Rational Expectations at the Racetrack : Testing Expected Utility Theory Using Betting Market Equilibrium (VERY PRELIMINARY DRAFT)

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Abstract

We present an adaptation of the Arrow-Debreu framework to model betting market equilibrium. Under very weak conditions on the distributions of risk preferences in the population, we show that there exists a unique fully revealing rational expectations equilibrium of the model. The inverse of the rational expectations pricing function represents the model's empirically observable equilibrium correspondence. We study the structure of this inverse pricing function for the case of one dimensional preference heterogeneity. This structure provides neccessary testable implications of the equilibrium model that are also sufficient for identifying the underlying distribution of preferences, and thereby can be used to test expected utility theory as well as a range of other preference theories for choice under risk.

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1 Introduction

A fundamental assumption about human behavior used in modern economic modelling is the expected utility hypothesis (EUH). In its most basic form, the EUH is a hypothesis about the nature of individual preferences for risky propsects, i.e. lotteries over a set of possible monetary outcomes. The EUH maintains that probability enters linearly into an economic agent's preference for risk, leading the agent to acts so as to maximize an expected utility of wealth when faced with a choice among lotteries. Today, the assumption of expected utility maximization is a central building block in the economics of uncertainty, with applications ranging from strategic games to competitive asset markets.

However, expected utility theory has also undergone an immense amount of scrutinty and criticism as a description of how individuals actually behave when faced with a risky decision. In particular, the assumption that probability enters linearly into the calculus of comparing lotteries has been shown to be routinely violated by subjects in experimental settings. An especially vivid demonstration of this phenomena is the well known Allais paradox.[1].

These negative experimental findings have motivated a number of attempts to relax expected utility theory in ways that better fit the experimental evidence (for a review of nonexpected utility theories, see Starmer [2000]). The most prominent of these attempts is cumulative prospect theory (CPT) [Tversky and Kahneman, 1992], which became one of the hallmark successes of the experimental approach to economics. In addition to allowing probability to nonlinearly enter preferences, CPT further relaxes EUT by also allowing an economic agent to treat losses and gains asymmetrically. The empirical superiority of CPT over EUT in explaining choice patterns uncovered in the laboratory have led to loud cries from the experimental community to abandon expected utility theory in applied economic modelling (e.g., Rabin and Thaler [2001]).

Despite these findings from the world of experiments, the development and use of economic models built on the basis of expected utility maximization continues largely unabated. The reason lies in the neoclassical defense that the "true" test of an economic theory lies in its capacity to explain the actual choices of economic actors in naturally occuring institutions (where opportunities for self selection, feedback from the environment, and learning are present), not the hypothetical choices of experimental subjects in the laboratory. The neoclassical defense has thus provided provided a buffer that has thus far kept expected utility theory relevant despite its experimental failings [Cubitt et al., 2001].

Thus a fundamental empirical question is whether market data rather than experimental data can be brought to bear on the debate between expected and non-expected utility theory. Do violations of EUT

[1]For a review of the experimental evidence against EUT, see Starmer [2000].

continue to persist when experienced economic agents in real markets are observed? Moreover, even if violations exist at the individual level, to what extent do they matter when individual level choices are aggregate to determine market outcomes. As Machina [1982] has shown, EUT can always be viewed as a first order approximation to an individual's non-EUT preferences. Is the approximation "sufficient" when the real interests is aggregate market behavior rather than individual choice?

The problem of identifying the nature of individual risk preferences from market outcomes is a difficult question to address directly. While financial and insurance markets would seem natural candidates for empirical analysis, the complexity of the underlying equilibrium in these markets confounds the issue of risk preferences. Financial markets are inherently dynamic, while insurance markets requires simultaneous analysis of demand with an imperfect supply side. In order to pursue equilibrium analysis of these markets, simplifying assumptions about the nature of risk preferences, such as time separability and constant relative risk aversion, are often times neccessary. This leaves little room for the possibility of specifying risk preferences "nonparametrically", and considering the observable consequence of different theories for choice under risk on the market equilibrium.

In this paper, we study the possibility of using betting/prediction markets prices to identify risk preferences. Betting markets come in a variety of popular forms, ranging from the odds market at racetracks, to the more recent online exchanges that have come to be nominally known as "prediction markets", such as Tradesports and the Iowa Electronic Market. As we shall further explain, the common thread tying together these markets is that they are single period ex-ante markets for the trade of a complete set of Arrow-Debreu securities. For example, at the racetrack, a "bet on horse i" is equivalent to a security that pays off 1 dollar in the event the i^{th} horse wins, and 0 dollars otherwise. Securities differ both by their chances that the underlying event will be realized (i.e., the probability that horse i wins) and by their market prices. In this way, betting markets come as close as any market to offering consumers a menu of simple lotteries akin to those used in choice experiments, providing a natural laboratory to test theories of individual choice under risk.

As with other markets, the central problem in using betting markets to estimate and test models of individual choice behavior is that individual bets are not readily observed. Instead, only prices across markets, i.e., the equilibrium correspondence that maps the primitives in each market to equilibrium prices, can be possibly observed. In order to fill this gap, an economic model of equilibrium pricing in the betting market is needed, with empirically distinct consequences for the pricing of bets arising from distinct assumptions about the nature of preferences. Of course, due to "textbook" one period, complete contingent claims nature of betting markets, we have such a theory close at hand in the form of the usual Arrow-Debreu price theory, the general form of which does not require us to make any assumptions about the structure of risk preferences beyond weak regularity conditions (such as continuity and monotinicity). The question is whether Arrow-Debreu theory has enough "content" to empirically distinguish between distinct utility theories of decision making under risk?

The goal of the present paper is to adapt the general Arrow-Debreu machinery to the particular features of betting markets, and thereby derive the empirically relevant equilibrium correspondence implied by the theory. Our equilibrium corespondence takes the form of a rational expectations pricing function, which is an invertible map from a set of suitably defined "states of nature" to market clearing prices. We then show what empirically distinct consequences different assumptions about the distribution of risk preferences in the population have for the equilibrium correspondence. More generally, we show that our adaptation of the Arrow-Debreu model implies certain neccessary tesable restrictions on the equilibrium correspondence that are sufficient for identifying the distribution of risk preferences. The analysis thus provides a method for recovering risk preferences from the equilibrium correspondence, which in turn can be estimating using "real" market data. Empirical application of the methodology is forthcoming.

2 Betting Markets and Market Clearing Prices

We shall use the language of horse races to describe betting markets more generally. Consider a race with n horses. An ex-ante market (i.e. before the race is run) is open for the trade of n Arrow-Debreu securities. A unit of security i buys 1 dollar in the event that horse i wins the race, and 0 dollars otherwise. Let r_i denote the price of security i, and let $M_i > 0$ denote the total number of dollars in the market spent on purchasing security i. Let τ denote the participatory tax per dollar bet, commonly called the track take. Then in order for markets to clear, i.e., in order for the supply of dollars to equal the demand of dollars in each of the possible n outcomes of the race, it must be the case that

$$\begin{aligned} & \frac{(1-\tau)M_i}{r_i} = (1-\tau)(M_1 + \cdots + M_n) \quad (\forall i) \\ \Rightarrow \quad & r_i = \frac{M_i}{M_1 + \cdots + M_n} \quad (\forall i) \\ \Rightarrow \quad & r_i = s_i = \text{The market share (of dollars) of horse } i \quad (\forall i) \\ \Rightarrow \quad & (r_1, \ldots, r_n) \in Int(\Delta^{n-1}). \end{aligned}$$

Prices at the racetrack are not customarily quoted in terms of the Arrow-Debreu prices r_i, but rather in terms of the odds R_i on horse i. The odds R_i are defined as net profit per dollar bet on horse i in the event i wins the race, and thus can be expressed in terms of the Arrow-Debreu price r_i as

$$R_i = \frac{(1-\tau)}{r_i} - 1.$$

Using the market clearing condition that $r_i = s_i (\forall i)$, we have that market clearing odds are

$$R_i = \frac{(1-\tau)}{s_i} - 1. \tag{1}$$

The market clearing condition (1) is in fact how betting odds are institutionally determined at the racetrack, and "parimutuel" betting systems more generally. It was purportedly "discovered" in 1865 by Pierre Oller, a French perfume maker, who found that setting odds according to (1) with positive track take τ offered an alternative to bookmakers with the advantage of assuring the "house" a profit.

While the prices determined by (1) are market clearing, Arrow-Debreu equilibrium requires that they also be consistent with utility maximizing behavior on the part of the bettors at the track. That is, equilibrium occurs at prices $\mathbf{r} = (r_1, \ldots, r_n)$ such that the aggregate demand for securities at $\mathbf{r}$ results in the horses having market shares $\mathbf{r}$. In order to model aggregate demand and explore equilibrium, we now turn to the issue of preferences.

3 Preferences

Suppose a bettor has beliefs $(p_1, \ldots, p_n) \in \Delta^{n-1}$ over the possible outcomes of the race, and is deciding which horse to back with the M dollars the bettor has alloted for the race. If the market odds are $(R_1, \ldots, R_n)$, then from the point of view of a bettor, each horse i in the race can be thought of as a simple gamble (R_i, p_i), which pays a net return of R_i per dollar invested with probability p_i, and pays a net return of -1 with probability $(1 - p_i)$. Thus the market offers the bettor faces a choice among a menu of n gambles $G = \{(R_1, p_1), \ldots, (R_n, p_n)\}$[2].

We postulate the existence of a *stable* (across races) continnum of consumers T. Each consumer $t \in T$ has a complete, continuous, transitive, and strictly monotonic preference relation $\succsim_t$ over simple gambles $(R, p) \in \mathbb{R}_+ \times [0, 1]$. Thus each consumer t's preference relation can be represented by a continuous utility

[2]More generally, we can allow the menu G to include the option of not betting, which is equivalent to a gamble that offers a net rate of returm zero with probability one.

function $V_t : \mathbf{R} \times [0,1] \to \mathbb{R}$ that is strictly increasing in a gamble's net rate of return from winning R (the first argument of V_t) and probability of winning p (the second argument of V_t). We further impose the restriction that each consumer t's utility function is strictly minimized whenever $p = 0$, i.e., $V_t(0, R) = V_t(0, R')$ and $V_t(0, R) < V(p, R')$ for any returns R, R' and any probability $p > 0$. Thus the worst gamble for any consumer is one that has no probability of winning, regardless of the return from winning (since this return is never realized).

Let $\mathbf{V} \subset \mathbb{R}^{\mathbf{R} \times [0,1]}$ be the set of all such utility functions. We endow $\mathbf{V}$ with the relative product topology, otherwise known as the topology of pointwise convergence. Let the measurable sets in $\mathbf{V}$ be the Borel subsets (the σ-algebra of subsets generated by the open sets) in this topology. Our population T gives rise to a probability measure $\mathbf{P}_V$ over the space $\mathbf{V}$. The probability measure $\mathbf{P}_V$ describes the distribution of consumer preferences for gambles (R, p).

4 Information

Following the suggestion of Figlewski [1979], we consider the information problem facing bettors as one involving a mixture of risk and uncertainty. In each race, there exists some "true" objective probability distribution $(p_1, \ldots, p_n)$ that determines the outcome of the race. That is, bettors face a well defined risk as to which horse will win. However betters are *uncertain* about the distribution $(p_1, \ldots, p_n)$.

In order to model the situation more precisely, we let S denote a set of "states of nature", and let $f : S \to \Delta^{n-1}$ map states of natures to probability distributions over horses. The interpretation is that once a state of nature has been determined, then via f this state determines the probability distribution according to which the outcome of the race is determined.

We shall assume that S is rich enough so as to be onto $Int(\Delta^{n-1})$, and without loss of generality (since we can always form equivalence classes), we take S to be one to one with $Int(\Delta^{n-1})$. Once again, WLOG and for ease of notation, we take $S = Int(\Delta^{n-1})$.

We can introduce information partitions I^t over S for each consumer $t \in T$. We assume that for any two distinct states s' and s, a non-measure zero mass of consumers can distinguish between the occurence of s and s'. Thus in a hypothetical world of pooled information, where every information signal is public knowledge, each bettor knows the state s and hence the distribution over outcomes $(p_1, \ldots, p_n)$.

We now turn to the problem of existence and uniqueness of a fully revealing rational expectations equilibrium for our model. Our strategy is two parts. First we assume a world of pooled information, and consider the existence and uniqueness of market clearing prices $(R_1, \ldots, R_n)$ for each state s. We then show that his pooled information pricing function $p(s)$ is 1-1, and thus a constitutes a fully revealing rational expectations equilibrium.

5 Pooled Information Equilibrium

We impose assumptions on the distribution of preferences, our fundamental primitive of interest, in order to study equilibrium. If the state s is commonly known, then given market prices $(R_1, \ldots, R_n)$, consumers face a common meny of n gambles

$$G = \{(R_1, p_1), \ldots, (R_n, p_n)\} \subset \mathbf{R} \times [0, 1].$$

The subset of the population T that prefers the i^{th} gamble from the set G is denoted

$$S_i = \{V \in \mathbf{V} : V(R_i, p_i) \geq V(R_j, p_j) \text{ for all } j \neq i\}.$$

The share of the population T that prefers the i^{th} gamble from the set G is thus

$$q_i(G) = q_i(R_1, \ldots, R_n; p_1, \ldots, p_n) = \mathbf{P}_V(S_i).^3 \tag{2}$$

We refer to q_i as the *market share* of the i^{th} gamble from G (i.e., if the market offered a choice of gambles from the set G, then q_i is the share of the population T that chooses the i^{th} gamble). Notice that for any n-tuple of distinct probabilities $(p_1, \ldots, p_n)$, q_i is a well defined function over $(R_1, \ldots, R_n) \in \mathbf{R}^n$. Likewise, for any n-tuple of returns $(R_1, \ldots, R_n)$, q_i is a well defined function over $(p_1, \ldots, p_n) \in [0, 1]^n$. In our use of the market share functions, we shall only consider variation in the n-tuple of probabilities $(p_1, \ldots, p_n)$ over the n dimensional unit simplex Δ^{n-1}.

We make two mild regularity assumptions on the distribution of preferences $\mathbf{P}_V$, which we refer to as *continuity* and *desireability*. Continuity requires that the probability measure $\mathbf{P}_V$ be sufficiently continuous, or atomless, so as to not permit a positive mass of consumers to be indifferent between two distinct gambles. Desireability requires that for any gamble $g = (p_g, R_g)$ with nonzero probability of winning $p_g > 0$, and for any finite set of gambles G with $g \in G$, it is always possible to induce some positive mass of the population to prefer g from G by making g's return from winning R_g sufficiently large. We now more formally define

[3] S_i is measurable based on the topology on $\mathbf{V}$.

continuity and desireability, and examine their consequences for the market share functions q_i.

5.1 Continuity

Continuity The probability measure $\mathbf{P}_V$ is continuous if for any two distinct gambles (R_i, p_i) and (R_j, p_j) with p_i or p_j greater than 0 (or both), the number of consumers indifferent between gamble i and j has a probability measure of zero. More precisely, if $p_i > 0$ or $p_j > 0$ then

$$\mathbf{P}_V(\{V \in \mathbf{V} : V(R_i, p_i) = V(R_j, p_j)\}) = 0.$$

Lemma 5.1 *If $\mathbf{P}_V$ is continuous, then for any finite set of gambles G with at least one gamble having nonzero probability of winning,*

$$\mathbf{P}_V(S_i \cap S_j) = 0 \text{ for every } i \neq j.$$

Proof If $p_i = 0$, then $S_i = \emptyset$ because each $V \in \mathbf{V}$ is strictly minimized at $p = 0$. Otherwise, $p_i > 0$, and for $i \neq j$, $S_i \cap S_j$ is a subset of

$$\{V \in \mathbf{V} : V(R_i, p_i) = V(R_j, p_j)\},$$

which by continuity has measure 0 under $\mathbf{P}_V$. ∎

Theorem 5.2 *If $\mathbf{P}_V$ is continuous, then for any finite set of distinct gambles G with at least one gamble in G having nonzero probability of winning, the sum of the market shares equals 1, i.e.,*

$$\sum_{i=1}^{n} q_i(R_1, \ldots, R_n; p_1, \ldots, p_n) = 1.$$

Proof Recall that

$$\sum_{i=1}^{n} q_i(R_1, \ldots, R_n; p_1, \ldots, p_n) = \sum_{i=1}^{n} \mathbf{P}_V(S_i).$$

Moreover, since G is finite, each $V \in \mathbf{V}$ attains a maximum over G, and thus

$$\bigcup_{i=1}^{n} S_i = \mathbf{V}.$$

However by Lemma 5.1, we have that for all $i \neq j$,

$$\mathbf{P}_V(S_i \cap S_j) = 0.$$

Thus

$$\sum_{i=1}^{n} \mathbf{P}_V(S_i) = \mathbf{P}_V\left(\bigcup_{i=1}^{n} S_i\right) = 1. \quad \blacksquare$$

We now establish the following central result.

Theorem 5.3 *If $\mathbf{P}_V$ is continuous, then for any n-tuple of distinct probabilities $(p_1, \ldots, p_n)$,*

$$q_i(R_1, \ldots, R_n; p_1, \ldots, p_n)$$

is a continuous function in $(R_1, \ldots, R_n) \in \mathbf{R}^n$. Furthermore, for any n-tuple of distinct returns $(R_1, \ldots, R_n)$,

$$q_i(R_1, \ldots, R_n; p_1, \ldots, p_n)$$

is a continuous function in $(p_1, \ldots, p_n) \in \Delta^{n-1}$.

Proof We prove the first part of the theorem (continuity in $(R_1, \ldots, R_n)$). The second part (continuity in $(p_1, \ldots, p_n)$) follows similarly to the first.

Let us fix any n-tuple of distinct probabilities $(p_1, \ldots, p_n)$. Now consider any n-tuple of returns $(R_1, \ldots, R_n)$. Define a function $F : \mathbf{V} \to \{0, 1\}$ as

$$F(V) = \prod_{j \neq i} \mathbf{1}\left[V(R_i, p_i) \geq V(R_j, p_j)\right],$$

where $\mathbf{1}(\cdot)$ is the indicator function. Then clearly

$$q_i(R_1, \ldots, R_n; p_1, \ldots, p_n) = \int F(V)\mathbf{P}_V(dV).$$

Now consider any sequence of n-tuples of returns $\{(R_1^t, \ldots, R_n^t)\}_{t \in \mathbb{N}_+}$ that converges to $(R_1, \ldots, R_n)$. For each $t \in \mathbb{N}_+$, define $F^t : \mathbf{V} \to \{0, 1\}$ as

$$F^t(V) = \prod_{j \neq i} \mathbf{1}\left[V(R_i^t, p_i) \geq V(R_j^t, p_j)\right],$$

and thus,

$$q_i(R_1^t, \ldots, R_n^t; p_1, \ldots, p_n) = \int F^t(V)\mathbf{P}_V(dV).$$

We need to establish that $q_i(R_1^t, \ldots, R_n^t; p_1, \ldots, p_n) \xrightarrow{t} q_i(R_1, \ldots, R_n; p_1, \ldots, p_n)$.

For every $V \in \mathbf{V} - S_i$ we have $F(V) = 0$. Thus for every $V \in \mathbf{V} - S_i$,

$$V(R_i, p_i) < V(R_j, p_j) \text{ for some } j \neq i.$$

By continuity of every utility function $V \in \mathbf{V}$, we have that for every $V \in \mathbf{V} - S_i$,

$$F^t(V) \xrightarrow{t} F(V).$$

On the other hand, for every $V \in S_i$, $F(V) = 1$. Since at least one $p_i > 0$, then by Lemma 5.1, for almost every $V \in S_i$,[4]

$$V(R_i, p_i) > V(R_j, p_j) \text{ for every } j \neq i.^5$$

Once again, by continuity of every utility function $V \in \mathbf{V}$, we have that for almost every $V \in S_i$,

$$F^t(V) \xrightarrow{t} F(V).$$

Thus for almost every $V \in \mathbf{V}$, $F^t(V)$ converges pointwise to $F(V)$. By Lesbesgue's dominated convergence theorem,

$$q_i(R_1^t, \ldots, R_n^t; p_1, \ldots, p_n) \xrightarrow{t} q_i(R_1, \ldots, R_n; p_1, \ldots, p_n).$$

The proof of continuity in $(p_1, \ldots, p_n)$ follows similarly. The requirement in the theorem that $(p_1, \ldots, p_n)$ range over Δ^{n-1} is overly restrictive. it is only used to ensure that $(p_1, \ldots, p_n) \in \Delta^{n-1}$ implies at least one $p_i > 0$, which allows all the steps from continuity in returns to be repeated. ∎

The last implication of the continuity assumption that we consider concerns the monitinicity of the market share functions q_i. We first define the relevant meaning of monitinicity, and then show that it is satisfied under continity of the distribution of preferences.

Monotinicity For any n-tuple of distinct probabilities $(p_1, \ldots, p_n)$, any n-tuple of returns $(R_1, \ldots, R_n)$, and any strict subset $\mathbf{I} \subset \{1, \ldots, n\}$, consider a change to the returns from winning appearing in the choice set G that weakly increases the returns of the gambles indexed by $\mathbf{I}$ and weakly decreases the returns of the

[4] The statement "for almost every $V \in S_i$" means "for all $V \in S_i$ except possibly in a subset $S \subset S_i$ with $\mathbf{P}_V(\mathrm{S}) = 0$".
[5] $S_i = \{V \in \mathbf{V} : V(R_i, p_i) > V(R_j, p_j) \text{ for every } j \neq i\} \cup_{j \neq i} (S_i \cap S_j)$.

remaining gambles. This change leads to a new choice set $G^* = \{(R_i^*, p_i)\}_{i\in\{1,\ldots,n\}}$ with

$$R_i^* \geq R_i \text{ for } i \in \mathbf{I}$$

and

$$R_i^* \leq R_i \text{ for } i \notin \mathbf{I}.$$

We say that the distribution of consumer preferences $\mathbf{P}_V$ satisfies *monotinicity in return* if the sum of the shares of the gambles indexed by $\mathbf{I}$ weakly increase as a result of the change in returns, and the sum of the shares of the gambles indexed by $\{1,\ldots,n\} - \mathbf{I}$ weakly decrease as a result of the change in returns. That is

$$\sum_{i\in\mathbf{I}} q_i^* \geq \sum_{i\in\mathbf{I}} q_i$$

and

$$\sum_{i\notin\mathbf{I}} q_i^* \leq \sum_{i\notin\mathbf{I}} q_i.$$

Remark The definition of *monotinicity in probability* is stated similarly, except for any n-tuple of distinct returns $(R_1,\ldots,R_n)$, and any n-tuple of probabilities $(p_1,\ldots,p_n) \in \Delta^{n-1}$[6], we consider a weak increase of the probabilities of winning of the gambles for a strict subset of the gambles, and a weak decreases of the probabilities of winning for the remaining gambles, producing a new n-tuple of probabilities $(p_1^*,\ldots,p_n^*) \in \Delta^{n-1}$. The distribution $\mathbf{P}_V$ satisfies *monotinicity in probability* if such a change results in an increase in the of sum the shares of the gambles for which the probabilities increased, and a decrease in the sum of the shares of the gambles for which the probabilities decreased.

Theorem 5.4 *If $\mathbf{P}_V$ is continuous, then it satisfies monitinicity in return and monitinicity in probability.*

Proof We prove the theorem for monitincity in returns. A similar argument follows for monitinicity in probability. Let $(p_1,\ldots,p_n)$, $(R_1,\ldots,R_n)$, and $(R_1^*,\ldots,R_n^*)$ be the n-tuples described in the definition of monitinicity. Similarly to the proof of Theorem 5.3, define

$$F_i(V) = \prod_{j\neq i} \mathbf{1}\left[V(R_i,p_i) \geq V(R_j,p_j)\right] \quad \text{and} \quad F_i^*(V) = \prod_{j\neq i} \mathbf{1}\left[V(R_i^*,p_i) \geq V(R_j^*,p_j)\right]$$

[6]We restrict the domain of n-tuple of probabilities to the simplex so as to ensure at least one probability is nonzero.

By the linearity of the integral operation

$$\sum_{i\in\mathbf{I}} q_i(R_1,\ldots,R_n;p_1,\ldots,p_n) = \int \sum_{i\in\mathbf{I}} F_i(V)\mathbf{P}_V(dV) \quad \textit{and} \quad \sum_{i\in\mathbf{I}} q_i(R_1^*,\ldots,R_n^*;p_1,\ldots,p_n) = \int \sum_{i\in\mathbf{I}} F_i^*(V)\mathbf{P}_V(dV)$$

Since at least one $p_i > 0$, then by Lemma 5.1, for almost every $V \in \mathbf{V}$, $\sum_{i\in\mathbf{I}} F_i(V)$ equals 0 or 1.[7] However by the monitinicicty of each $V \in \mathbf{V}$, $\sum_{i\in\mathbf{I}} F_i(V) = 1$ implies $\sum_{i\in\mathbf{I}} F_i^*(V) \geq 1$. Thus for almost every $V \in \mathbf{V}$,

$$\sum_{i\in\mathbf{I}} F_i^*(V) \geq \sum_{i\in\mathbf{I}} F_i(V)$$

and since the integral is an increasing linear operation,

$$\sum_{i\in\mathbf{I}} q_i(R_1^*,\ldots,R_n^*;p_1,\ldots,p_n) \geq \sum_{i\in\mathbf{I}} q_i(R_1,\ldots,R_n;p_1,\ldots,p_n). \quad \blacksquare$$

Thus to summarize,

$$q_i(R_1,\ldots,R_n,p_1,\ldots,p_n)$$

is the share of population T that chooses the i^th gamble from the choice set of distinct gambles $G = \{(R_1,p_1),\ldots,(R_n,p_n)\}$. The continuity assumption on the probability distribution $\mathbf{P}_V$ of consumer preferences carried three important consequences :

1. For any such choice set G, so long as there is some gamble with $p_i > 0$, the market shares sum to 1, i.e.,

$$\sum_{i=1}^{n} q_i(R_1,\ldots,R_n;p_1,\ldots,p_n) = 1.$$

2. For an n-tuple of distinct probabilities $(p_1,\ldots,p_n)$,

$$q_i(R_1,\ldots,R_n;p_1,\ldots,p_n)$$

is continuous in $(R_1,\ldots,R_n)$ over $\mathbf{R}^n$. Likewise, for any n-tuple of distinct returns, $(R_1,\ldots,R_n)$,

$$q_i(R_1,\ldots,R_n;p_1,\ldots,p_n)$$

is continuous in $(p_1,\ldots,p_n)$ over Δ^{n-1}.

[7] The set of V for which $\sum_{i\in\mathbf{I}} F_i(V) > 0$ equals $\cup_{i\neq j;i,j\in\mathbf{I}}(S_i \cap S_j)$

3. For any distinct n-tuple of distinct probabilities $(p_1, \ldots, p_n)$, and any n-tuple of returns $(R_1, \ldots, R_n)$, and any strict subset $\mathbf{I} \subset \{1, \ldots, n\}$,

$$R_i^* \geq R_i \text{ for } i \in \mathbf{I} \quad \text{and} \quad R_i^* \leq R_i \text{ for } i \notin \mathbf{I}$$

implies $\sum_{i \in \mathbf{I}} q_i^* \geq \sum_{i \in \mathbf{I}} q_i$ (i.e., monitinicicty in return). In addition, monitinicity in probability also holds true.

5.2 Desireability

The final assumption that we wish to place on the distribution of preferences is a formalization of the idea that when the return from winning offered by a gamble is made large enough, then it is always possible to induce some positive mass of consumers to prefer this gamble over all other gambles being offered by the market. This is an essential requirement for equilibrium. Otherwise, it would be possible that the return offered by a bet on a horse with low probability of winning could not be made high enough in order to compensate any positive mass of consumers to bet on the horse. Thus any finite return on such a gamble would would be inconsistent with an equilibrium, for as we shall see, equilibrium in the betting market requires some positive mass of bettors to gamble on each horse in a race.

Desireability A continuous distribution of preferences $\mathbf{P}_V$ satisfies *desireability* if for any n-tuple of distinct probabilities $(p_1, \ldots, p_n)$, and any n-tuple of returns $(R_1, \ldots, R_n)$, $p_i > 0$ implies that for any nondecreasing sequence of returns $\{R_i^t\}_{t \in \mathbb{N}}$ with $\lim R_i^t = \infty$,

$$\lim_{t \to \infty} q_i(R_1, \ldots, R_i^t, \ldots, R_n; p_1, \ldots, p_n) > 0.$$ [8]

From desireability we can deduce the following useful lemma.

Lemma 5.5 *Consider any n-tuple of distinct, nonzero probabilities $(p_1, \ldots, p_n)$, and any subset $\mathbf{I} \subset \{1, \ldots, n\}$. If $\{(R_1^t, \ldots, R_n^t)\}_{t \in \mathbb{N}}$ is a sequence of n-tuples of returns with $\{R_i^t\}_{t \in \mathbb{N}}$ for $i \in \mathbf{I}$ nondecreasing and converging to ∞, and $\{R_i^t\}_{t \in \mathbb{N}}$ for $i \notin \mathbf{I}$ converging to $\bar{R}_i$, then then there exists a positive integer M such that for all $t > M$,*

$$\sum_{i \in \mathbf{I}} q_i^t > 0.$$

[8] Notice the limit in this case neccesarily exists because the market share of the i^{th} gamble is nondecreasing in own return R_i by monitinicity in return, and thus q_i^t is a monotone sequence in $[0, 1]$.

That is, at least one of the gambles indexed by $\mathbf{I}$ *has a market share greater than* 0 *for all* n*-tuples of returns far along enough in the sequence.*

Proof Consider fixing $R_i = \bar{R}_i$ for $i \notin \mathbf{I}$, and let the returns for the gambles indexed by $i \in \mathbf{I}$ follow the sequence $\{R_i^t\}_{t \in \mathbb{N}}$. The resulting sequence of market shares, which we denote as $\bar{q}_i^t$ can be shown by desireability and monitinicity in return to satisfy

$$\lim_{t \to \infty} \sum_{i \in \mathbf{I}} \bar{q}_i^t > 0.^9$$

Thus there exists a positive integer N such that for all $t \geq N$,

$$\sum_{i \in \mathbf{I}} \bar{q}_i^t > 0.$$

In particular then,

$$\sum_{i \in \mathbf{I}} \bar{q}_i^N > 0.$$

Since the q_i functions are continuous in the n-tuple of returns, we can find an $\epsilon > 0$ such that $|\hat{R}_i - \bar{R}_i| < \epsilon$ for all $i \notin \mathbf{I}$ implies

$$\sum_{i \in \mathbf{I}} \hat{q}_i^N > 0.$$

By assumption we can find an N' such that $t > N'$ implies $|R_i^t - \bar{R}_i| < \epsilon$ for all $i \notin \mathbf{I}$. Taking $M = \max\{N, N'\}$ thus ensures that $t > M$ implies

$$\sum_{i \in \mathbf{I}} q_i^t > 0.^{10} \quad \blacksquare$$

5.3 Existence and Uniqueness

The market odds $(R_1, \ldots, R_n)$ and the public probabilities $(p_1, \ldots, p_n)$ together give rise to a set of n gambles $G = \{(R_1, p_1), \ldots, (R_n, p_n)\}$. The consumer population T of bettors in turn gives rise to market shares q_i for $i = 1, \ldots, n$ for the gambles in G. These market shares in turn feedback to determine new market clearing odds by way of the "parimutuel" condition (1), and thus equilibrium in the market is attained at odds that are consistent with the market shares they produce.

That is, The the set of gambles available in the market has the form $G = \{(R(s_i), p_i)\}_{i \in \{1,\ldots,n\}}$. Note

[9] Once again we know by monitinicity in return that the limit exists.

[10] More precisely, $t > M$ implies $\sum_{i \in \mathbf{I}} q_i(R_{\mathbf{I}}^t, R_{-\mathbf{I}}^t) \geq \sum_{i \in \mathbf{I}} q_i(R_{\mathbf{I}}^N, R_{-\mathbf{I}}^t) > 0$

that if the the market share $s_i = 0$, i.e., a zero mass of bettors bet on the i^{th} horse, then since our population is a continuum, the partimutuel mechansim rewards the entire bet pool to this zero mass of bettors, and thus $R(s_i) = \infty$. Of course, in our model, preferences are only defined over gambles having finite returns, which is a difficulty that we handle in the analysis to follow.

Given such a set G, the market share of the i^{th} gamble, as we have already examined, is given by

$$q_i(R(s_1), \ldots, R(s_n); p_1, \ldots, p_n).$$

Thus the market is in equilibrium when, for some market shares $(s_1^*, \ldots, s_n^*)$,

$$s_i^* = q_i(R(s_1^*), \ldots, R(s_n^*); p_1, \ldots, p_n) \quad \text{for} \quad i = 1 \ldots, n. \tag{3}$$

In words, the market is in equilibrium when bettors choose among the gambles in proportions that sustain the odds of the gambles.

We now come to a central result of the paper.

Theorem 5.6 *If the probabilities $(p_1, \ldots, p_n) \in \Delta^{n-1}$ are distinct and nonzero, and the distribution of consumer preferences satisfies continuity and desireability, then the parimutuel market has unique equilibrium odds $(R(s_1^*), \ldots, R(s_n^*))$, with market shares $(s_1^*, \ldots, s_n^*) \in \Delta^{n-1}$ distinct and nonzero.*

Proof We prove the result in three steps. In the first step, we introduce an upper bound $\bar{R}$ on the odds payable by a gamble in the market, and show that an equilibrium exists under this restriction by Brouwer's fixed point theorem. This follows from continuity of $\mathbf{P}_V$ which drives the continuity of the q_i. In the second step, we show that it is always possible to raise the upper bound $\bar{R}$ high enough such that it is not binding in equilibrium, and thus an equilibrium of the form (10) exists. This result is driven by the desireability assumption. Lastly we show that the equilibrium is unique, which is driven by monotinicity in return (a consequence of continuity).

We shall assume there is an upper bound $\bar{R}$ on the net returns payable by a gamble. Under this "restriction" to the parimutuel mechanism, the the market shares s_i determine the market returns $\bar{R}_i$ through

$$\bar{R}_i(s_i) = \min\left(\frac{1-\tau}{s_i} - 1, \bar{R}\right).$$

Thus whenever $s_i \leq (1-\tau)/(1+\bar{R})$, the restriction $\bar{R}$ on the odds is binding.

Since the return vector $(\bar{R}(s_1), \ldots, \bar{R}(s_n))$ is clearly a continuous function of the market shares $(s_1, \ldots, s_n)$, and since the market share function q_i are continuous in returns by Theorem 5.3, we have $f : \Delta^{n-1} \to \Delta^{n-1}$ given by

$$f_i(s_1, \ldots, s_n) = q_i(\bar{R}(s_1), \ldots, \bar{R}(s_n); p_1, \ldots, p_n) \quad \text{for} \quad i = 1 \ldots, n,$$

is a continuous function. By the Brouwer fixed point theorem, the map f has a fixed point $(\bar{s}_1, \ldots, \bar{s}_n)$, which is thus an equilibrium of the restricted parimutuel market.

If the upper bound $\bar{R}$ is not binding for any of the $\bar{s}_i$, then clearly the fixed point satisfies the property (10) of being an equilibrium in the unrestricted parimutuel market. We now show that it is possible to raise the bar $\bar{R}$ sufficiently high so that it is not binding for the corresponding restricted equilibrium $(\bar{s}_1 \ldots, \bar{s}_n)$.

Suppose that this was not true. Then there exists a sequence of uppper bounds $\{\bar{R}^t\}$ monotonically converging to ∞ with a corresponding sequence of equilibria $\{(\bar{s}_1^t, \ldots, \bar{s}_n^t)\} \subset \Delta^{n-1}$ where for each t the upper bound $\bar{R}^t$ is binding for at least one $\bar{s}_i$. Since this sequence of market shares lives in a compact space, we can find a convergent subsequence $\{(\bar{s}_1^{t_k}, \ldots, \bar{s}_n^{t_k})\}$ converging to $(\bar{s}_1, \ldots, \bar{s}_n)$, with $\bar{s}_i = 0$ for at least one i (which follows from the fact that the restriction is binding for each t_k).

Let $\mathbf{I} \subset \{1, \ldots, n\}$ be the strict subset of indices i for which $\bar{s}_i = 0$. Then for each $i \in \mathbf{I}$, the sequence $\{\bar{R}(\bar{s}_i^{t_k})\}$ converges to ∞,[11] and without loss of generality we can say it converges to ∞ monotonically.[12]

However by desireability, this situation is not possible. It would be mean that there is a sequence of n-tuples of returns $\{(R_1^m, \ldots, R_n^m\}$ with $\{R_i^m\}$ monotonically converging to ∞ for $i \in \mathbf{I}$ and $\{R_i^m\}$ converging to finite R_i for $i \notin \mathbf{I}$, and

$$\lim_{m \to \infty} \sum_{i \in \mathbf{I}} q_i(R_1^m, \ldots, R_n^m; p_1, \ldots, p_n) = 0,$$

which contradicts lemma 5.5. Thus it must be the case that we can find a large enough upper bound $\bar{R}$ such that $\bar{R}$ is not binding for the equilibrium $(\bar{s}_1, \ldots, \bar{s}_n)$. These market shares thus satisfy the condition for $(s_1^*, \ldots, s_n^*)$ in (10). Moreover, these equilibrium market shares must also be located in the interior of Δ^{n-1} (because $\bar{R}$ is nonbinding), i.e., $\bar{s}_i > 0$ for all i. It also follows that since the probability distribution $(p_1, \ldots, p_n)$ involved distinct probabilities, the equilibrium market shares $(\bar{s}_1, \ldots, \bar{s}_n)$ must be distinct, since otherwise one gamble in the market would dominate another, thereby causing the latter to have zero market share, which would contradict the fact the equilibrium shares lies in the interior of the simplex.

We now address uniqueness. Suppose that there exist two n-tuples of market shares $(\bar{s}_1, \ldots, \bar{s}_n)$ and

[11] Since $\min\left(\frac{1-\tau}{s_i^{t_k}} - 1, \bar{R}^{t_k}\right)$ goes to ∞.

[12] We can always take a subsequence to assure monotonic convergence

$(s_1^*, \ldots, s_n^*)$ that satisfy the equilibrium condition (10). Then for $i = 1, \ldots, n$,[13]

$$\bar{s}_i = q_i(R(\bar{s}_1), \ldots, R(\bar{s}_n)) \quad \text{and} \quad s_i^* = q_i(R(s_1^*), \ldots, R(s_n^*)).$$

Since both equilibrium tuples are located in the simplex, it must be the case that for some nonempty strict subset $\mathbf{I} \subset \{1, \ldots, n\}$, $s_i^* \leq \bar{s}_i$ for all $i \in \mathbf{I}$ with a strict inequality for at least one $i \in \mathbf{I}$, and and $s_i^* \geq \bar{s}_i$ for all $i \notin \mathbf{I}$ with a strict inequality for at least one $i \notin \mathbf{I}$. This imples that

$$\sum_{i \in \mathbf{I}} s_i^* < \sum_{i \in \mathbf{I}} \bar{s}_i,$$

However we also have that $R(s_i^*) \geq R(\bar{s}_i)$ for $i \in \mathbf{I}$ and $R(s_i) \leq R(\bar{s}_i)$ for $i \notin \mathbf{I}$, which by monotinicity in return of $\mathbf{P}_V$, implies that

$$\sum_{i \in \mathbf{I}} s_i^* \geq \sum_{i \in \mathbf{I}} \bar{s}_i.$$

This is a contradiction, and thus the equilibrium is unique. ∎

6 Fully Revealing Rational Expectations

We have established that for any n-tuple of distinct and nonzero public probabilities $(p_1, \ldots, p_n) \in \Delta^{n-1}$, there exists a unique n-tuple of equilibrium odds $(R(s_1^*), \ldots, R(s_n^*))$ with distinct and nonzero market shares $(s_1^*, \ldots, s_n^*) \in \Delta^{n-1}$. We now establish that this pricing function from states to prices represents a fully revealing REE. That is, the function is 1-1.

The questions is thus the following : given equilibrium odds, and given the distribution of consumer preferences $\mathbf{P}_V$, if all bettors in the market hold common beliefs $(p_1, \ldots, p_n)$ about the chances of each horse winning, then can we uniquely recover the $(p_1, \ldots, p_n)$? That is, for any observed n-tuple of distinct equilibrium odds $(R(s_1^*), \ldots, R(s_n^*))$, with the implicitly observed n-tuple of distnct and nonzero market shares $(s_1^*, \ldots, s_n^*) \in \Delta^{n-1}$, can we uniquely solve the system of equations in $(p_1, \ldots, p_n) \in \Delta^{n-1}$,

$$s_i^* = q_i(R(s_1^*), \ldots, R(s_n^*); p_1, \ldots, p_n). \quad \text{for} \quad i = 1, \ldots, n \tag{4}$$

Theorem 6.1 *If the odds $(R(s_1^*), \ldots, R(s_n^*))$ are distinct, and the distribution of consumer preferences satisfies continuity, then there exists a unique probability distribution $(p_1^*, \ldots, p_n^*) \in \Delta^{n-1}$, consisting of distinct*

[13]We supress the probabilities $(p_1, \ldots, p_n)$ in the following notation because they remain the same.

and nonzero probabilities, that solves (4).

Proof Since the R_i^* are assumed distinct, and $\mathbf{P}_V$ satisfies continuity, then we that for any $(p_1, \ldots, p_n) \in \Delta^{n-1}$,

$$\sum_{i=1}^{n} s_i^* - q_i(R_1^*, \ldots, R_n^*; p_1, \ldots, p_n) = 0.^{14} \tag{5}$$

Furthermore, by continuity once again, $q_i(R_1^*, \ldots, R_n^*; p_1, \ldots, p_n)$ is continuous in $(p_1, \ldots, p_n)$ over Δ^{n-1}. Now consider the following continous self map over Δ^{n-1} (where for simplicity we write $R^* = (R_1^*, \ldots, R_n^*)$). For $(p_1, \ldots, p_n) \in \Delta^{n-1}$,

$$p_i \mapsto \frac{p_i + \max(0, s_i^* - q_i(R^*; p_1, \ldots, p_n))}{\sum_{j=1}^{n}(p_j + \max(0, s_j^* - q_j(R^*; p_1, \ldots, p_n)))} \quad \text{for} \quad i = 1, \ldots, n. \tag{6}$$

By the Brouwer fixed point theorem, this map must have a fixed point $(p_1^*, \ldots, p_n^*) \in \Delta^{n-1}$.

Moreover, this fixed point must satisfy $q_i(R^*, p_1^*, \ldots, p_n^*) = s_i^*$ for $i = 1, \ldots, n$. If these are equalities are not satisfied, then by (5) we have that for at least one i we have $s_i^* > q_i(R^*, p_1^*, \ldots, p_n^*)$, and for at least one j we have $s_j^* < q_j(R^*, p_1^*, \ldots, p_n^*)$. Thus

$$\sum_{i=1}^{n} p_i^* + \max(0, s_i^* - q_i(R^*, p_1^*, \ldots, p_n^*)) > 1,$$

and under the the mapping (6), p_j^* must get sent to a strictly smaller number, which violates the fact that $(p_1^*, \ldots, p_n^*)$ is a fixed point. Thus $(p_1^*, \ldots, p_n^*)$ solves (4)).

Since each s_i^* is nonzero and distinct, it must be the case that each p_i^* is nonzero and distinct. Otherwisem some gamble, indexed by i say, would be dominated (either it has zero probability or it has the same probability as another gamble but lower return), and thus $q_i(R^*, p_1^*, \ldots, p_n^*) = 0$, which is inconsistent with the fact the market share of the i^{th} gamble is $s_i^* > 0$.

The uniquness of $(p_1^*, \ldots, p_n^*)$ follows from monotinicity in a manner parallel to that used in proving the uniquness in Theorem 5.6, except exploiting monitinicity in probability instead of monitinicity in return (both of which recall follows from continuity). ∎

[14] Note that $\sum s_i^* = 1$ because the s_i are (observed) market shares.

7 The Equilibrium Correspondence

Thus by way of Theorems 5.6 and 6.1, we have established that for any horse race or parimutuel betting market with $n \geq 2$ states of the world, and a distribution of consumer preferences $\mathbf{P}_V$ over gambles satisfying continuity, there exists an *invertible equilibrium pricing function* $R_i^*(p_1, \ldots, p_n)$ for $i = 1, \ldots, n$. This pricing function maps any distinct and nonzero n-tuple of probabilities $(p_1, \ldots, p_n) \in \Delta^{n-1}$ to a n-tuple of returns $(R(s_1^*, \ldots, R(s_n^*))$, with distinct and nonzero market shares $(s_1^*, \ldots, s_n^*) \in \Delta^{n-1}$, that solves the equilibrium condition (10). We express the inverse equilibrim pricing function as

$$p_i^*(R(s_1^*), \ldots, R(s_n^*)) \quad \text{for} \quad i = 1, \ldots, n,$$

which maps any n-tuple of returns $(R(s_1^*), \ldots, R(s_n^*))$ with distinct and nonzero market shares $(s_1^*, \ldots, s_n^*) \in \Delta^{n-1}$ to a distinct and nonzero n-tuple of probabilities $(p_1, \ldots, p_n) \in \Delta^{n-1}$. It is straightforward to show that this inverse pricing function satisfies the symmetry condition

$$p_i^*(R_i, R_{-i}) = p_i^*(R_i, Q_{-i}),$$

where Q_{-i} is a permutation of the elements in R_{-i}.

This inverse equilibrium pricing function is the empirically observable equilibrium correspondencel. Our data consist of a sample of races where the winning horse and the odds on each horse and the track take can be observed. One approach to the analysis is to estimate the inverse equilibrium pricing function nonparametrically from the winners and odds data. Then for any parametric assumption about the distribution of preferences, we can estimate the parameters so as to minimizes the distance between observed shares and predicted shares.

Alternatively, we can mimic the empirical strategy of Jullien and Salanie [2000] and numerically solve for the inverse probabilities for any given value of the model parameters and the observed odds. Using these probabilities for each race along with the the observed winner of each race allows us to construct a likelihood function over model parameters. Further work on the empirically implementing the model is currently underway.

8 Identifying Risk Preferences

In this section, I present the model of parimutuel market equilibrium, which was presented in full generality in the previous sections, for the special case that heterogeneiety across bettors in preferences for risk can be reduced to a single dimensional type satisfying a single crossing condition. I show that in this special setting, the $p_i(R_1, \ldots, R_n)$ functions for $i = 1, \ldots, n$ have a specific structure that serve as the basis nonparametric identification and establishing testable implications. In addition, when preferences also satisfy expected utility, we can actually analytically solve for $p_i(R_1, \ldots, R_n)$, thus allowing us to estimate the model in the straightforward fashion. In this note, I present the model of parimutuel market equilibrium (which was presented in full generality in an earlier note) for the special case where heterogeneiety in risk preference across bettors can be reduced to a single dimensional type satisfying a single crossing condition. I show that in this setting, the $p_i(R_1, \ldots, R_n)$ functions for $i = 1, \ldots, n$ have a very specific structure that can be used for establishing testable implications of the model, etc. In addition, when preferences also satisfy expected utility theory , we can actually analytically solve for the $p_i(R_1, \ldots, R_n)$, thereby allowing us to estimate the model in the straightforward fashion.

8.1 Preferences

Our population model consist of two components.

- We have a parameterized utility function over gambles

$$V : (-1, \infty) \times [0, 1] \times \mathbb{R} \to \mathbb{R},$$

  where $V(R, p, \theta)$ is the utility that a person of type θ recieves from consuming the gamble (R, p). We assume that V is continuous in (R, p, θ), and for each $\theta \in \mathbb{R}$, $V(R, p, \theta)$ is strictly increasing in p and R, and striclty minimized at $p = 0$. For the purposes of this note, we shall assume that the domain of θ is $\mathbb{R}$, however any interval valued domain is equally applicable.

- The type θ is distributed according to a continuous cdf F that is strictly increasing over the support of θ, (and thus F admits a strictly increasing inverse cdf F^{-1}).

Now consider a menu of n distinct gambles $G = \{(R_1, p_1), \dots, (R_n, p_n)\}$, and let the subset of the population who chooses (R_i, p_i) from G be denoted

$$I_i = \left\{ \theta \in \mathbb{R} : V(R_i, p_i, \theta) = \max_{j \in \{1, \dots, n\}} V(R_j, p_j, \theta) \right\}.$$

Thus the share of the population who choose (R_i, p_i) from G is $q_i(G) =$

$$q_i(R_1, \dots, R_n; p_1, \dots, p_n) = \mathbf{P}_\theta(I_i). \tag{7}$$

Clearly, each I_i is a closed set.

We now define a single crossing condition for our population model.

Single Crossing The function $V(R, p, \theta)$ satisfies the single crossing condition if for any two gambles (R_1, p_1) and (R_2, p_2) with $p_1 > p_2$, and for some θ such that

$$V(R_2, p_2, \theta) \geq V(R_1, p_1, \theta),$$

then $\theta' > \theta$ implies that

$$V(R_2, p_2, \theta') > V(R_1, p_1, \theta').$$

Assuming that our population model satisfies the single crossing condition (as is the case for CARA, CRRA, and all the standard single parameter models of risk preferences), then the market shares $q_i(G)$ can be simplified beyond (7) in a manner that is especially useful for our purposes. In order to show this simplification, we first state the following lemma with the proof ommited.

Lemma 8.1 *If the population model satisfies the single crossing condition, then the following hold :*

1. *For $i \neq j$, $I_i \cap I_j$ is at most a singleton.*

2. *For every i, I_i is convex.*

3. *$p_i < p_j$ implies $I_j \leq I_i$ (i.e. if $x \in I_j$ and $y \in I_i$, then $x \leq y$.*

Now suppose that $S = \{i_1, \dots, i_m\} \subset \{1, \dots, n\}$ indexes the gambles in G that recieve non-zero market share, i.e., $i \in S$ iff $q_i(G) > 0$. Then based on the lemma, it is fairly starightforward to show the following result.

Theorem 8.2 *If the population model satisfies the single crossing condition, then there exists* $-\infty < \theta_1 < \cdots < \theta_{m-1} < \infty$ *such that*

$$I_{i_1} = (-\infty, \theta_1], I_{i_2} = [\theta_1, \theta_2], \ldots, I_{i_m} = [\theta_m, \infty)$$

and

$$V_{\theta_1}(R_{i_1}, p_{i_1}) = V_{\theta_1}(R_{i_2}, p_{i_2}), \ldots, V_{\theta_{m-1}}(R_{i_{m-1}}, p_{i_{m-1}}) = V_{\theta_{m-1}}(R_{i_m}, p_{i_m}). \tag{8}$$

As a simple corollary to the theorem, the market shares $q_i(G)$ for $i \in S$ can be expressed as

$$\begin{aligned} q_{i_1}(G) &= F(\theta_1) \\ q_{i_2}(G) &= F(\theta_2) - F(\theta_1) \\ &\vdots \\ q_{i_m}(G) &= 1 - F(\theta_{m-1}). \end{aligned}$$

Thus in summary, given a menu of gambles G, and the indices $S = \{i_1, \ldots, i_m\}$ of gambles recieving non-zero market share, the market shares $q_{i_j}(G)$ for $j = 1, \ldots, m$ are determined by the locations $\theta_1, \ldots, \theta_{m-1}$ of the *marginal bettors*, where the marginal bettor θ_j satisfies the *marginal condition* of being indifferent between the gambles (R_{i_j}, p_{i_j}) and $(R_{i_{j+1}}, p_{i_{j+1}})$.

8.2 Equilibrium

In a parimutuel betting market, the market share s_i alloted to each horse i determines its return R_i through

$$R_i = R(s_i) = \frac{1-\tau}{s_i} \quad \text{where} \quad \sum_{i=1}^{n} s_i = 1. \tag{9}$$

In market equilibrium, the share alloted to each horse is such that

$$s_i = q_i(R(s_1), \ldots, R(s_n); p_1^*, \ldots, p_n^*) \quad \text{for} \quad i = 1, \ldots, n. \tag{10}$$

I now show if we observe the returns $(R_1, \ldots, R_n)$ in a parimutuel market, which by (9) implies that we also observe the market shares $(s_1, \ldots, s_n)$, then we can recover the locations $\theta_1, \ldots, \theta_{n-1}$ of the marginal bettors. We can thus solve for the unobserved probabilities $p_i^* = p_i(R_1, \ldots, R_n)$ satisfying (10) by finding the probability distribution $(p_1^*, \ldots, p_n^*)$ that satisfies the marginal conditions (8). Let us see how this works.

Step 1 : Recovering the Marginal Bettors

Suppose we observe the returns $(R_1, \ldots, R_n)$ in a market, which are ordered such that

$$R_1 < \cdots < R_n.$$

As a neccessary condition for equilibrium, it must be that

$$p_1^* > \cdots > p_n^*.$$

Since all the returns are finite, each horses recieves a positive market share. Thus there exist marginal bettors $\theta_1, \ldots, \theta_{n-1}$ such that

$$I_1 = (-\infty, \theta_1], I_2 = [\theta_1, \theta_2], \ldots, I_n = [\theta_{n-1}, \infty).$$

and thus

$$\begin{aligned} s_1 &= F(\theta_1) \\ s_2 &= F(\theta_2) - F(\theta_1) \\ &\vdots \\ s_n &= 1 - F(\theta_{n-1}). \end{aligned} \tag{11}$$

Since the shares $s_1, \ldots, s_n$ are observable via (9), we can invert the system (11) to locate the marginal bettors $\theta_1, \ldots, \theta_{n-1}$, i.e.,

$$\begin{aligned} \theta_1 &= F^{-1}(s_1) \\ \theta_2 &= F^{-1}(s_1 + s_2) \\ &\vdots \\ \theta_{n-1} &= F^{-1}(s_1 + \cdots + s_{n-1}). \end{aligned} \tag{12}$$

Step 2 : Recovering the Probabiliies

Having recovered the marginal bettors $\theta_1, \ldots, \theta_{n-1}$, we can find the probabilities

$$p_1^* = p_1(R_1, \ldots, R_n), \ldots, p_n^* = p_n(R_1, \ldots, R_n)$$

that satisfy the equilibrium condition (10) by finding the probability distribution $(p_1^*, \ldots, p_n^*)$ that satisfies the marginal conditions (8). From the results of my previous note, which related to a more general setting, we know a unique such solution exists. However we can use a more direct argument for our special setting to show a unique solution.

For any choice of p_1^*, there are unique values of $p_2^*(p_1^*), \ldots, p_n^*(p_1^*)$ that satisfiy the marginal conditions (8). Notice that the functions $p_i^*(p_1^*)$ for $i = 2, \ldots, n$ are continuous and increasing. Our problem is to find p_1^* such that

$$p_1^* + p_2^*(p_1^*) + \cdots + p_n^*(p_1^*) = 1. \tag{13}$$

Notice that $p_1^* = 0$ imples that $p_2^*(p_1^*) = 0, \ldots, p_n^*(p_1^*) = 0$ since for every θ, $V_\theta(R, p)$ is strictly minimized at $p = 0$. Likewise, it is clear that $p_1^* = 1$ implies

$$p_1^* + p_2^*(p_1^*) + \cdots + p_n^*(p_1^*) > 1.$$

Thus for a unique $p_1^* \in (0, 1)$ we have that (13) is satisfied. We can numerically locate this solution p_1^* by way of bisection over $(0, 1)$. Furthermore, if we can produce a closed form solution to $p_i^*(p_1^*)$ for $i = 2, \ldots, n$, then locating p_1^* becomes all the more tractible. Let us now turn to some examples of empirical interest.

8.3 Example : Expected Utility Theory

If bettor preferences follow expected utility theory, then not only do the $p_i^*(p_1^*)$ functions admit a closed form, but we can also analytically solve for p_1^* in (13). Let us see why -

Suppose preferences in the population satisfy expected utility theory. Then

$$V_\theta(R, p) = pU_\theta(R) + (1 - p)U_\theta(-1).$$

Now consider the marginal condition for the i^{th} marginal bettor θ_i, namely

$$\begin{aligned} & V_{\theta_i}(R_i, p_i^*) = V_{\theta_i}(R_{i+1}, p_{i+1}^*) \\ \Rightarrow \quad & p_i^* U_{\theta_i}(R_i) + (1 - p_i^*)U_{\theta_i}(-1) = p_{i+1}^* U_{\theta_i}(R_{i+1}) + (1 - p_{i+1}^*)U_{\theta_i}(-1) \\ \Rightarrow \quad & p_{i+1}^* = \left(\frac{U_{\theta_i}(R_i) - U_{\theta_i}(-1)}{U_{\theta_i}(R_{i+1}) - U_{\theta_i}(-1)} \right) p_i^*. \end{aligned}$$

For $i = 1, \ldots, (n-1)$, define

$$c_i = \prod_{j=1}^{i} \frac{U_{\theta_j}(R_j) - U_{\theta_j}(-1)}{U_{\theta_j}(R_{j+1}) - U_{\theta_j}(-1)}.$$

Then we have that

$$\begin{aligned} p_2^*(p_1^*) &= c_1 p_1^* \\ p_3^*(p_1^*) &= c_2 p_1^* \\ &\vdots \\ p_n^*(p_1^*) &= c_{n-1} p_1^*. \end{aligned}$$

Thus (13) becomes

$$p_1^* + c_1 p_1^* + \cdots + c_{n-1} p_1^* = 1,$$

which yields

$$p_1^* = \frac{1}{1 + c_1 + \cdots + c_{n-1}}.$$

8.4 Example : Anticipated Utility

If bettor preferences satisfy anticipated utility (i.e. rank dependent expected utility), a level of generalization beyond expected utility theory, then the $p_i^*(p_1^*)$ functions admit a closed form. Let us see why -

Suppose preferences in the population satisfy RDEU. Then (maintaining the structure of single dimensional type heterogeneity)

$$V_\theta(R, p) = G(p) U_\theta(R) + (1 - G(p)) U_\theta(-1),$$

where G is a continuous increasing function mapping the unit interval into itself. Now consider the marginal condition for the i^{th} marginal bettor θ_i, namely

$$\begin{aligned} & V_{\theta_i}(R_i, p_i^*) = V_{\theta_i}(R_{i+1}, p_{i+1}^*) \\ \Rightarrow \quad & \frac{G(p_{i+1}^*)}{G(p_i^*)} = \left(\frac{U_{\theta_i}(R_i) - U_{\theta_i}(-1)}{U_{\theta_i}(R_{i+1}) - U_{\theta_i}(-1)} \right). \end{aligned}$$

Then defining c_i for $i = 1, \ldots, (n-1)$ as we did for expected utility theory, we have that

$$\begin{aligned} p_2^*(p_1^*) &= G^{-1}(c_1 G(p_1^*)) \\ p_3^*(p_1^*) &= G^{-1}(c_2 G(p_1^*)) \\ &\vdots \\ p_n^*(p_1^*) &= G^{-1}(c_{n-1} G(p_1^*)). \end{aligned}$$

These closed forms make solving (13) for p_1^* a computationally "inexpensive" proposition (so long as closed form for the inverse of G is available).

8.5 Example : Cumulative Prospect Theory

If bettor preferences satisfy cumulative prospect theory, namely

$$V_\theta(R, p) = G(p)U_\theta(R_i) + H(1 - p_i)U_\theta(-1),$$

then we find ourselves in a similar diffuclty to Jullien and Salanie [2000] in that we cannot generally analytically solve for $p_i^*(p_1^*)$. Thus numerically solving (13) requires numerically evaluating the $p_i^*(p_1^*)$ for any proposed candidate p_1^*, which (potentially) adds considerable computational burden, even though the overall root finding procedure is straightforward (bisection on the interval $(0, 1)$).

9 Conclusion

We have presented an adaptation of the Arrow-Debreu framework to model betting market equilibrium. Under very weak conditions on the distributions of preferences, we have shown that there exists a unique fully revealing rational expectations equilibrium of the model. The inverse of the rational expectations pricing function represents the empirically observable equilibrium correspondence implied by the model. We study the structure of this inverse pricing function for the case of one dimensional preference heterogeneity. This structure provides neccessary testable implications of the equilibrium model that are also sufficient for identifying the underlying distribution of preferences, and thereby can be used to test expected utility theory.

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