

# Evolution of Wealth and Asset Prices in Markets with Case-Based Investors<sup>1</sup>

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I analyze whether case-based decision makers (CBDM) can survive in an asset market in the presence of expected utility maximizers. Conditions are identified, under which the CBDM retain a positive mass with probability one. CBDM can cause predictability of asset returns, high volatility and bubbles. It is found that the expected utility maximizers can disappear from the market for a finite period of time, if the mispricing of the risky asset caused by the case-based decision-makers aggravates too much. Only in the case of logarithmic expected utility maximizers do the case-based decision makers disappear from the market for all parameter values.

Keywords: asset pricing, bubbles, volatility, case-based decision theory, evolution.

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# 1 Introduction

The empirical literature in financial markets has identified multiple phenomena which contradict the joint hypothesis of expected utility maximization and rational expectations. Bubbles, high volatility and predictability of returns are commonly observed in real and experimental markets, see e.g. Kindelberger (1978), Sunder (1995), Jegadeh (1990) and Shiller (1981). The explanations proposed in the literature usually rely on the introduction of "noise traders", whose misperceptions lead to deviation of prices from the theoretical predictions, see Shleifer (2000).

The recent literature on evolution in financial markets, however, casts doubt on the viability of these explanations. In the class of expected utility maximizers, only those with correct beliefs survive in complete markets with perfect foresight, see Blume and Easley (2001) and Sandroni (2000). Hence, noise traders can have no influence on prices in the long-run.

However, the results of Blume and Easley (2001) and Sandroni (2000) do not rule out an alternative way to explain the behavior of asset prices: introducing agents whose preferences significantly deviate from expected utility maximization. One of the approaches taken in the literature consists in modelling financial markets in which Choquet expected utility maximizers are present, see Mukerji and Tallon (2004) for an overview of the literature. However, Condie (2005) show that Choquet expected utility maximizers can survive in a complete financial market only if their behavior perfectly mimics the behavior of an expected utility maximizer with correct beliefs.

In this paper, I provide a model in which expected utility maximizers interact with investors whose behavior is consistent with the case-based decision theory proposed by Gilboa and Schmeidler (1995). Hence, their perception of the world significantly differs from the framework proposed by Savage (1954). It is shown that these traders not only survive in the presence of expected utility maximizers, but can also cause asset prices to deviate from prices under rational expectations. Equilibrium prices exhibit excess volatility, bubbles and predictability of returns. The model further shows that under certain conditions, the expected utility maximizers can be driven out of the market for a finite period of time, during which an asset with a positive fundamental value is traded at a 0-price. Hence, it can provide an explanation for price crashes occurring on assets with positive fundamentals.

Case-based decision-makers are not assumed to have knowledge of possible states of nature or of the distribution of state-dependent outcomes. Instead they learn from experience. The evaluation of an alternative depends on the performance of similar alternatives in the past. Past utility realizations increase or decrease the evaluation of an alternative, depending on whether they exceed a certain bench-mark called aspiration level.

In a stationary environment, a case-based decision maker who adapts his aspiration level in a manner that is both ambitious and realistic, learns to imitate an expected utility maximizer with correct expectations in the long-run, see Gilboa and Schmeidler (1996). However, up to my knowledge, there are no results evaluating the relative performance of case-based decision makers in a market environment.

In Guerdjikova (2003), it is shown that an asset market populated only by case-based investors can exhibit 0-price equilibria and a price dynamic featuring excess volatility, bubbles and predictability of asset returns. This paper applies these results to study the evolutionary dynamic of wealth in a market populated by both case-based decision-makers and expected utility maximizers. The proportion of the two types of investors and, therefore, their wealth share, evolve according to the relative success of both groups. This endogenizes the wealth of the investors and allows to assess the relative performance of these two strategies.

I address two main issues: first, I analyze whether case-based investors can retain a positive share in the economy over time and, second, I determine the influence of their behavior on prices.

The results depend on the risk-attitude of the agents in the economy: for all coefficients of relative risk-aversion below 1, it is possible to indicate a range of parameters, for which *the case-based investors not only survive, but also influence the asset prices in a non-trivial way*. The case-based investors disappear with certainty for all parameter values, only if logarithmic expected utility maximizers are present in the market.

Previous research indicates that the relationship between survival and risk-attitude is not unequivocal: De Long, Shleifer, Summers and Waldmann (1990, 1991) show that noise traders choosing risky portfolios can outperform rational traders with high risk-aversion. Their main result is derived in a static setting, in which the shares of both types of traders remain constant

over time. De Long, Shleifer, Summers and Waldmann (1990) introduce an imitation dynamic and show that for low rates of adjustment, the results derived in the static context still hold.

In contrast, Blume and Easley (1992), Hens and Schenk-Hoppé (2001), Evstigneev, Hens and Schenk-Hoppé (2002, 2003) derive their results by directly studying the wealth dynamic in an asset market. They prove that the most successful strategy consists in maximizing a logarithmic expected utility function with correct beliefs. Should a logarithmic utility maximizer be absent from the market, Blume and Easley (1992) show that the market selects for investors with relative risk-aversion closest to 1. The findings of the model I present are consistent with these results.

The major part of the literature on evolution in financial markets is concerned with deriving the strategy which accumulates the entire wealth in the limit and, hence, completely determines prices. There is still little research into how different investment rules perform relative to each other. The relevance of such comparative analysis stems from the fact that the use of optimal rules might be limited by informational and computational costs. In this case, a decision-maker might have to compare simple rules with respect to their survival chances. One such issue is addressed by Sciubba (2001), who analyzes the relative performance of the CAPM rule, as compared to the logarithmic utility maximization with correct beliefs and the mean-variance utility maximization. She shows, that the CAPM-traders vanish, whereas those maximizing a mean-variance utility imitate the logarithmic utility maximizers and therefore survive.

In a similar spirit, the present paper abstracts from determining the optimal strategy and concentrates on the comparison of two decision rules. The expected utility maximizers in this model can be seen as fundamentalists who rely only on their information about the dividend stream. At the same time, the case-based investors might be viewed as a very stylized version of chartists, who extrapolate the past performance of an asset into the future. Empirical studies show that both rules enjoy popularity under investors in financial markets, see e.g. Taylor and Allen (1992), Mentkhoff (1997), (1998) and Lui and Mole (1998).

Most of the literature which analyzes the interaction between fundamentalists and chartists, see Day and Huang (1990), Farmer (2002), Farmer and Joshi (2002), neglects the wealth dynamics or uses an imitation dynamic with respect to an exponential utility function, see Lux (1995, 1998) and Brock and Hommes (1997). In contrast, the current paper relies on the equilibrium wealth

dynamic to derive conditions for the survival of the case-based investors. The conclusions of the paper could be used to draw comparisons between these two rules and to better understand the price process and the evolution of wealth in markets, in which both types of traders are present.

The paper is organized as follows: section two gives a description of the economy and introduces the evolutionary dynamic of investor types. In section three, the evolutionary dynamic is analyzed in order to determine whether case-based investors can survive in a financial market. Section four discusses how empirically observed phenomena, such as bubbles or price crashes can emerge in the presence of case-based decision makers in the market. Section five concludes. The proofs of all propositions are stated in the appendix.

## 2 The Economy

The economy consists of a continuum of investors, uniformly distributed on  $[0; 1]$ . Time is discrete:  $t = 0, 1, \dots$ . In period  $t$ , a proportion  $e_t$  of the investors are expected utility maximizers (EUM), whereas the rest,  $c_t = 1 - e_t$  are case-based decision makers (CBDM). No population growth is considered.

The model has an OLG structure. Each investor lives for two periods. Investors consume only in the second period of their life. Preferences are represented by a CRRA utility function

$$\begin{aligned} u_\beta(x) &= x^\beta, \beta \in (0; 1] \\ u_\beta(x) &= \ln x, \beta = 0, \end{aligned}$$

which is identical for all investors.  $(1 - \beta)$ , therefore, denotes the coefficient of relative risk aversion. There is one consumption good in the economy with price normalized to 1. The initial endowment of the investors consists of one unit of the consumption good in the first period and is 0 in the second period.

Consumption can be transferred between two periods by either using a riskless storage technology  $b$ , or investing in a risky asset  $a$ . The payoff of  $b$  is  $(1 + r)$  per unit of consumption good stored. It is available in a perfectly elastic supply at a price of 1.

The supply of  $a$  is  $A = 1$ . Its payoff in period  $t$  is:

$$\delta_t = \left\{ \begin{array}{ll} \delta & \text{with probability } q \\ 0 & \text{with probability } 1 - q \end{array} \right\},$$

where  $\delta_t$  is i.i.d. Its price is  $p_t$ . New emissions are not considered, since I am interested in the

behavior of prices on the secondary asset market. Let the payoffs satisfy  $1 > \delta > r > 0$ .

Short sales are not permitted. Therefore, the set of available acts reduces to:

$$\gamma_t^i \in [0; 1],$$

with  $\gamma$  denoting the share of endowment invested into  $a$  and  $i \in \{eu; cb\}$ , where  $eu$  and  $cb$  identify the EUM and the CBDM, respectively. Since diversification is possible, I will assume that all investors of a given type choose identical portfolios. This amounts to replacing each of the types by a representative investor.

Given  $\gamma_{t-1}^i$ , the indirect utility from consumption of  $i$  at time  $t$  can be written as:

$$v_t(\gamma_{t-1}^i) = u_\beta \left( \gamma_{t-1}^i \left( \frac{p_t}{p_{t-1}} + \frac{\delta_t}{p_{t-1}} \right) + (1 - \gamma_{t-1}^i) (1 + r) \right). \quad (1)$$

Note, that  $v_t(\gamma_{t-1}^i)$  depends on  $p_t$ , and hence, on the decisions of the young investors at time  $t$ .

## 2.1 Information and Individual Decisions

The individual decision-making process will predetermine the evolution of asset prices, as well as of the shares of different investor types in the economy.

### 2.1.1 Case-Based Decision Makers

First consider the CBDM, as introduced in Gilboa and Schmeidler (2001). Their description of the situation contains the investment problem, they have to solve, as well as the available alternatives,  $\gamma_t^{cb} \in [0; 1]$ . Unlike EUM, CBDM do not use information about possible states of nature, state-contingent outcomes and their probability distribution. Therefore, they can only base their decisions on the experience of previous generations, they know about. This information is called memory. I assume that the memory consists only of the act chosen and utility realized by the CBDM in the previous generation. The experience of the EUM is not taken into account. Hence, the memory at time  $t$  can be written as

$$M_t = ((\gamma_{t-1}^{cb}; v_t(\gamma_{t-1}^{cb})))$$

The assumption that CBDM base their decision on the observation of a single case might seem to be very limiting. However, as will become clear, it is possible to show that CBDM can survive and influence prices even with such limited knowledge about the environment they are facing. It, therefore seems intuitive that the results for survival of the CBDM and their influence on prices should be robust to extending the length and the contents of their memory.

The utility obtained from  $\gamma_{t-1}^{cb}$  is then compared to an aspiration level  $\bar{u}$  assumed to be identical for all CBDM and constant over the time. If an act is considered satisfactory, it is chosen again, else, it is abandoned and a different act is chosen next.

The perceived similarity among acts allows the decision-maker to evaluate acts that were not previously chosen. Since the available acts are elements of  $[0; 1]$ , I assume that the similarity between  $\gamma$  and  $\gamma'$ ,  $s(\gamma; \gamma')$ , is a strictly decreasing function of the Euclidean distance<sup>3</sup>  $\|\gamma - \gamma'\|$ .

Each act is evaluated according to its cumulative utility:

$$U_t(\gamma) = [v_t(\gamma_{t-1}^{cb}) - \bar{u}] s(\gamma; \gamma_{t-1}^{cb})$$

and

$$\gamma_t^{cb} = \arg \max_{\gamma \in [0;1]} U_t(\gamma).$$

Hence, the decision of a CBDM takes the form:

$$\gamma_t^{cb} = \left\{ \begin{array}{ll} \gamma_{t-1}^{cb} & \text{if } v_t(\gamma_{t-1}^{cb}) \geq \bar{u} \\ \arg \max_{\gamma \in [0;1]} U_t(\gamma) & \text{if } v_t(\gamma_{t-1}^{cb}) \leq \bar{u} \end{array} \right\}. \quad (2)$$

The following lemma obtains directly from (2).

**Lemma 1** *Let  $\gamma_0^{cb} \in (0; 1)$ . If  $\bar{u}$  satisfies*

$$u_\beta(\gamma_0^{cb} + (1 - \gamma_0^{cb})(1 + r)) \geq \bar{u},$$

*then  $\gamma_t^{cb} = \gamma_{t-1}^{cb} = \gamma_0^{cb}$  for all  $t$ . If*

$$u_\beta(\gamma_0^{cb} + (1 - \gamma_0^{cb})(1 + r)) < \bar{u},$$

*then  $\gamma_t^{cb} \in \{0; 1\}$  almost surely holds for all  $t \geq \bar{t}$  for some finite  $\bar{t}$ .*

To derive the share of the endowment of CBDM invested into  $a$ , denote by  $\tilde{p}_t$  the price of  $a$ , for which the CBDM are indifferent among all portfolios (if such a price exists):

$$\tilde{p}_t : u_\beta \left( \gamma_{t-1}^{cb} \left( \frac{\tilde{p}_t}{p_{t-1}} + \frac{\delta_t}{p_{t-1}} \right) + (1 - \gamma_{t-1}^{cb})(1 + r) \right) - \bar{u} = 0$$

Case 1: As long, as  $\tilde{p}_t > 0$  and  $\gamma_{t-1}^{cb} \geq \frac{1}{2}$

$$\gamma_t^{cb} = \left\{ \begin{array}{ll} \gamma_{t-1}^{cb} & \text{if } p_t > \tilde{p}_t \\ [0; \gamma_{t-1}^{cb}] & \text{if } p_t = \tilde{p}_t \\ 0 & \text{if } p_t < \tilde{p}_t \end{array} \right\}.$$

Case 2: For  $\gamma_{t-1}^{cb} \in (0; \frac{1}{2})$ ,

$$\gamma_t^{cb} = \left\{ \begin{array}{ll} \gamma_{t-1}^{cb} & \text{if } p_t > \tilde{p}_t \\ [0; \gamma_{t-1}^{cb}] & \text{if } p_t = \tilde{p}_t \\ 1 & \text{if } p_t < \tilde{p}_t \end{array} \right\}.$$

<sup>3</sup> The empirical literature on similarity in economics is still sparse, but see Buschena and Zilberman (1995, 1999) and Zizzo (2002) for indications that the similarity is a decreasing function of the Euclidean distance.

Case 3: For  $\gamma_{t-1}^{cb} = 0$ ,  $\tilde{p}_t$  exists only if  $u(1+r) = \bar{u}$ :

$$\gamma_t^{cb} = \begin{cases} 0 & \text{if } u(1+r) > \bar{u} \\ [0; 1] & \text{if } u(1+r) = \bar{u} \\ 1 & \text{if } u(1+r) < \bar{u} \end{cases}.$$

Figure 1 illustrates the three cases.

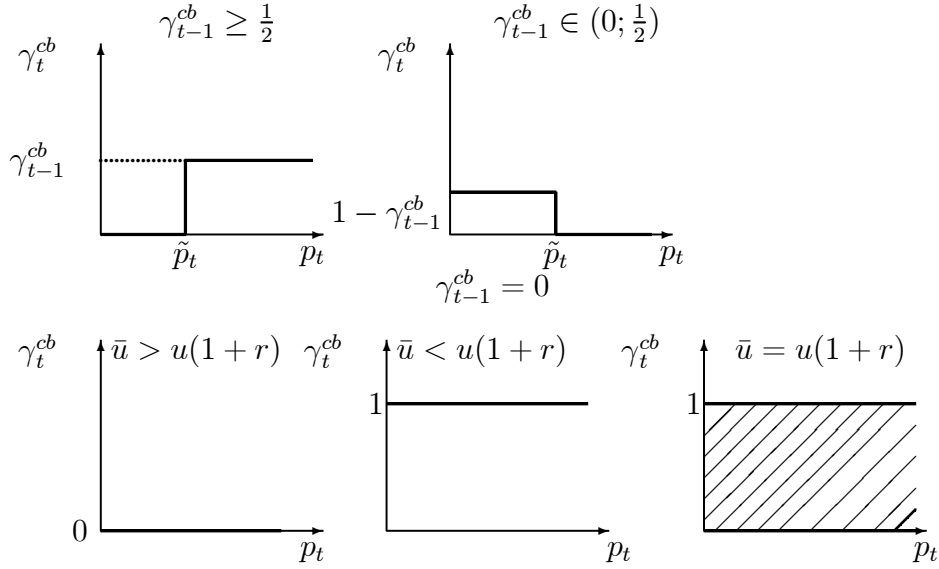


Figure 1

Note that  $\gamma_t^{cb}(p_t)$  is a non-empty, convex-valued, upper hemicontinuous correspondence.

The three cases demonstrate that the share of the endowment invested into the risky asset by CBDM is monotonically increasing in the price  $p_t$ , except if  $\gamma_{t-1}^{cb} \in (0; \frac{1}{2})$ . Since, for reasons to become obvious later, the main interest will be on the case of relatively high aspiration levels, which imply that diversified portfolios are held only for a finite number of periods, cases 1 and 3 will describe the demand of CBDM for assets. Hence, the relationship between  $p_t$  and  $\gamma_t^{cb}(p_t)$  will be positive. This will be the main difference between the CBDM and the EUM, whose share of endowment invested into the risky asset will decrease in  $p_t$ .

### 2.1.2 Expected Utility Maximizers

Now I turn to the description of the EUM. I assume, as usual, that EUM have expectations about the state-contingent payments of each of the assets. Especially, the EUM are informed about the correct distribution of the dividends of the risky asset and of the returns of the safe technology. However, they are unable to predict the influence of the CBDM on asset prices. Instead, they act

boundedly rational, forming their expectations about the price as if the economy consisted only of EUM, identical to themselves. Note that although in general, the EUM will not have rational expectations, their beliefs will be correct in the limit, when  $e \rightarrow 1$ . I discuss the implications of introducing EUM with rational expectations in the last section, where I argue that the results on survival of CBDM are robust in this respect.

Let  $p_\beta^{eu}$  denote the price which would emerge in a stationary equilibrium if only EUM were present in the market. To compute  $p_\beta^{eu}$ , first write the maximization problem of an investor who expects that  $p_t = p_{t+1} = p$ ,

$$\max_{\tilde{\gamma}_t^{eu} \in [0;1]} q \left[ \left(1 + \frac{\delta}{p}\right) \tilde{\gamma}_t^{eu} + (1+r)(1 - \tilde{\gamma}_t^{eu}) \right]^\beta + (1-q) [\tilde{\gamma}_t^{eu} + (1+r)(1 - \tilde{\gamma}_t^{eu})]^\beta$$

for  $\beta \in (0; 1]$  and

$$\max_{\tilde{\gamma}_t^{eu} \in [0;1]} q \ln \left[ \left(1 + \frac{\delta}{p}\right) \tilde{\gamma}_t^{eu} + (1+r)(1 - \tilde{\gamma}_t^{eu}) \right] + (1-q) \ln [\tilde{\gamma}_t^{eu} + (1+r)(1 - \tilde{\gamma}_t^{eu})]$$

for  $\beta = 0$ . For  $\beta \in [0; 1)$  the optimum is:

$$\tilde{\gamma}_t^{eu}(p) = \left\{ \begin{array}{l} 0, \\ \frac{(1+r) \left[ \left( \frac{(1-q)r}{q \left( \frac{\delta}{p} - r \right)} \right)^{\frac{1}{\beta-1}} - 1 \right]}{\frac{\delta}{p} + r \left[ \left( \frac{(1-q)r}{q \left( \frac{\delta}{p} - r \right)} \right)^{\frac{1}{\beta-1}} - 1 \right]}, \\ 1, \end{array} \right. \left. \begin{array}{l} \text{if } \frac{(1+r) \left[ \left( \frac{(1-q)r}{q \left( \frac{\delta}{p} - r \right)} \right)^{\frac{1}{\beta-1}} - 1 \right]}{\frac{\delta}{p} + r \left[ \left( \frac{(1-q)r}{q \left( \frac{\delta}{p} - r \right)} \right)^{\frac{1}{\beta-1}} - 1 \right]} < 0 \\ \text{if } \frac{(1+r) \left[ \left( \frac{(1-q)r}{q \left( \frac{\delta}{p} - r \right)} \right)^{\frac{1}{\beta-1}} - 1 \right]}{\frac{\delta}{p} + r \left[ \left( \frac{(1-q)r}{q \left( \frac{\delta}{p} - r \right)} \right)^{\frac{1}{\beta-1}} - 1 \right]} \in [0; 1] \\ \text{if } \frac{(1+r) \left[ \left( \frac{(1-q)r}{q \left( \frac{\delta}{p} - r \right)} \right)^{\frac{1}{\beta-1}} - 1 \right]}{\frac{\delta}{p} + r \left[ \left( \frac{(1-q)r}{q \left( \frac{\delta}{p} - r \right)} \right)^{\frac{1}{\beta-1}} - 1 \right]} > 0 \end{array} \right\} \quad (3)$$

Obviously, for  $\beta \in [0; 1)$ ,  $p_\beta^{eu}$  is the fixed point of  $\tilde{\gamma}_t^{eu}(p)$ :

$$\tilde{\gamma}_t^{eu}(p_\beta^{eu}) = p_\beta^{eu}.$$

Two special cases are of interest. If  $\beta = 1$ , i.e.  $u(x) = x$  then

$$\tilde{\gamma}_t^{eu}(p) = \left\{ \begin{array}{ll} 0 & p > \frac{q\delta}{r} \\ [0; 1] & p = \frac{q\delta}{r} \\ 1 & p < \frac{q\delta}{r} \end{array} \right\}$$

and

$$p_1^{eu} = \min \left\{ \frac{q\delta}{r}; 1 \right\},$$

hence the price equals the fundamental value of the asset. If  $\beta = 0$ , i.e.  $u(x) = \ln x$ , then

$$p_0^{eu} = \min \{ p_{\log}^{eu}; 1 \} \text{ with}$$

$$p_{\log}^{eu} = \frac{1 + r + \delta - \sqrt{(1 + r + \delta)^2 - 4q\delta(1 + r)}}{2r}.$$

Since the EUM perceive  $p_{\beta}^{eu}$  to be the "true" price of the risky asset, the share of their endowment invested into the risky asset  $\gamma_t^{eu}$  is determined as a solution to

$$\max_{\gamma_t^{eu} \in [0;1]} q \left[ \left( \frac{p_{\beta}^{eu} + \delta}{p_t} \right) \gamma_t^{eu} + (1 + r)(1 - \gamma_t^{eu}) \right]^{\beta} + (1 - q) \left[ \frac{p_{\beta}^{eu}}{p_t} \gamma_t^{eu} + (1 + r)(1 - \gamma_t^{eu}) \right]^{\beta},$$

respectively

$$\max_{\gamma_t^{eu} \in [0;1]} q \ln \left[ \left( \frac{p_{\beta}^{eu} + \delta}{p_t} \right) \gamma_t^{eu} + (1 + r)(1 - \gamma_t^{eu}) \right] + (1 - q) \ln \left[ \frac{p_{\beta}^{eu}}{p_t} \gamma_t^{eu} + (1 + r)(1 - \gamma_t^{eu}) \right]$$

and is decreasing in  $p_t$ . Note that if  $p_{\beta}^{eu} = 1$ ,  $\gamma_t^{eu}(p_t) = 1$  for all  $p_t$ .

$\gamma_t^{eu}(p_t)$  is a continuous function for  $\beta \in [0; 1)$ , whereas for  $\beta = 1$  it is a non-empty, convex-valued and upper hemicontinuous correspondence. The value of demand for  $a$  of the whole population is obtained as:

$$d_t(p_t) = e_t \gamma_t^{eu}(p_t) + (1 - e_t) \gamma_t^{cb}(p_t). \quad (4)$$

It is a correspondence, which also has the characteristics stated above and maps the interval  $[0; 1]$  into  $[0; 1]$ .

## 2.2 The Evolution of Investor Types

After describing the decision process of the investors in the economy, I now introduce the selection dynamic. The fitness of a given type of investors is measured by the actual average return they achieve relative to the average returns of the society as a whole. This gives rise to a replicator dynamic, in which the share of the type of investors who perform better grows. A higher wealth share for a particular type of investors in the economy then implies greater influence on market processes, see equation (4).

The OLG structure of the model does not allow for a natural wealth dynamic to arise. However, since each investor is born with the same initial endowment of 1 unit of the consumption good, the share of a type of investors can be identified with the total income of the investors of this type. Hence, the replicator dynamic can be interpreted as a wealth dynamic in the model at hand. In fact, replacing the investors in this model by their infinitely-lived, but myopic counterparts would lead to a well-defined wealth process which is identical to the one obtained from the replicator dynamic.

### 2.2.1 Replicator Dynamic

Differently from the usual approach in evolutionary game theory, the replicator dynamic in this model is applied not to the portfolio strategy chosen by an individual, but to the "meta"-strategies used by the two types of investors, hence to the performance of case-based decision-making versus expected utility maximization.

The following replicator dynamic is introduced, following Weibull (1995, pp. 124-125)<sup>4</sup>.

Denote by

$$\begin{aligned}\tilde{v}_t^i &= \gamma_{t-1}^i \frac{p_t + \delta_t}{p_{t-1}} + [1 - \gamma_{t-1}^i] (1 + r) \\ \tilde{v}_t &= e_{t-1} \left[ \gamma_{t-1}^{eu} \frac{p_t + \delta_t}{p_{t-1}} + [1 - \gamma_{t-1}^{eu}] (1 + r) \right] \\ &\quad + (1 - e_{t-1}) \left[ \gamma_{t-1}^{cb} \frac{p_t + \delta_t}{p_{t-1}} + [1 - \gamma_{t-1}^{cb}] (1 + r) \right]\end{aligned}\quad (5)$$

the average returns achieved by an investor of type  $i \in \{eu; cb\}$  and by the society as a whole.

The replicator dynamic is written as:

$$e_t = \frac{\tilde{v}_t^{eu}}{\tilde{v}_t} e_{t-1} \quad (6)$$

Hence, the equilibrium share of EUM becomes:

$$e_t^* = \frac{\left[ \frac{p_t^*(e_t^*) + \delta_t}{p_{t-1}} \gamma_{t-1}^{eu} + (1 + r) (1 - \gamma_{t-1}^{eu}) \right] e_{t-1}}{\frac{p_t^*(e_t^*) + \delta_t}{p_{t-1}} (\gamma_{t-1}^{eu} e_{t-1} + \gamma_{t-1}^{cb} (1 - e_{t-1})) + (1 + r) (1 - (\gamma_{t-1}^{eu} e_{t-1} + \gamma_{t-1}^{cb} (1 - e_{t-1})))}. \quad (7)$$

The numerator represents the wealth of the old EUM at time  $t$ , the denominator is the wealth of the old investors in the economy at  $t$ . Hence, the share of young investors following a decision rule at time  $t$  is equal to the relative share of wealth held by the old investors following the strategy. I therefore claim, that the replicator dynamic can be interpreted as a relative wealth dynamic in the sense of Blume and Easley (1992, 2001), Hens and Schenk-Hoppé (2001) and Evstigneev, Hens and Schenk-Hoppé (2002, 2003).

The analysis in section 1 shows that if  $e_{t-1} = 0$ ,  $p_{t-1} = 0$  might obtain, see figure 1. To insure that  $e_t^*$  is always well defined, compute the limit of (7) as  $e_{t-1}$  and  $p_{t-1}$  converge to 0. Obviously,

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<sup>4</sup> The choice of the replicator dynamic is made for computational simplicity. In fact, any payoff-monotone dynamic which is regular for all  $e \neq 0$  would lead to the same stability properties of  $e = 1$  as the replicator dynamic introduced below.

$p_{t-1} = \gamma_{t-1}^{eu} e_{t-1}$  and  $\gamma_{t-1}^{eu} = 1$  will hold for prices near 0. Substituting in (7) gives:

$$\lim_{e_{t-1} \rightarrow 0} e_t^* = \lim_{e_{t-1} \rightarrow 0} \frac{\frac{p_t^*(e_t^*) + \delta_t}{e_{t-1}} e_{t-1}}{\frac{p_t^*(e_t^*) + \delta_t}{e_{t-1}} e_{t-1} + (1+r)(1-e_{t-1})} = \frac{p_t^*(e_t^*) + \delta_t}{p_t^*(e_t^*) + \delta_t + 1 + r},$$

which is well defined. Note that  $e_t = 0$  can be interpreted as a situation in which the EUM present in the market have an initial endowment of 0. In this situation, equilibrium prices are determined by the CBDM alone and might allow for arbitrage, especially,  $p_t = 0$  might obtain. The EUM can then obtain the risky asset at no cost in period  $t$ . Then if  $\delta_{t+1} > 0$  obtains, the wealth share of the EUM becomes strictly positive in period  $t + 1$ . On the other hand, if the initial mass of the CBDM is 0, then it remains 0 in all subsequent periods.

### 2.2.2 Temporary Equilibrium With Replicator Dynamic

**Definition 1** Given  $(e_{t-1}; \gamma_{t-1}^{eu}; \gamma_{t-1}^{cb}; p_{t-1})$ , a temporary equilibrium with replicator dynamic at time  $t$  is defined as a vector:  $(e_t^*; \gamma_t^{*eu}; \gamma_t^{*cb}; p_t^*)$ , such that:

- (i)  $\gamma_t^{*eu} = \gamma_t^{eu}(p_t^*)$ ;
- (ii)  $\gamma_t^{*cb} = \gamma_t^{cb}(p_t^*)$ ;
- (iii)  $p_t^*(e_t^*)$  clears the market for the risky asset given  $e_t^*$ ;
- (iv)  $e_t^*$  is determined by (7).

In appendix B, it is shown that such an equilibrium exists in each period of time as long as the initial state  $(e_{t-1}; \gamma_{t-1}^{eu}; \gamma_{t-1}^{cb}; p_{t-1})$  is an equilibrium. The evolution of the system is therefore well defined.

## 3 The Evolution of Wealth

The definition of a temporary equilibrium with evolutionary dynamic, combined with the dividend process determine the evolution of the system. I first discuss the stationary states.

### 3.1 Stationary states

**Proposition 2** (i)  $e_t^* = 1, p_t^* = \gamma_t^{eu} = p_\beta^{eu}$  is a stationary state.

(ii)  $e_t^* = 0, p_t^* = \gamma_t^{cb} = \gamma_0^{cb}$  is a stationary state, if  $\gamma_0^{cb} > 0$  and

$$\bar{u} < u_\beta (\gamma_0^{cb} + (1+r)(1-\gamma_0^{cb}))$$

hold.

If only EUM are present in the market, the equilibrium prices and allocations coincide with those in an equilibrium with rational expectations. If  $e = 0$ , the price  $p_t^* = \gamma_0^{cb}$  is constant over time, but, in general  $p_\beta^{eu} \neq p_t^*$ . Hence, arbitrage opportunities might remain unused in the market. However, the state in which only CBDM are present in the market is stationary only if the aspiration level is relatively low so that  $\gamma_t^{cb} > 0$  holds for all  $t$ . As discussed above,  $p_t^* = 0$  leads almost surely to  $e_T^* > 0$  for some finite  $T > t$ , contradicting the assumption of stationarity.

**Proposition 3** *Let*

$$\bar{u} < u_\beta (p_\beta^{eu} + (1+r)(1-p_\beta^{eu})).$$

*Then each  $e \in [0; 1]$  is a stationary state, if the portfolios held and the price of a fullfil:*

$$\gamma_t^{eu} = \gamma_t^{cb} = p_\beta^{eu} = p_t^*$$

*for all  $t \geq 0$ .*

Proposition 3 identifies stationary states, in which both types of traders coexist. The equilibrium price and allocations coincide with the REE. By mimicking the EUM, CBDM with relatively low aspiration levels are thus able to survive in a financial market. However, they will not influence prices and it would not be possible to empirically reject the hypothesis of rational expectations and expected utility maximization in such a market. It is, therefore, interesting, whether a positive share of case-based decision-makers can survive if the portfolio strategies of EUM and CBDM differ.

### 3.2 Can the Case-Based Investors Survive?

It has been shown, that CBDM with a relatively low aspiration level can survive in a market, without influencing prices if  $\bar{u}$  is relatively low.

The further discussion will concentrate on the case, in which the stationary states in which CBDM are present in the market, as described in propositions 2 and 3 do not occur, i.e. on

$$\bar{u} > u_\beta (\gamma_0^{cb} + (1+r)(1-\gamma_0^{cb}))$$

for  $\gamma_0^{cb} > 0$ , since then the CBDM will change their portfolio holdings over time generating a non-trivial dynamic. The discussion of the results for asset markets without EUM in Guerdjikova (2003) shows, that the dynamic of the system crucially depends on the aspiration level of the CBDM. Two cases will be of importance:

$$\bar{u} \in (u_\beta (1+r); u_\beta (1+\delta)),$$

referred to as *high aspiration level* and

$$\bar{u} \in (u_\beta (\gamma_0^{cb} + (1+r)(1-\gamma_0^{cb})); u_\beta (1+r)),$$

the case of *low aspiration level*.

For

$$\bar{u} \in (u_\beta (\gamma_0^{cb} + (1+r)(1-\gamma_0^{cb})); u_\beta (1+r)),$$

it is easy to see, that the return of  $b$  is satisfactory for the CBDM, whereas the return of  $a$ , given that the dividend is 0 and the price of  $a$  remains unchanged or falls is not satisfactory. Hence,  $\gamma^{cb} = 0$  obtains and holds forever.

For

$$\bar{u} \in (u_\beta (1+r); u_\beta (1+\delta)),$$

the return of  $a$  is considered satisfactory, when the dividend is high and the price of  $a$  weakly increases, whereas the return of  $b$  and the return of  $a$  if its dividend is low, are regarded as unsatisfactory. Hence, the CBDM will switch infinitely often between  $\gamma^{cb} = 1$  and  $\gamma^{cb} = 0$ .

Since the influence of CBDM on prices depends positively on their share  $c$ , in the following I examine the stability of the stationary state  $e = 1$  to determine whether the CBDM can survive and influence the prices in the market.

### 3.2.1 The case of high aspiration levels

Consider first the case of high aspiration level:

$$\bar{u} \in (u_\beta (1+r); u_\beta (1+\delta)).$$

**Proposition 4** *Let*

$$\bar{u} \in (u_\beta (1+r); u_\beta (1+\delta)).$$

1. If

$$q \in \left[ \frac{r}{r + (\delta - r)(1 + \delta)^{\beta-1}}; \frac{(1 + \delta)r}{(1 + r)\delta} \right),$$

then for each  $\beta \in (0; 1]$ , there exists an  $\tilde{e} \in (0; 1)$ , such that  $e_t^*$  is a submartingale for  $e_{t-1}^* < \tilde{e}$  and a supermartingale for  $e_{t-1}^* \geq \tilde{e}$ .

2. For all values of  $\beta \in [0; 1]$  if

$$q \in \left[ \frac{(1 + \delta)r}{(1 + r)\delta}; 1 \right],$$

$e_t^*$  is a submartingale for all  $e_{t-1} \in [0; 1]$ . The CBDM disappear with probability 1.

Proposition 4 establishes conditions under which the stationary state  $e = 1$  is not stable, in the sense that the replicator dynamics does not converge to it with probability 1.

To see the intuition behind the result, consider the case of  $\beta = 1$ . The condition on  $q$  in the first part of proposition 4 insures that  $p_1^{eu} = 1$ , hence, the EUM hold  $a$  in each period of time, independently of  $p_t$ . The high aspiration level implies that the CBDM are constantly switching between  $\gamma^{cb} = 1$  and  $\gamma^{cb} = 0$ . The replicator dynamic of  $e_t^*$  is concave in the returns of the EUM. Therefore, it selects for the less risky strategy, given that the expected returns of two strategies are identical. When  $\gamma^{cb} = 0$ , the portfolio of the CBDM is less risky, than the portfolio of the EUM,  $\gamma^{eu} = 1$ . Moreover, if  $p_1^{eu} = \frac{q\delta}{r} = 1$ , as  $c \rightarrow 0$  the expected returns of both portfolios are the same and the replicator dynamic selects for the less risky one — those of the CBDM. By continuity, the same result holds in some surrounding of  $e = 1$  and in some surrounding of  $q = \frac{r}{\delta}$ . Hence, as long as  $q$  is not very large, a positive share of CBDM survives.

If, however,  $q$  exceeds  $\frac{(1+\delta)r}{(1+r)\delta}$ , the excess return of the EUM becomes sufficiently high to compensate for the higher risk of their portfolio. In this case, they accumulate the whole market wealth with probability 1. If the utility function is logarithmic, only this case is relevant and the CBDM almost surely disappear. This result is consistent with the findings cited in section 2.

Higher values of  $p_\beta^{eu}$  coincide with higher values of  $q$  ceteris paribus. The probability of high dividends has two effects on the evolutionary dynamic. On the one hand, higher  $q$  implies higher expected returns of  $a$  and therefore higher profits for the investors holding it, i.e. for the EUM. On the other hand, higher values of  $q$  cause the CBDM to switch less frequently between the two undiversified portfolios and to hold the risky asset during a larger share of time, hence to behave in a less risk-averse manner. These two effects work in the same direction, making the strategy of the EUM more successful.

A result similar to the one of proposition 4 can be derived for lower fundamental values of the risky asset, when the utility function is linear.

**Proposition 5** *Suppose that  $\beta = 1$ . Let the aspiration level satisfy*

$$\bar{u} \in (1 + r; 1 + \delta).$$

Then there is a critical value  $\tilde{p}_1^{eu} \in (\frac{1}{2}; 1)$  such that  $E [e_{t+2}^* | e_t^*] < e_t^*$  holds for

$$e_t^* \in \left[ \max \left\{ p_1^{eu}; 1 - p_1^{eu} + \frac{r p_1^{eu^2}}{1+r} \right\}; 1 \right)$$

if  $p_1^{eu} \in (\tilde{p}_1^{eu}; 1)$ .

Although the result is stated for the case of a linear utility function, the argument can be extended to all  $\beta > 0$ . Indeed, because of proposition 4, we know that  $e = 1$  is unstable at

$$q = \frac{r}{r + (\delta - r)(1 + \delta)^{\beta-1}}.$$

Since the dynamic of the system is continuous w. r. t.  $q$ , it follows that this property still holds in some surrounding of  $p_\beta^{eu} = 1$  and especially for

$$q \in \left( \tilde{q}; \frac{r}{r + (\delta - r)(1 + \delta)^{\beta-1}} \right)$$

for some  $\tilde{q} \in \left( 0; \frac{r}{r + (\delta - r)(1 + \delta)^{\beta-1}} \right)$ .

For lower values of  $p_\beta^{eu}$  (even in the case of a linear utility function), the results are not clear. For  $\beta = 1$ , it can be shown that

$$E [e_{t+2}^* | e_t^*, \gamma_t^{*cb} = 1] < 0$$

and

$$E [e_{t+2}^* | e_t^*, \gamma_t^{*cb} = 0] > 0$$

hold for  $e_t$  sufficiently close to 1. It is, à priori, not clear which of these two effects will dominate. Nevertheless, the intuition suggests that for sufficiently low  $p_\beta^{eu}$ , the CBDM will disappear. Let, e.g.  $\delta = 0$ , so that  $p_1^{eu} = 0$ . In this case, the CBDM, hold a strictly dominated asset with positive frequency and disappear. By continuity, this result holds in some surrounding of  $p_1^{eu} = 0$  ( $\delta = 0$ ) and, therefore, CBDM with high aspiration level cannot survive for low fundamental values of the risky asset.

To summarize, if the fundamental value of the risky asset is neither too high, nor too low, there is a positive probability that the CBDM will not disappear from the market. This result can be made even stronger:

**Proposition 6** *Suppose that  $e_t^*$  is a supermartingale on some interval  $[\tilde{e}; 1]$ . Then*

$$\Pr\{e_t^* \rightarrow 1\} = 0.$$

The share of CBDM thus remains almost surely positive, as long as it can be shown, that  $e_t^*$  is a

supermartingale near 1. This result can be interpreted in terms of the definition of survival and dominance introduced by Blume and Easley (1992). In their terminology survival requires that the share of an investor type, say of CBDM, fulfills:

$$\Pr \left\{ \limsup_{t \rightarrow \infty} c_t > 0 \right\} = 1, \quad (8)$$

whereas the CBDM dominate the market, if

$$\Pr \left\{ \liminf_{t \rightarrow \infty} c_t > 0 \right\} = 1 \quad (9)$$

is satisfied. Proposition 6 implies that both (8) and (9) are fulfilled, as long as  $e_t^*$  is a supermartingale on some interval  $[\tilde{e}; 1]$ .

### 3.2.2 The case of low aspiration level

Now suppose that the CBDM have an aspiration level which satisfies

$$u_\beta (\gamma_0^{cb} + (1+r)(1-\gamma_0^{cb})) < \bar{u} < u_\beta (1+r),$$

implying that the CBDM hold  $b$  in each period of time. Again, it is possible to identify values of the parameters, for which the state  $e = 1$  is not stable and the CBDM almost surely survive in positive proportion.

**Proposition 7** *If*

$$\bar{u} \in (u_\beta (\gamma_0^{cb} + (1+r)(1-\gamma_0^{cb})); u_\beta (1+r))$$

*and*

1.  $q \in \left[ \frac{r}{r+(\delta-r)(1+\delta)^{\beta-1}}; \frac{(1+\delta)r}{(1+r)\delta} \right)$ , then for each  $\beta \in (0; 1]$ , there exists an  $\hat{e} \in (0; 1)$ , such that  $e_t$  is a supermartingale above  $\hat{e}$  and a submartingale below  $\hat{e}$ .
2. If  $q \in \left[ \frac{(1+\delta)r}{(1+r)\delta}; 1 \right]$ , for any  $\beta \in [0; 1]$ , the share of the CBDM converges to 0 almost surely.

For low aspiration levels, the CBDM survive for exactly the same values of  $q$ , which were found in proposition 4. Although in the case of low aspiration level,  $q$  influences the selection only by increasing the average return of the EUM and not through the less risk-averse behavior on the side of the CBDM, in the limit when  $c \rightarrow 0$ , the conditions for survival of the CBDM are identical in both cases.

However, the cut-off values  $\tilde{e}$  (as defined in proposition 4) and  $\hat{e}$  from proposition 7 reflect the fact that the strategy of the CBDM is riskier in the case of high aspiration level. Hence, CBDM with low aspiration levels are likely to survive in a higher proportion than CBDM with high

aspiration levels, as the following proposition demonstrates:

**Proposition 8**  $\tilde{e}$ , as defined in proposition 4 and  $\hat{e}$  from proposition 7 satisfy:

$$\tilde{e} > \hat{e}.$$

**Proposition 9** Suppose that

$$\bar{u} \in (u_\beta (\gamma_0^{cb} + (1+r)(1-\gamma_0^{cb})); u_\beta (1+r))$$

Let

$$q < \frac{r}{r + (\delta - r)(1 + \delta)^{\beta-1}}.$$

Then for all  $\beta \in (0; 1]$ , there is an  $\check{e}(\beta) \in (0; 1)$  such that  $e_t^*$  is a supermartingale on  $[\check{e}(\beta); 1]$ .

The result of proposition 6 applies in this case as well, implying that the share of CBDM remains positive with probability 1, as long as  $e_t^*$  is a supermartingale in some interval  $[\check{e}; 1]$ . Note that as in the case of high aspiration levels, CBDM cannot survive in the presence of logarithmic EUM.

## 4 Asset Prices in the Presence of Case-Based Decision Makers

The results of section 3 show that CBDM can survive in strictly positive proportion in the presence of EUM. This section analyzes the effect of their behavior on asset prices.

Consider first the case of high aspiration level. If  $p_\beta^{eu} < 1$ , the CBDM can influence prices and cause bubbles, excessive volatility and predictability of returns, as long as their share is sufficiently high. First note that in an OLG model with constant initial endowments and no population growth, the equilibrium under rational expectations predicts  $p_t^* = p_\beta^{eu}$  for all  $t$ . This would obviously obtain for  $e = 1$ . Denote by

$$\hat{p}_\beta^{eu} = \min \{p \mid \gamma^{eu}(p) = 0\}$$

the minimal price at which the EUM hold only bonds. Suppose now that  $c_t > \hat{p}_\beta^{eu}$  holds. Since the CBDM switch between  $a$  and  $b$  infinitely often, the price of  $a$  will fluctuate depending on the share of CBDM and on their behavior, exhibiting excessive volatility. Moreover, the returns of  $a$  are predictable. Especially, if in a certain period only EUM hold  $a$ , an external observer could predict, that the price of  $a$  in the next period will (weakly) rise, since the young CBDM will buy  $a$  in  $t$ , independently of the dividend paid by the risky asset.

CBDM can cause a bubble to emerge and to persist in the market for several periods. Suppose for instance that the share of EUM is lower than  $(1 - \hat{p}_\beta^{eu})$  at some time  $t$  and that CBDM choose

$\gamma_t^{cb} = 1$ . Then the equilibrium price of  $a$  will satisfy

$$p_t^* = (1 - e_t^*) > \hat{p}_\beta^{eu} \geq p_\beta^{eu}.$$

Moreover, if  $\delta_{t+1} = \delta$ , then the returns of the CBDM will exceed those of the EUM and  $e_{t+1}^* < e_t^*$  will hold. Since the young CBDM choose  $\gamma_{t+1}^{cb} = 1$ ,

$$p_{t+1}^* = (1 - e_{t+1}^*) > p_t^* > \hat{p}_\beta^{eu} \geq p_\beta^{eu}$$

holds in equilibrium. Hence, the price increases above the price under rational expectations, as long as the dividend of the risky asset remains positive. In the first period  $t'$ , such that  $\delta_{t'} = 0$ , the bubble will burst, since the CBDM will switch to  $\gamma^{cb} = 0$  and their share will decrease. Moreover, the price of the risky asset might even fall below the price under rational expectations  $p_\beta^{eu}$ . For instance, in the case of a linear utility function this would happen, if

$$(1 - e_{t'}^*) > \frac{(p_1^{eu} - 1)^2}{p_1^{eu} (1 + r)},$$

hence, if the bubble has lasted long enough to decrease substantially the share of the EUM.

Now consider the case of low aspiration levels, hence,  $\gamma_t^{*cb} = 0$  holds for each  $t$ . It turns out that CBDM with low aspirations can drive the EUM out of the market for a finite number of periods.

Denote by

$$\check{p}_\beta^{eu} = \max \{p \mid \gamma^{eu}(p) = 1\}.$$

Let  $e_t \leq \check{p}_\beta^{eu}$  so that  $\gamma_t^{eu} = 1$ ,  $p_t^* = e_t^*$  and let the next period dividend be low,  $\delta_{t+1} = 0$ . The equilibrium share of the EUM is given by the solution of the equation:

$$e_{t+1}^* = \frac{\frac{e_{t+1}^*}{e_t^*} e_t^*}{\frac{e_{t+1}^*}{e_t^*} e_t^* + (1 + r)(1 - e_t^*)} = \frac{e_{t+1}^*}{e_{t+1}^* + (1 + r)(1 - e_t^*)}. \quad (10)$$

(10) has two solutions:  $e_{t+1}^* = 0$  and

$$e_{t+1}^{**} = e_t^* (1 + r) - r < e_t^*$$

Hence, if the initial share of the EUM is relatively small:

$$e_t^* < \min \left\{ \frac{r}{1 + r}; 1 - \check{p}_\beta^{eu} \right\},$$

$e_{t+1}^{**} < 0$  and  $e_{t+1}^* = 0$  obtains in equilibrium.

The EUM can vanish, if they hold the risky asset, hoping that it is valuable, but if there are not enough of their type to prevent prices from falling, when the dividend of the asset is low. This effect is similar to the noise trader risk identified by De Long, Shleifer, Summers and Waldmann

(1990). Although the EUM don't have rational expectations in this model, they suffer from an undervaluation of the risky asset, caused by the CBDM. If further the returns of the EUM are relatively low compared to those of the population as a whole, then the share of the CBDM will increase causing the undervaluation of the risky asset to become even more severe.

The effect arises, because of the dependence of the replicator dynamic on the price of the risky asset and therefore indirectly on  $e_t$  itself. It shows, that even in markets, in which EUM are à priori present, the price of an asset with positive fundamental value may fall to 0 and remain so for few periods.

The EUM will, however, not disappear forever. Since they will hold an asset with positive fundamental value, their share in the population will become positive in the first period, in which the dividend of the risky asset becomes strictly positive. Hence, the price of the asset becomes positive in finite time.

The results of this section imply that some of the phenomena empirically observed in financial markets could be attributed to the presence of CBDM in the economy. However, the emergence of bubbles or price crashes requires a relatively high proportion of CBDM in the market. Although the probability of such events is positive, it is not clear, whether its analytical computation is possible. Future work will therefore have to deal with simulations of the model, from which the frequency of such phenomena could be estimated.

## 5 Conclusion

The analysis of the model answers the two questions stated in the introduction by identifying conditions under which CBDM survive in the presence of EUM and discussing their influence on prices. It is shown that CBDM can survive for certain ranges of the parameters if the coefficient of relative risk aversion is less than 1. Case-based investors are cause predictability of asset prices, high volatility and bubbles, as well as price crashes when the share of EUM in the market is relatively low.

A final note has to be made on the issue of introducing EUM with rational expectations. It is straightforward to see, that in the case of a linear utility function, as long, as the short-sale constraints are not binding, the expected returns of the two assets will be identical at each period

of time. Therefore, the replicator dynamic will select for the less risky strategy in each period and the CBDM would retain a positive proportion in the market. Since the replicator dynamic and the demand of the investors are continuous with respect to  $\beta$ , it can be expected that similar results will obtain in a surrounding of  $\beta = 1$ . Hence, the results about the instability of the stationary state  $e = 1$  would remain valid at least for a range of coefficients of relative risk-aversion close to 1.

# Appendix A

## Proof of Proposition 2

(i) Let  $e_{t-1} = 1$ . Since

$$e_t^* = \frac{\left[ \frac{p_t^*(e_t^*) + \delta_t}{p_{t-1}} \gamma_{t-1}^{eu} + (1+r)(1 - \gamma_{t-1}^{eu}) \right] e_{t-1}}{\frac{p_t^*(e_t^*) + \delta_t}{p_{t-1}} \gamma_{t-1}^{eu} e_{t-1} + (1+r)(1 - \gamma_{t-1}^{eu} e_{t-1})} = 1,$$

the claim of the proposition obtains.

(ii) If  $p_{t-1} > 0$  and  $e_{t-1} = 0$ , then  $e_t^* = \frac{\tilde{v}_t^{eu \cdot 0}}{\tilde{v}_t} = 0$ . Therefore, if  $\gamma_t^{*cb} > 0$  for all  $t \geq 1$ ,  $e_t = 0$  will obtain for all  $t$ . Since  $\gamma_0^{cb} > 0$ , in  $t = 1$ , all CBDM remember the case  $(\gamma_0^{cb}; v_t(\gamma_0^{cb}))$  and observe a return of at least

$$u_\beta(\gamma_0^{cb} + (1+r)(1 - \gamma_0^{cb})) > \bar{u},$$

if  $p_0 = p_1$ , implying  $\gamma_1^{cb} = \gamma_0^{cb}$ . By induction,

$$\gamma_t^{cb} = p_t^* = \gamma_0^{cb}$$

obtains in each period  $t$ . ■

## Proof of Proposition 3:

Let  $p_t = p_\beta^{eu}$  for all  $t$ . Then, the assumption

$$\bar{u} < u(p_\beta^{eu} + (1+r)(1 - p_\beta^{eu}))$$

guarantees that  $\gamma_t^{cb} = \gamma_0^{cb} = p_\beta^{eu}$  for all  $t$ . Indeed, if a CBDM remembers  $(\gamma^{cb} = p_\beta^{eu}; v_t(p_\beta^{eu}))$ ,

$$v_t(p_\beta^{eu}) \geq u_\beta(p_\beta^{eu} + (1+r)(1 - p_\beta^{eu})) > \bar{u}$$

holds and  $\gamma_t^{cb} = p_\beta^{eu}$ . Moreover,  $\gamma^{eu}(p_\beta^{eu}) = p_\beta^{eu}$  is the optimal choice of the EUM. Since  $\gamma_t^{eu} = \gamma_t^{cb}$ ,  $\tilde{v}_t^{eu} = \tilde{v}_t^{cb} = \tilde{v}$ , hence  $e_t = \text{const.}$  ■

## Proof of proposition 4

**Lemma 10** If  $q \geq \frac{r(1+\delta)}{\delta(1+r)}$ ,  $p_\beta^{eu} = 1$  holds.

## Proof of lemma 10

Let first  $\beta = 0$ . Observe that  $p_0^{eu}$  is strictly decreasing in  $q$  and

$$p_0^{eu}(q) = 1$$

obtains at  $q = \frac{r(1+\delta)}{\delta(1+r)}$ . Hence,  $p_0^{eu} \geq 1$  is inconsistent with the condition necessary for  $c_t^*$  to be a submartingale.

Now let  $\beta \in (0; 1]$ . The value of  $q$ , for which  $\gamma_t^{eu} = p_\beta^{eu} = 1$  is an interior solution of (3) is given by:

$$\frac{(1+r) \left[ \left( \frac{(1-q)r}{q(\delta-r)} \right)^{\frac{1}{\beta-1}} - 1 \right]}{\delta+r \left[ \left( \frac{(1-q)r}{q(\delta-r)} \right)^{\frac{1}{\beta-1}} - 1 \right]} = 1,$$

which simplifies to

$$q = \frac{r}{r + (\delta - r)(1 + \delta)^{\beta-1}}. \quad (11)$$

Denote

$$\left( \frac{(1-q)r}{q \left( \frac{\delta}{p} - r \right)} \right)^{\frac{1}{\beta-1}} =: z.$$

From  $\frac{\partial \tilde{\gamma}_t^{eu}}{\partial z} > 0$  and

$$\frac{\partial z}{\partial q} = \frac{1}{\beta-1} \left( \frac{(1-q)r}{q \left( \frac{\delta}{p} - r \right)} \right)^{\frac{2-\beta}{\beta-1}} \frac{(-r(\delta-r))}{q^2(\delta-r)^2} > 0,$$

it follows that  $\frac{\partial \tilde{\gamma}_t^{eu}}{\partial q} > 0$  and since  $\tilde{\gamma}_t^{eu}$  falls in price, it follows that the equilibrium price is increasing in the probability of high dividend  $q$ . Therefore, for values of  $q$ , higher than (11), the price under rational expectations (for a given  $\beta$ ) is equal to 1, see figure 2. ■

Hence,  $\gamma_t^{*eu} = 1$  whenever  $p_t \leq 1$ . Hence, for

$$q \in \left[ \frac{r}{r + (\delta - r)(1 + \delta)^{\beta-1}}; 1 \right]$$

$\gamma_t^{*eu} = 1$  for all  $t \geq 1$ . The dynamics of prices and asset holdings are given by:

$$\begin{aligned} \gamma_t^{*eu} &= 1 \text{ for each } t; \\ \gamma_t^{*cb} &= \left\{ \begin{array}{l} 1, \text{ if } \gamma_{t-1}^{cb} = 1 \text{ and } \delta_t = \delta \text{ or} \\ \gamma_{t-1}^{cb} = 0 \\ 0, \text{ if } \gamma_{t-1}^{cb} = 1 \text{ and } \delta_t = 0 \end{array} \right\}; \\ p_t^* &= \left\{ \begin{array}{l} 1, \text{ if } \gamma_{t-1}^{cb} = 1 \text{ and } \delta_t = \delta \text{ or} \\ \gamma_{t-1}^{cb} = 0 \\ e_t^*, \text{ if } \gamma_{t-1}^{cb} = 1 \text{ and } \delta_t = 0 \end{array} \right\}. \end{aligned}$$

In periods such that  $\gamma_t^{*eu} = \gamma_t^{*cb} = 1$ ,  $\tilde{v}_t^{eu} = \tilde{v}_t^{cb} = \tilde{v}_t$ , hence

$$e_t^* = e_{t-1}^*, \text{ if } \gamma_{t-1}^{*cb} = 1.$$

These periods have, therefore, no effect on the dynamic of  $e_t$ . It is, hence sufficient to analyze

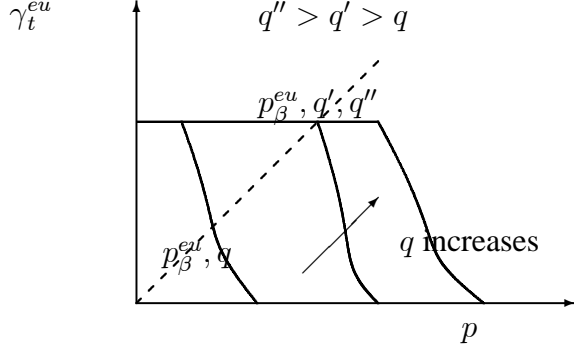


Figure 2

$e_t$  in periods in which  $\gamma_t^{*eu} \neq \gamma_t^{*cb}$  holds. In such periods we have:

$$E [c_t^* | c_{t-1}^*] = q \frac{(1+r) c_{t-1}}{(1+r) c_{t-1}^* + \frac{(1+\delta)}{p_{t-1}^*} e_{t-1}^*} + (1-q) \frac{(1+r) c_{t-1}}{(1+r) (1 - e_{t-1}^*) + \frac{1}{p_{t-1}^*} e_{t-1}^*}.$$

Since  $\gamma_{t-1}^{*eu} = 1$  and  $\gamma_{t-1}^{*cb} = 0$ ,  $p_{t-1}^* = e_{t-1}^*$ . Therefore,

$$E [c_t^* | c_{t-1}^*] = q \frac{(1+r) c_{t-1}^*}{(1+r) c_{t-1}^* + (1+\delta)} + (1-q) \frac{(1+r) c_{t-1}^*}{(1+r) c_{t-1}^* + 1} \geq c_{t-1}^{**}$$

$$\Leftrightarrow$$

$$r(1+\delta) + r(1+r) c_{t-1}^* - q\delta(1+r) \geq (1+\delta)(1+r) c_{t-1}^* + (1+r)^2 c_{t-1}^{*2}$$

$$r(1+\delta + (1+r) c_{t-1}^*) - (1+r) c_{t-1}^* (1+\delta + (1+r) c_{t-1}^*) - q\delta(1+r) \geq 0$$

$$(1+\delta + (1+r) c_{t-1}^*) [r - (1+r) c_{t-1}^*] - q\delta(1+r) \geq 0. \quad (12)$$

For  $c_{t-1}^* \geq \frac{r}{1+r}$ ,  $E [c_t^* | c_{t-1}^*] < c_{t-1}^*$ , therefore  $c_t^*$  is a supermartingale. Hence,  $e_t^*$  is a sub-

martingale, if  $e_{t-1}^* \leq \frac{1}{1+r}$ . On the other hand, if  $c_{t-1}^* \rightarrow 0$ , the l.h.s. of (12) becomes:

$$(1 + \delta)r - q\delta(1 + r) > 0, \text{ if } q < \frac{(1 + \delta)r}{(1 + r)\delta}.$$

If  $q < \frac{(1+\delta)r}{(1+r)\delta}$ , the continuity of the l.h.s. of (12) guarantees, that  $E[c_t^* | c_{t-1}^*] > c_{t-1}^*$  (and hence  $c_t^*$  is a submartingale) for  $c_{t-1}^*$  close to 0. It follows that for some  $\tilde{c} \in (0; \frac{r}{1+r})$ ,  $E[c_t^* | c_{t-1}^* = \tilde{c}] = \tilde{c}$ . The assertion of the first part of the proposition now follows by defining  $\tilde{e} = 1 - \tilde{c}$ .

If  $q > \frac{(1+\delta)r}{(1+r)\delta}$ , the l.h.s. of (12) is negative for all  $c_{t-1}^* \in [0; 1]$  and, therefore  $E[e_t^* | e_{t-1}^*] > e_{t-1}^*$  for all  $e_{t-1}^* \in [0; 1]$ . It follows, that  $e_t^*$  is a submartingale on  $[0; 1]$  and converges almost surely.

Hence, on almost each dividend path

$$\lim_{t \rightarrow \infty} \frac{e_t^*}{e_{t-1}^*} = \lim_{t \rightarrow \infty} \frac{(1 + \delta_t)}{1 + (1 - e_t^*)r + \delta_t} = 1$$

must hold, which is only possible, if  $e_t^* \rightarrow 1$  with probability 1.

It remains to show that the condition  $q < \frac{(1+\delta)r}{(1+r)\delta}$  is consistent with the assumption that  $p_\beta^{eu} = 1$  only for  $\beta \in (0; 1]$ , but not for  $\beta = 0$ .

**Lemma 11**  $q < \frac{(1+\delta)r}{(1+r)\delta}$  and

$$q \geq \frac{r}{r + (\delta - r)(1 + \delta)^{\beta-1}}$$

can hold simultaneously for all  $\beta \in (0; 1]$ , but not for  $\beta = 0$ .

Indeed,

$$\frac{r}{r + (\delta - r)(1 + \delta)^{\beta-1}} < \frac{(1 + \delta)r}{(1 + r)\delta}$$

is equivalent to

$$(\delta - r) \left[ (1 + \delta)^\beta - 1 \right] > 0,$$

which is always satisfied for  $\beta > 0$ . The logarithmic utility function represents the limit case,  $\beta = 0$ , in which the equality holds. ■

**Proof of proposition 5:**

Assume that  $e_t \in \left[ \max \left\{ p_1^{eu}; 1 - p_1^{eu} + \frac{p_1^{eu} r^2}{1 - p_1^{eu}} \right\}; 1 \right)$ .

The proposition is proved separately for periods  $t$  with  $\gamma_t^{cb} = 1$  and those with  $\gamma_t^{cb} = 0$ .

Let first  $\gamma_t^{cb} = 1$ .  $\gamma_{t+1}^{cb} = 1$  holds only if  $\delta_{t+1} = \delta$  and then:

$$p_{t+1} = p_t = p_1^{eu}.$$

By assumption,

$$c_t^* = 1 - e_t^* \leq 1 - p_1^{eu} < p_1^{eu},$$

since  $p_1^{eu} < \frac{1}{2}$ . Therefore,

$$\begin{aligned} \tilde{v}_{t+1}^{cb} &= 1 + \frac{\delta}{p_1^{eu}} \text{ and} \\ \tilde{v}_{t+1} &= p_1^{eu} + \delta + (1+r)(1-p_1^{eu}) = 1+r-rp_1^{eu} + \delta, \end{aligned}$$

as long as  $e_{t+1}^* < p_1^{eu}$  holds. Furthermore, since  $1 + \frac{\delta}{p_1^{eu}} > 1 + \delta > \bar{u}$ ,  $\gamma_{t+1}^{cb} = 1$ .

Alternatively, if  $\delta_{t+1} = 0$ ,  $v_{t+1}(\gamma_t^{cb}) \leq 1 < \bar{u}$ , hence  $\gamma_{t+1}^{cb} = 0$  and  $\tilde{v}_{t+1}^{cb} \leq 1$ . Since

$$\tilde{v}_{t+1} = p_1^{eu} + (1+r)(1-p_1^{eu}) = 1+r-rp_1^{eu} > 1,$$

$c_{t+1}^* < c_t^*$  and  $p_{t+1}^* = p_1^{eu}$ .

If  $\delta_{t+1} = \delta$ , the returns and the behavior of the investors in  $(t+2)$  is described exactly as in  $(t+1)$ , except if  $c_{t+1}^* > p_1^{eu}$ , since then  $p_{t+2}^* > p_1^{eu}$  may obtain. This is the case if:

$$c_{t+1}^* = \frac{\left(1 + \frac{\delta}{p_1^{eu}}\right)}{1+r-rp_1^{eu} + \delta} c_t^* > p_1^{eu}. \quad (13)$$

However, if  $p_1^{eu}$  is sufficiently large,  $c_{t+1}^* < p_1^{eu}$  holds for all  $c_t^* \in (0; 1 - p_1^{eu})$ . Indeed, rewrite (13) as

$$c_t^* > \frac{p_1^{eu}(1+r-rp_1^{eu} + \delta)}{1 + \frac{\delta}{p_1^{eu}}}.$$

To exclude the case, in which the inequality in (13) holds, it is necessary that:

$$\frac{p_1^{eu}(1+r-rp_1^{eu} + \delta)}{1 + \frac{\delta}{p_1^{eu}}} > 1 - p_1^{eu},$$

or that

$$-rp_1^{eu^3} + p_1^{eu^2}(2+r+\delta) - (1-\delta)p_1^{eu} - \delta > 0. \quad (14)$$

Note first that for  $\delta = \frac{r}{2q}(p_1^{eu} = \frac{1}{2})$ , the l.h.s. is negative and that for  $\delta = \frac{r}{q}(p_1^{eu} = 1)$ , the l.h.s. is positive. Using now the fact that  $p_1^{eu} = \frac{q\delta}{r}$ , rewrite (14) as:

$$q^2\delta^2(1-q) + \delta q(2q+qr+r) - r > 0$$

and since the l.h.s. of this expression is a convex quadratic function, there exists a  $\hat{\delta}$ , such that for every  $\delta > \hat{\delta}$  (14) is satisfied.

Then<sup>5</sup>:

$$E [c_{t+2}^* | c_t^*, \gamma_t^{cb} = 1] = c_t^* q \frac{\left(1 + \frac{\delta}{p_1^{eu}}\right)}{R + \delta} \left[ q \frac{1 + \frac{\delta}{p_1^{eu}}}{R + \delta} + (1 - q) \frac{1}{R} \right] + c_t^* (1 - q) \frac{1}{R} \left[ q \frac{(1 + r)}{R + \delta} + (1 - q) \frac{(1 + r)}{R} \right],$$

where  $R = 1 + r - rp^{eu}$ . Using simple algebra and the fact that  $p_1^{eu} = \frac{q\delta}{r}$  shows that

$$E [c_{t+2}^* | c_t^*, \gamma_t^{cb} = 1] > c_t^*,$$

if and only if

$$(q + r) R (R(1 + r) + \delta(1 - q)) + (1 - q)(1 + r)(R + \delta(1 - q))(R + \delta) > (R + \delta)^2 R^2 \quad (15)$$

holds. If

$$p_1^{eu} = \frac{q\delta}{r} = 1,$$

meaning that  $R = 1$  and  $q\delta = r$ , condition (10) simplifies to:

$$(1 + \delta)r(\delta - r + qr) > 0,$$

which is always true. On the other hand, for

$$p_1^{eu} = \frac{q\delta}{r} = \frac{1}{2}$$

and, hence,  $R = 1 + \frac{r}{2}$  and  $q\delta = \frac{r}{2}$ , (15) is equivalent to

$$\frac{qr}{2} + \frac{qr^2}{4} + \frac{3r^3}{16} - \frac{1}{2} - r - \delta r - \delta^2 r - \delta r^2 > 0,$$

which is never satisfied, since

$$\begin{aligned} \frac{qr}{2} &< \frac{r}{2} < \frac{1}{2} \\ \frac{qr^2}{4} &< r^2 < r \\ \frac{3r^3}{16} &< r^3 < r^2 < \delta r. \end{aligned}$$

Therefore,

$$E [c_{t+2}^* | c_t^*, \gamma_t^{cb} = 1] > c_t^*$$

holds for  $\delta = \frac{r}{q}$  and since the expected value of  $c_{t+2}^*$  is continuous in  $\delta$ , it follows that the process  $c_t^*, c_{t+2}^*, c_{t+4}^* \dots$  is a submartingale in some surrounding of  $\delta = \frac{r}{q}$ . At the same time,

$$E [c_{t+2}^* | c_t^*, \gamma_t^{cb} = 1] < c_t^*$$

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<sup>5</sup> In fact, as above, it should be taken into account that the share of the case-based decision-makers might exceed  $p^{eu}$  in  $(t + 2)$  if the risky asset pays a high dividend. However, this will only increase the expected value of  $c_{t+2}^*$ . Since the argument relies on showing that the expected value of  $c_{t+2}^*$  exceeds  $c_t^*$ , neglecting this effect has no influence on the results.

holds for  $\delta = \frac{r}{2q}$ . By continuity of the expected value of  $c_{t+2}^*$ , there is, therefore, a value for  $\delta$ ,  $\bar{\delta} \in \left(\frac{r}{2q}; \frac{r}{q}\right)$  such that the expected value of  $c_{t+2}^*$  exceeds  $c_t^*$  for  $\delta > \bar{\delta}$ .

Let now  $\gamma_t^{cb} = 0$ .

$$\begin{aligned} E [c_{t+2}^* | c_t^*, \gamma_t^{cb} = 0] &= c_t^* q \frac{(1+r)}{R+\delta} \left[ q \frac{\left(1 + \frac{\delta}{p_1^{eu}}\right)}{R+\delta} + (1-q) \frac{1}{R} \right] + \\ &+ c_t^* (1-q) \frac{(1+r)}{R} \left[ q \frac{\left(1 + \frac{\delta}{p_1^{eu}}\right)}{R+\delta} + (1-q) \frac{1}{R} \right]. \end{aligned}$$

Again, one should take into account that  $c_{t+1}^* > p_1^{eu}$  might obtain if  $\delta = 0$ . This requires that:

$$\frac{(1+r)}{1+r-rp_1^{eu}} c_t^* > p_1^{eu},$$

or, equivalently

$$e_t^* < 1 - p_1^{eu} + \frac{rp_1^{eu2}}{1+r},$$

which is excluded by the assumptions made.

Using simple algebra and the fact that  $p_1^{eu} = \frac{q\delta}{r}$  shows that

$$E [c_{t+2}^* | c_t^*, \gamma_t^{cb} = 0] > c_t^*$$

holds if

$$(1+r) [R(1+r) + \delta(1-q)] [R + \delta(1-q)] > (R+\delta)^2 R^2. \quad (16)$$

For  $p_1^{eu} = 1$ , (16) is equivalent to

$$(1+r)(1+\delta-r) > 0,$$

which is always satisfied. For  $p_1^{eu} = \frac{1}{2}$ , (16) becomes

$$\frac{3r^2}{8} + \frac{r}{4} + \frac{r^2\delta}{2} + \frac{r\delta}{2} + \frac{\delta}{2} > 0,$$

which is obviously satisfied for all positive values of  $r$  and  $\delta$ . Since the expected value of  $c_{t+2}^*$  is continuous in  $\delta$ , it follows that there is a  $\check{\delta} \in \left[\frac{r}{2q}; \frac{r}{q}\right)$  such that

$$E [c_{t+2}^* | c_t^*, \gamma_t^{cb} = 0] > c_t^*$$

for all  $\delta > \check{\delta}$ . Now choose the maximal of the three values  $\hat{\delta}$ ,  $\check{\delta}$ ,  $\bar{\delta}$  and denote it by  $\tilde{\delta}$ . Let  $\tilde{p}_1^{eu} = \frac{q\tilde{\delta}}{r}$ . It follows that  $\tilde{p}_1^{eu} \in \left(\frac{1}{2}; 1\right)$  and that

$$E [c_{t+2}^* | c_t^*] > c_t^*,$$

for  $p_1^{eu} > \tilde{p}_1^{eu}$  and

$$e_t \in \left[ \max \left\{ p_1^{eu}; 1 - p_1^{eu} + \frac{p_1^{eu} r^2}{1 - p_1^{eu}} \right\}; 1 \right).$$

Obviously, then:

$$\begin{aligned} E [e_{t+2}^* | e_t^*, \gamma_t^{cb} = 0] &< e_t^* \\ E [e_{t+2}^* | e_t^*, \gamma_t^{cb} = 1] &< e_t^*, \end{aligned}$$

if  $p_1^{eu} > \tilde{p}_1^{eu}$  and

$$e_t \in \left[ \max \left\{ p_1^{eu}; 1 - p_1^{eu} + \frac{p_1^{eu} r^2}{1 - p_1^{eu}} \right\}; 1 \right)$$

are fulfilled simultaneously. ■

### Proof of proposition 6:

In Lemma 19 in Sciubba (1999, p. 40) it is demonstrated, that a supermartingale bounded between  $[0; 1]$  and starting below 1 cannot converge to its upper boundary with probability 1. The following argument follows closely the proof of Proposition 17 in Sciubba (1999, pp. 40-41). Suppose that  $e_t^*$  converges to 1 with strictly positive probability and denote the event on which this happens by  $\Theta$ . Now consider  $e_t^*$  on the event  $\Theta$  and suppose that on  $\Theta$ ,  $\Pr\{e_t^* \rightarrow 1\} = 1$ . Denote by  $\Theta_0 \subseteq \Theta_1 \subseteq \dots \Theta_t \subseteq \dots \Theta$  the natural filtration of  $\Theta$ . Since  $\Pr\{\Theta\} > 0$ , and since the process of the dividends is i.i.d., the law of large numbers applies and the distribution of dividends on  $\Theta$  coincides with the distribution of the dividends on  $\Omega$ , the set of all possible dividend paths. Especially,  $\Pr\{\delta_t = \delta \mid \Theta_{t-1}\} = \Pr\{\delta_t = \delta\} = q$ . Therefore the process  $e_t^*$  on  $\Theta$  can be described in exactly the same way, as the process  $e_t^*$  on  $\Omega$  and therefore  $e_t^*$  is a supermartingale on  $\Theta$ . But, according to Lemma 19 in Sciubba (1999, p. 40)  $\Pr\{e_t^* \rightarrow 1 \mid \Theta\} \neq 1$ , since  $e_t^*$  is a supermartingale bounded above by 1. Therefore, there is no event with positive probability, on which  $e_t^* \rightarrow 1$  occurs almost surely. Hence,  $\Pr\{e_t^* \rightarrow 1\} = 0$  and the CBDM survive with probability 1. ■

### Proof of proposition 7:

As demonstrated in lemma 10,

$$q \geq \frac{r}{r + (\delta - r)(1 + \delta)^{\beta-1}}$$

implies  $p_\beta^{eu} = 1$  and  $\gamma_t^{eu} = 1$  for all  $t$ . For  $\bar{u} \in (1; (1+r))$ ,  $\gamma_t^{cb} = 0$  for all  $t$ . Hence,  $p_t^* = e_t^* = 1 - c_t^*$ .  $v_t^{cb} = \tilde{v}_t^{cb} = (1+r)$ , whereas

$$\tilde{v}_t = e_t^* + \delta_t + (1 - e_t^*)(1+r) = 1 + \delta_t + c_t^* r$$

for all  $t$ . Hence,  $E [c_{t+1}^* | c_t^*]$  becomes:

$$E [c_{t+1}^* | c_t^*] = \left[ q \frac{(1+r)}{1+c_t^*r+\delta} + (1-q) \frac{(1+r)}{1+c_t^*r} \right] c_t^* \begin{matrix} \geq \\ \leq \end{matrix} c_t^*$$

This simplifies to:

$$(1-c_t^*)r(1+c_t^*r+\delta) - q(1+r)\delta \begin{matrix} \geq \\ \leq \end{matrix} 0. \quad (17)$$

For  $c_t \rightarrow 0$  the l.h.s. becomes:

$$r(1+\delta) - q(1+r)\delta > 0, \text{ iff } q < \frac{(1+\delta)r}{(1+r)\delta}.$$

For  $c_t^* = 1$ , the l.h.s. is negative. Since the left-hand side of (17) is a concave quadratic function of  $c_t^{*2}$ , it follows that for  $q < \frac{(1+\delta)r}{(1+r)\delta}$ , there exists a unique  $\hat{c} \in (0; 1)$ , for which the l.h.s. of (17) is 0. For  $c_t^* > \hat{c}$ ,  $c_t^*$  is a supermartingale and vice versa. Defining  $\hat{e} =: 1 - \hat{c}$  gives the desired result.

If  $q \geq \frac{(1+\delta)r}{(1+r)\delta}$ , then  $c_t^*$  is a supermartingale on  $[0; 1]$ . Hence,  $e_t^*$  is a submartingale on  $[0; 1]$  and converges almost surely. On almost each dividend path,

$$\lim_{t \rightarrow \infty} \frac{e_t^*}{e_{t-1}^*} = \lim_{t \rightarrow \infty} \frac{\frac{(e_t^* + \delta_t)}{e_t^* + \delta_t + (1+r)(1-e_{t-1}^*)}}{\frac{(e_{t-1}^* + \delta_{t-1})}{e_{t-1}^* + \delta_{t-1} + (1+r)(1-e_{t-2}^*)}} = 1.$$

must hold. Let  $e_t^* = e_{t-1}^* = e_{t-2}^*$ , then it follows that:

$$\lim_{t \rightarrow \infty} \frac{(e_t^* + \delta_t)}{e_t^* + \delta_t + (1+r)(1-e_t^*)} = \lim_{t \rightarrow \infty} \frac{(e_t^* + \delta_{t-1})}{e_t^* + \delta_{t-1} + (1+r)(1-e_t^*)}.$$

Since  $\delta_t$  is a stochastic process, this can only hold, if  $e_t^* \rightarrow 1$  almost surely.

Use lemma 11 to show that  $q < \frac{(1+\delta)r}{(1+r)\delta}$  is consistent with  $p_\beta^{eu} = 1$  only for  $\beta \in (0; 1]$ , but not for  $\beta = 0$ . ■

### Proof of proposition 8:

Rewrite conditions (12) and (17) as:

$$-(1+r)^2 c_t^{*2} - (1+r) c_t^* (1+\delta-r) + (1+\delta)r - q\delta(1+r)$$

and

$$-r^2 c_t^{*2} - (1+r) c_t^* (1+\delta-r) + (1+\delta)r - q\delta(1+r),$$

respectively. One easily sees that

$$\begin{aligned} & -(1+r)^2 c_t^{*2} - (1+r) c_t^* (1+\delta-r) + (1+\delta)r - q\delta(1+r) \\ & < -r^2 c_t^{*2} - (1+r) c_t^* (1+\delta-r) + (1+\delta)r - q\delta(1+r) \end{aligned}$$

always holds. Hence, the sole positive root of (12)  $\tilde{e}$  is always greater than the sole positive root

of (17),  $\hat{e}$ . ■

### Proof of proposition 9:

Let first  $\beta = 1$ . For each  $p_1^{eu}$  one can choose  $e_t^*$  to be sufficiently high so that  $p_{t+1} = p_1^{eu}$ . The restrictions on  $\bar{u}$  implies  $\gamma_t^{cb} = 0$  for all  $t$ . Hence,  $\tilde{v}_t^{cb} = (1+r)$ , whereas  $\tilde{v}_t = 1 + \delta_t + r - rp_1^{eu}$ .

Hence,

$$E [c_{t+1}^* | c_t^*] = \left[ q \frac{(1+r)}{1+r-rp_1^{eu}+\delta} + (1-q) \frac{(1+r)}{1+r-rp_1^{eu}} \right] c_t^* \stackrel{\geq}{\leq} c_t^*$$

is equivalent to:

$$-q(1+r)\delta \stackrel{\geq}{\leq} -rp_1^{eu}(1+r-rp_1^{eu}+\delta) \quad (18)$$

and using  $p_1^{eu} = \frac{q\delta}{r}$  one obtains that  $c_t^*$  is a submartingale, if

$$q\delta^2(1-q) > 0,$$

which is obviously true for  $q, \delta > 0$ .<sup>6</sup> Hence,  $e_t^*$  is a supermartingale on  $\left[ \max \left\{ p_1^{eu}; \frac{p_1^{eu}(1+r-rp_1^{eu})}{(1+r)} \right\}; 1 \right]$ .

Now consider  $\beta \in (0; 1]$ . In the limit, when  $c_t^* \rightarrow 0$ , the condition for  $c_t^*$  to be a submartingale is determined by equation (18) and is equivalent to:

$$p_\beta^{eu} \in \left( (1+r+\delta) - \sqrt{(1+r+\delta) - 4q\delta(1+r)}; (1+r+\delta) + \sqrt{(1+r+\delta) - 4q\delta(1+r)} \right).$$

Since  $p_\beta^{eu}$  is increasing in  $\beta$  and since  $p_1^{eu}$  satisfies this condition, it follows that

$$p_\beta^{eu} < (1+r+\delta) + \sqrt{(1+r+\delta) - 4q\delta(1+r)}$$

for all  $\beta \in [0; 1]$ . Moreover, for  $\beta = 0$ ,

$$p_0^{eu} = (1+r+\delta) - \sqrt{(1+r+\delta) - 4q\delta(1+r)},$$

hence, for all  $\beta \in (0; 1]$ , the condition is satisfied and  $c_t^*$  is a submartingale in some surrounding of  $c_t = 0$ , implying that there exists an  $\check{e}(\beta) \in (0; 1)$  such that  $e_t^*$  is a supermartingale on  $[\check{e}(\beta); 1]$ . ■

## Appendix B

In this appendix, I show that the definition of a temporary equilibrium with replicator dynamic as defined in section 3.2.2 is meaningful, in the sense that starting from an equilibrium state of the economy, an equilibrium exists in each period of time  $t \geq 1$ .

**Proposition 12** *Suppose that  $(e_{t-1}; \gamma_{t-1}^{eu}; \gamma_{t-1}^{cb}; p_{t-1})$  is such that*

$$p_{t-1} = e_{t-1}\gamma_{t-1}^{eu}(p_{t-1}) + (1-e_{t-1})\gamma_{t-1}^{cb}(p_{t-1}).$$

*Given such  $(e_{t-1}; \gamma_{t-1}^{eu}; \gamma_{t-1}^{cb}; p_{t-1})$ , at time  $t$ , a temporary equilibrium with replicator dynamic*

<sup>6</sup> For  $q\delta = 0$ ,  $\gamma_t^{eu} = \gamma_t^{cb} = 0$  and  $v_t^{cb} = v_t^{eu}$  for all  $t$ . Hence,  $c_t = \text{const}$ .

exists.

**Proof of proposition 12<sup>7</sup>:**

In order to show the existence of an equilibrium, it is sufficient to demonstrate that the system of equations formulated in conditions (i), (ii), (iii) and (iv) of definition 1 has a solution:

$$e_t^* = \frac{\tilde{v}_t^{*eu}}{\tilde{v}_t^*} e_{t-1} \quad (19)$$

$$\gamma_t^{*eu} = \gamma_t^{eu}(p_t^*) \quad (20)$$

$$\gamma_t^{*cb} = \gamma_t^{cb}(p_t^*)$$

$$\gamma_t^{*eu} e_t^* + \gamma_t^{*cb} (1 - e_t^*) = p_t^* \quad (21)$$

First note that since

$$p_{t-1} = \gamma_{t-1}^{eu} e_{t-1} + \gamma_{t-1}^{cb} (1 - e_{t-1}),$$

(19) can be written as:

$$e_t^* = \frac{\left[ \frac{p_t^* + \delta_t}{p_{t-1}} \gamma_{t-1}^{eu} + (1+r) (1 - \gamma_{t-1}^{eu}) \right] e_{t-1}}{p_t^* + \delta_t + (1+r) (1 - \gamma_{t-1}^{eu} e_{t-1} - \gamma_{t-1}^{cb} (1 - e_{t-1}))} =: f(p_t^*), \quad (22)$$

where  $p_t^*$  is determined from (20) and (21), taking  $e_t$  as given. It follows that  $f(p_t^*)$  depends on  $e_t$  through  $p_t^*$ . Moreover,  $f(p_t^*)$  is continuous in  $p_t^*$  and can only take values between 0 and 1 for all possible values of  $e_t$  and, thus, of  $p_t^*$  between 0 and 1. Therefore, it suffices to show that  $f(p_t^*(e_t))$  has a fixed point in order to prove the existence of an equilibrium. Since for a given  $e_t$ , multiple equilibria can emerge, it will be shown that in each possible case, equilibria can be selected in such a way that a fixed point argument applies.

Consider the function  $\gamma_t^{eu}(p_t)$  determining the share of the income of the EUM invested in asset  $a$ . It is easy to show that  $\gamma_t^{eu}(p_t)$  is a decreasing function of the price such that there are values  $\check{p}_\beta^{eu}$  and  $\hat{p}_\beta^{eu}$  and

$$p_t \leq \check{p}_\beta^{eu}$$

implies  $\gamma_t^{eu}(p_t) = 1$ , whereas

$$p_t \geq \hat{p}_\beta^{eu}$$

implies  $\gamma_t^{eu}(p_t) = 0$ .

It is necessary to consider several cases depending on the values of the parameters  $\check{p}_\beta^{eu}$ ,  $\hat{p}_\beta^{eu}$  and

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<sup>7</sup> The proof is given for  $\beta \in [0; 1)$  and uses the fact that  $\gamma_t^{eu}(p_t)$  is a continuous function. The proof for  $\beta = 1$  follows the same lines, taking into account that  $\gamma_t^{eu}(p_t)$  is an upper-hemi continuous correspondence.

$\bar{u}$ . I will use the notation  $\varpi := u^{-1}(\bar{u})$  to denote the amount of the consumption good necessary to render utility exactly equal to the aspiration level.

### Case 1

Let first  $\check{p}_\beta^{eu} \geq 1$  and let  $\gamma_{t-1}^{cb} = 1$ . It follows that  $p_{t-1}^* = 1$ . Now, if  $1 + \delta_t \geq \varpi$  holds, then  $p_t^*(e_t) = 1$  for all  $e_t$  and is, thus, continuous in  $e_t$ , which guarantees the existence of an equilibrium according to the Brouwer's fixed point theorem, Mas-Collel, Whinston and Green (1995, p. 952).

If  $1 + \delta_t < \varpi$ , then  $\gamma_t^{cb} = 0$  and  $p_t^*(e_t) = e_t$ , which is again a continuous function.

Now let  $\gamma_{t-1}^{cb} \in [\frac{1}{2}; 1]$ . The decision of the CBDM depends on the comparison

$$\gamma_{t-1}^{cb} \left( \frac{p_t^* + \delta_t}{p_{t-1}} \right) + (1 - \gamma_{t-1}^{cb}) (1 + r) \lesseqgtr \varpi.$$

Note that

$$\gamma_t^{cb} = \begin{cases} \gamma_{t-1}^{cb} & \text{if } \gamma_{t-1}^{cb} \left( \frac{\gamma_{t-1}^{cb}(1-e_t)+e_t+\delta_t}{p_{t-1}} \right) + (1 - \gamma_{t-1}^{cb}) (1 + r) \geq \varpi \\ 0 & \text{if } \gamma_{t-1}^{cb} \left( \frac{\gamma_{t-1}^{cb}(1-e_t)+e_t+\delta_t}{p_{t-1}} \right) + (1 - \gamma_{t-1}^{cb}) (1 + r) < \varpi \end{cases}$$

since

$$p_t^* = (1 - e_t) \gamma_t^{*cb} + e_t.$$

Hence, there exists a constant  $\bar{n}$  such that<sup>8</sup>

$$\gamma_t^{cb} = \begin{cases} \gamma_{t-1}^{cb} & \text{if } e_t \geq \bar{n} \\ 0 & \text{if } e_t < \bar{n} \end{cases}$$

and, therefore

$$p_t^*(e_t) = \begin{cases} (1 - e_t) \gamma_{t-1}^{cb} + e_t & \text{if } e_t \geq \bar{n} \\ e_t & \text{if } e_t < \bar{n} \end{cases}.$$

$p_t^*(e_t)$  is continuous but for upward jumps<sup>9</sup> and this property is preserved if  $p_t^*(e_t)$  is subjected to a monotone transformation, see Milgrom and Roberts (1994, p. 445). Since  $\check{p}_\beta^{eu} \geq 1$ ,  $\gamma_t^{*eu} = 1$  holds for each  $t$  and therefore,  $\gamma_t^{*eu} \geq \gamma_t^{*cb}$  for all  $t$ . This implies that  $e_t^*$  is increasing in the price of the risky asset  $p_t$ ,  $\frac{\partial f}{\partial p_t^*} \geq 0$  and hence,  $f(\cdot)$  is also continuous but for upward jumps. Moreover,  $f$  transfers  $[0; 1]$  into  $[0; 1]$ . Theorem 1 of Milgrom and Roberts (1994, p. 446) ascertains that such functions have a fixed point. Therefore, an equilibrium share  $e_t^*$  of EUM exists in this case.

<sup>8</sup> If  $\bar{c} > 1$ , then  $\gamma_t^{cb} = 0$  for all  $e_t$ .

<sup>9</sup> A function  $g : [0; 1] \rightarrow [0; 1]$  is continuous but for upward jumps, if for all  $x' \in [0; 1]$

$$\limsup_{x \uparrow x'} g(x) \leq g(x') \leq \liminf_{x \downarrow x'} g(x)$$

holds, see Milgrom and Roberts (1994, p. 445).

If  $\gamma_{t-1}^{cb} \in [0; \frac{1}{2}]$ , then two cases must be considered: if

$$\gamma_{t-1}^{cb} \left( \frac{1 + \delta_t}{p_{t-1}} \right) + (1 - \gamma_{t-1}^{cb}) (1 + r) \leq \varpi,$$

then

$$\gamma_{t-1}^{cb} \left( \frac{\gamma_{t-1}^{cb} (1 - e_t) + e_t + \delta_t}{p_{t-1}} \right) + (1 - \gamma_{t-1}^{cb}) (1 + r) < \varpi$$

and therefore

$$\gamma_t^{*cb} = 1.$$

Hence,

$$p_t^*(e_t) = 1$$

and a fixed point of  $f(\cdot)$  exists.

If, in contrast

$$\gamma_{t-1}^{cb} \left( \frac{1 + \delta_t}{p_{t-1}} \right) + (1 - \gamma_{t-1}^{cb}) (1 + r) > \varpi$$

and

$$\gamma_{t-1}^{cb} \left( \frac{\gamma_{t-1}^{cb} (1 - e_t) + e_t + \delta_t}{p_{t-1}} \right) + (1 - \gamma_{t-1}^{cb}) (1 + r) \geq \varpi, \quad (23)$$

then

$$p_t(e_t) = \gamma_{t-1}^{cb} (1 - e_t) + e_t.$$

Condition (23) is satisfied if  $e_t \geq \bar{n}$ , where  $\bar{n}$  is some constant determined by the parameters.

Otherwise, if  $e_t < \bar{n}$ ,

$$p_t(e_t) = \gamma_t^{*cb}(e_t) (1 - e_t) + e_t,$$

where  $\gamma_t^{*cb}(e_t)$  is chosen such that condition (23) is satisfied with equality. Furthermore, at  $e_t = \bar{n}$ , all values of  $\gamma_t^{cb}(e_t) \in [\gamma_{t-1}^{cb}; \gamma_t^{*cb}(\bar{n})]$  are optimal and therefore,  $p_t(e_t)$  becomes:

$$p_t(e_t) = \left\{ \begin{array}{ll} \gamma_{t-1}^{cb} (1 - e_t) + e_t & \text{if } e_t > \bar{n} \\ [\gamma_{t-1}^{cb} (1 - e_t) + e_t; \gamma_t^{*cb}(\bar{n}) (1 - e_t) + e_t] & \text{if } e_t = \bar{n} \\ \gamma_t^{*cb}(\bar{n}) (1 - e_t) + e_t & \text{if } e_t < \bar{n} \end{array} \right\}.$$

Obviously,  $p_t(e_t)$  is an upper hemi-continuous, non-empty and convex valued correspondence which maps  $[0; 1]$  into  $[0; 1]$ . Replacing the values of  $p_t(e_t)$  into the continuous  $f(\cdot)$  preserves these properties. Hence,  $f(\cdot)$  has a fixed point.

## Case 2

Now consider the case of  $\hat{p}_\beta^{eu} < 1$  and  $0 < \check{p}_\beta^{eu} < 1$ .

Denote by  $\tilde{k}$  the critical value of the price  $p_t$  such that:

$$\gamma_{t-1}^{cb} \left( \frac{\tilde{k} + \delta_t}{p_{t-1}} \right) + (1 - \gamma_{t-1}^{cb}) (1 + r) = \varpi,$$

hence if  $p_t \geq \tilde{k}$ ,  $\gamma_t^{*cb} = \gamma_{t-1}^{cb}$  will hold. In this case, the equilibrium price is determined according to:

$$p_t^* = e_t \gamma_t^{eu}(p_t^*) + (1 - e_t) \gamma_{t-1}^{cb}.$$

Since  $\gamma_t^{eu}(p_t^*)$  is continuous and differentiable on the intervals  $[0; \check{p}_\beta^{eu})$ ,  $(\check{p}_\beta^{eu}; \hat{p}_\beta^{eu})$  and  $(\hat{p}_\beta^{eu}; 1]$ , one can apply the implicit function theorem and the fact that  $\frac{\partial \gamma_t^{eu}(p_t)}{\partial p_t} \leq 0$ , whenever  $\gamma_t^{eu}(p_t)$  is differentiable to show that  $p_t^*(e_t)$  is continuous. Moreover,  $p_t^*$  is increasing in  $e_t$ . Hence, there is a critical value of  $e_t$ ,  $\tilde{n}$  such that

$$\begin{aligned} p_t(e_t) &\geq \tilde{k} \text{ iff} \\ e_t &\geq \tilde{n} \end{aligned}$$

and  $p_t(e_t)$  is continuous for  $e_t \geq \tilde{n}$ . Now suppose that  $e_t < \tilde{n}$  and let first  $\gamma_{t-1}^{cb} \geq \frac{1}{2}$ . Then

$$p_t^* = e_t \gamma_t^{eu}(p_t^*) \text{ if } e_t < \tilde{n},$$

which also implies a continuous solution  $p_t(e_t)$ . However, a discontinuity arises at  $e_t = \tilde{n}$ .

Note, however that if there is a set of values of  $e_t \in (\tilde{n}; \tilde{n} + \xi)$  such that the solution of

$$p_t = e_t \gamma_t^{eu}(p_t) + (1 - e_t) \gamma_{t-1}^{cb}$$

satisfies

$$p_t > \tilde{k},$$

whereas the solution of

$$p_t = e_t \gamma_t^{eu}(p_t)$$

satisfies

$$p_t < \tilde{k}.$$

Hence, the economy has at least two possible equilibria at  $e_t \in (\tilde{n}; \tilde{n} + \xi)$ , in the one of which the CBDM choose  $\gamma_t^{cb} = \gamma_{t-1}^{cb}(p_t > \tilde{k})$ , whereas in the other  $\gamma_t^{cb} = 0$ , ( $p_t > \tilde{k}$ ). Moreover, for every  $e_t \in (\tilde{n}; \tilde{n} + \xi)$ , it is possible to choose a  $\gamma_t^{cb}(e_t)$  in such a way that the equilibrium price

$$p_t = e_t \gamma_t^{eu}(p_t) + (1 - e_t) \gamma_t^{cb}(e_t)$$

satisfies

$$p_t = \tilde{k}.$$

It follows that on  $e_t \in (\tilde{n}; \tilde{n} + \xi)$ , there is a continuum of equilibria for each possible value of  $e_t$ .

Hence, the correspondence  $p_t(e_t)$  has the form depicted in figure 3 and it is easily seen that it possesses a fixed point, since it is upper hemi-continuous, non-empty and convex valued and

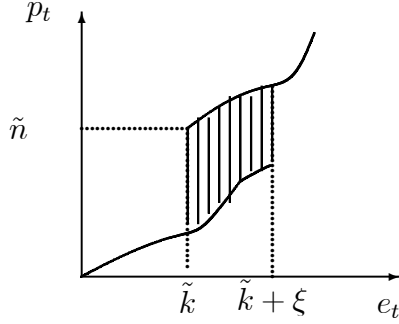


Figure 3

transforms  $[0; 1]$  into  $[0; 1]$ . A monotonic transformation of the correspondence by  $f(\cdot)$  does not change this property, implying that an equilibrium exists.

Now suppose that  $\gamma_{t-1}^{cb} < \frac{1}{2}$ . The decision of the CBDM for  $e_t < \tilde{n}$  depends on whether

$$e_t \gamma_t^{eu}(p_t^*) + (1 - e_t) = p_t^*$$

has a solution  $p_t^*(e_t) > \tilde{k}$ . Let  $\hat{n}$  denote the corresponding critical value of  $e_t$ . If  $p_t^*(e_t) \leq \tilde{k}$ , then

$$\gamma_t^{cb} = 1,$$

if  $p_t^*(e_t) > \tilde{k}$ , then  $\gamma_t^{cb}$  is chosen in such a way that

$$e_t \gamma_t^{eu}(\tilde{k}) + (1 - e_t) \gamma_t^{cb} = \tilde{k}.$$

Hence,

$$p_t^*(e_t) = \left\{ \begin{array}{ll} e_t \gamma_t^{eu}(p_t^*) + (1 - e_t) \gamma_{t-1}^{cb} & \text{if } e_t \geq \tilde{n} \\ \tilde{k} & \text{if } e_t \in [\tilde{n}; \hat{n}] \\ e_t \gamma_t^{eu}(p_t^*) + (1 - e_t) & \text{if } e_t > \hat{n}. \end{array} \right\}.$$

This function is continuous in  $e_t$  and so is  $f(\cdot)$ , hence, an equilibrium exists.

### Case 3

Now consider the case in which  $\hat{p}_\beta^{eu} \leq 0$ . In this case,  $\gamma_t^{*eu} = 0$  and the price becomes:

$$p_t(e_t) = \begin{cases} (1 - e_t) \gamma_{t-1}^{cb} & \text{if } \gamma_{t-1}^{cb} (1 - e_t) > \bar{n} \\ 0 & \text{if } \gamma_{t-1}^{cb} (1 - e_t) \leq \bar{n} \end{cases}.$$

This function is continuous but for downward jumps. Moreover, since  $\gamma_{t-1}^{eu} = 0$  must hold, it follows that

$$\gamma_{t-1}^{cb} \geq \gamma_{t-1}^{eu}$$

and, therefore,  $f(\cdot)$  is decreasing in  $p_t$ . Hence,  $f(e_t)$  is continuous but for upward jumps and possesses a fixed point, which insures the existence of an equilibrium. ■

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