha)) \right) = \lim_{\alpha \to 1} \left( \frac{2 \lambda u_s (\alpha)}{\phi (\alpha + 1)} e^{2 \lambda u_s (\alpha)} + 2 \lambda u_s (\alpha) (\alpha^2 + 1) \right)
\]

\[
= \lim_{\alpha \to 1} \left( \frac{\lambda u_s (\alpha)}{\phi (\alpha + 1)} \phi (1 - \alpha) \right)
\]

\[
= \frac{\lambda}{4 \phi^2} \left( \lim_{\alpha \to 1} u_s (\alpha) \right)^2
\]

which completes the proof of (ii). For (iii), we have

\[
F^{\alpha}_{1c}(u) - \tilde{V}^\alpha(u) = \left( 1 - \frac{\alpha}{2 \alpha} \right) \left( \Pi - \frac{(1 - \alpha) c}{\lambda} \right) e^{\frac{-\lambda u}{(1 - \alpha)}} + C_1^{\alpha} e^{\frac{-\lambda u}{(1 - \alpha)}} - \left( \Pi - \frac{2 c}{\lambda} \right) e^{-\frac{\lambda u}{2 \phi}}
\]

\[
\leq \left( 1 - \frac{\alpha}{2 \alpha} \right) \left( \Pi - \frac{(1 - \alpha) c}{\lambda} \right) + \left| C_1^{\alpha} \right| - \left( \Pi - \frac{2 c}{\lambda} \right)\left| e^{-\frac{\lambda u}{(1 - \alpha)}} - e^{-\frac{\lambda u}{2 \phi}} \right|
\]

\[
\leq \left( 1 - \frac{\alpha}{2 \alpha} \right) \Pi + \left| C_1^{\alpha} \right| - \left( \Pi - 2 c / \lambda \right)\left| e^{-\frac{\lambda u}{(1 - \alpha)}} - e^{-\frac{\lambda u}{(1 + \alpha)}} \right|
\]

where the first inequality uses the triangle inequality and \( e^{-|x|} \leq 1 \) and the second uses the fact that \( e^{-\frac{\lambda u}{(1 + \alpha)}} - e^{-\frac{\lambda u}{(1 - \alpha)}} \) is hump-shaped and achieves its maximum at \( x^*(\alpha) \) (see Proof of Proposition 7.4). Clearly all three terms converge to 0 as \( \alpha \to 1 \). □