Dividing and Discarding:
A Procedure for Taking Decisions with Non-transferable Utility*

(Preliminary Draft)

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Abstract

We consider a setting in which two players must take a single action. The analysis is done within a private values model in which (i) the players' preferences over actions are private information, (ii) utility is quadratic (non-transferable), (iii) implementation is bayesian and (iv) the welfare criterion is utilitarian. We characterize an optimal monotonic allocation rule. Instead of asking the agents to directly report their types, this allocation can be implemented dynamically. The agents are asked if they are to the left or to the right of the midpoint of the interval of possible types ($eg.1/2$ for the initial interval $[0,1]$). If both reports agree, the section of the interval which none preferred is discarded and the remaining interval is divided in two parts and the process continued until one agent chooses left and the other right. In that case, the midpoint of this remaining interval is implemented. This implementation can be carried out by a Principal who lacks commitment, implying this process is an optimal communication protocol.

1 Introduction

Many situations of interest require two agents to take a joint action. Examples abound– managers of two different divisions within a firm, tariff negotiations in a trade block, Monetary Union members deciding on monetary policy, parties in a political coalition, and others.

Before reaching a decision on the action or policy to be implemented, it is common for the agents to be involved in long conversations or negotiations that can take several rounds. Typically, a broad set of alternatives is considered at first, and slowly the set of alternatives "on the table" is refined until a decision is reached.

A conflict naturally arises between the agents' incentive to share information so that a better decision is taken and their fear that, if they reveal too much information, the other party might take advantage of it. A way around this problem is to reveal information very coarsely at first, and slowly refining it as agents

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learn that their interests are more aligned, and that, by sharing more information, a better decision can be attained. In contrast, once the agents learn that their positions are clearly in conflict there is no more scope for further communication.

In this paper, we consider an environment and an *optimal* decision rule that captures the features described above. We analyze the case in which the players' preferences over actions are private information and uniformly distributed, utility is non-transferable and the common action to be chosen belongs to an interval.[1]

The environment we consider makes the task of finding an optimal joint action hard because the scope for the players to misreport their preferences is very large and the instruments available to induce truthfulness are very limited. We further restrict attention to the case in which the Agent's preferences only depend on their own private information, so that, unlike other papers that analyze decisions in committees, there is no advantage in sharing information to uncover some underlying truth.[2] Note that, again, this assumption lowers the incentives for agents to be truthful about their types.

The optimal allocation rule we propose is very simple. It can be implemented by having a Principal or Mediator simultaneously asking the players if they are to the left or to the right of the midpoint over the remaining choice set. If they both agree on the side of the coarse partition they prefer, we discard the section of the interval which none preferred, and continue dividing the remaining interval in this way until one chooses left and the other right. In that case, the midpoint of the last remaining interval is implemented. We therefore name this mechanism the Divide and Discard mechanism (DD for short).

We find the DD appealing for a number of reasons. In spite of the complexity brought up by the lack of side payments, the DD is an extremely simple mechanism. Most importantly, its dynamic implementation resembles many real world situations (such as bilateral trade agreements) in which there are many rounds of negotiations and alternatives are successively discarded until an agreement is reached. The principal or mediator provides a way to mitigate the conflict that arises between the agents' incentives to share their information in order to achieve a better allocation and their fear that if they reveal too much information about their preferred actions the other player may manipulate the allocation to his advantage. This is resolved by having information being released very coarsely at first and gradually refined as agents learn that their interests are partially aligned. Indeed, the DD hints on why contracting parties that negotiate sequentially could commit not to consider choices that were eliminated in previous rounds, i.e., agree to rule out – Hart and Moore (2007) –: as players move along further rounds of negotiation, they can be confident about the alignment of their interests. Therefore, whenever an agreement is reached, it will necessarily deliver an outcome that cannot be Pareto dominated by those that were ruled out.

An additional advantage is that the amount of communication required by the DD is very low. In fact, with the dynamic implementation, each player just needs to communicate one bit of information per round. Furthermore, as soon as there is disagreement there is no need to keep conveying more refined information about the players' preferences. In settings for which, in addition to efficiency, a designer is concerned with informational requirements of the candidate mechanisms, such a feature is desirable.[3]

[1] If the decision were contractible and choices binary, a simple voting mechanism is able to attain the efficient outcome. See, for example, the analysis of enforceable voting by Maggi and Morelli (2006). Alternatively, if transfers were possible and players had quasilinear utility, the problem could be easily solved using the expected externality mechanisms proposed by Arrow (1979), and d'Aspremont and Gerard-Varet (1979).

[2] See, for example, Persico (2004).

[3] For communication requirements of arbitrary social choice rules, see Segal (2006).

We show that the DD can be implemented by a principal who lacks commitment. In many circumstances, it might be difficult for a mediator or principal to commit to a mechanism. Within a firm, for instance, it is not clear that a CEO with authority will commit to not overrule the divisions' managers. Therefore, the fact that it can be implemented without commitment is an important feature of the DD allocation.

The implementation of the DD allocation may call for several rounds of communication. Remarkably, the same expected value can be attained with just one round of cheap talk. The allocation that attains this value was derived by Alonso, Dessein and Matouschek (2008) (ADM from now on).[4]

An advantage of the long cheap talk is that the resulting allocation is renegotiation proof. Instead, with just one round of cheap talk both players could be reporting that their preferred allocation is in the same partition but would not be allowed to divide this partition further into smaller subdivisions in search of a better allocation. Also, as mentioned before, it seems natural that as the Agents learn that their interests are more aligned they are willing to reveal more and more details of their true preferences. For this, long conversations are necessary.

For the case in which transfers are available (and utility is quasi-linear), one can solve for the optimal allocation using the virtual utility representation of the players´ preferences and applying standard maximization techniques. Once an optimal is found, one can back up the necessary transfers to satisfy incentive compatibility.[5] In general, solving for optimal mechanisms with non-transferable utility and Bayesian implementation is a very hard problem. Nonetheless, we prove that the DD is optimal in the class of monotonic mechanisms by showing that it cannot be improved upon by any monotonic allocation rule. To establish this we combine standard methods, in setting up the problem, with a guess and verify approach to solve it. The proof consists of two main steps. First, we show that we can weakly improve upon any mechanism that does not have $\frac{1}{2}$ as the allocation when the preferred actions of the players are on different sides of $\frac{1}{2}$. Second, if allocation has this property, we show that the problem is separable and that hence it must be self-replicating. Self replication results in the DD allocation.

Moulin (1980), Barberà and Jackson (1994), and Barberà (2001) have studied the implementation of social choice functions in dominant strategies in more general settings than ours. In contrast to their work, we just require the allocations to be Bayesian Incentive Compatible. On the one hand, this makes the task of finding an optimal rule difficult as it is somewhat hard to pin down the set of all allocation that are interim Incentive Compatible. On the other hand, it allows for a better outcome for the players.

Our work also relates to the cheap talk literature. ADM (2008) extend to a two experts setting, Crawford and Sobel (1982) result, that if communication takes place just once players would communicate their private information coarsely by reporting intervals rather than precise types. In the spirit of Krishna and Morgan (2004) and Aumann and Hart (2003), who analyze multistage communication, we allow for long cheap talk. By showing that it is an optimal monotonic mechanism, we prove that the communication protocol induced by DD is optimal. As the ADM allocation generates the same expected value as the DD, talking through partitions is also an optimal communication protocol. Goltsman et al (2008) solve for optimal communication protocols in the Crawford and Sobel (1982) model. They study three different processes: (i) (possibly long) cheap- talk (negotiations), (ii) non-binding recommendations by a third party (mediation), and (iii) binding recommendations by a third party (arbitration). They show that, if the misalignment of incentives is low,

[4] Their very interesting paper focuses mainly on the issue of centralized vs. decentralized decision making. In both cases, they consider only decision making with no commitment and one round of communication.

[5] See, for example, Myerson (1981).

negotiation and mediation lead to the same outcome. Moreover, only two rounds of cheap talk are needed to obtain the mediation outcome. For most parameter values, however, arbitration always dominates the other protocols.

In the next section we introduce the model. In Section 3 we characterize the DD allocation and show it is implementable even if the principal lacks commitment. Section 3 also compares the DD implementation to an implementation with just one round of cheap talk. In Section 4 we show the DD is optimal even we allow for commitment. All proofs are relegated to the Appendix.

2 The Model

We consider a setting in which two ex-ante symmetric players, $i = 1, 2$, have to take a joint action a. Player $i's$ type is determined by his favorite action, θ_i, which is uniformly distributed over $U[0,1]$. Types are independent and privately known by the players. Their (Bernoulli) utility function is

$$u_i(a, \theta_i) = -(a - \theta_i)^2.$$

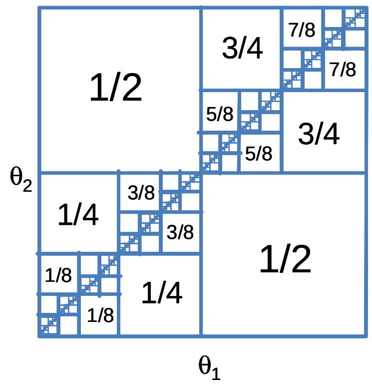
We also allow for a benevolent mediator who can elicit information for the players and has the power implement an action. His objective is to maximize the ex-ante sum of expected utilities. We will start by assuming the mediator cannot commit to an allocation rule. This implies that if he reaches a point in which he cannot solicit any more information from the agents he will simply choose the action that maximizes the expected sum of the agents utilities conditional on his information set which we denote by F.

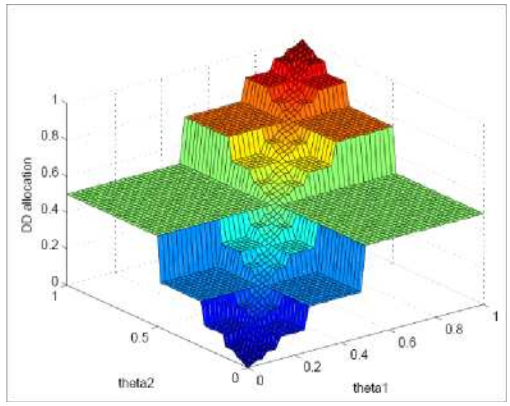
$$\bar{a} = \arg\max_a E\left[\sum_i -(a - \theta_i)^2 |F\right]$$

3 The DD Allocation

The mediator starts by simply asking agents to simultaneously report if they prefer an allocation below or above the midpoint of the interval of possible types ($\frac{1}{2}$ for the initial interval $[0, 1]$). If their reports fall on different sides of the midpoint, then he implements the midpoint. If they both report to be on the same side of the midpoint then he restarts the process considering only the interval they both preferred. For example the $[0, \frac{1}{2}]$ interval if they both reported "below" or the $[\frac{1}{2}, 1]$ interval if they reported "above". This process is iterated until agents eventually report to be on different sides of the relevant midpoint.[6] The DD captures in a simple way the property that negotiations often take place in rounds, and choices are sequentially eliminated until an agreement is reached. The resulting allocation from this mechanism is graphed below.

[6]With zero probability the types are the same. In that case, the procedure described above would never stop, then simply take the limiting point as the allocation. Also, if at any point a player is indifferent between reporting either "below" or "above" we assume he flips a fair coin to decide.





We now verify that the agents have incentives to report truthfully in every round.

Proposition 1 *DD allocation is Incentive Compatible.*

The proof is in the Appendix but the result is straightforward. At each stage, reporting the truth on average brings the allocation closer to the Agent's preferred allocation. The only types who are indifferent are the cutoff types.

Additionally, we must check that this allocation rule can be implemented without commitment. Indeed, we next verify that once it is common knowledge that the agents are on different sides of a midpoint, the mediator cannot extract any more beneficial information from them so he will choose the last midpoint as the allocation.

Proposition 2 *Suppose* $\underline{\theta} \leq \theta_i < \frac{\underline{\theta}+\bar{\theta}}{2} < \theta_{-i} \leq \bar{\theta}$ *then* :

$$\begin{aligned} a(\theta) &= \frac{\underline{\theta}+\bar{\theta}}{2} \in \arg\max_{a(\tilde{\theta})} E\left[\sum_i -\left(a\left(\tilde{\theta}\right)-\theta_i\right)^2 | \underline{\theta} \leq \theta_i < \frac{\underline{\theta}+\bar{\theta}}{2} < \theta_{-i} \leq \bar{\theta}\right] \\ & s.t \\ \theta_i &\in \arg \max_{\underline{\theta} \leq \tilde{\theta}_i < \frac{\underline{\theta}+\bar{\theta}}{2}} E_{\theta_{-i}}\left[-\left(a\left(\tilde{\theta}_i, \theta_{-i}\right)-\theta_i\right)^2 | \frac{\underline{\theta}+\bar{\theta}}{2} < \theta_{-i} \leq \bar{\theta}\right] \\ \theta_i &\in \arg \max_{\frac{\underline{\theta}+\bar{\theta}}{2} < \tilde{\theta}_{-i} \leq \bar{\theta}} E_{\theta_i}\left[\sum_i -\left(a\left(\theta_i, \tilde{\theta}_{-i}\right)-\theta_{-i}\right)^2 | \underline{\theta} \leq \theta_i < \frac{\underline{\theta}+\bar{\theta}}{2}\right] \end{aligned}$$

The Proposition above is stronger than required since it establishes that even with the ability to commit, once the agents know that they are on opposite sides of the midpoint, it is not efficient for the principal to extract any more information from them. It is then easy to verify that if all the mediator knows is that the players are in different sides of a given midpoint, his optimal choice for an allocation is the midpoint itself.

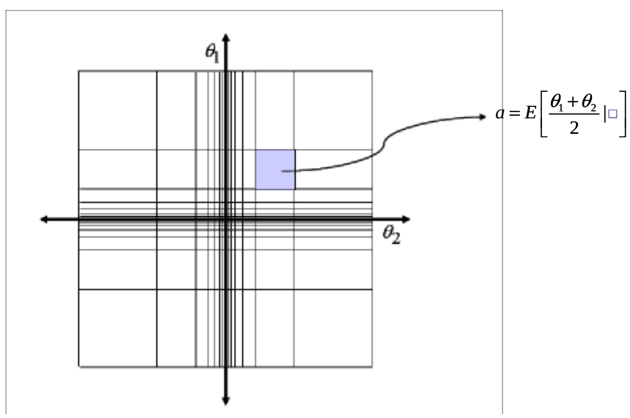
Fully reveling their types directly in one round of communication would fail if the mediator cannot commit. Once she learns the agents' type she would then want to deviate from whatever was promised and implement the first best allocation $a = \frac{\theta_i+\theta_{-i}}{2}$. Restricting the amount of information conveyed to the mediator is a way to prevent her from perturbing the allocation rule ex-post. Similar forces are behind the partition equilibria, in Crawford and Sobel (1982). In the next section we compare the DD allocation to the one that ADM obtain using partition equilibria as in Crawford and Sobel (1982)

Short and Long Cheap Talk In recent work ADM studied the optimal allocation without commitment restricting their analysis to only one round of cheap talk by the Agents. As they acknowledge:

> "It is well known in the literature on cheap talk games that repeated rounds of communication may expand the set of equilibrium outcomes even if only one player is informed. However, even for a simple cheap talk game such as the leading example in Crawford and Sobel (1982), it is still an open question as to what is the optimal communication protocol."

Surprisingly, the value attained with the allocation they characterize (for the extreme case that corresponds to our model) is exactly the same as the one we attain with the DD allocation. Since we show in the next section that the DD is an optimal mechanism, this implies that, in this environment, one round of communication is actually sufficient.[7] Hence, the conclusions derived by ADM are actually much stronger since the limitation of their analysis to one round of communication is actually of no consequence in terms of ex-ante payoffs.

Nonetheless, we believe that the dynamic implementation of the DD allocation with several rounds of communication has some additional appealing features. Before making our case it is useful to recall what the allocation characterized by ADM looks like. Essentially, the type space is partitioned, each agent reports the element of the partition to which his favorite action belongs, and then the Principal implements as an allocation the average type given the reported rectangle. Partitions are very fine close to the middle of the interval since incentive constraints are not very binding for those types and progressively become coarser towards the extremes. Below we replicate Figure 2 from their paper.



ADM Allocation

The first important thing to note is that the DD allocation is renegotiation proof, while the ADM allocation is not. For example, if both players report to be in the shaded area in the figure above they would have a strong incentive to communicate further. Essentially, they are facing the same situation they were facing originally albeit within a smaller range. Instead, with the DD allocation as long as agents report to be in the same quadrant they would keep on refining their reports until it is clear their interests are in conflict. This happens when there is a value (the midpoint) that objectively separates both types.

Second, the amount of communication that is needed to be transmitted to implement the DD allocation is lower than the one needed for the ADM allocation. With the DD rule, only two bits of information need

[7] See for example Aumann and Hart (2003) or Krishna and Morgan (2004) for more on the potential benefits of long cheap talk.

to be transmitted each round with a total expected information requirement of just 4 bits. The lower bound on the expected amount of communication required by the ADM allocation is 6 bits.[8]

Lastly, although ADM provide a very simple difference equation to characterize their allocation we find the simplicity of the DD mechanism very appealing.

4 The DD is an Optimal Monotonic Mechanism:

A natural question is whether the Mediator could improve upon the DD with an alternative mechanism. In this section, we show that the DD is optimal in the class of *all* monotonic Incentive Compatible Mechanisms. Hence, irrespective of wether he can commit or not, the Mediator cannot do better than the DD (if he restricts attention to monotonic allocations).

The Mediator' objective is to maximize the ex-ante sum of their utilities

$$\sum_i E\left[u_i\left(a,\theta_i\right)\right]$$

by choosing an action schedule (an enforceable contract) that maps the players' reported types $\tilde{\theta}$, and an independent randomization device $x \sim U\left[0,1\right]$ into an action:[9],[10]

$$a\left(\tilde{\theta}_i,\tilde{\theta}_{-i};x\right):\left[0,1\right]^2\times\left[0,1\right]\rightarrow\Re.$$

Before proceeding toward solving the above problem, it is convenient to recast it in terms of the agents' virtual utilities.[11]

Preliminaries

The objective is to find the incentive compatible allocation rule $a(\theta;x)$ that solves the following problem:

$$\begin{aligned}&\max_{a(\theta;x)} E_{\theta,x}\left(u(\theta_i,a)+u(\theta_{-i},a)\right)\\ s.t.\theta_i &\in \arg\max_{\tilde{\theta}_i} E_{\theta_{-i},x}\left[-\left(a(\tilde{\theta}_i,\theta_{-i};x)-\theta_i\right)^2\right]\ \forall i\end{aligned}$$

In order to get the virtual utility representation, we start by making use of the following standard result:

Lemma 1 (IC Representation) *Letting*

$$U(\theta_i,a\left(\theta;x\right))=\max_{\tilde{\theta}} E_{\theta_{-i}}\left[-\left(a(\tilde{\theta},\theta_{-i};x)-\theta_i\right)^2\right]=E_{\theta_{-i}}\left[-\left(a(\theta_i,\theta_{-i};x)-\theta_i\right)^2\right],$$

Incentive Compatibility is equivalent to:

[8] This is achieved by using one bit to signal if the agent is above or below 1/2. From then on, a bit with a value of 1 implies the agent is in the coarsest remaining partition. For example to signal a type in the partition $(3/4,1)$ the agent would report $1,1$. The next interval to the left would be transmitted as $1,0,1$ and so on.

[9] As we assume the Mediator is committed in this section, we can, without loss, restrict attention to Direct Mechanisms. This follows from the Revelation Principle (Myerson, 1981).

[10] The randomization device is simply used to preserve the symmetry in the allocation. There is no need for randomization device from an efficiency standpoint.

[11] See Myerson (1981) and Myerson's notes on virtual utility at http://home.uchicago.edu/~rmyerson/research/virtual.pdf

$$U(\theta_i, a(\theta;x)) = \begin{cases} U(\theta', a(\theta;x)) + 2\int_{\theta'}^{\theta_i} E_{\theta_{-i},x}\left[a(\tau,\theta_{-i};x) - \tau\right] d\tau, \text{ if } \theta_i > \theta' \\ U(\theta', a(\theta;x)) - 2\int_{\theta_i}^{\theta'} E_{\theta_{-i},x}\left[a(\tau,\theta_{-i};x) - \tau\right] d\tau, \text{ if } \theta_i < \theta' \end{cases},$$

where $\theta' \in [0,1]$, *and* $E_{\theta_{-i},x}\left[a(\theta_i,\theta_{-i};x)\right]$ *non-decreasing in* θ_i.

The proof follows from Milgrom and Segal (2002) and is detailed in the Appendix.

Incentive compatibility only requires that allocations be non-decreasing in expectation. We restrict a bit further the set feasible allocations by considering only schedules that are non-decreasing, that is, schedules that satisfy:

$$a(\theta_i, \theta_{-i}; x) \text{ non-decreasing in } \theta_i \text{ for all } \theta_{-i}. \qquad \text{(Monotonicity)}$$

Beyond the fact that this condition facilitates our analysis in this section it also allows us to directly connect our results with those of optimal allocations with no commitment. When the planner cannot commit, he will always choose an allocation that is monotonic on his beliefs of the player's types.[12]

After integrating by parts the representation of $U(\theta_i, a(\theta;x))$ induced by Lemma (1), one can recast our program of interest as:

$$\max_{a(\theta;x)} \sum_{i=1}^{2} \left\{ \begin{array}{c} -E_{\theta_{-i}}\left[a\left(\theta',\theta_{-i};x\right) - \theta'\right]^2 \\ +2\Pr(\theta_i > \theta')\, E_\theta\left[\left(a(\theta;x) - \theta_i\right)(1-\theta_i)\,|\theta_i > \theta'\right] \\ -2\Pr(\theta_i < \theta')\, E_\theta\left[\left(a(\theta;x) - \theta_i\right)\theta_i|\theta_i < \theta'\right] \end{array} \right\}, \qquad (1)$$

$$s.t.\ a(\theta;x) \text{ being Incentive Compatible}$$

$$\text{and satisfying Monotonicity} \qquad (2)$$

where $\theta' \in [0,1]$ can be chosen arbitrarily.

As mentioned in the introduction, we do not allow for transfers. This greatly increases the difficulty of the problem. In settings with quasi-linear preferences and side payments, any allocation schedule can be made IC by a proper choice of a transfer function. Therefore, in such cases, it is correct to proceed by simply maximizing pointwise the Incentive Compatible representation of the players' utility, as it can always be made equivalent to the players' expected utility by choosing the right side payments. The fact that players cannot make side payments forces us to consider the Incentive Compatibility Constraints explicitly, which, in turn, makes the problem fairly hard.[13]

Given the difficulties outlined above instead of trying to characterize the whole allocation directly in one step we proceed by showing that it is without loss to set the allocation to 1/2 when the players are on different sides of 1/2. This is shown formally in Lemma (2). In turn, we show using Lemma (3) that this then allows us to completely describe the optimal mechanism.

The Optimal Monotonic Mechanism

The symmetry of the problem makes it natural to pick $\theta' = \frac{1}{2}$ as the reference type in the optimization

[12] Note that, although we restrict our attention to monotonic allocations, we are quite general in the sense of not requiring the allocation to be differentiable nor continuous.

[13] Among other difficulties, the set of IC schedules is not convex. Hence, we cannot directly rely on Lagrangian Theorems to find the optimal contract.

problem. Using this together with the Law of Iterated Expectations, we can, ignoring constant terms, write the objective functional in (1) as:

$$\left(\begin{array}{c}\underbrace{\sum_i\left[-\frac{1}{4}E_{\theta_{-i},x}\left[\left(a\left(\frac{1}{2},\theta_{-i};x\right)-\frac{1}{2}\right)^2|\theta_{-i}\leq\frac{1}{2},x<\frac{1}{2}\right]-\frac{1}{2}E_{\theta,x}\left[a\left(\theta;x\right)\left(\theta_i\right)|\theta_i,\theta_{-i}<\frac{1}{2}\right]\right]}_{A}\\ \underbrace{\sum_i\left[-\frac{1}{4}E_{\theta_{-i}}\left[\left(a\left(\frac{1}{2},\theta_{-i};x\right)-\frac{1}{2}\right)^2|\theta_{-i}>\frac{1}{2},x<\frac{1}{2}\right]+\frac{1}{2}E_{\theta,x}\left[a\left(\theta;x\right)\left(1-\theta_i\right)|\theta_i,\theta_{-i}>\frac{1}{2}\right]\right]}_{B}\\ \underbrace{\sum_i\frac{1}{2}\left[E_{\theta,x}\left[\left[a\left(\theta_i,\theta_{-i};x\right)\right]\left(1-\theta_i\right)|\theta_i>\frac{1}{2},\theta_{-i}<\frac{1}{2}\right]-E_{\theta,x}\left[\left[a\left(\theta_i,\theta_{-i};x\right)\right]\theta_i|\theta_i\leq\frac{1}{2},\theta_{-i}>\frac{1}{2}\right]\right]}_{C}\\ \underbrace{\sum_i\left[-\frac{1}{4}E_{\theta_{-i},x}\left[\left(a\left(\frac{1}{2},\theta_{-i}\right)-\frac{1}{2}\right)^2|\theta_{-i}\leq\frac{1}{2},x>\frac{1}{2}\right]-\frac{1}{4}E_{\theta_{-i},x}\left[\left(a\left(\frac{1}{2},\theta_{-i};x\right)-\frac{1}{2}\right)^2|\theta_{-i}>\frac{1}{2},x>\frac{1}{2}\right]\right]}_{D}\end{array}\right), \tag{3}$$

Now, consider the terms C and D.

For any non-decreasing schedule $a\left(\theta_i,\theta_{-i}\right)$, one has that

$$\begin{aligned}&E_{\theta,x}\left[a\left(\theta_i,\theta_{-i};x\right)\left(1-\theta_i\right)|\theta_{-i}<\frac{1}{2}<\theta_i\right]\\ \leq\;&E_{\theta,x}\left[a\left(\theta_i,\theta_{-i};x\right)|\theta_{-i}<\frac{1}{2}<\theta_i\right]E_\theta\left[\left(1-\theta_i\right)|\theta_i>\frac{1}{2}\right],\end{aligned}$$

for the expected value of the product of a non-decreasing function and a decreasing function is no larger than the product of the expected values.

A similar reasoning can be applied to the other term in C, so it follows that

$$\begin{aligned}&-E_{\theta,x}\left[\left[a\left(\theta_i,\theta_{-i};x\right)-\theta_i\right]\theta_i|\theta_i\leq\frac{1}{2}<\theta_{-i}\right]\\ \leq\;&-E_{\theta,x}\left[a\left(\theta_i,\theta_{-i};x\right)|\theta_i\leq\frac{1}{2}<\theta_{-i}\right]E_\theta\left[\theta_i|\theta_i\leq\frac{1}{2}\right].\end{aligned}$$

This discussion suggests that the term C – that is associated with the region in which $\theta_i<1/2<\theta_{-i}$ – would be maximized by the choice of a constant action. Note that a constant action of $\frac{1}{2}$ in this region also maximizes the term D in the objective. From the work in social choice theory by Moulin (1980) we know that if we required instead ex-post incentive compatibility the optimal allocation would also have 1/2 of diagonals.[14]

Nonetheless, once we consider Bayesian implementation, it is not obvious that it is optimal to set 1/2 as the allocation in these regions. We could expect that by perturbing the allocation slightly in these off-diagonal regions one could improve the attainable values on the on-diagonals $\left(\left(\frac{1}{2},1\right)^2\ and\ \left(0,\frac{1}{2}\right)^2\right)$ once incentive constraints are taken explicitly into account. Suppose we were to carry out such a perturbation in the region where $\theta_i<\frac{1}{2}<\theta_{-i}$. Note that we start from $a\left(\theta\right)=\frac{1}{2}<\theta_{-i}$ hence, if we were to make the allocation strictly increasing in θ_{-i} in this region player $-i$ would have more incentives to claim his type is higher than it actually is. This would not help us bring the allocation in $\left(\frac{1}{2},1\right)^2$ any closer to first best since

[14] See also Barberà and Jackson (1994), and Barberà (2001) for detailed discussions of strategy-proof social choice functions.

the problem with the first best allocation is exactly that types in this region would want to pretend they are higher than they actually are. Therefore, within the class of weakly increasing allocation rules it is best to set a constant $\left(a(\theta)=\frac{1}{2}\right)$ on the off-diagonals. The following lemma establishes this formally.

Lemma 2 (1/2 off-diagonals) *Given any symmetric incentive compatible allocation $a(\theta;x)$ that satisfies Monotonicity we can find an alternative incentive compatible allocation $\tilde{a}(\theta;x)$ which is weakly better and satisfies $\tilde{a}(\theta;x)=\frac{1}{2}$ for all x, when $\theta_i>\frac{1}{2}>\theta_{-i}$, and $\tilde{a}\left(\frac{1}{2},\theta_{-i};x\right)=\frac{1}{2}$ when $x>\frac{1}{2}$.*

This is a very powerful result towards the full characterization of the optimal allocation. Once one knows that setting $\frac{1}{2}$ off-diagonals is optimal – so that this region plays no role in terms of providing incentives over the main diagonal – the problem is separable. The following result states this in a precise way.

Lemma 3 (Separability) *Let $a^*(\theta,x)$ be an allocation that solves the program of interest. If $a^*(\theta;x)=\frac{1}{2}$ for all $\theta\in\left[0,\frac{1}{2}\right)\times\left(\frac{1}{2},1\right]$, and $a^*\left(\frac{1}{2},\theta_{-i},x\right)=\frac{1}{2}$ for all θ_{-i} whenever $x>\frac{1}{2}$ then:*

$$\begin{aligned}\textit{for }\theta &\in (0,1/2)^2,\ a^*(\theta,x)\in\arg\max_{a()}\sum_i E\left[u_i(a,\theta_i)\,|\theta\in(0,1/2)^2\right]\\ \textit{s.t. IC for }i &= 1,2\ \textit{given }\theta_{-i}\in(0,1/2)\ \textit{and monotonicity}\end{aligned}$$

and

$$\begin{aligned}\textit{for }\theta &\in (1/2,1)^2,\ a^*(\theta,x)\in\arg\max_{a()}\sum_i E\left[u_i(a,\theta_i)\,|\theta\in(1/2,1)^2\right]\\ \textit{s.t. IC for }i &= 1,2\ \textit{given }\theta_{-i}\in(1/2,1)\ \textit{and monotonicity}\end{aligned}$$

Furthermore, the problem over $\left[0,\frac{1}{2}\right]^2$,and respectively $\left[\frac{1}{2},1\right]^2$ is, subject to rescaling, exactly the same as the original problem (the problem over $[0,1]^2$). So we can sequentially apply appropriately rescaled versions of Lemmas (2) and (3) to those regions.

As the resulting allocation is IC, the above discussion proves

Theorem 1 *The DD allocation is optimal in the class of non-decreasing Incentive Compatible allocations.*

5 Concluding Remarks

This paper considered a setting in which two agents have to take a commonly agreed action for the case in which the players' preferences over actions are private information, there are no transfers, the action to be chosen rather than being binary belongs to an interval, and the welfare criterion is utilitarian.

The main results are as follows. The optimal allocation (which we label the DD allocation) can be implemented by simultaneously asking the players if they are to the left or to the right of the midpoint over the remaining choice set. If they both agree on the side of the coarse partition they prefer, we discard the section of the interval which none preferred and continue dividing the remaining interval in this way until one chooses left and the other right. In that case, the midpoint of the remaining interval is implemented. The DD allocation can also be implemented by a Principal without commitment, and, surprisingly, yields the same expected value as the one attained with just one round of cheap talk between the Agents, as in the mechanism in ADM.

References

[1] Alonso, R., Dessein, W. and Matouschek, N., "When Does Coordination Require Centralization?", *American Economic Review,* forthcoming.

[2] Arrow, K., 1979. "The Property Rights Doctrine and Demand Revelation under Incomplete Information", *Economics and Human Welfare.* Academic Press.

[3] Athey, S. , Atkeson, A, and Kehoe, P., 2005. "The Optimal Degree of Monetary Policy Discretion", *Econometrica*, Vol. 73 (5), pp. 1431-1476.

[4] Aumann, R.J. and Hart, S., 2003. "Long Cheap Talk", *Econometrica* Vol. 71 (6), pp. 1619–1660.

[5] Barberà, S. and Jackson, M., 1994. "A Characterization of Strategy-proof Social Choice Functions for Economies with Pure Public Goods", *Social Choice and Welfare,* 11, pp. 241-252.

[6] Barberà, S., 2001. "An Introduction to Strategy-proof Social Choice Functions", *Social Choice and Welfare,* pp. 619-653.

[7] Crawford, V. and Sobel, J., 1982. "Strategic Information Transmission", *Econometrica*, Vol. 50(6), pp. 1431-1451.

[8] d'Apresmont, C. and Gerard-Varet, L., 1979. "Incentives and Incomplete Information", *Journal of Public Economics* 11, pp. 25-45.

[9] Goltsman, M., Hörner, J., Pavlov, G., and Squintani, F., 2008, "Arbitration, Mediation and Cheap Talk", mimeo, Department of Economics, Northwestern University.

[10] Hart, O. and Moore, J., 2004. "Agreeing Now to Agree Later: Contracts that rule out but do not rule in", mimeo, Department of Economics, Harvard University.

[11] Krishna, V. and Morgan, J., 2004. "The art of conversation: eliciting information from experts through multi-stage communication", *Journal of Economic Theory*, Vol. 17(2), pp. 147-179.

[12] Maggi, G. and Morelli, M., 2006. "Self-Enforcing Voting in International Organizations", *American Economic Review,* Vol. 96(4), pp. 1137-1158.

[13] Milgrom, P. and Segal, I., 2002. "Envelope Theorems for Arbitrary Choice Sets", *Econometrica* Vol. 70(2), pp. 583-601.

[14] Moulin, H., 1980. "On Strategy Proofness and Single Peakedness", *Public Choice,* 35, pp. 437–55.

[15] Myerson, R., 1981. "Optimal Auction Design", *Mathematics of Operations Research*, Vol. 6(1) , pp. 58-73.

[16] Persico, N., 2004. "Committee Design with Endogenous Information", *Review of Economic Studies,* 71, pp. 165-91.

[17] Rosenlicht, M., 1999. *Introduction to Analysis,* Prentice-Hall.

[18] Segal, I.,2006. Communication in Economic Mechanisms, in Advances in Economics and Econometrics: Theory and Application, *Ninth World Congress (Econometric Society Monographs)*, ed. by Richard Blundell, Whitney K. Newey, and Torsten Persson, Cambridge University Press

6 Appendix

Appendix A: Preliminary Results

In this appendix, we prove some of the preliminary results we will need to prove the optimality of the DD.

Proof Lemma (IC Representation)

Proof. The proof is standard. For necessity, just notice that the integral formula is implied by Milgrom and Segal's (2002) Envelope Theorem. The monotonicity condition follows because, if $a(\theta)$ is Incentive Compatible, for any $\theta' > \theta''$, one must have

$$E_{\theta_{-i}}\left[-\left(a(\theta',\theta_{-i})-\theta'\right)^2\right] \geq E_{\theta_{-i}}\left[-\left(a(\theta'',\theta_{-i})-\theta'\right)^2\right] \quad (\mathrm{IC}_{\theta'\theta''})$$

and

$$E_{\theta_{-i}}\left[-\left(a(\theta'',\theta_{-i})-\theta''\right)^2\right] \geq E_{\theta_{-i}}\left[-\left(a(\theta',\theta_{-i})-\theta''\right)^2\right] \quad (\mathrm{IC}_{\theta''\theta'})$$

Combining both expressions one has, after a few algebraic manipulations, that

$$2\left[\theta'-\theta''\right]\left\{E_{\theta_{-i}}\left[a(\theta',\theta_{-i})\right]-E_{\theta_{-i}}\left[a(\theta'',\theta_{-i})\right]\right\} \geq 0$$

which implies that

$$E_{\theta_{-i}}\left[a(\theta',\theta_{-i})\right] \geq E_{\theta_{-i}}\left[a(\theta'',\theta_{-i})\right],$$

For sufficiency, let $\theta_i > 0.5$, *and consider* $0.5 < \widehat{\theta} < \theta_i$.

$$\begin{aligned} U_i(\theta_i)-U_i(\widehat{\theta}) &= 2\int_{\widehat{\theta}}^{\theta_i} E_{\theta_{-i}}\left[a(\tau,\theta_{-i})-\tau\right]d\tau \geq \\ 2\int_{\widehat{\theta}}^{\theta_i} E_{\theta_{-i}}\left[a\left(\widehat{\theta},\theta_{-i}\right)-\tau\right]d\tau &= 2\left[E_{\theta_{-i}}\left(a\left(\widehat{\theta},\theta_{-i}\right)\right)\left[\theta_i-\widehat{\theta}\right]-\left[\frac{\theta_i^2}{2}-\frac{\widehat{\theta}^2}{2}\right]\right] = \\ & E_{\theta_{-i}}\left[-\left(a(\widehat{\theta},\theta_{-i})-\theta_i\right)^2\right]+E_{\theta_{-i}}\left[\left(a(\widehat{\theta},\theta_{-i})-\widehat{\theta}\right)^2\right], \end{aligned}$$

where the first inequality follows from the expected monotonicity of the allocation.

It then follows that

$$U_i(\theta_i) \geq E_{\theta_{-i}}\left[-\left(a(\widehat{\theta},\theta_{-i})-\theta_i\right)^2\right].$$

The analysis for all other cases is analogous. ■

To prove Proposition1, it is convenient to show the following

Lemma 4 ((Dynamic IC)) *Thruthtelling is IC given the "dynamic" implementation of the DD allocation rule.*

Proof. Consider a truncated version of the DD allocation rule in which for N rounds the players are asked if they are above or below the the midpoint. If in any of these rounds the announcements fell in different sides of the relevant midpoint then that is the allocation chosen. If for N rounds they have always picked the same side then at round $N+1$ the players are asked to announce their types. The allocation chosen is then the median announcement where there is phantom announcement in the midpoint of the remaining interval. Clearly as $N \to \infty$ this alternative mechanism converges to the DD mechanism. We will show that for any $N \geq 0$ this mechanism is IC and therefore the DD mechanism is IC.

$N = 0$: It is clearly IC to reveal the true type since your type only affects the allocation when the allocation equals your type. Note that this is regardless of where the phantom voter is chosen.

$N = 1$: Consider a player with $\theta_i < 1/2$ (the other case is symmetric). If he announces bottom then with probability 1/2 the game is ended $\theta_j \geq 1/2$ and the allocation is $\frac{1}{2}$. If he announces top then also with probability 1/2 the game is ended $\theta_j < 1/2$. Now suppose that the game is not ended after his first announcement then if he was truthful the phantom voter in the second round is set at 1/4 if he lied in the first round the phantom voter will be set at 3/4. Clearly given the allocation rule in the final round he wants to reveal truthfully and the phantom voter being at 3/4 gives him lower expected utility.

The induction step. Suppose that when there were $N > 0$ rounds the players had no incentive to lie. Consider a new game with $N+1$ rounds. If you are truthful in the first round and the game does not end then we have already shown that the player will not lie in the remaining rounds since the game looks the same except that the type space is a smaller interval. Actually, even if he deviates by mistake in the first round he won't have an incentive to deviate in the remaining rounds. Hence, we can simply check if he would benefit from deviating only once. Deviating only once is not profitable because it will make the possible stopping points be further away in expectation from the Agent's true type. To illustrate consider a type close to 1/2 (but lower) if this type lies in the first round and the game does not end he will be truthful from then on hence with probability 1/4 the allocation will be 3/4 with probability 1/8 it will be 5/8 and so on. Suppose instead he was truthful in the first round and the game didn't end then with probability 1/4 the allocation would be 1/4 with probability 1/8 it would be 3/8 and so on. Note that the expected allocation is closer to the Agent's type if he is truthful hence he wouldn't want to lie at the first stage either. ■

Proof of Proposition 1. Lemma (Dynamic IC) shows that, at any given round, truthtelling is optimal given the player´s information set at that round (what he knows about the other player's type). By the Law of Iterated Expectations, it follows that the player´s ex-ante (i.e., before having more information about his opponent's type) payoff is maximized when he announces truthfully. ■

Proof of Proposition 2. Using the constraints and Milgrom and Segal's (2002) Envelope Theorem, one has that, for $\underline{\theta} \leq \theta_1 < \frac{\underline{\theta}+\bar{\theta}}{2}$:

$$-E_{\theta_2}\left[\left(a\left(\theta\right)-\theta_1\right)^2|\theta_2 > \frac{\underline{\theta}+\bar{\theta}}{2}\right]$$

$$= \quad -E_{\theta_2}\left[\left(a\left(\frac{\underline{\theta}+\bar{\theta}}{2},\theta_2\right)-\frac{\underline{\theta}+\bar{\theta}}{2}\right)^2|\theta_2 > \frac{\underline{\theta}+\bar{\theta}}{2}\right] - 2\int_{\theta_1}^{\frac{\underline{\theta}+\bar{\theta}}{2}} E_{\theta_2}\left[\left(a\left(\tau,\theta_2\right)-\tau\right)|\theta_2 > \frac{\underline{\theta}+\bar{\theta}}{2}\right].$$

By the same token, for $\frac{\underline{\theta}+\bar{\theta}}{2} < \theta_2 \leq \bar{\theta}$

$$\begin{aligned} & -E_{\theta_1}\left[\left(a\left(\theta\right)-\theta_2\right)^2 |\theta_1 < \frac{\underline{\theta}+\bar{\theta}}{2}\right] \\ = & -E_{\theta_1}\left[\left(a\left(\theta_1,\frac{\underline{\theta}+\bar{\theta}}{2}\right)-\frac{\underline{\theta}+\bar{\theta}}{2}\right)^2 |\theta_1 < \frac{\underline{\theta}+\bar{\theta}}{2}\right] + 2\int_{\frac{\underline{\theta}+\bar{\theta}}{2}}^{\theta_2} E_{\theta_1}\left[\left(a\left(\theta_1,\tau\right)-\tau\right)|\theta_1 < \frac{\underline{\theta}+\bar{\theta}}{2}\right] \end{aligned}$$

Integration of both expressions by parts allows us to write the objective as (we ignore constant terms)

$$\begin{aligned} & -E_{\theta_2}\left[\left(a\left(\frac{\underline{\theta}+\bar{\theta}}{2},\theta_2\right)-\frac{\underline{\theta}+\bar{\theta}}{2}\right)^2 |\theta_2 > \frac{\underline{\theta}+\bar{\theta}}{2}\right] - 2\left[E_{\theta}\left[a\left(\theta\right)\left[\theta_1+\underline{\theta}\right]|\theta_1 < \frac{\underline{\theta}+\bar{\theta}}{2} < \theta_2\right]\right] \\ & -E_{\theta_1}\left[\left(a\left(\theta_1,\frac{\underline{\theta}+\bar{\theta}}{2}\right)-\frac{\underline{\theta}+\bar{\theta}}{2}\right)^2 |\theta_1 < \frac{\underline{\theta}+\bar{\theta}}{2}\right] + 2E_{\theta}\left[\left(\bar{\theta}-\theta_2\right)a\left(\theta\right)|\theta_1 < \frac{\underline{\theta}+\bar{\theta}}{2} < \theta_2\right] \end{aligned}$$

Note that, for all non-decreasing $a\left(.\right),$

$$\begin{aligned} & E_{\theta}\left[\left(\bar{\theta}-\theta_2\right)a\left(\theta\right)|\theta_1 < \frac{\underline{\theta}+\bar{\theta}}{2} < \theta_2\right] \\ \leq \;& E_{\theta}\left[\left(\bar{\theta}-\theta_2\right)|\theta_1 < \frac{\underline{\theta}+\bar{\theta}}{2} < \theta_2\right] E_{\theta}\left[a\left(\theta\right)|\theta_1 < \frac{\underline{\theta}+\bar{\theta}}{2} < \theta_2\right] \end{aligned}$$

and

$$\begin{aligned} & -\left[E_{\theta}\left[a\left(\theta\right)\left[\theta_1+\underline{\theta}\right]|\theta_1 < \frac{\underline{\theta}+\bar{\theta}}{2} < \theta_2\right]\right] \\ \leq \;& -E_{\theta}\left[a\left(\theta\right)|\theta_1 < \frac{\underline{\theta}+\bar{\theta}}{2} < \theta_2\right] E_{\theta}\left[\left[\theta_1+\underline{\theta}\right]|\theta_1 < \frac{\underline{\theta}+\bar{\theta}}{2} < \theta_2\right]. \end{aligned}$$

Hence, those terms if one picks a constant. The best constant over those regions is $\frac{\underline{\theta}+\bar{\theta}}{2}$. Such constant also maxmizes the terms

$$-E_{\theta_2}\left[\left(a\left(\frac{\underline{\theta}+\bar{\theta}}{2},\theta_2\right)-\frac{\underline{\theta}+\bar{\theta}}{2}\right)^2 |\theta_2 > \frac{\underline{\theta}+\bar{\theta}}{2}\right]$$

and

$$-E_{\theta_1}\left[\left(a\left(\theta_1,\frac{\underline{\theta}+\bar{\theta}}{2}\right)-\frac{\underline{\theta}+\bar{\theta}}{2}\right)^2 |\theta_1 < \frac{\underline{\theta}+\bar{\theta}}{2}\right]$$

of the objective. Hence, setting $a\left(\theta\right) = \frac{\underline{\theta}+\bar{\theta}}{2}$ for $\theta_1 < \frac{\underline{\theta}+\bar{\theta}}{2} < \theta_2$ is optimal, as claimed. ■

Appendix B: The Optimality of the DD

In this appendix, we show that the DD is optimal among the class on non-decreasing Incentive Compatible allocations. Our strategy of proof involves two main steps.

The first step shows that, for any given class of allocations, if it is optimal to set $\frac{1}{2}$ off-diagonals, then the DD must be an optimal allocation within that class. The idea of the proof is straightforward. Given an uniform distribution, the problem of finding an optimal allocation, *conditional* on both agents having their types on the same region – say, above $\frac{1}{2}$ – and given that a constant action is being chosen when they are in different regions, is exactly the same as the original problem. In other words, if a constant action is chosen off-diagonals, so that the schedule off-diagonals plays no role in the provision of incentives, the problem self-replicates. Hence, the DD must be optimal.

In the second step, we show that if a non-decreasing schedule does not have $\frac{1}{2}$ off-diagonals, it cannot be optimal. We proceed in the following way. Starting with an arbitrary Incentive Compatible, non-decreasing, and continuous schedule $a(.)$, we show, using the same arguments as in the text, and ignoring Incentive Compatibility issues, that an improvement can be attained if one sets $\frac{1}{2}$ off-diagonals. We then move towards showing that further modifications of $a(.)$ along the main diagonals can be made so to guarantee both Incentive Compatibility, and that the new Incentive Compatible allocation still fares better than $a(.)$. We then use limiting arguments to argue that the same reasoning applies to starting schedules which are non-decreasing, but discontinuous.

Steps 1 and 2 together prove Theorem1.

Separability and the Optimality of $\frac{1}{2}$ Off-Diagonals

Proof of Lemma Separability. The program of interest is:

$$\underset{a(.,.)\text{ is IC over }[0,1]^2}{\arg\max}\left(\begin{array}{c}\sum_i\left[-\frac{1}{4}E_{\theta_{-i}}\left[\left(a\left(\frac{1}{2},\theta_{-i}\right)-\frac{1}{2}\right)^2|\theta_{-i}\leq\frac{1}{2},x<\frac{1}{2}\right]-\frac{1}{2}E_\theta\left[a\left(\theta\right)\left(\theta_i\right)|\theta_i,\theta_{-i}<\frac{1}{2}\right]\right]\\ \sum_i\left[-\frac{1}{4}E_{\theta_{-i}}\left[\left(a\left(\frac{1}{2},\theta_{-i}\right)-\frac{1}{2}\right)^2|\theta_{-i}>\frac{1}{2},x<\frac{1}{2}\right]+\frac{1}{2}E_\theta\left[a\left(\theta\right)\left(1-\theta_i\right)|\theta_i,\theta_{-i}>\frac{1}{2}\right]\right]\\ \sum_i\left[-\frac{1}{4}E_{\theta_{-i}}\left[\left(a\left(\frac{1}{2},\theta_{-i}\right)-\frac{1}{2}\right)^2|\theta_{-i}\leq\frac{1}{2},x>\frac{1}{2}\right]-\frac{1}{4}E_{\theta_{-i}}\left[\left(a\left(\frac{1}{2},\theta_{-i}\right)-\frac{1}{2}\right)^2|\theta_{-i}>\frac{1}{2},x>\frac{1}{2}\right]\right]\\ \sum_i\frac{1}{2}\left[E_\theta\left[\left[a\left(\theta_i,\theta_{-i}\right)\right]\left(1-\theta_i\right)|\theta_i>\frac{1}{2},\theta_{-i}<\frac{1}{2}\right]-E_\theta\left[\left[a\left(\theta_i,\theta_{-i}\right)\right]\theta_i|\theta_i\leq\frac{1}{2},\theta_{-i}>\frac{1}{2}\right]\right]\end{array}\right)$$

Given $a^*(\theta,x)$ sets 1/2 off-diagonals what remains to be proven is that

$$a^*(\theta,x)\in\underset{a(.,.)}{\arg\max}\left(\begin{array}{c}\sum_i\left[-\frac{1}{4}E_{\theta_{-i}}\left[\left(a\left(\frac{1}{2},\theta_{-i}\right)-\frac{1}{2}\right)^2|\theta_{-i}\leq\frac{1}{2},x<\frac{1}{2}\right]-\frac{1}{2}E_\theta\left[a\left(\theta\right)\left(\theta_i\right)|\theta_i,\theta_{-i}<\frac{1}{2}\right]\right]\\ \sum_i\left[-\frac{1}{4}E_{\theta_{-i}}\left[\left(a\left(\frac{1}{2},\theta_{-i}\right)-\frac{1}{2}\right)^2|\theta_{-i}>\frac{1}{2},x<\frac{1}{2}\right]+\frac{1}{2}E_\theta\left[a\left(\theta\right)\left(1-\theta_i\right)|\theta_i,\theta_{-i}>\frac{1}{2}\right]\right]\\ s.t.\ IC\ for\ i=1,2\ given\ \theta_{-i}\in(0,1)\ and\ monotonicity\\ and\ a(\theta,x)=1/2\text{ off-diagonals}\end{array}\right)$$

Since any schedule which is incentive compatible over $[0,1]^2$ and has a constant off-diagonals must, in fact, be incentive compatible over $\left[0,\frac{1}{2}\right]^2$ and $\left[\frac{1}{2},1\right]^2$. The program above implies:

$$\begin{aligned}for\ \theta &\in (0,1/2)^2,\ a^*(\theta,x)\in\arg\max_{a()}\sum_i E\left[u_i(a,\theta_i)|\theta\in(0,1/2)^2\right]\\ s.t.\ IC\ for\ i &= 1,2\ given\ \theta_{-i}\in(0,1/2)\ and\ monotonicity\end{aligned}$$

and

$$\begin{aligned}for\ \theta &\in (1/2,1)^2,\ a^*(\theta,x)\in\arg\max_{a()}\sum_i E\left[u_i(a,\theta_i)|\theta\in(1/2,1)^2\right]\\ s.t.\ IC\ for\ i &= 1,2\ given\ \theta_{-i}\in(1/2,1)\ and\ monotonicity\end{aligned}$$

as desired. ■

Lemma (1/2 off-diagonals): *Given any incentive compatible allocation $a(\theta;x)$ that satisfies Monotonicity we can find an alternative incentive compatible allocation $\tilde{a}(\theta;x)$ which is weakly better and satisfies $\tilde{a}(\theta;x)=\frac{1}{2}$ for all x, when $\theta_i>\frac{1}{2}>\theta_{-i}$, and $\tilde{a}\left(\frac{1}{2},\theta_{-i};x\right)=\frac{1}{2}$ when $x>\frac{1}{2}$.*

Proof Lemma (1/2 off-diagonals): Without loss of generality we focus on the case in which the starting $a(\theta;x)$ is symmetric across players and around $\frac{1}{2}$ i.e.

$$\begin{aligned} a(\theta_i,\theta_{-i};x) &= a(\theta_{-i},\theta_i;x) \\ a(\theta_i,\theta_{-i};x) &= 1-a(1-\theta_i,1-\theta_{-i};x) \end{aligned}$$

We first consider the case where the starting $a(\theta;x)$ is continuous and show later (in Step 3) the result extends to non-continuous allocations. Furthermore, let us first point out that it is without loss to start with a schedule that is not constant off-diagonals. Indeed, if $a(\theta;x)=c\in\Re$ for all θ in $\left[0,\frac{1}{2}\right]\times\left[\frac{1}{2},1\right]$ and $\left[\frac{1}{2},1\right]\times\left[0,\frac{1}{2}\right]$, we could set $\frac{1}{2}$ off-diagonals (and $\frac{1}{2}$ fifty percent of time whenever a player announces $\frac{1}{2}$) without affecting incentives and such a change would generate a gain.

We now proceed to prove the result through a sequence of steps.

Step 1 (1/2 off-diagonals generates an improvement): *In this first step we show that a strict improvement can be attained if we replace the off-diagonal values in the original allocation by* 1/2. Formally,

$$a_{1/2}(\theta,x)=\begin{cases}\frac{1}{2} & \text{if } \theta_i>\frac{1}{2}>\theta_{-i} \ \textit{ or if } \theta_i=\frac{1}{2} \textit{ and } x>\frac{1}{2} \\ a(\theta;x) & \text{otherwise}\end{cases}$$

We should note however that $a_{1/2}(\theta,x)$ may not be IC (we will address this in the next step).

By picking $\theta'=\frac{1}{2}$, and ignoring constant terms we can write the objective functional as:

$$V(a)=\left(\begin{array}{c} \sum_i\left[-\frac{1}{4}E_{\theta_{-i},x}\left[\left(a\left(\frac{1}{2},\theta_{-i};x\right)-\frac{1}{2}\right)^2|\theta_{-i}\leq\frac{1}{2},x<\frac{1}{2}\right]-\frac{1}{2}E_{\theta,x}\left[a(\theta;x)(\theta_i)|\theta_i,\theta_{-i}<\frac{1}{2}\right]\right] \\ \sum_i\left[-\frac{1}{4}E_{\theta_{-i},x}\left[\left(a\left(\frac{1}{2},\theta_{-i};x\right)-\frac{1}{2}\right)^2|\theta_{-i}>\frac{1}{2},x<\frac{1}{2}\right]+\frac{1}{2}E_{\theta,x}\left[a(\theta;x)(1-\theta_i)|\theta_i,\theta_{-i}>\frac{1}{2}\right]\right] \\ \sum_i\left[-\frac{1}{4}E_{\theta_{-i},x}\left[\left(a\left(\frac{1}{2},\theta_{-i};x\right)-\frac{1}{2}\right)^2|\theta_{-i}\leq\frac{1}{2},x>\frac{1}{2}\right]-\frac{1}{4}E_{\theta_{-i},x}\left[\left(a\left(\frac{1}{2},\theta_{-i};x\right)-\frac{1}{2}\right)^2|\theta_{-i}>\frac{1}{2},x>\frac{1}{2}\right]\right] \\ \sum_i\frac{1}{2}\left[E_{\theta,x}\left[\left[a(\theta_i,\theta_{-i};x)\right](1-\theta_i)|\theta_i>\frac{1}{2},\theta_{-i}<\frac{1}{2}\right]-E_{\theta,x}\left[\left[a(\theta_i,\theta_{-i};x)\right]\theta_i|\theta_i\leq\frac{1}{2},\theta_{-i}>\frac{1}{2}\right]\right] \end{array}\right)$$

For any non-decreasing $a(\theta,x)$, one has that

$$\begin{aligned} & E_{\theta,x}\left[a(\theta_i,\theta_{-i})(1-\theta_i)|\theta_i>\frac{1}{2},\theta_{-i}<\frac{1}{2}\right] \\ \leq\ & E_\theta\left[a(\theta_i,\theta_{-i})|\theta_i>\frac{1}{2},\theta_{-i}<\frac{1}{2}\right]E_\theta\left[(1-\theta_i)|\theta_i>\frac{1}{2},\theta_{-i}<\frac{1}{2}\right], \end{aligned}$$

and

$$\begin{aligned} & -E_\theta\left[a(\theta_i,\theta_{-i})\theta_i|\theta_i\leq\frac{1}{2},\theta_{-i}>\frac{1}{2}\right] \\ \leq\ & -E_\theta\left[a(\theta_i,\theta_{-i})|\theta_i\leq\frac{1}{2},\theta_{-i}>\frac{1}{2}\right]E_\theta\left[\theta_i|\theta_i>\frac{1}{2},\theta_{-i}<\frac{1}{2}\right]. \end{aligned}$$

with strict inequality holding whenever $a(\theta_i,\theta_{-i})$ is not constant over the region under analysis.

Hence, over

$$\left[0,\frac{1}{2}\right)\times\left(\frac{1}{2},1\right]\cup\left(\frac{1}{2},1\right]\times\left[0,\frac{1}{2}\right)$$

the optimal schedule is constant. Moreover, by setting the constant to $\frac{1}{2}$ when one announces $\frac{1}{2}$, and $x > \frac{1}{2}$, the term

$$-\frac{1}{4}E_{\theta_{-i}}\left[\left(a\left(\frac{1}{2},\theta_{-i}\right)-\frac{1}{2}\right)^2|\theta_{-i}\leq\frac{1}{2},x>\frac{1}{2}\right]-\frac{1}{4}E_{\theta_{-i}}\left[\left(a\left(\frac{1}{2},\theta_{-i}\right)-\frac{1}{2}\right)^2|\theta_{-i}>\frac{1}{2},x>\frac{1}{2}\right]$$

is also maximized.

Therefore the modified allocation has (a) a constant action off-diagonals and (b) prescribes, at least half of the time, type $\frac{1}{2}'s$ most preferred actions. Hence,

$$V\left(a_{1/2}\left(\theta,x\right)\right)>V\left(a\left(\theta;x\right)\right).$$

■

Step 2 (Restoring IC): In this step we show that we can modify $a_{1/2}(\theta,x)$ in a way that restores IC, preserves 1/2 off-diagonals and does strictly better than the original allocation $a(\theta,x)$. Throughout, we specify the changes only for the top quadrant $(\left[\frac{1}{2},1\right]^2)$ as the required changes over the bottom quadrant are similar.

We start by constructing an alternative schedule as follows.

For an integer N, consider the partition of $\left[\frac{1}{2},1\right]$ given by $\{A_i\}_i$, where $i\in\{1,...,N\}$; and, for $i<N-1$, $A_i=\left[\frac{1}{2}+\frac{i-1}{2N},\frac{1}{2}+\frac{i}{2N}\right)$ and $A_N=\left[\frac{1}{2}+\frac{N-1}{2N},1\right]$. Let $a_{ij}=E_{\theta,x}\left[a\left(\theta;x\right)|\theta\in A_i\times A_j\right]$ denote the expected action in each square $A_i\times A_j$ under the original allocation. Now consider, the schedule

$$\overline{a}_N\left(\theta,x\right)=\begin{cases}\frac{1}{2} & \text{if } \theta_i>\frac{1}{2}>\theta_{-i} \textit{ or if } \theta_i=\frac{1}{2} \textit{ and } x>\frac{1}{2}\\ a_{ij} & \text{if } \theta\in A_i\times A_j\subset\left(\frac{1}{2},1\right]^2\\ a_{1j} & \text{if } \theta_i=\frac{1}{2},\ \theta_j\in A_j \textit{ and } x<\frac{1}{2}\end{cases}\tag{A1}$$

There exists a $\overline{N}$ and a $\gamma>0$ so that for all $N>\overline{N}$,

$$V\left(\overline{a}_N\left(\theta,x\right)\right)>V\left(a\left(\theta,x\right)\right)+\gamma.$$

This follows from noting that $\overline{a}_N(\theta,x)$ converges to $a(\theta;x)$ over $\left(\frac{1}{2},1\right]^2$ when N goes to the infinity, so that:

$$\lim_{N\to\infty}\overline{a}_N\left(\theta,x\right)=a_{1/2}\left(\theta,x\right),$$

Since

$$V\left(a_{1/2}\left(\theta,x\right)\right)>V\left(a\right),$$

making use of the Dominated Convergence Theorem, it then follows that there exists a $\overline{N}$ and a $\gamma>0$ so that if $N>\overline{N}$

$$V\left(\overline{a}_N\left(\theta,x\right)\right)>V\left(a\left(\theta,x\right)\right)+\gamma$$

as stated.

Although better than $a(\theta;x)$ for N large, $\overline{a}_N(\theta,x)$ might not be IC.

To re-establish incentive compatibility, we need all the "cutoff types" $\frac{1}{2}+\frac{i}{2N}$, $i\in\{1,..,N-1\}$, to be indifferent between reporting "left" or "right" together with expected monotonicity of the allocation. As a first step we show that it is sufficient to modify the allocation by adding (respectively subtracting) constants δ_i, along the diagonal squares $A_i\times A_i$, $i\geq1$ (resp. $i\leq-1$) to satisfy the indifference conditions. As a second step, we show that such δ_i can be chosen to be positive and such that the resulting allocation fares strictly better than $a(\theta;x)$. Finally, the last step shows that expected monotonicity is indeed satisfied. In what follows, we restrict attention to the region $\left[\frac{1}{2},1\right]^2$. The analysis for the region $\left[0,\frac{1}{2}\right]^2$ is analogous.

Step 2A:

In order for the IC constraints to be satisfied, the $\{\delta_i\}_i$ must be chosen to guarantee

$$\begin{aligned} & -\frac{1}{N}\sum_{j=1,j\neq i}^{N}\left(a_{i,j}-\left(\frac{1}{2}+\frac{i}{N}\right)\right)^2-\frac{1}{2N}\left(a_{ii}+\delta_i-\left(\frac{1}{2}+\frac{i}{N}\right)\right)^2 \\ = & -\frac{1}{N}\sum_{j=1,j\neq i+1}^{N}\left(a_{i+1,j}-\left(\frac{1}{2}+\frac{i}{N}\right)\right)^2-\frac{1}{2N}\left(a_{i+1i+1}+\delta_{i+1}-\left(\frac{1}{2}+\frac{i}{N}\right)\right)^2, \end{aligned}$$

so that the cut-off types are indifferent between reporting left and right.

The above conditions induce a difference equation of the following form

$$\begin{aligned} & -\frac{1}{N}\sum_j a_{i,j}^2+2\left(\frac{1}{2}+\frac{i}{N}\right)\frac{1}{N}\sum_j a_{i,j}-\frac{1}{N}\delta_i^2+2\frac{1}{N}\left(\frac{1}{2}+\frac{i}{N}\right)\delta_i-2\frac{1}{N}a_{ii}\delta_i \\ = & -\frac{1}{N}\sum_j a_{i+1,j}^2+2\frac{1}{N}\left(\frac{1}{2}+\frac{i}{N}\right)\sum_j a_{i+1,j}-\frac{1}{N}\delta_{i+1}^2+2\frac{1}{N}\left(\frac{1}{2}+\frac{i}{N}\right)\delta_{i+1}-2\frac{1}{N}a_{i+1i+1}\delta_{i+1} \end{aligned} \tag{A2}$$

or

$$\delta_{i+1} = \frac{1}{2}+\frac{i}{N}-a_{i+1i+1}\pm\frac{N}{2}\left(\begin{array}{c} \frac{4}{N^2}\left(\frac{1}{2}+\frac{i}{N}-a_{i+1i+1}\right)^2 \\ -4\frac{1}{N}\left[\begin{array}{c} \frac{2\delta_i}{N}\left[\left(\frac{1}{2}+\frac{i}{N}\right)-a_{ii}\right]-\frac{1}{N}\delta_i^2+ \\ \frac{1}{N}\sum_j\left(a_{i+1,j}^2-a_{i,j}^2\right) \\ -\frac{2}{N}\left(\frac{1}{2}+\frac{i}{N}\right)\sum_j\left(a_{i+1,j}-a_{i,j}\right) \end{array}\right] \end{array}\right)^{1/2}$$

$$\delta_{i+1} = \frac{1}{2}+\frac{i}{N}-a_{i+1i+1}\pm\frac{1}{2}\left(\begin{array}{c} 4\left(\frac{1}{2}+\frac{i}{N}-a_{i+1i+1}\right)^2 \\ -4\left[\begin{array}{c} 2\delta_i\left[\left(\frac{1}{2}+\frac{i}{N}\right)-a_{ii}\right]-\delta_i^2+ \\ \sum_j\left(a_{i+1,j}^2-a_{i,j}^2\right) \\ -2\left(\frac{1}{2}+\frac{i}{N}\right)\sum_j\left(a_{i+1,j}-a_{i,j}\right) \end{array}\right] \end{array}\right)^{1/2},$$

(where we have dropped the superscripts in the summation sign).

Hence

$$\frac{\delta_{i+1}}{\sqrt{N}} = \begin{array}{c} \frac{1}{\sqrt{N}}\left(\frac{1}{2}+\frac{i}{N}-a_{i+1i+1}\right) \\ \pm\frac{1}{2}\left(\frac{4}{N}\left(\frac{1}{2}+\frac{i}{N}-a_{i+1i+1}\right)^2-4\left[\begin{array}{c} \frac{2\delta_i}{N}\left[\left(\frac{1}{2}+\frac{i}{N}\right)-a_{ii}\right]-\frac{\delta_i^2}{N} \\ +\frac{1}{N}\sum_j\left(a_{i+1,j}^2-a_{i,j}^2\right) \\ -\frac{2}{N}\left(\frac{1}{2}+\frac{i}{N}\right)\sum_j\left(a_{i+1,j}-a_{i,j}\right) \end{array}\right]\right)^{1/2} \end{array} \tag{A3}$$

We next show that, for a properly chosen δ_1, one can find a sequence of $\{\delta_i\}_i$ with $\delta_i\geq 0$ for all i.

Step 2B: There exists a sequence of non-negative $\{\delta_i\}$ that (i) solve Equation $(A3)$, and (ii) lead to a schedule that improves strictly upon the initial $a(\theta;x)$. Moreover, the $\{\delta_i\}$ can be chosen to be $O\left(\sqrt{N}\right)$.

We establish this result in 3 Claims.

Claim 1: If δ_1 is strictly positive and $O\left(\sqrt{N}\right)$, one can find, for all $i\geq 2$, a sequence of strictly positive$\{\delta_i\}_{i\geq 2}$, where each δ_i is also $O\left(\sqrt{N}\right)$.

Proof: Since $a(\theta, x)$ is a continuous function over a compact set, it is uniformly continuous. Therefore, for any $\varepsilon > 0$, there exists a N'so that, if $N > N'$,

$$|a_{i,j} - a_{i+1,j}| < \varepsilon$$

and

$$\left|a_{i,j}^2 - a_{i+1,j}^2\right| < \varepsilon.$$

for *all* i,j.

Hence,

$$\frac{1}{N}\left|\sum_j \left(a_{i,j}^2 - a_{i+1,j}^2\right)\right| \leq \frac{1}{N}\sum_j \left|\left(a_{i,j}^2 - a_{i+1,j}^2\right)\right| < \varepsilon$$

and

$$\frac{1}{N}\left|\sum_j \left(a_{i,j} - a_{i+1,j}\right)\right| \leq \frac{1}{N}\sum_j \left|\left(a_{i,j}^2 - a_{i+1,j}^2\right)\right| < \varepsilon$$

This implies that, for all i, j,

$$\frac{1}{N}\left|\sum_j \left(a_{i,j}^2 - a_{i+1,j}^2\right)\right| = O\left(\frac{1}{N}\right)$$

and

$$\frac{1}{N}\left|\sum_j \left(a_{i,j} - a_{i+1,j}\right)\right| = O\left(\frac{1}{N}\right).$$

Now, consider A3 when $i = 1$.

If δ_1 is $O\left(\sqrt{N}\right)$, δ_1^2 is $O(N)$. It then follows that $-2\frac{1}{2N}\left(a_{11} - \left(\frac{1}{2} + \frac{1}{N}\right)\right)\delta_1 = O\left(\frac{1}{\sqrt{N}}\right)$, and $-\frac{1}{2N}\delta_1^2 = O(1)$

Moreover, from the discussion in the beginning of the proof, one has that

$$-\frac{1}{N}\sum_j a_{1,j}^2 + \frac{1}{N}\sum_j a_{2,j}^2 = O\left(\frac{1}{N}\right),$$

and

$$2\left(\frac{1}{2} + \frac{1}{N}\right)\frac{1}{N}\left[\sum_j \left(a_{1,j} - a_{2,j}\right)\right] = O\left(\frac{1}{N}\right).$$

Therefore, whenever δ_1 is $O\left(\sqrt{N}\right)$,

$$\frac{1}{\sqrt{N}}\left[\left(\tfrac{1}{2} + \tfrac{1}{N} - a_{22}\right)\right] \pm \left(\sqrt[2]{\tfrac{1}{N}\left(\tfrac{1}{2} + \tfrac{1}{N} - a_{22}\right)^2 - 2\left[\begin{array}{c} \frac{\delta_1}{N}\left[\left(\frac{1}{2} + \frac{1}{N}\right) - a_{11}\right] - \frac{1}{2}\frac{\delta_1^2}{N} \\ +\frac{1}{2}\frac{1}{N}\sum_j \left(a_{2,j}^2 - a_{1,j}^2\right) - \frac{1}{N}\left(\frac{1}{2} + \frac{1}{N}\right)\sum_j \left(a_{2,j} - a_{1,j}\right) \end{array}\right]}\right)$$

is $O(1)$ (and well defined for N large, as the term inside the square root will be strictly positive). Hence, $\frac{\delta_2}{\sqrt{N}}$ must be $O(1)$, which implies that δ_2 must also be $O\left(\sqrt{N}\right)$. It is easy to see that δ_2 can be chosen to be positive by picking the positive term of the square root.

Proceeding inductively, the result follows.■

Denote by $\hat{N}$ the value such that for $N > \hat{N}$ the sequence $\{\delta\}$ is well defined and positive.

Claim 2: There exists a strictly positive δ_1, which is $O\left(\sqrt{N}\right)$ so that the schedule defined by

$$\widetilde{a}_1(\theta, x) = \begin{cases} \overline{a}_N(\theta, x) + \delta_1 \text{ if } \theta \in A_1 \times A_1 \\ \overline{a}_N(\theta, x) \ \textit{otherwise} \end{cases}$$

satisfies

$$V(\widetilde{a}_1(\theta; x)) \geq V(a(\theta; x)).$$

Proof: This follows immediately from Step 1. In fact, since for all $N > \max\left(\overline{N}, \hat{N}\right)$,

$$V(\overline{a}_N(\theta, x)) > V(a((\theta; x))) + \gamma$$

for γ strictly positive.

By adding a strictly positive number over $A_1 \times A_1$, one will decrease type $\frac{1}{2}'$s payoff.

Since, from type $\frac{1}{2}'s$ perspective, the harm caused by such change will occur with probability $\frac{1}{N}$, and

$$V(\overline{a}_N(\theta, x)) > V(a((\theta; x))) + \gamma$$

the positive δ_1 necessary to satisfy

$$V(\widetilde{a}_1(\theta; x)) \geq V(a(\theta; x)).$$

can be made $O\left(\sqrt{N}\right)$.■

Claim 3: One can find a schedule that satisfies the indifference condition (Equation $(A3)$) and fares strictly better than $a(\theta; x)$.

Proof: For some $N > \overline{\overline{N}} = \max\left(\overline{N}, \hat{N}\right)$, define a new schedule that is equal to $\overline{a}_N(\theta, x)$ except at the squares $A_i \times A_i$, where it is equal to $a_{ii} + \delta_i$ for $i \geq 1$, where the δ_i is given by the sequence defined in Claim 1 for the δ_1.in Claim 2. Denoting this schedule by $\widetilde{a}(., x)$ one has that

$$V(\widetilde{a}(\theta; x)) > V(\widetilde{a}_1(\theta; x)) \geq V(a(\theta; x)).$$

This follows because (i) $V(.)$ is linear in $a(.)$ for $\theta \in \left[\frac{1}{2}, 1\right]^2$, and, finally, (ii) the adding of the $\delta_i' s$ for $i \geq 2$ does *not* affect the utility of type $\frac{1}{2}$.■

We have just shown that starting from an arbitrary continuous and non-decreasing schedule a, we can construct an alternative schedule $\widetilde{a}$ that has $\frac{1}{2}$ off-diagonals, satisfies Local Incentive Compatibility and fares better than the initial a. What is left to show is that $\widetilde{a}$ satisfies expected monotonicity. We know argue that this is in fact the case.

Step 2C (Monotonicity): For all i, there is $\widetilde{N}$ so that, for $N > \widetilde{N}$,

$$\sum_{j=1}^{N} \widetilde{a}_{i+1j} \geq \sum_{j=1}^{N} \widetilde{a}_{ij}$$

Proof: First note that the indifference condition Eq.[$A2$] can be read as

$$0 = \sum_j \left(a_{i+1,j}^2 - a_{i,j}^2\right) - 2\left(\frac{1}{2} + \frac{i}{N}\right) \sum_j (a_{i+1,j} - a_{i,j})$$
$$- \left(\delta_i^2 - \delta_{i+1}^2\right) + 2\left(\frac{1}{2} + \frac{i}{N} - a_{i+1i+1}\right)(\delta_i - \delta_{i+1}) - 2\delta_i(a_{ii} - a_{i+1i+1})$$

Now we do a Taylor series expansion of $a_{i+1,j}^2$ and δ_{i+1}^2,

$$a_{i+1,j}^2 = a_{i,j}^2 + 2a_{i,j}\left[a_{i+1,j} - a_{i,j}\right] + O\left(\left(a_{i+1,j} - a_{i,j}\right)^2\right),$$

$$\delta_{i+1}^2 = \delta_i^2 + 2\delta_i\left[\delta_{i+1} - \delta_i\right] + O\left(\left(\delta_{i+1} - \delta_i\right)^2\right)$$

Therefore, one can write the indifference condition as

$$\begin{aligned} 0 &= \sum_j\left(2a_{i,j}\left[a_{i+1,j} - a_{i,j}\right] + O\left(\left(a_{i+1,j} - a_{i,j}\right)^2\right)\right) - 2\left(\frac{1}{2} + \frac{i}{N}\right)\sum_j\left(a_{i+1,j} - a_{i,j}\right) \\ &\quad + 2\delta_i\left[\delta_{i+1} - \delta_i\right] + O\left(\left(\delta_{i+1} - \delta_i\right)^2\right) + 2\left(\frac{1}{2} + \frac{i}{N} - a_{i+1i+1}\right)\left(\delta_i - \delta_{i+1}\right) - 2\delta_i\left(a_{ii} - a_{i+1i+1}\right) \end{aligned}$$

or

$$\begin{aligned} 0 &= \sum_j\left(2a_{i,j}\left[a_{i+1,j} - a_{i,j}\right] + O\left(\left(a_{i+1,j} - a_{i,j}\right)^2\right)\right) - 2\left(\frac{1}{2} + \frac{i}{N}\right)\sum_j\left(a_{i+1,j} - a_{i,j}\right) \\ &\quad - 2\left[\delta_i - \delta_{i+1}\right]\left[\delta_i - \left(\frac{1}{2} + \frac{i}{N} - a_{i+1i+1}\right)\right] + O\left(\left(\delta_{i+1} - \delta_i\right)^2\right) - 2\delta_i\left(a_{ii} - a_{i+1i+1}\right) \end{aligned} \quad (4)$$

Now, assume, towards a contradiction, that expected monotonicity is violated i.e. there is an i such that for all N :

$$\sum_{j=1}^N \widetilde{a}_{ij} > \sum_{j=1}^N \widetilde{a}_{i+1j}$$

This, in turn, implies that

$$\delta_i - \delta_{i+1} > \sum_{j=1}^N\left(a_{i+1j} - a_{ij}\right) \geq 0.$$

For N large and δ of order $O\left(\sqrt{N}\right)$:

$$\delta_i - \left(\frac{1}{2} + \frac{i}{N} - a_{i+1i+1}\right) > 0,$$

and therefore:

$$-2\left[\delta_i - \delta_{i+1}\right]\left[\delta_i - \left(\frac{1}{2} + \frac{i}{N} - a_{i+1i+1}\right)\right] < -2\left[\delta_i - \left(\frac{1}{2} + \frac{i}{N} - a_{i+1i+1}\right)\right]\sum_{j=1}^N\left(a_{i+1j} - a_{ij}\right). \quad (5)$$

Also, since $\left[a_{i+1,j} - a_{i,j}\right] \geq 0\ \forall j$

$$\sum_j\left(a_{i,j}\left[a_{i+1,j} - a_{i,j}\right]\right) \leq \max_j a_{i,j}\sum_j\left(a_{i+1,j} - a_{i,j}\right). \quad (6)$$

From Eq. [4] and the last two inequalities we get:

$$\begin{aligned} 0 &< 2\sum_j\left(a_{i+1,j} - a_{i,j}\right)\left[\max_j a_{i,j} - \left(\frac{1}{2} + \frac{i}{N}\right) - \left[\delta_i - \left(\frac{1}{2} + \frac{i}{N} - a_{i+1i+1}\right)\right]\right] \\ &\quad + \sum_j O\left(\left(a_{i+1,j} - a_{i,j}\right)^2\right) + O\left(\left(\delta_{i+1} - \delta_i\right)^2\right) - 2\delta_i\left(a_{ii} - a_{i+1i+1}\right) \end{aligned}$$

Finally show that for large N the right hand side of this inequality is smaller than zero, which leads to the desired contradiction. Towards this note that for large N the following are true:

1.
$$\left[\max_j a_{i,j} - \left(\frac{1}{2} + \frac{i}{N}\right) - \left[\delta_i - \left(\frac{1}{2} + \frac{i}{N} - a_{i+1i+1}\right)\right]\right] \sum_{j=1}^{N} (a_{i+1j} - a_{ij}) < 0$$

and of order $O\left(\sqrt{N}\right)$. This follows from $\left[\max_j a_{i,j} - \left(\frac{1}{2} + \frac{i}{N}\right) - \left[\delta_i - \left(\frac{1}{2} + \frac{i}{N} - a_{i+1i+1}\right)\right]\right]$ being $O\left(\sqrt{N}\right)$, and $\sum_{j=1}^{N} (a_{i+1j} - a_{ij})$ being $O(1)$.

2. $-2\delta_i (a_{ii} - a_{i+1i+1})$ is $O\left(\frac{1}{\sqrt{N}}\right)$, since δ_i is $O\left(\sqrt{N}\right)$ and $(a_{ii} - a_{i+1i+1})$ is $O\left(\frac{1}{N}\right)$.

3. $\sum_j O\left((a_{i+1,j} - a_{i,j})^2\right)$ is $O\left(\frac{1}{N}\right)$.

4. $O\left((\delta_{i+1} - \delta_i)^2\right)$ is $O(1)$.

Hence, there exists an N large such that the right hand side of the inequality is negative which implies, that for all i, there must be an $\widetilde{N}$ so that, for $N > \widetilde{N}$,

$$\sum_{j=1}^{N} \widetilde{a}_{i+1j} \geq \sum_{j=1}^{N} \widetilde{a}_{ij}. \blacksquare$$

All the results above establish the proof for the case in which the initial $a(.)$ is continuous.

In the next Step 3 we deal with the case in which $a(.)$ is non-decreasing but potentially discontinuous.

Step 3: *Given any incentive compatible allocation $a(\theta; x)$ that satisfies Monotonicity we can find an alternative incentive compatible allocation $\tilde{a}(\theta; x)$ which is weakly better and satisfies $\tilde{a}(\theta; x) = \frac{1}{2}$ for all x, when $\theta_i > \frac{1}{2} > \theta_{-i}$, and $\tilde{a}\left(\frac{1}{2}, \theta_{-i}; x\right) = \frac{1}{2}$ when $x > \frac{1}{2}$.*

Consider an initial allocation $a(\theta, x)$ which satisfies monotonicity and incentive compatibility but is not continuous in θ. Lemmas 5 and 6 below imply that there exists a sequence of non-decreasing continuous functions $\{f_n\}_n$ such that

$$f_n(\theta, x) \to a(\theta, x) \text{ for all } \theta.$$

Now applying the Steps 1 & 2 detailed above to each function f_n we get a sequence of functions $\left\{\tilde{f}_n\right\}_n$ with the property that:

$$V\left(\tilde{f}_n\right) > V(f_n) \quad \forall n$$

Since (by continuity and the fact that $[0,1]^2$ is compact) there exists $k < \infty$ so that

$$\max_\theta |f_n(\theta)| < k \text{ for all } n,$$

one has, by the Dominated Convergence Theorem,

$$V(f_n) \to V(a) \tag{A9}$$

This implies that there exists an $\bar{n}$ such that for $n > \bar{n}$

$$V\left(\tilde{f}_n\right) \geq V(a).$$

Finally note that for all $\tilde{f}_n$ the values set in the off-diagonals is $\frac{1}{2}$. Hence, setting $\frac{1}{2}$ off-diagonal is weakly better even if the starting allocation was not continuous.$\blacksquare$

We now show that, for any non-decreasing function $g : [0,1]^2 \to \Re$, we can find a sequence of continuous non-decreasing functions $\{g_m\}_m$ which converge pointwise to $g(.)$. In order to do so, we first use the following result, which proof can be found in Rosenlicht (1968, pages 237 and 238)

Lemma 5 *For a given N, consider the partition of $[0,1]$ given by $\{A_i\}_i$, where $A_i = \left[\frac{1}{2} + \frac{i}{2N}, \frac{1}{2} + \frac{i+1}{2N}\right)$ whenever $i \in \{-N, .., N-2\}$, and $A_{N-1} = \left[\frac{1}{2} + \frac{N-1}{2N}, 1\right]$. Consider a non-decreasing simple function $g : [0,1]^2 \to \Re$; that is, a function*

$$g(\theta) = c_{ij} \in \Re \text{ whenever } \theta \in A_i \times A_j.$$

with $c_{i+1j+1} \geq c_{i+1j+1} \geq c_{ij}$.

One can then find a sequence of non-decreasing continuous functions that converge to $g(.)$ pointwise.

With the above result in hands, we are now ready to prove

Lemma 6 *Let $g : [0,1]^2 \to \Re$ be a non-decreasing function. Then there is a sequence of non-decreasing, continuous functions that converge pointwise to g.*

Proof. For an integer N, consider the partition of $[0,1]$ given by $\{A_i\}_i$, where $A_i = \left[\frac{1}{2} + \frac{i}{2N}, \frac{1}{2} + \frac{i+1}{2N}\right)$ whenever $i \in \{-N, .., N-2\}$, and $A_{N-1} = \left[\frac{1}{2} + \frac{N-1}{2N}, 1\right]$. Define $g_N(.)$ as follows:

$$g_N(\theta) = E_\theta\left[g(\theta) | A_i \times A_j\right] \text{ if } \theta \in A_i \times A_j.$$

Clearly, for all θ,

$$||g_N(\theta) - g(\theta)|| \to 0$$

as $N \to \infty$.

Now, fixing a N, one has that $g_N(\theta)$ is a non-decreasing simple function. Hence, by 5, one can find, for each N, a sequence of non-decreasing continuous functions $\{g_m^N(.)\}_m$ so that

$$\left|\left|g_m^N(\theta) - g_N(\theta)\right|\right| \to 0$$

as $m \to \infty$. Since

$$\left|\left|g_m^N(\theta) - g(\theta)\right|\right| \leq \left|\left|g_m^N(\theta) - g_N(\theta)\right|\right| + ||g_N(\theta) - g(\theta)||,$$

we have that for all θ

$$\left|\left|g_m^N(\theta) - g(\theta)\right|\right| \to 0$$

as $m, N \to \infty$. ■