Endogenous Risk Sharing Groups and Dynamic Selection in Mechanism Design*

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Abstract

We create a dynamic theory of endogenous risk sharing groups, with good internal information, and their coexistence with relative performance, individualistic regimes, which are informationaly more opaque. Inequality and organizational form are determined simultaneously. Numerical techniques and succinct re-formulations of mechanism design problems with suitable choice of promised utilities allow the computation of a stochastic steady state and its transitions. Regions of low inequality and moderate to high wealth (utility promises) produce the relative performance regime, while regions of extremely high inequality and low wealth produce the risk sharing group regime. If there is a cost to prevent coalitions, risk sharing groups emerge at high wealth levels also. Transition from the relative performance regime to the group regime tend to occur when rewards to observed outputs exacerbate inequality, while transitions from the group regime to the relative performance regime tend to

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come with a decrease in utility promises. Some regions of inequality and wealth deliver long term persistence of organization form and inequality, while other regions deliver high levels of volatility.

INTRODUCTION

Risk sharing groups would seem to have many advantages. Individuals are supposed to have good information about one another efforts, to coordinate their efforts for good production decisions, and to share the risk of idiosyncratic income fluctuations. Given a contract or arrangement of such a network with outsiders, it would always be better to coordinate in this way. Further, with no internal incentive constraints, a relatively high level of within group inequality can be tolerated. A poor agent can be required to work hard with little consumption compensation. There is one disadvantage of groups, however. Members can collude against the outsider and coordinate actions to undercut presumed individual incentive schedules. The problem is that the shared common information in the group is private to the group and not known to the outsider.

An alternative to a risk sharing group is a more individualistic, relative performance regime. In this regime, there is quite limited private information on individual efforts and limited sharing of risk. Neither other working agents nor the outsider know an individual’s effort level. Only outputs are public. If outputs are correlated, then an agent will be punished when his output is observed to be low while the other’s is high. That is, the inference would be that the low output agent might have been shirking, and so punishment is given in order to help overcome the moral hazard problem. These high powered individual incentives can generate an overall surplus which is higher than the one obtained with groups. One gets this result even allowing for a cost of the consumption good that prevents side payments and the secret formation of a collusive group. There is a limitation, however. Very low utilities in which an
agent might be required to work hard and receive little consumption are not incentive compatible.

Earlier research as in Holmstrom and Milgrom (1990), Ramakrishman and Thakor (1991), and Itoh (1993) focus on a comparison of the two regimes. High correlation is a force for relative performance, for example. In turn, Prescott and Townsend (2002) have shown that when preferences are not Gorman aggregable, as when individuals work on their own project exclusively, then risk sharing groups with good internal information can constitute a Pareto optimal arrangement when either inequality of the initially promised utilities is high or the average level of these promises is low. Likewise, the relative performance regime, with more limited information overall and imperfect risk sharing, Pareto dominates when there less inequality, along a 45% line in the map of utility promises for the two working agents, and moderate to high levels of utility overall.

Networks in this sense are endogenous but linked to observables such as overall wealth and inequality. This is a theory of networks which is testable. Ahlin and Townsend (2006) associate a network in the theory with a joint liability group in data, that is, two (or more) individuals who have cosigned loaned and are jointly responsible for paying off the lender. Thai villages which are relatively poor and which have more variety in the within-village wealth distribution are more likely to have such groups than middle-income villages with little inequality.

There in fact considerable interest in the development literature on risk sharing and its extent. Townsend (1994) tested for risk sharing as if entire Indian villages were likely risk sharing groups. Risk sharing was indeed pervasive but there were exceptions. Fafchamps and Lund (1998) argue that it is the network of family related households in Philippine villages that is really the appropriate group. Yet in the Thai village data, Chiappori, Schulholfer-Wohl, Samphantharak and Townsend (2006) are finding that risk sharing varies both across villages and across kinship groups. It is
hard to find an initial hypothesis which is uniformly successful.

There is a parallel literature in corporate finance. Samphantharak (2002) meticulously constructs family-related firms through relationships and cross share holdings, then tests for whether investment is sensitive to cash flow for firms individually and also for firms in such groups. Some groups are more likely to act as the theory predicts: those with a large number of members, those with firms not listed on the stock exchange, and those with a financial intermediary in the group. A related literature on tunneling of Bertrand, Mehta and Mullainathan (2002) examines how shocks make their way through a network of related firms, though their interpretation is that dominant partner is stealing from subsidiaries.

If one tracks over time the membership of village or corporate groups, one turns up other anomalies. The quite well defined cropping groups surveyed by Townsend and Mueller (1998) show considerable exit, entry, and churning in membership at the end of the season. Likewise, the well know Chibol groups of South Korea have considerable turnover. Few of the theories described above deal with these phenomena. Indeed many authors have expressed discomfort that maintained hypotheses about what should or should not be a risk sharing group are harder to maintain when one scrutinizes the data.

The theory here, when made dynamic, can begin to address these issues. We find that the level of overall wealth, inequality, and organization form are all endogenous and can evolve in interesting ways over time. Suppose a local economy starts with high utilities groups and some inequality. Then if one of the agents who is working hard has an unfortunate low output, the entire group is punished. The lower levels of wealth albeit at similar inequality moves the local economy to a relative performance regime. Likewise, starting at equal and low utilities, success can move economy upward to relative performance. The transition from relative performance and equality occur when outputs are different across the individuals, raising inequality, and justifying
the adoption of the group regimes.

Under some parameter configurations there is a nontrivial steady state distribution of regimes, with some local economies having a high degree of equality and intensive competition while other local economies have an extreme degree of inequality and the risk sharing groups. There are intermediate possibilities but these receive extremely low mass. Likewise in some regions there is great persistence of the contemporary regime. But in other regions there is organizational volatility and the probability of back and forth transitions from one regime to another is quite high.

Various other authors have taken up the challenge of producing a theory of groups or networks, that is, making groups or group size endogenous. Kranton and Bramoullé (2005) model networks as the costly formation of bilateral links. In their equilibrium there is an "externality" which is not internalized. Their setup allows heterogenous treatment of otherwise identical individuals. In a dynamic version with exogenous pairwise meetings probabilities, the size of a dynamic network grows and shrinks over time with agents at the fringe at risk of being cut-off entirely. Related is Jackson and Watts (2005) who focus on self-improving and myopic paths. In contrast, Genicot and Ray (2005) consider a static model with a core notion of outcomes, that is, the possibility that subgroups of individuals may destabilize insurance arrangements among the larger group. Self-enforcing risk-sharing agreements are robust not only to single-person deviations but also to potential deviations by subgroups. However, such deviations must be credible, in the sense that the subgroup must pass exactly the same test that is applied to the entire group. Stable groups have bounded size and the degree of risk sharing is non-monotonic in risk.

Also delivering clusters but with varying internal risk sharing is the paper of Murgai, Winters, Sadoulet and deJanvry (2002). High association costs combined with low extraction costs lead to clusters with full insurance, while low association costs with high extraction costs will lead to community-level partial insurance. Empirical results
from data on water exchanges among households along irrigation canals in Pakistan are used to support this proposition.

Here, in our paper, we take a dynamic, foresighted, mechanism design approach with information as an explicit impediment to trade. This connects us to a mechanism design literature with its focus on information and theories of inside and outside monies. Cavalcanti and Wallace (1999) study implementable allocations in a random matching model in which some people have publicly known histories (banker) and others have private histories (non-bankers). Bankers can issue bank notes, to be compared with outside money. Though outside money dominates, Mills (2000) shows that information lags for bankers can deliver the combined use of both kinds of monies. For us groups have good internal information though the outsider does not, while individuals in relative performance have limited information as with the principal. Our results show that diverse information structures and organization forms can coexist and evolve in interchangeable forms.

We characterize the solution to the multiagent moral hazard model problem as a linear program. At this point, a curse of dimensionality emerges - a common problem in the linear programing solutions of contract problems, exacerbated here by the presence of more than one agent. We develop a formulation that makes computational implementation less demanding. The basic idea of this reformulation is to solve the moral hazard problem using a variable that summarizes utilities from consumption and promises to the future, instead of using consumption and promises directly. This formulation can be useful for multiagent dynamic moral hazard problems in general.

The paper is organized as follows. In section 2 we present the model with variables in a finite grid. This is basically a dynamic extension of the formulation of Prescott and Townsend (2002). In section 3 we present the model in a continuous setting, and prove some propositions about the dynamics of the model. In particular, we show that incentives for efforts make the dynamics depend on outputs, and that
the presence of outside options for agents and the outsider contractor can make the feasible set compact and rule out degenerate steady states. Section 4 presents and discusses numerical results obtained. The numerical procedure adopted, including some computationally convenient reformulations of the problem, is discussed in details in the appendix.

THE MODEL

Environment:

A local economy has three individuals, two agents living for multiple periods, with preferences as discounted expected utilities over consumption and effort, and a principal, whose objective function is the present value of the surplus of production over the agents’ consumption. The utility of agent \(i\) at period \(t\) is \(w^t_i \equiv E\{\sum_{s=t}^{T} \beta^{s-t}[U(c^s_i) + V(e^s_i)]\}\), where \(c^t_i\) and \(e^t_i\) are, respectively, the consumption and the effort of agent \(i\) at period \(t\), and \(\beta\) is a subjective discount factor. \(U\) is strictly concave and increasing and \(\lim_{c \to 0} U'(c) = \infty\). The function \(V(e)\) is decreasing with \(e\), meaning that agents prefer to make low effort. The production technology is characterized by \(p(q^t_1, q^t_2 \mid e^t_1, e^t_2) > 0\), a probability distribution of outputs at \(t\) that depends on the effort levels of both individuals within the period. This allows correlated outputs. The present value of the principal’s surplus flow at \(t\) is given by \(S^t \equiv \sum_{s=t}^{T} (1 + r)^{s-t} [q^s_1 + q^s_2 - c^s_1 - c^s_2]\), where \(r\) is an exogenous interest rate. So, this is a small open economy. We assume that \(\frac{1}{1+r} = \beta\). For expositional convenience we use the notation \(x^t \equiv (x^t_1, x^t_2)\) for the variables \(c^t_i, e^t_i\) and \(q^t_i\). The sets of possible individual consumption, effort and output for each individual in each period are denoted respectively by \(C\), \(E\) and \(Q\). Thus consumption, effort and output pairs are, respectively, in \(C^2\), \(E^2\) and \(Q^2\).

We study allocations that are efficient conditional on the following moral hazard
problem: the principal does not observe efforts of agents, although it is possible to allow one agent to observe the effort of the other. This implies that the decision of how much effort is done is ultimately taken by the agents. The decision process depends on how the agents are organized. Two regimes are available: a group regime in which individuals are allowed to communicate and thus to collude and jointly define levels of effort and share risks, and a relative performance regime in which individuals are not allowed to communicate and collude. Thus, in the relative performance regime, each individual decide how much effort to do. In the group regime this decision is taken jointly, according to Pareto weights inside the group. We assume that the implementation of the relative performance regime has a fixed cost of $k$ that represents the cost of avoiding collusion and side payments among individuals. The type of collective organization at a given date is characterized by the regime (relative performance or groups), and in the case of groups, by the Pareto weights. We define $O$ as the set of types of organization available. The elements of $O$ characterize the regime, and for the groups regime, the Pareto weight inside groups. We denote the set of possible Pareto weights by $M$.

**The Mechanism Design Program**

We formulate the Pareto problem as a maximization of the expected surplus of an outsider contractor, that we call the principal, conditional on promised utility levels for the two agents. The Mechanism design problem defines the type of organization of the individuals, and their vectors of consumption, effort and output, conditional on a set of incentive and technological constraints, and a vector of initial expected utilities. We follow Phelan and Townsend (1991) and solve the problem sequentially. At each period the distribution of consumption, effort, organization and future expected utilities are defined by a principal-agent problem. We require that the arrangements do not allow gains for all individuals resulting from renegotiation between the principal
and the agents.

At any period, \( t \), the principal chooses a probability distribution on the elements of \( \Gamma_t \equiv C^2 \times Q^2 \times E^2 \times O \times W_t^{t+1} \), which are vectors expressing consumption, output and effort vectors, the type of organization that individuals are part of at \( t \), and the pair of expected future utilities. Notice that we allow the set of expected future utilities, or future promises, \( W_{t+1} \) to depend on time: in a finite periods problem, the set of possible promises may depend on how many periods are left until the final period. The distribution of effort, consumption, output and the type of organization in future periods is implied by the choices of promised future utilities \( w_1^{t+1} \) and \( w_2^{t+1} \). For notational convenience, we sometimes omit the index \( t \) on variables and refer to the initial pair of expected utilities as \( w \) and the pair of promised utilities as \( w' \).

![Timeline 1](image)

Figure 1

We assume that \( \Gamma_t \) is a finite set, that is, \( C, Q, E, O \) and \( W_{t+1} \) are all finite sets. For efforts and outputs, a finite grid has a natural interpretation: \( E \) has two elements \( e_h \) and \( e_l \), denoting high and low effort respectively, and \( Q \) has two elements, \( q_h \) and \( q_l \) denoting respectively success and failure. Other objects, such as consumption and...
utility promises are more natural on a continuum, that is approximated here with a fine grid. We may also assume that the set of possible promised utilities $W$ has an arbitrary lower bound. This is the case when individuals have an outside option. For example they could be entirely on their own. The utility from production in autarky, or possibly a lower value, if there is some punishment or social stigma from abandoning the link with the principal and the other agent, could be lower bounds on the possible promises. Similarly, we assume that the principal have the possibility to walk away to some outside option, so arrangements are subject to a lower bound in the level of surplus to the principal.

The sequence of events in period $t$ is presented in figure 1. First, is as if the principal assigns individuals to a type of organization: agents will be assigned to the groups or the relative performance regime. When they are assigned to the groups regime, the Pareto weights inside the groups is defined. Then, the agents or the group decide their amount of effort $(e_{1t}, e_{2t})$. Following the employment of effort, an output pair $(q_{1t}, q_{2t})$ is obtained and observed by the principal. Finally, conditional on the output, a consumption pair $(c_{1t}, c_{2t})$ and a pair of promised expected utilities $(w^{t+1}_{1t}, w^{t+1}_{2t})$ is defined.

This sequence is determined by a probability function $\pi^t : \Gamma \to [0, 1]$, that maximizes the surplus of the principal at $t$. This choice is subject to a set of incentive and technological constraints. Throughout the remainder of the paper, we call $(w^t_{1t}, w^t_{2t})$ the ex-ante utility pair at $t$, and $(w^{t+1}_{1t}, w^{t+1}_{2t})$ the set of promised (future) utilities in $t$.

**Group Constraints** Inside a group, each agent can enforce actions defined in an agreement (or contingent plan) made with the principal. But the members of a group could collude and take actions and consumption that are not recommended in the initial plan if this can benefit one agent without hurting the other. This is
anticipated in the original plan, which is defined so that the agents have no possibility of such Pareto improvements given assigned Pareto weights $\mu$. Therefore, the plan must be such that the actions taken maximize the sum of the $\mu$ weighted agents utilities. Similarly, the plan must be such that there is no gain from renegotiations between the group of agents and the principal. This implies that the plan chosen must be surplus maximizer among all feasible plans that produce at least the same sum of $\mu$ weighted utilities.

Part of the domain of the probability function $\pi^t$ characterizes the distribution of outcomes under the groups regime. These elements are subject to a set of constraints that are typical of the group regimes. We denote $\pi^t_g(c, q, e, w', \mu)$ the joint probability that the groups regime is chosen, and $c, q, e, w'$ and $\mu$ are respectively the values of consumption, output, effort, promises and Pareto weights inside groups. The incentive constraints determining that it will be optimal for a group with Pareto weight $\mu$ to do the recommended level of effort is:

$$\sum_{c,q,w'} \pi^t_g(c, q, e, w' \mid \mu) \sum_i \mu_i [U(c_i) + V(e_i) + \beta w'_i] \geq \sum_{c,q,w'} \pi^t_g(c, q, e, w' \mid \mu) \left( \frac{p(q \mid e')}{p(q \mid e)} \right) \sum_i \mu_i [U(c) + V(e_i) + \beta w'_i],$$

$\forall e, e'$, where $\pi^t_g(c, q, e, w' \mid \mu)$ is the probability of $c, q, e$ and $w'$ conditional on the adoption of the group regime with Pareto weights $\mu$. This is equal to $\pi^t_g(c, q, e, w', \mu)$ divided by the probability that groups with Pareto weights $\mu$ are chosen, $\pi(\mu)$. In the constraints that follow, $\pi^t_g(c, q, e, w', \mu)$ can be obtained form $\pi^t_g(c, q, e, w' \mid \mu)$ by multiplying it by $\pi(\mu)$. Given $\mu$, the distribution of output must be consistent with the technology $p$. This condition is implied by the technological constraints:

$$\sum_{c,w'} \pi^t_g(c, q, e, w' \mid \mu) = p(q \mid e') \sum_{c,q,w'} \pi^t_g(c, q, e, w' \mid \mu), \forall q, e, \mu$$

The set of feasible arrangements that are preferred by a group with internal Pareto weight $\mu$ to the plan implied by $\pi^t$ is:
\[ \Psi_g(\mu \mid \pi^t_g) \equiv \{ \pi(c, q, e, w') : \pi \text{ is positive, sums to 1 and satisfies (1) and (2)} \} \]

for any \( e \) given \( \mu \) and \( \sum_{c,q,w',e} \pi(c, q, e, w') \mu \sum_i [U(c_i) + V(e_i) + \beta w'_i] > W_\mu(\pi^t_g) \}, \]

where \( W_\mu(\pi^t_g) = \sum_{c,q,w'} \pi^t_g(c, q, e, w' \mid \mu) \sum_i \mu_i [U(c_i) + V(e_i) + \beta w'_i] \)

In the definition of the set \( \Psi_g \), the notation \( w_1 > w_2 \) means that all elements in \( w_1 \) are not smaller than the corresponding element in \( w_2 \), and at least one element in \( w_1 \) is greater than the correspondent element of \( w_2 \). The condition determining that there is no possibility of gains from renegotiation between the group and the principal is that for any \( \pi \) in \( \Psi_g(\mu \mid \pi^t_g) \),

\[ \sum_{c,q,w'} \pi^t_g(c, q, e, w' \mid \mu) \bigg[ q_1 + q_2 - k - c_1 - c_2 + \beta S^{t+1}(w') \bigg] > \sum_{c,q,e,w'} \pi(c, q, e, w') \bigg[ q_1 + q_2 - k - c_1 - c_2 + \beta S^{t+1}(w') \bigg] \tag{3} \]

Notice that this implies that the surplus function is strictly decreasing: higher utility for both agents cannot produce lower surplus. Notice also that (3) implies that inside a group there is risk sharing on aggregate consumption. The consumption of individuals depend only on the aggregate consumption and on the Pareto weights inside the group. Specifically, agent levels of consumption at any period \( t \) must satisfy \( c^t_i = c_i(\mu, c_a) \), where the function \( c_i(\mu, c_a) \) is defined by the following subproblem generating the groups risk sharing rule:

\[ (c_1(\mu, c_a), c_2(\mu, c_a)) = \arg \max_{(c_1, c_2) \in C^2} \mu_1 U(c_1) + \mu_2 U(c_2), \]

s.t.

\[ c_1 + c_2 = c_a, \]

where, again, \( c_a \) is the aggregate consumption (with values on \( C + C = 2C \)). This property gives us the interpretation of a group as a risk sharing network.

**Relative Performance Constraints** In the relative performance regime, agents are prevented from communicating to coordinate efforts and make side payments.
that could possibly mitigate incentives. Since outputs are correlated, differences in output among agents may be an indication of different levels of effort. Comparative performance can be used as an incentive tool, and it may be useful for incentive schemes that individuals consume what they are assigned and are prevented from coordinating actions.

We use the notation \( \pi_r^t(c, q, e, w') \) to define the joint probability that agents are assigned to the relative performance regime, \( r \), and the vectors of consumption, outputs, efforts and promised utilities in period \( t \) are respectively \( c, q, e \) and \( w' \). The incentive constraints for the relative performance regime in \( t \) are:

\[
\sum_{c,q,e_1,e_2,w'} \pi_r^t(c, q, e_1, e_2, w') [U(c_1) + V(e_1) + \beta w_1'] \\
\geq \sum_{c,q,e_1,e_2,w'} \pi_r^t(c, q, e_1, e_2, w') \frac{p(q | \hat{e}_1, e_2)}{p(q | e_1, e_2)} [U(c_1) + V(\hat{e}_1) + \beta w_1'],
\]

\( \forall e_1, \hat{e}_1 \in E \), and

\[
\sum_{c,q,e_1,e_2,w'} \pi_r^t(c, q, e_1, e_2, w') [U(c_2) + V(e_2) + \beta w_2'] \\
\geq \sum_{c,q,e_1,e_2,w'} \pi_r^t(c, q, e_1, e_2, w') \frac{p(q | e_1, \hat{e}_2)}{p(q | e_1, e_2)} [U(c_2) + V(\hat{e}_2) + \beta w_2'],
\]

\( \forall e_2, \hat{e}_2 \in E \). Note that there is one constraint per agent. These constraints determine that agents have no incentive to deviate from an effort level defined in the arrangement with the principal. The technological constraints defining that the technology is given by \( p \) are:

\[
\sum_{c,w'} \pi_r^t(c, \tilde{q}, \tilde{e}, w') = p(\tilde{q} | \tilde{e}) \sum_{c,q,w'} \pi_r^t(c, q, \tilde{e}, w'), \forall \tilde{q}, \tilde{e}.
\]

As in the group regime, we impose the constraint that there is no possibility of improvements for all individuals resulting from renegotiation. The set of feasible
relative performance arrangements that provide at least the same level of utility for both agents as \( \pi_t \) and is preferred by at least one of the agents is:

\[
\Psi_r(\pi_t) \equiv \{ \pi(c, q, e, w') : \pi \text{ is positive, sums to 1 and satisfies (4) and (5) and (6)} \text{ for any } e, \text{ and } \sum_{c,q,w',e} \pi^t_r(c, q, e, w') [U(c_i) + V(e_i) + \beta w'_i] > w^t_{i,r}(\pi^t_r) \text{ for } i=1,2 \},
\]

where \( \pi^t_r(c, q, e, w' | r) \) is the probability of \( c, q, e \) and \( w' \) conditional on the adoption of the relative performance regime.

In order for \( \pi^t_r \) to be such that no improvement can be obtained from renegotiation, it must be the case that it maximizes the surplus of the principal in \( \Psi_r(\pi^t_r) \). Therefore, it must be the case that:

\[
\begin{align*}
&\sum_{c,q,e,w} \pi^t_r(c, q, e, w) [q_1 + q_2 - c_1 - c_2 + \beta S^{t+1}(w')] \\
&> \sum_{c,q,e,w'} \pi(c, q, e, w') [q_1 + q_2 - c_1 - c_2 + \beta S^{t+1}(w')] 
\end{align*}
\]

for any \( \pi \) in \( \Psi_r(\pi^t_r) \).

**The Program** The principal-agent problem at \( t \) is:

**Program1**

\[
S^t(w_1, w_2) \equiv \max_{\pi^t_r, \pi^t_g} \sum_{c,q,e,w'} \pi^t_r(c, q, e, w') [q_1 + q_2 - c_1 - c_2 + \beta S^{t+1}(w')] \\
+ \sum_{c,q,e,w',\mu} \pi^t_g(c_a, q, e, w', \mu) [q_1 + q_2 - c_1 - c_2 + \beta S^{t+1}(w')] 
\]

subject to the promise keeping constraint

\[
\begin{align*}
&\sum_{c,q,e,w'} \pi^t_r(c, q, e, w') [U(c_i) + V(e_i) + \beta w'_i] \\
&+ \sum_{c_a,q,e,w',\mu} \pi^t_g(c_a, q, e, w', \mu) [U(c_i(\mu, c_a)) + V(e_i) + \beta w'_i] = w_i,
\end{align*}
\]
for $i = 1, 2$

$$\sum_{c,q,e,w'} \pi^t_r(c, q, e, w') + \sum_{c_a,q,e,w',\mu} \pi^t_g(c_a, q, e, w', \mu) = 1, \quad (10)$$

$$\pi^t_r(c, q, e, w') \geq 0.$$

$\pi^t_r$ satisfies (1), (2) and (3) for any $\mu$ in $M$ and $\pi^t_r$ satisfies (4), (5), (6) and (7).

Notice that whenever the regime is relative performance a cost of $k$ is subtracted from the surplus. Again, this is the cost to avoid collusion and communication among agents.

Program 1 defines the surplus function at $t$, $S^t$, given the surplus function at $t + 1$, $S^{t+1}$. This characterizes a functional equation $S^t = F(S^{t+1})$. As in Phelan and Townsend (1992), a fixed point in this functional equation, $S = F(S)$ is a solution for the infinite periods problem. The solution of program 1 starting from this function reveals the choice of regime in an infinite periods problem conditional on initial promises. This choice will depend on the surplus that each regime is capable of obtaining in contracts that generate utilities equal to the initial promises.

The Space of Promises and the Dynamics of the Model

The space of possible promises will depend on outside options and also on incentive compatibility. Under relative performance, each individual needs incentives for effort. Suppose consumption were to zero to some agent. His effort would then be the least possible, and his utility would be $V(e_l)/(1 - \beta)$. Any relative performance contracts must offer at least this utility level. However, if there is no outside option for agents, the group regimes is capable of sustaining arrangements with utility as low as $V(e_h)/(1 - \beta)$ for (at most) one of the agents. That is, if the regime is groups it is possible to impose high effort, and zero consumption when $\mu_i = 0$, which provides
utility equal to $V(e_h)/(1 - \beta) < V(e_l)/(1 - \beta)$.

Some areas of the utilities map are not achievable by any regime. For instance, extremely low utility for both individuals cannot be obtained. This would require high effort by both individuals, and it is impossible to enforce that without significant rewards in good states. This is illustrated for some numerical examples in the numerical section below.

The solution to program 1 determines the dynamic behavior for utility promises, and thus of output, consumption and effort levels. As in Rogerson (1985), optimal incentives make the realization of observable outputs impact on promises for the future and thus on the dynamics. These dynamics in turn determines the distribution of consumption, inequality and organizational form in the long run. The literature on dynamic incentive problems shows a tendency for immiseration with a growing fraction of the population converging to absolute misery (as in Thomas and Worrall (1990)), and possibly a vanishing fraction of the population converging to extreme prosperity (Atkeson and Lucas (1992), Phelan (1998)). In order to rule out degenerate steady states and absorbent points, we impose a lower bound on utility promises, denoted by $w$ and on Surplus denoted by $\mathcal{S}$. These lower bounds could be interpreted as outside options for the agents and the principal, respectively. Given these boundaries, the set of promises is constrained to a compact subset of $W^2$. We call it $\widetilde{W} \subseteq R^2$. The following propositions show how imposing these boundaries can rule out the possibility of degenerate steady states or absorbent points. Following the related literature, and in order to make sure that the results are not merely an artifact from discretization, we develop these results for the continuous case, which is a straightforward extension of the discrete formulation above where the sets $C$, and $W$ are assumed to be continuous respectively in $\mathbb{R}_+$ and $\widetilde{W}$. For simplicity, we assume that the set of available Pareto weights inside a group is finite, so that $O$ is finite.
Characterization of the Problem for Continuous of \( w \) and \( c \). In order to characterize the model with promises and consumption on a continuum, it is useful to remind the reader about the timing of events. First, there is the choice of organization, second choice of effort, third realization of output, fourth consumption and finally promises. Both consumption and promises may depend on outputs. From the concavity of \( S_{t+1} \) and \( U(c) \), if there is randomization of consumption and promises after outputs are realized, it is possible to increase surplus keeping the same level of individual utilities conditional on efforts, thus keeping incentives for efforts. So, randomization of consumption and promise for a given realization organizational form (\( \mu \) and regime), efforts and output is suboptimal. Therefore, for each initial promises, \( w \), consumption and promises for the future can be written as a functions of organizational form, effort and output, denoted \( c(e, q, o \mid w) \) and \( w'(e, q, o \mid w) \), respectively. Here, \( o \in O \) represents the organizational form, and we use the notation \( o = \mu \) when the regime is groups with internal Pareto weights \( \mu \), and \( o = r \), when the regime is relative performance.

The other choice variables of the problem are the probabilities of organizational form, and the probabilities of efforts for each organizational form, denoted by, \( \Pr(o \mid w) \) and \( \Pr(e \mid o, w) \), respectively. These decisions are summarized by the joint probability distribution:

\[
\Pr(e, q, o \mid w) = \Pr(o \mid w) \cdot \Pr(e \mid o, w) \cdot p(q \mid e),
\]

where \( p(q \mid e) \) is determined by the technology. For expositional convenience, we characterize the problem in terms of the joint distribution \( \Pr(o, e, q \mid w) \), but for this to be consistent with (11), the choice of \( \Pr(o, e, q \mid w) \) is simply:

\[
\frac{\Pr(e, q, o \mid w)}{\Pr(o \mid w)} = \Pr(e, q \mid o, w) = \Pr(e \mid o, w) \cdot p(q \mid e)
\]

\( \forall (e, q, o) \in E^2 \times Q^2 \times O \equiv Y \). The choice variable is thus \( \{\Pr(e, q, o \mid w), c(e, q, o \mid w)\} \).
$w), w'(e, q, o \mid w) \in X$. Where $X$ is the set of functions from $Y$ to $[0, 1] \times C^2 \times W^2$.

Any choice $x = \{\text{Pr}(e, q, o), c(e, q, o), w'(e, q, o)\}$ must be subject to a set of constraints.

**Group constraints:** The incentive constraints determining that it will be optimal for a group with Pareto weight $\mu$ to do the recommended level of effort is:

$$
\sum_q \text{Pr}(e, q \mid \mu, w) \sum_i \mu_i [U(c_i(e, q, \mu \mid w) + V(e_i) + \beta w_i'(e, q, \mu \mid w))] \geq \sum_q \text{Pr}(e, q \mid \mu, w) \frac{p(q \mid \tilde{e})}{p(q \mid e)} \sum_i \mu_i [U(c(e, q, \mu \mid w)) + V(c_i) + \beta w'_i(e, q, \mu \mid w)], \forall e, \tilde{e}, \mu.
$$

The set of feasible arrangements that are preferred by a group with internal Pareto weight $\mu$ to the plan implied by $x \equiv \{\text{Pr}(y), c(y), w'(y)\}$ for any $y$ in $Y$.

$\Psi_g(\mu \mid x) \equiv \{\tilde{x} : \tilde{x} = \{\text{Pr}(e, q), \tilde{c}(e, q), \tilde{w}'(e, q)\}$ satisfies (12) and (13) for any $e$ given $\mu$ and $\sum_{e,q} \text{Pr}(e, q) \sum_i \mu_i [U(c_i(e, q)) + V(e_i) + \beta \tilde{w}'_i(e, q)] > W_\mu(x)$, where $W_\mu(x) = \sum_{e,q} \text{Pr}(e, q \mid \mu) \sum_i \mu_i [U(c_i(e, q, \mu)) + V(e_i) + \beta w'_i(e, q, \mu)]$

The condition determining that there is no possibility of gains from renegotiation between the group and the principal is that for any $\tilde{x}$ in $\Psi_g(\mu \mid x)$,

$$
\sum_{e,q} \text{Pr}(e, q) [\tilde{q} + \tilde{q}_2 - \tilde{c}_1 - \tilde{c}_2 + \beta S^{t+1}(\tilde{w}'(y))]
< \sum_{e,q,\mu} \text{Pr}(e, q \mid \mu, w) [\tilde{q}_1(e, q, \mu) + \tilde{q}_2(e, q, \mu) - \tilde{c}_1(e, q, \mu) - \tilde{c}_2(e, q, \mu) + \beta S^{t+1}(\tilde{w}'(e, q, \mu))]
$$

**Relative Performance Constraints:** The incentive constraints for the relative performance regime are:

$$
\sum_{q, e_2} \text{Pr}((e_1, e_2, q \mid r) [U(c_1(e_1, e_2, q, r) + V(e_1) + \beta w'_1(q, e_1, e_2, r))] 
\geq \sum_{e_1, e_2} \text{Pr}((e_1, e_2, q \mid r) \frac{p(q \mid \tilde{e}_1, e_2)}{p(q \mid e_1, e_2)} [U(c_1(e_1, e_2, q, r)) + V(c_1) + \beta w'_1(e_1, e_2, q, r)],
$$
\[ \forall e_1, \hat{e}_1 \in E, \text{ and} \]

\[ \sum_{q, e_1} \Pr((e_1, e_2, q | r) [U(c_1(e_1, e_2, q, r)) + V(e_2) + \beta w'_2(e_1, e_2, q, r)]) \geq \sum_{q, e_1} \Pr((e_1, e_2, q | r) \frac{p(q | \hat{e}_1, e_2)}{p(q | e_1, e_2)} [U(c_1(e_1, e_2, q, r)) + V(\hat{e}_1) + \beta w'_2(e_1, e_2, q, r)]) , \]

As in the group regime, we impose the constraint that there is no possibility of improvements for one individuals that does not hurt the others resulting from renegotiation. Given an arrangement \( x \), the set of feasible relative performance arrangements that provide at least the same level of utility for both agents and is preferred to \( x \) by at least one agent is:

\[ \Psi_r(x) \equiv \{ \hat{x} : \hat{x} = \{ \hat{\Pr}(e, q), \hat{c}(e, q), \hat{w}'(e, q) \} \} , (12) \text{ and } (15) \text{ and } (16) \text{ for any } e, \text{ and} \]

\[ \sum_{q, e} \hat{\Pr}(q, e) [U(\hat{c}(i, e, q)) + V(e_i) + \beta \hat{w}'(i, q, e)] > w'^{i}_{1, r}(x) \text{ for } i=1,2 \], where

\[ w'^{i}_{1, r}(x) = \sum_{q, e} \Pr(e, q | r) [U(c_i(e, r, q)) + V(e_i) + \beta w'_i(e, r, q)]. \]

In order for \( x \) to be such that no improvement can be obtained from renegotiation, it must be the case that it maximizes the surplus of the principal in \( \Psi_r(x) \). Therefore, it must be the case that:

\[ \sum_{q, e} \hat{\Pr}(q, e) [q_1 + q_2 - \hat{c}_1(q, e, r) - \hat{c}_2(q, e, r) + \beta S^{t+1}(\hat{w}'(q, e, r))] \]

\[ < \sum_{q, e} \Pr(q, e | r) [q_1 + q_2 - c_1(q, e, r) - c_2(q, e, r) + \beta S^{t+1}(w'(q, e, r))] \]

for any \( \hat{x} \) in \( \Psi_r(x) \).

**Program 1c**

\[ S^t(w_1, w_2) \equiv \max_{\{\Pr, c, w'\} \in X} \sum_{q, e} \Pr(q, e, r) [q_1 + q_2 - c_1(q, e, r) - c_2(q, e, r) + \beta S^{t+1}(w'(q, e, r))] + \sum_{q, e, \mu} \Pr(q, e, \mu) [q_1 + q_2 - c_1(q, e, \mu) - c_2(q, e, \mu) + \beta S^{t+1}(w'(q, e, \mu))] \]

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subject to the promise keeping constraint

$$\sum_{q,e} \Pr(q,e,r) \left[ U(c_i(q,e,r)) + V(e_i) + \beta w_i'(q,e,r) \right] + \sum_{q,e,\mu} \Pr(q,e,\mu) \left[ U(c_i(q,e,\mu)) + V(e_i) + \beta w_i'(q,e,\mu) \right] = w_i,$$

for $i = 1, 2$

$$\sum_{q,e} \Pr(q,e,r) + \sum_{q,e,\mu} \Pr(q,e,\mu) = 1,$$

$\Pr(q,e,r), \Pr(q,e,\mu) \geq 0,$

and $\{\Pr, c, w'\}$ satisfies (12),(13),(14),(15),(16) and (17). Program 1c defines the surplus function at $t$, $S^t$, given the surplus function at $t+1$, $S^{t+1}$. This characterizes a functional equation $S^t = F(S^{t+1})$. A fixed point in this functional equation, $S = F(S)$ is a solution for the infinite periods problem.

Since agents prefer to make the low level of effort, whenever both agents are required to make high effort, it must be the case that different levels of output generate different rewards, through consumption, promises or both. The same happens when only one agent is required to make high effort, when this agent has non-zero Pareto weight $\mu$ inside a group. This difference in the levels of rewards conditional on outputs is what motivates individuals to make the high level of effort. So, for any $o$ in $O$, whenever the maximum effort is adopted by at least one agent, and the regime is not a group with $\mu_i = 0$, there are two different values of $q$, $q_1$ and $q_2$ such that either $c(q_1,e,o) \neq c(q_2,e,o)$ or $w'(q_1,e,o) \neq w'(q_2,e,o)$ or both. Since $p(q \mid e) > 0$ for any $q$ and $e$, this guarantees that either consumption or promises or both are not constant across possible contingencies.

High effort for at least one agent is crucial to rule out absorbent points and degenerate steady states. But it depends on high efforts, that is a result of the model and
its parameters, and not an assumption. However, these results are useful in the numerical analysis. We, know, from the numerical analysis, that low efforts can happen in areas of high utilities for both agents, implying low surplus. We can thus impose a lower bound on the surplus, which by its turn imposes upper bounds on promises, and verify, in the numerical solution, that low efforts are never reached. This can be tested by comparing the surplus obtained from the solution, with the trivial solution of constant consumption and low effort. In our numerical results, we did not obtain low efforts for both, and the surplus obtained by this solution is higher than the one with constant consumption and low effort. This is an indication that low effort is actually never reached in the solution.

Lemma 1 determines sufficient conditions for output dependency of consumption or promises.

Lemma 1 Suppose that $W$ has a lower bound $w > V(e_h)/(1 - \beta)$, and that in the optimal solution given initial promises $w$, at least one agent always makes the high amount of effort. Then, in the optimal solution for $w$ either consumption or future promises or both depend on outputs.

Proof. In the relative performance regime, agents have separate incentive constraints, so if any agent makes high effort, he will require consumption and promises to depend on output, or otherwise his incentive constraint will not be valid. Let us now consider the groups regime. If $\mu_i = 0$ for some agent $i$, the solution for a group will always have zero consumption, high effort and promises equal to $w$ for $i$. Therefore, the utility of $i$ will be $V(e_h) + \beta w < w$. So, if the groups regime is chosen, $\mu_i > 0$ for both agents. This implies that, even if groups are chosen and one of the agents is expected to do low effort, consumption and(or) promises must be dependent on outputs, or otherwise the incentive constraints will not be satisfied. ■

The following lemma and proposition show a result analogous to the one presented.
by Rogerson (1985). Randomness in rewards implies randomness in promises. In order to present these propositions, we use the following notation: $Z$ is the set of possible values $(q, e, o)$ that are reached with positive probabilities in the solution. $\pi_z(w) \equiv \Pr(z \mid w)$ is the probability of state $z \in Z$. $u_z(w) = U(c(z \mid w)) + V(z \mid w)$ is the vector of current utilities (from current consumption and effort), $w'_z(w) = w'(z \mid w)$ be the vector of future promises in state $z$ and $s(u, e, q) = q_1 + q_2 - U^{-1}(u_1 - V(e_1)) - U^{-1}(u_2 - V(e_2))$ is the current surplus given current utility $u$, effort level $e$ and output $q$.

**Lemma 2** Let $\tilde{W}$ be the set on which the infinite periods surplus function $S(w)$ is defined. Suppose optimal consumption is always nonzero for any $w$ in the interior of $\tilde{W}$. Then $S(w)$ is concave and differentiable, and for any $w$ in the interior of $\tilde{W}$, its gradient is given by

$$S'(w) = \mathbb{E}[s_u(u_z, e_z, q_z)]$$

**Proof.** We follow the Benveniste and Scheinkman theorem as presented by Stokey and Lucas. The infinite periods surplus function $S(w)$ is:

$$S(w) = \sum_{z \in Z} \pi_z(w)[s(u_z(w), e_z(w), q_z(w)) + \beta S(w'_z)]$$

Since randomization is allowed, $S$ is concave. Adding an amount $\varepsilon$ (a vector with two elements) of utility in every state through consumption would not change incentives for effort by agents and would change instant utility by an amount $\varepsilon$. But from optimization and from the non-renegotiation constraint, it must be the case that the surplus obtained this way (increasing current utilities by $\varepsilon$) is not higher than the surplus from the solution of program 1c with initial promises $w + \varepsilon$. Therefore:

$$\sum_{z \in Z} \pi_z(w)[s(u_z(w) + \varepsilon, e_z(w), q_z(w)) + \beta S(w'_z)] \leq S(w + \varepsilon) \quad (21)$$
Since \( S(w) \) is concave and not lower than the expression on the left hand side of (21), it must also be differentiable in any interior \( w \), with its gradient at equal to the gradient of the left hand side with respect to \( \varepsilon \) at \( \varepsilon = 0 \). Therefore:

\[
S'(w) = E[s_u(u_z,e_z,q_z)]
\] (22)

**Proposition 3** Suppose the conditions of the Lemma 2 are valid, and let \( w \) be such that in the optimal solution incentives are always needed, (or putting it differently, either both efforts are high or the agent with high effort is not assigned to a group with \( \mu = 0 \)). Then \( w' \) depends on output.

**Proof.** We start from the case with the initial promise, \( w \), in the interior of \( \tilde{W} \) where, (22) must be valid. Incentives for efforts will not change if, at any realization of uncertainty, \( z \), consumption is subtracted in the current period so that \( u_z \) is replaced by \( u_z - y \) and \( w'_z \) is replaced by \( w'_z + y/\beta \), where \( y \) is also a vector with 2 elements that is common for any state in \( Z \). The optimal choices for any \( z \) must thus solve:

\[
\max_y s(u_z - y, e_z, q_z) + \beta S(w'_z + y/\beta).
\]

Two cases are possible. First, if \( w_z \) is in the interior of \( \tilde{W} \) for any \( z \) in \( Z \). In this case, \( s_u(u_z,e_z,q_z) = S'(w'_z) \). Since incentives are always needed, given any \( e \), different levels of \( q \) imply different levels of either \( u \) (and thus consumption) or \( w' \). But from this last equation, for any \( e \), different levels of consumption imply different levels of of promises, implying that \( w' \) is depends on \( z \). A second possible case is in a border of the feasible set, \( \tilde{W} \). Suppose, for instance, that \( w'_{iz} = \overline{w}_i \) for some agent \( i \), where \( \overline{w}_i \) is an upper bound on the possible promises to agent \( i \). In this case, \( s_{u_i}(u_z,e_z,q_z) \leq S'(\overline{w}_i) \). If agent \( i \) needs incentives (and thus has consumption or promises dependent on output) it cannot be the case that \( w'_{iz} \) is always equal to \( \overline{w}_i \). Incentives would require different
values for $u_z$, which would imply that $S'(w) > E[s_u(u_z, e_z, q_z)]$, violating (22). So, $w'$ depends on $z$. Analogous argument can be used to prove that $w'$ depends on $z$ even when some lower bound is reached with positive probability.

Let us now consider the case where initial promise, $w$ is not in the interior of $\tilde{W}$. Suppose some agent $i$ that needs incentives for effort starts from promises in an upper bound $\overline{w}_i$, the space of possible promises. Suppose incentives to $i$ are given only through consumption. From (21) $S'_i(\overline{w}_i) \leq \sum_{z \in Z} \pi_z s_{ui}(u_z, e_z, q_z) < s_{ui}(u_z, e_z, q_z)$, where $\underline{z}$ is the state with maximal value of $s_{ui}$. Because Then increasing $u_{z\underline{}}$ by $\varepsilon$ and decreasing $w'_{z\underline{}}$ from $\overline{w}$ to $\overline{w} - \frac{\varepsilon}{2}$ would not change the optimal choice of efforts for agents and would increase surplus. It must be the case that $w'_{z\underline{}} \neq \overline{w}_i$. But for the reasoning presented above, if $w'_{z\underline{}}$ is in the interior for any $z$, $s_{ui}(u_z, e_z, q_z) = S'_i(w'_{z\underline{}})$ is valid, so $w'$ depends on $z$. If a lower bound $\underline{w}$ is reached with probability 1, than for some $\overline{z}$, $S'_i(\underline{w}) < s_{ui}(u_{\overline{z}}, e_{\overline{z}}, q_{\overline{z}})$, and there is some role for improvement. So $w'$ must depend on $z$. . Analogous argument can be used to show that there can be no degenerate distribution of $w'$ when $w$ is in the lower bound of the space of promises.

This proposition determines that part of incentives for effort are given through promises. Since optimal incentives depend on the regime, it is possible that the characteristics of the dynamics depend on the regime to which agents are assigned. The impact of regimes on the dynamics will be evaluated in the numerical section below.

A combination of Lemma 1 and Proposition 3 determine that a lower bound on promises and high efforts for at least one agent, any point in $\tilde{W}$, $w$ produces, with positive probability, a value of promises $w' \neq w$. However, this is not enough to guarantee that there is no convergence in probability, or almost sure convergence to any point in $\tilde{W}$. In principle, it is possible that although there is always some randomization, the there is either almost sure convergence or convergence in probability.
to some point in $\hat{W}$ (in which case, the variance of the distribution converges to zero). The following steps show that there exist a positive numbers, $\rho$ and $\delta$, such that for any $w$ in $\hat{W}$, the probability that $|w - w'| \geq \delta$ is bigger than $\rho$. This would rule out convergence in probability.

**Definition 4** The surplus specific to organization $\sigma$, $S^\sigma_{w}(w)$ is the solution to program 1c with the additional constraint that $Pr(\sigma, e \mid w) = 1$. $\hat{W}_T^\sigma$ is the set of all values of $w$ for which this last program is well defined. The solution graph for organization $\sigma$ is $\Delta^\sigma(\hat{W}_T^\sigma) = \{x = [(\hat{Pr}(q, e, \sigma), \tilde{c}(q, e, \sigma), \hat{w}'(q, e, \sigma), w)] : \hat{Pr}(q, e, \sigma), \tilde{c}(q, e, \sigma), \hat{w}'(q, e, \sigma) \text{ solves the organization specific problem given } w \text{ in } \hat{W}_T^\sigma\}$. 

**Lemma 5** $S^t(w)$ can always be obtained as a randomization of organization and effort specific surpluses. More specifically: $S^t(w) = \sum_{o,e} Pr(o,e \mid w) \cdot S^t_{o,e}(w_{o,e})$, where $w_{o,e} = \Pr(q \mid e, o, w).[U(c_i(q, e, o)) + V(e_i) + \beta w_i'(q, e, o)]$.

**Proof.** Whenever a type of organization and effort is adopted and , the choices conditional on this organization must be surplus maximizers conditional on the utilities they generate. Otherwise, it would be possible to increase the overall surplus by increasing the surplus for this particular kind of organization and keeping the probability of organizational forms constant. ■

**Lemma 6** Suppose $S^{t+1}(w')$ is a concave, surplus function well defined in some compact set $\hat{W}_{t+1}$. Suppose $W_t = [\underline{w}, \infty)$ for some $\underline{w} \in \mathbb{R}$. Then, given the additional constraint $S^t(w) \leq S$, the set for which the solution to problem 1 is well defined, $\hat{W}_t$, is compact. Further, both $\hat{W}_t^{\sigma,e}$, and $\Delta^{\sigma,e}(\hat{W}_t^{\sigma,e})$ are compact sets .

**Proof.** Since there is a lower bound on $W_t$, and infinite current utility requires infinite consumption, $\hat{W}_t$ is bounded. Also, the feasible set of Pr is compact by (20) and since $S^t(w) \leq S$, we can restrict the domain of $c$ to a compact subset of $C$. First we prove
that for any kind of organization \( \sigma \), the set of values of \( w \) for which \( S^w_{\text{e}}(w) \) is well defined, \( \tilde{W}_t^\sigma \) is compact. This follows because if we take a sequence \( \{w_n\} \) converging to \( \tilde{w} \), there is a corresponding subsequence \( \{\text{Pr}_{ni}(q, e, \sigma), c_{ni}(q, e, \sigma), w'_{ni}(q, e, \sigma)\} \) converging to \( \{\tilde{\text{Pr}}(q, e, \sigma), \tilde{c}(q, e, \sigma), \tilde{w}'(q, e, \sigma)\} \) that generates utility pair \( \tilde{w} \). Constraints (12),(13),(15) and (16) are clearly valid in this limit.

It remains to prove that the non-renegotiation constraint is valid for this limiting point. Suppose one of the renegotiation constraints is not valid. Then, there is some utility level \( w > \tilde{w} \) that is incentive compatible and has a surplus level \( S \) that is at least as large as the one implied by \( \{\tilde{\text{Pr}}(q, e, \sigma), \tilde{c}(q, e, \sigma), \tilde{w}'(q, e, \sigma)\} \), \( \tilde{S} \). \( S \) cannot be larger than \( \tilde{S} \), as this would make the non renegotiation constraint invalid for \( ni \) large enough, since the utility vector converges to \( \tilde{w} \) as \( ni \) converges to infinite. So \( S = \tilde{S} \).

But then, for some solution with \( w^* \) between \( w \) and \( \tilde{w} \), non renegotiation would require that the surplus \( t \) is smaller than \( \tilde{S} \). But this would imply that for \( ni \) high enough \( \{\text{Pr}_{ni}(q, e, \sigma), c_{ni}(q, e, \sigma), w'_{ni}(q, e, \sigma)\} \) violates non renegotiation, contradiction. So the function is well defined for \( \tilde{w} \) under regime \( \sigma \), and the set \( \tilde{W}_t^\sigma \) is compact. Notice that this also guarantees that \( \Delta^\sigma(\tilde{W}_t^\sigma) \) is compact. But from the lemma before, the set of values of \( w \) for which \( \tilde{W}_t \) is well defined is the set of possible randomizations between values of \( w \) produced by some regime. Since this last set is compact, \( \tilde{W}_t \) must be compact. ■

From the last Lemma, if we take as a starting point some function defined in a compact set, and start iterating it through program 1c, solving it for any possible \( w \) until a fixed point is reached, the set of values for which the limiting function \( S(w) \) has a solution that is compact.

**Proposition 7** Suppose the conditions on Lemma 2 are valid, that the set \( \tilde{W} \) on which \( S(w) \) is defined is compact, that in the solution to problem 1 there is always high effort by at least one agent, and that there is a lower bound on \( W, w > V(e_h)/(1 - \beta) \). Then the distribution of \( w' \) does not converge to any point in \( \tilde{W} \).
Proof. For any \( \mathfrak{c} \) and \( \mathfrak{d} \) in \( E^2 \times O \), we can define continuous functions from the elements of \( \Delta \mathfrak{c}(W) \cap \Pr(e, e_1 = 0) \) to \( \Re \) as \( f_{\mathfrak{c}}(\Pr(q, e, e_1), \mathfrak{c}(q, e, e_1), w'(q, e, e_1, w)) = \max \{ w'(q, e, e_1) - w | w \} \). Since \( \Delta \mathfrak{c}(W) \) is compact, \( f_{\mathfrak{c}} \) has a minimum value, that we call \( d_{\mathfrak{c}} \), that is higher than zero (otherwise proposition 3 would be contradicted). We define \( \delta = \min \{ \alpha : \Pr(\alpha) > 0 \} \). This implies that for any \( w, \) and \( o \) there is some value of outcome that generates a value of \( w' \) that has a distance from \( w \) bigger than \( \delta \). The probability of this must be at least as high as \( \rho = \min_{q,e} p(q | e) \). So, there is no convergence. \( \blacksquare \)

From this last theorem, we know that the adoption of lower bounds on promises and surplus can overcome the problem of immiseration.

**NUMERICAL RESULTS**

This section presents some results of numerical solutions of the model. Program 1 is a typical linear programing problem. It has the form:

\[
\max_b \mathfrak{c} \cdot b \\
\text{s.t. } \mathfrak{A} \cdot b \leq \mathfrak{c}.
\]

This kind of problem can be solved by existing numerical software. However, depending on the dimension of the problem (the size of the vector \( b \) and the constraint matrix \( \mathfrak{A} \)), the numerical solution may demand an unreasonably high amount of time and memory.

In the case of Program 1, the dimension of the program depends on the number of elements on \( C, \) \( Q, \) \( E \) and \( M, \) and on the number of constraints.

If we compute the problem using \( Q \) and \( E \) with 2 elements each, \( C \) with 18 elements and \( M \) with 101 elements, and 30 possible promised utilities in \( W \), the choice vector

\footnote{This is the size of the grids used in our final computations: 18 elements in \( C, \) 30 in \( W, \) 2 in \( Q, \) 2 in \( E \) and 101 in \( M. \) If the grids on \( C, W \) and \( M \) are too thick solutions tend to be less smooth.}
\( \pi_g \) has 52,358,400 elements, and \( \pi_r \) has 4,665,600 elements (the decision vector \( b \) has thus 57,024,000 elements). There are 2 constraints in (4) and (5), 16 in (6), 202 in (1), 1616 in (2), and 2 in (9) and (10). So, we have a total number of 1832. Given the high dimension of the problem, the computational implementation is very slow and requires extremely high RAM memory. As the dynamic iterations of the problem require many computations of a whole grid (with 900 elements in the grid when \( W \) has 30 elements), it is important that program 1 can be computed rapidly. In order to make computation faster, we reformulated the problem, breaking it in smaller subproblems. This reformulation is explained in detail in the Numerical Appendix.

For computational purpose, we assumed that the set of possible consumption levels for each agent, \( C \), has 18 elements between 0 and 40. The set of possible outputs per agent is \( Q = \{2, 20\} \), and the set of possible Pareto weights inside groups, with 101 elements, is \( M = \{0, 0.01, 0.02, ..., 1\} \). We analyze in detail the numerical solution of the problem for the following specification: the utility of consumption and the disutility of effort are characterized by \( U(c) = e^{0.5} \) and \( V(e) = -e^{0.5} \). The set of possible effort levels per agent (that determines the disutility of high effort) is \( E = \{2, 6\} \). In our baseline specification, the cost of blocking collusion, \( k \), is equal to 0.3. In order to have rapid convergence to the steady state, we assume a low discount rate \( \beta \) of 0.7. In order to verify the robustness of the results, we also computed the model for 2 alternative values for \( \beta \) and \( k \). As in Prescott and Townsend (2002) the technology \( P(q_1, q_2 \mid e_1, e_2) \), is given by the following table:

\[
\begin{array}{cccc}
q = (2, 2) & q = (20, 2) & q = (2, 20) & q = (20, 20) \\
\hline
e = (2, 2) & 0.6979 & 0.0021 & 0.0021 & 0.2979 \\
e = (6, 2) & 0.2991 & 0.4009 & 0.0009 & 0.2991 \\
e = (2, 6) & 0.2991 & 0.0009 & 0.4009 & 0.2991 \\
e = (6, 6) & 0.2979 & 0.0021 & 0.0021 & 0.6979 \\
\end{array}
\]
Notice that this matrix implies that outputs of agents are correlated: if they do the same amount of effort, it is very unlikely that their outputs will be different. In the computations, we allowed a grid of 30 values for \( w_i \), and \( w'_i \). We assumed a lower bound in \( W \), representing the value of an outside option for each agent, of -3.5. This is smaller than the autarky utility for each agent\(^2\). A possible interpretation for this lower bound smaller than the autarky utility is that agents can move to autarky as an outside option but they have some punishment when they do it, that could come from legal enforcement or social stigma. This particular lower bound has been chosen to make the ergodic distribution of contracts more interesting. Without a lower bound on promised utilities, the ergotic distribution has only group contracts at the axis region, with one of the agents having the lower level of utilities possible\(^3\). For the parametrization and the lower bound on promises used, the ergodic distribution has significant probability of both assignment to the groups regime and the relative performance regime.

**Solutions on the ex-ante utilities map**

We used the numerical procedure presented in the appendix to find the maximum surplus for a grid of utility pairs in the infinite periods problem. We started with a surplus function \( S^{T+1}(u_1^{T+1}, u_0^{T+1}) = 0 \), and iterated until the function \( S^t \) converged\(^4\)

\(^2\)Notice that the distribution of outputs for one individual does not depend on the effort of the other. The autarky utility is the utility of making the maximum effort (which is the maximizing level of effort) and consuming the obtained level of output without any kind of consumption smoothing across time or states.

\(^3\)When the lower bound is too high (e.g. the autarky utility), the relative performance regime is observed only for low levels of utility for both individuals.

\(^4\)After 15 iterations, the maximum difference between \( S^t \) and \( S^{t+1} \) was smaller than \( 10^{-5} \), and we assumed that \( S^t \) was a reasonable approximation of the fixed point characterizing the infinite periods surplus function.
to a fixed point, $S$.

At each pair of ex-ante utilities there is a distribution of efforts, outputs, interim utilities, and corresponding consumption and promised utilities that characterize the solution to the maximization of surplus conditional on utility levels. Figure 2 presents the expected values of effort, consumption and promised future utilities for agent 1 at each ex-ante utility pair (in the horizontal axes) for all parameters at the baseline case and $\beta = 0.5$. Another variable expressed in figure 2 is what we call interim utilities, the utilities that come both from consumption and promises after output is realized. The interim utility of agent $i$ is denoted by $v_i \equiv U(c_i) + \beta w_i'$. This variable is useful for computation\textsuperscript{5} and provides a good summary of how rewards and punishments react to outputs. Notice that agent 1 tends to make the high level of effort when his ex-ante utility is low, and the low level of effort when it is high. And the expected values of interim utility, consumption and promised utility of agent 1 tend to increase with his level of ex ante utility $w_1$, except in the region of transition from high effort to low effort.

\textsuperscript{5}See appendix.
Figure 2

Figure 3, shows, for 3 different values of $\beta$, the difference between the infinite periods surplus that can be obtained under the relative performance regime and the group regime for each utility pair.
Surplus Under R. P. - Surplus Under Groups

\[ \beta = 0.5 \]

\[ \beta = 0.7 \]

\[ \beta = 0.85 \]

**Figure 3**

The horizontal axes represent ex-ante (before effort is employed) expected levels of utilities of the two agents. Notice that for very low utility levels and high inequality with respect to utility, the group regime tends to produce higher surpluses so the net difference is negative. For intermediate utility levels and low inequality, the relative performance regime produce higher surpluses. For very high utility levels (very low surplus for the principal) the optimal effort level for both agents is low, that is like as no moral hazard problem and apart from the cost to block collusion, \( k \), the surplus under relative performance virtually equals the surplus under groups. That is, the
surplus under groups is higher than the surplus under relative performance by an amount equal to $k$.

Notice from Figure 2, that for very high levels of utility, that correspond to very low levels of surplus, both agents make low effort. In order to avoid degenerate steady states, we impose a lower limit for surplus, $S = -4$. We then verify that for all parametrization used, low effort is never used by both agents\(^6\).

Figure 4 shows the areas, in the map of ex-ante utilities, $w$, where each regime dominates. This is shown for two alternative values of $\beta$ and $k$. The red area represents the region where agents are assigned to the relative performance regime with greater than 50% of probability. In the blue area, there is a probability higher than 50% that they are assigned to the group regime. The yellow areas are those in which the problem has no solution, either because surpluses are lower than $S$ or because the non renegotiation constraints are not valid, as in the extreme southwest of the utility maps\(^7\). Notice again that the relative performance regime tends to dominate for intermediate levels of utility and low levels of inequality. These characteristics were observed in all the parameterization of the problem that we computed.

\(^6\) We also checked that the surplus given by these solutions is higher than the one resulting from the trivial absorbent solution in which individuals always have the same consumption and make the low level of effort.

\(^7\) In this area, The surplus function is increasing, which violates non renegotiation.
R.P. Dominates(Red) X Groups Dominate(blue)X No solution (yellow)

\[ \beta = 0.5 \quad \beta = 0.7 \quad \beta = 0.85 \]

\[ k = 0 \quad k = 0.6 \]

Figure 4

Steady State

Promised utilities at period \( t \) characterize the state in period \( t + 1 \). In this model, the initial state variable in each period is the ex-ante utility pair. If agents have an initial utility pair of \( (w^0_1, w^0_2) \), the solution of the surplus maximization problem at this point determines a distribution of \( (w^1_1, w^1_2) \). Similarly, the solution of the problem in each point makes it possible to characterize, starting from the probability distribution of \( (w^t_1, w^t_2) \), denoted by \( H_t \), the distribution of \( (w^{t+1}_1, w^{t+1}_2) \), \( H_{t+1} \). In other words, the solution of the surplus maximization for the entire grid determines an operator \( T \), such that \( H_{t+1} = TH_t \).
Steady State Distribution of Promised Utility Pairs

\[ \beta = 0.7 \ (59.47\% \text{R.P.}) \]

\[ \beta = 0.5 \ (81.54\% \text{R.P.}) \]

\[ \beta = 0.85 \ (32.54\% \text{R.P.}) \]

\[ k = 0 \ (73.93\% \text{R.P.}) \]

\[ k = 0.6 \ (32.91\% \text{R.P.}) \]

Figure 5

Successive iteration of the operator \( T \) from a starting distribution \( H_0 \) produces convergence to an invariant probability measure \( H^* \), that can be interpreted as a
steady state distribution of utility pairs. The steady state distribution of utility pairs in the example presented above is illustrated in figure 5. This steady state distribution has significant presence of both the relative performance and the group regimes. Another feature that can be observed in figure 5 is that the distribution tend to concentrate in two separate sets: one, in the central area of the picture, where relative performance dominates, and another near the axes where group dominates. The area between these two sets has zero or positive low mass in the steady state distribution because it is not efficient to assign agents to utility pairs in this area, or keep them there very long. A possible explanation for that is that instead of promising utilities in this area, it is preferable to randomize between offering a utility pair in the area where groups dominate, and offering utilities in the central area with assignment to relative performance. The function max\(\{S_r^t(w_1, w_2), S_g^t(w_1, w_2)\}\), where \(S_r^t(w_1, w_2)\) and \(S_g^t(w_1, w_2)\) are the maximum surplus produced under the group and the relative performance regimes, respectively, is not concave (although both \(S_r^t(w_1, w_2)\) and \(S_g^t(w_1, w_2)\) are each concave). An illustration of this is provided by Figure 6, the shows the maximum surplus in each regime for different levels of ex-ante expected.

\(^8\)We sometimes found small positive probabilities in this area but we conjectured that those probably result from imperfections in the numerical results.
utility for agent 1, $w_1$, when the utility of agent 2 is fixed.

**Surplus Under Both Regimes For Fixed $w_2$**

![Graph showing surplus under both regimes for fixed $w_2$](image)

**Figure 6**

**Transitions between regimes**

From the structure presented in Figure 1, one can observe that given an initial vector of expected utilities in $t$, $w^t$, there is a corresponding distribution of regime assignments with associated promises. The regime specific programs, in turn, determine distributions of interim utilities that are associated with future promises, $w^{t+1}$. These future promises are also delivered by regime assignment, in $t+1$. For some values of $w^t$, it is possible that the regime to which they are assigned in $t$ is not the same to which they are assigned in $t+1$. When this happens, there is regime switching. Figure 7 presents the probability of transition between regimes at each point of the ex-ante utilities map. Transitions between regimes occur near the frontier between the areas in which each regime dominate.
Tables 1 and 2 present a detailed description of the solution to the surplus maximization problem for selected points with more than 0.5% probability in the steady state distribution. They show the expected values of efforts, interim utilities, consumption and interim and promised future utilities conditional on output. Figures 8 and 9 show each of these points as a red spot in the regime prevalence map. Table 1 and Figure 8 refer to points where there is 100% of probability of Relative Performance, and Table 2 and Figure 9 to points where there is 100% of probability of group assignment.

---

9 Approximately 94% of the 900 points in the utility map have less than 0.5% probability in the steady state.

10 We did not use the fact that there is 100% of assignment to one regime as a criterion to select these particular points. Most of the points with significant probability (more that 5%) in the steady state distribution have 100% of assignment to one regime.
### Table 1 – Expected Values Given Outputs for Selected Points With 100% R.P. Assignment

**Point 1: w=(-3.5068,-3.5068)**

<table>
<thead>
<tr>
<th>q1</th>
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<th>p(q1,q2)</th>
<th>E(e1)</th>
<th>E(e2)</th>
<th>E(v1)</th>
<th>E(v2)</th>
<th>E(c1)</th>
<th>E(c2)</th>
<th>E(w1)</th>
<th>E(w2)</th>
<th>P(switch)</th>
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</tr>
<tr>
<td>2.0000</td>
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<td>0.0021</td>
<td>6.0000</td>
<td>6.0000</td>
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<td>-0.5400</td>
<td>0.4088</td>
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<td>-2.8495</td>
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**Point 2: w=(2.6085,2.6085)**

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<td>5.1365</td>
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</table>

**Point 3: w=(2.6085,-3.5068)**

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<th>p(q1,q2)</th>
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<th>E(e2)</th>
<th>E(v1)</th>
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<th>E(w2)</th>
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---

**Selected Starting Points With 100% of Relative Performance Assignment**

**Point 1**

**Point 2**

**Point 3**

![Figure 8](image-url)
### Table 2 – Expected Values Given Outputs for Selected Points With 100% Groups

Point 4: \( w = (-1.2136, -5.8) \)

<table>
<thead>
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<th>p(q1,q2)</th>
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<th>E(e2)</th>
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Point 5: \( w = (7.1949, -0.44915) \)

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Point 6: \( w = (5.6661, 2.6085) \)

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<td>4.9799</td>
<td>13.6667</td>
<td>9.5417</td>
<td>5.1694</td>
<td>2.7129</td>
<td>0</td>
</tr>
</tbody>
</table>

Selected Starting Points With 100% of Groups Assignment

Point 4  
Point 5  
Point 6

Figure 9

Point one represents the utility pair \((w_1, w_2) = (-3.5068, -3.5068)\). In this case, both agents have 100% of probability of current assignment to the relative performance regime and the high effort level is always recommended to both agents. When their outputs are the same, there is no switching. But when the output of one agent
is higher than the other, there is almost 100% probability of switching. In order to give incentives for high effort, the principal punishes agents with low output. If outputs are unequal, the rewards to agents are also unequal. But part of the reward is achieved with promised utilities, and unequal utilities in the second period tend to be produced with a higher surplus under the group regime. It is as if the principal switches agents to a group in order to extract the maximum surplus in a situation in which agents are treated unequally.

Point 2, utility pair \((w_1, w_2) = (2.6085, 2.6085)\) is another example of 100% of probability of assignment to the relative performance regime. As in the other examples, unequal outcomes produce an exacerbation of inequality and a significant probability of switching (close to 38%). There may also be switching when both individuals are rewarded for high outputs.

Point 3 represents utility pair \((w_1, w_2) = (2.6085, -3.5068)\). Again, there is a 100% probability of assignment to relative performance and agents make maximum effort. But in this case, agent 1 has higher ex-ante utility than agent 2, and a higher output for agent 2 than for agent 1 does not make inequality high enough to justify switching to groups. However, switching happens with almost 100% probability when the output of agent 1 is higher than the output of agent 2. Notice that there is some probability of switching to groups when both outputs are high: the two agents are rewarded with higher promises and they may move to the area where the group regime dominates.

Point 4, \((w_1, w_2) = (-1.236, -5.8)\), is an example of 100% of probability of assignment to groups. Notice that in this case, both agents make high effort. When both outputs are high, there is about 8.5% of probability of switching to relative performance. In points 5 and 6, agent 1 makes low effort and agent 2 makes high effort. Notice that when the output of agent 2 is low, the whole group is punished for a low achievement of an agent who is supposed to do high effort. As the group takes
decision jointly and aggregate rewards are shared, the whole group is punished, not only the agent with low output. Indeed, a low output of agent 2 produces a significant decrease in reward to agent 1 even when his own output is high. Interim promises to 1 are lower whenever the output of 2 is low. In point 6, the punishment a punishment to both individuals can move them to an area where relative performance dominates. This produces significant probability of switching. Points 5 is an example where the probability of immediate switching is close to zero, a common property in the area where groups dominate.

In summary, numerical solutions reveal that transitions from groups to relative performance tend to occur when there is a reward or a punishment to both agents in a group. Transitions from the relative performance to the groups regime may also come when either rewards to different outcomes generate an exacerbation of inequality that justifies the adoption of the groups regime.

**Dynamic Paths**

Figures 10 to 15 present a detailed analysis of the dynamics of the solutions, starting from the Points selected for tables 1 and 2 and figures 8 and 9. They show the evolution over time of the expected values and variances of consumption and the Theil index\(^{11}\) of inequality with respect to consumption\(^{12}\). They also show the evolution of

\[^{11}\text{These are the expected values and variance of the Theil index inside pairs of agents, or local economies. The Theil index on consumption for a pair of individuals is } Th(c_1,c_2) = \sum_{i=1}^{2} \left( \frac{c_i}{\sum_{j=1}^{2} c_j} \ln \frac{c_i}{c_j} \right).\]

\[^{12}\text{We start with a given promised utilities pair with probability one. Successive iteration of the operator } T, \text{ produces an increasing number of points with positive probability. We calculate the expected value of promises over this increasing number of events using the corresponding probabilities. Note that the evolution of promised utilities is equal for both individuals when the starting level of promised utilities is equal for both of them. This follows from the symmetry of the problem across agents.}\]
the probability of assignment to groups.

For the starting points studied, the expected value of the Theil index of inequality tend to evolve monotonically to the steady state values. The variance of inequality tends to evolve monotonically, and this monotonicity is lost only in point 4. The probability of group assignment, on the other hand, is non-monotonic starting from many of the points selected. There is high diversity of dynamic behavior of consumption in the utilities map. There are both cases in which average consumption evolve monotonically, and cases where it is not monotonic. Another feature that can be observed in the consumption dynamics starting from points far from the diagonal, is that the variance of the consumption of the agent with high initial promise tends to increase faster than the one of the agent with low promises. This is especially clear for point 4. This is related to the fact that far from the diagonal, in a region where the groups regime is adopted, the reward of the high promise agent (that normally has higher internal Pareto weight and less risk aversion) tends to be more responsive to different outputs. The variance of promises (and thus consumption) of the agent that starts in better position tends to be much higher.

Another interesting aspect of these pictures is the diversity in regime persistence. In point 1, the probability of groups assignment increases slowly and steadily, which differs greatly from the very dramatic changes in the other relative performance points selected.

Figures 16 and 17 show the probabilities that there is zero, one, two, three, four or more than five transitions over time\textsuperscript{13}. This is the probability that a pair starting at

\textsuperscript{13} The procedure used to generate these distributions is the following: start with two state variables. A transition matrix (that comes from the solution) determines the joint probability distribution of $w'_1$ and $w'_2$ (future promises) given the joint distribution of $w_1$ and $w_2$. We added another state variable: number of transitions so far. The joint distribution of $w_1$, $w_2$ and the number of transitions until $t$ determines its joint distribution in $t+1$. For dimensional reasons, only up to 5 transitions are studied, so one of the possible states is 5 transitions or more. Of course, over time, the probability
a given point switched n times after t periods. We plotted these distributions for 1 to 100 periods.

The distributions of the number of transitions provide an even clearer picture of how regime persistence varies over the utility map. In unstable areas, pairs of agents move in and out of regimes. Part of the pairs that are in the initial regime after n periods are people that moved out and than back into the original regime. These transitions cannot be captured by the cross sectional distribution of regimes over time. But they are captured in the distributions of number of transitions.

The difference between observing just the evolution of the probability of regime assignment and the distribution of number of transitions is particularly clear for point 6 in Figure 17. There is a sharp drop in the probability of zero transitions over time. After 20 periods there is less than 5% probability that agents remains in the original regime without any transition. This is not as clear from the simple probability of regime assignment and its variance. The probability of regime assignment average does not reveal more instability starting from point 6 than from point 5 but observation of figure 17 show that, at least in the long run, point 6 is more unstable.

In general, the distribution of number of transitions shows considerable variability. While starting from points 2, and 6 there is low persistence in the initial regime, with the probability of zero transitions dropping fast (reaching less than 5% after 100 periods) and the probability of more than 5 transitions raising rapidly, from points 4 and specially 1 there is high persistence.

The decrease in permanence in the original regime without transitions follows different patterns depending on the departure point. In point 4 there is initially a sharp decrease in the probability of zero switches, but then this probability tends to stay relatively stable. After the 20 periods, the drop in the probability of zero transitions that there is 5 or more transitions converge to 1 as in the steady state there is transit from one point to any other point in the support.
tends to be slow: those pairs that resist the initial instability in the early unstable periods, tend to be relatively persistent in the original regime. This contrasts with point 6, where the probability of zero transitions converges rapidly to zero.

Figures 18 and 19 provide an illustration of how the distribution of promises evolve over time on the utility map starting from different points (point 2 in figure 18 and point 5 in figure 19). The bright spots are points with high probability. Those pictures show that there is some persistence in areas of the utility map. They also illustrate how the relative performance areas (point 2) and and the groups areas (point 5) may produce different dynamic behavior. While starting from the groups regime in point 5, short term shifts in promises tend to punish or reward the agent with higher promises, when we start from point 2 there is the possibility of considerable shifts in inequality due to differences in relative rewards. It is possible that after one period one agent gain while the other looses.

**EMPIRICAL IMPLICATIONS**

Although the numerical analysis reveals the possibility of great variability in dynamic paths, it suggests some regularities that are potentially testable. First, the cross sectional results obtained by Prescott and Townsend (2002) are maintained in our formulation: the relative performance regime dominates for intermediate levels of wealth and low inequality. We also obtain some dynamic observable implications: switches from relative performance to groups can be accompanied by big shifts in inequality, while transitions from groups to relative performance are related to changes in wealth (promised utilities). Economies that switch to relative performance tend to come from higher levels of inequality than those that switch to groups. Also, persistence in groups seems to be higher when there is high inequality than when there is low inequality.
Dynamic Paths - Starting From Point 1 - 100% Relative Performance

Expected Consumption

Variance Consumption

Expected Theil Index

Variance Theil Index

Probability of Group Assignment

Figure 10
Dynamic Paths - Starting From Point 2 - 100% Relative Performance

Figure 11
Dynamic Paths - Starting From Point 3 - 100% Relative Performance

Expected Consumption

Variance Consumption

Expected Theil Index

Variance Theil Index

Probability of Group Assignment

Figure 12
Dynamic Paths - Starting From Point 4 - 100% Groups

Expected Consumption

Variance Consumption

Expected Theil Index

Variance Theil Index

Probability of Group Assignment

Figure 13
Dynamic Paths - Starting From Point 5 - 100% Groups

Expected Consumption

Variance Consumption

Expected Theil Index

Variance Theil Index

Probability of Group Assignment

Figure 14 - Expectation and var. of consumption and Theil index are normalized to steady state values.
Dynamic Paths - Starting From Point 6 - 100% Groups

Expected Consumption

Variance Consumption

Expected Theil Index

Variance Theil Index

Probability of Group Assignment

Figure 15
Distribution Number of Transitions - Points With 100% Relative Perf. Assignment

Point 1

Figure 16

Point 2

Point 3

Figure 16
Distribution Number of Transitions - Points With 100% Group Assignment

Point 4 (100 periods)

Point 5 (100 periods)

Point 6 (100 periods)

Figure 17
Evolution of Distributions in the Utility Map - Point 2

Figure 18
Evolution of Distributions in the Utility Map - Point 5

Figure 19
REFERENCES


APPENDIX A

Reformulated Problem With Interim Utilities.—

One fact that is useful to reduce the dimension of the problem is that the joint distribution of \( U(c_t^1 + \beta w^{t+1}_1) \) and \( U(c_t^2 + \beta w^{t+1}_2) \) conditional on output is sufficient to characterize the effort decision of agents or groups at \( t \). The distributions of \( c_t^1, c_t^2, w_t^1 \) and \( w_t^2 \) are relevant in these decisions only through their effects on the composite terms \( U(c_t^1 + \beta w^{t+1}_1) \equiv v_1 \) and \( U(c_t^2 + \beta w^{t+1}_2) \equiv v_2 \), that represent the utility that comes both from consumption and future contracts. Throughout the remaining of the paper, we call \((v_t^1, v_t^2)\) the interim utilities pair. In order to diminish the size of the decision vector, we formulate the moral hazard problem making \( v_t^1 \) and \( v_t^2 \) decision variables, not \( c_t \) and \( w_t \) separately.

This strategy requires another maximization stage. For a fixed vector \((v_t^1, v_t^2)\), different distributions of \( c_t \) and \( w_t \) imply different expected values for the ex-post (after output in \( t \) is realized) surplus \( -c_t^1 - c_t^2 + \frac{1}{1+\gamma} S^{t+1}(w_t^1, w_t^2) \), denoted \( \tilde{S}^t \). In order to determine the cost (or benefit) of offering an utility pair \((v_t^1, v_t^2)\), we need to solve the problem of choosing vectors of consumption in \( t \) and promised future utilities that maximize the expected value of \( \tilde{S}^t \) conditional on \((v_t^1, v_t^2)\).

Also, instead of determining the probabilities of outcomes under groups and relative performance simultaneously as in program 1, we brake the problem in two, one for each regime: we first solve the problem assuming that only the relative performance regime is available, and then solve it using only the group regime. These separate programs allow us to construct a grid of surplus maximizing contracts for each regime. The general program in which both regimes are allowed can be approximated by choices (possibly random) over these grids. This procedure of solving the problem separately for each regime makes computation easier (we solve smaller programs) and allows us
to compare the performance of the regimes in each area of the utility pairs map.

Figure 20

The structure of the reformulated problem is illustrated in figure 20. At the beginning of period $t$, the principal takes as given a level of promises $w$. In order to deliver these promises, he defines a joint distribution of the regime assignment and utility pairs $(\bar{w}_1, \bar{w}_2)$. These utility pairs are delivered by an assignment of first, internal Pareto weights $\mu$ if the regime is groups, then levels of effort levels of effort $(e)$, and finally interim utilities $(v)$ conditional on outputs $(q)$. These interim utilities are delivered by current consumption $(c)$ and promises for future periods $(w')$.

The optimal policy by the principal involves 3 maximization stages, that can be solved backward. In a first stage (the last one chronologically), the principal defines the choices between consumption $(c)$ and promises $(w')$ that produce each interim utility pair with maximal surplus, taking as given the function determining the surplus associated with future promises, $S^{t+1}(w')$. This defines an interim surplus function $\tilde{S}^t(v)$. In another stage, that depends on the regime adopted, the principal takes the interim surplus function as given, and chooses the joint distribution of internal Pareto
weights (if the regime is group), efforts, output and interim promises that maximizes surplus subject to the \((\tilde{w}_1, \tilde{w}_2)\) promises, and the incentive and technological constraints. This stage defines regime specific surplus functions for groups \((S_g^t(\tilde{w}_1, \tilde{w}_2))\) and relative performance \((S_r^t(\tilde{w}_1, \tilde{w}_2))\). Finally, taking \(S_g^t\) and \(S_r^t\) as given, the principal chooses a lottery determining the probabilities that the two agents are assigned to the groups regime with expected utilities \((\tilde{w}_1, \tilde{w}_2)\), denoted by \(\tilde{\pi}_g^t(\tilde{w}_1, \tilde{w}_2)\), and that they are assigned to the relative performance regime with expected utilities \((\tilde{w}_1, \tilde{w}_2)\), denoted by \(\tilde{\pi}_r^t(\tilde{w}_1, \tilde{w}_2)\) such that initial promises \(w\) are kept and a maximum surplus is obtained. This defines the overall surplus function at \(t\), \(S^t(w)\).

The maximization program that determines the distribution of consumption and promised future utilities that generate interim utilities \((v_1^t, v_2^t)\), with maximum surplus is:

**Program 2**

\[
\tilde{S}^t(v_1, v_2) \equiv \max_{\pi^t} \sum_{w_1^t, w_2^t, c_1, c_2} \tilde{\pi}^t(w_1^t, w_2^t, c_1, c_2)(-c_1 - c_2 + \beta S^{t+1}(w_1^t, w_2^t))
\]

st.

\[
\sum_{w_1^t, w_2^t, c_1, c_2} \tilde{\pi}^t(w_1^t, w_2^t, c_1, c_2)(u(c_1) + \beta w_1^t) = v_i, i = 1, 2,
\]

\[
\sum_{w_1^t, w_2^t, c_1, c_2} \tilde{\pi}^t(w_1^t, w_2^t, c_1, c_2) = 1.
\]

where \(S^{t+1}(w_1, w_2)\) is assumed to be known from before.

The surplus that can be obtained in \(t\) over the relative performance regime, conditional on the utility pair \((\tilde{w}_1, \tilde{w}_2)\), is defined by:

**program 3-r**

\[
S^t_r(\tilde{w}_1, \tilde{w}_2) \equiv \max_{\pi^t_r} \sum_{q, e, v} \pi^t_r(q, e, v)[q_1 + q_2 + \tilde{S}^t(v_1, v_2)]
\]
st.
\[
\sum_{q,e,v} \pi^t_r(q, e, v)(v_i + V(e_i)) = \tilde{w}_i, \; i = 1, 2, \quad (26)
\]

\[
\sum_{q,v} \pi^t_r(q, e, v)[v_i + V(e_i)] \geq \sum_{q,v} \pi^t_r(q, e, v) \frac{p(q | \hat{e}_i; e_i)}{p(q | e)}[v_i + V(\hat{e}_i)], \; \forall e_i, \hat{e}_i, i = 1, 2, \quad (27)
\]

\[
\sum_{v} \pi^t_r(\hat{q}, \hat{e}, v) = p(\hat{q} | \hat{e}) \sum_{v,q} \pi^t_r(q, \hat{e}, v), \; \forall \hat{q}, \hat{e},
\]

\[
\sum_{q,e,v} \pi^t_r(q, e, v) = 1. \quad (29)
\]

\[
\sum_{q,e,v} \pi^t_r(q, e, v) \left[q_1 + q_2 - \tilde{S}^t(v_1, v_2)\right] \geq \sum_{q,e,v} \pi(q, e, v) \left[q_1 + q_2 - \tilde{S}^t(v_1, v_2)\right] \quad (30)
\]

for any \( \pi \) in the set:

\[
\tilde{\Psi}_r(\pi^t_r) \equiv \{ \pi(c, q, e, w^t_r) : \pi \text{ is positive, sums to one and satisfies (27) and (6)} \text{ for any } e \text{ and } i, \text{ and } \sum_{q,e,v} \pi^t_r(q, e, v)[v_i + \beta w^t_r] \geq w^t_{i,r}(\pi^t_r) \text{ for } i=1,2 \},
\]

where

\[
w^t_{i,r}(\pi^t_r) = \sum_{q,e_i,v} \frac{\pi^t_r(q, e, v)}{\sum_{q,e,v} \pi^t_r(q, e, v)}[v_i + V(e_i)]
\]

The analogous program that generates the maximum surplus under the groups regime\(^\text{14}\) is:

**Program 3-g**

\[
S^t_{g}(\tilde{w}_1, \tilde{w}_2) \equiv \max_{\pi^t_g} \sum_{q,e,v,\mu} \pi^t_g(q, e, v, \mu)[q_1 + q_2 + \tilde{S}^t(v_1, v_2)] \quad (31)
\]

\(^\text{14}\)Notice that although we do not impose the risk sharing rule explicitly here, it still must be valid (approximately since we work with a limited grid). This follows because the utility function is separable on consumption and utilities are weighted by the Pareto weights inside groups \( \mu \).
\[
\sum_{q,e,v,\mu} \pi_g^t(q,e,v,\mu)(v_i + V(e_i)) = \bar{w}_i, i = 1, 2,
\] (32)

\[
\sum_{q,v} \pi_g^t(q,e,v | \mu) \sum_i \mu_i[v_i + V(e_i)] \geq \sum_{q,v} \pi_g^t(q,e,v | \mu) \frac{p(q | \bar{e})}{p(q | e)} \sum_i \mu_i[v_i + V(e_i)], \forall e, \bar{e}, \mu.
\] (33)

\[
\sum_v \pi_g^t(q,\bar{e},v | \mu) = p(q | \bar{e}) \sum_{v,q} \pi_g^t(q,\bar{e},v | \mu), \forall q, \bar{e}, \mu
\] (34)

\[
\pi_g^t \geq 0 \text{ and } \sum_{q,e,v,\mu} \pi_g^t(q,e,v,\mu) = 1.
\] (35)

\[
\sum_{q,e,v} \pi_g^t(q,e,v | \mu) \left[ q_1 + q_2 - \widetilde{S}^t(v_1, v_2) \right] \geq \sum_{q,e,v} \pi(q,e,v) \left[ q_1 + q_2 - \widetilde{S}^t(v_1, v_2) \right]
\] (36)

for any \( \pi \) in the set:

\[\tilde{\Psi}_r(\mu, \pi_g^t) \equiv \{ \pi(q,e,v) : \pi \text{ is positive, sums to 1 and satisfies (33) and (2) for any } e \text{ given } \mu \text{ and } \sum \pi(q,e,v,\mu) \sum_i \mu_i[v_i + V(e_i)] \geq W_\mu(\pi_g^t) \}, \text{ where } W_\mu(\pi_g^t) = \sum_{q,v} \pi_g^t(q,e,v | \mu) \sum_i \mu_i[v_i + V(e_i)] \]

With the surplus functions in \( t \) for groups and relative performance, we are finally able to provide an approximation to the surplus function at \( t \), \( S^t(w_1, w_2) \). The surplus is approximated by:

**Program 4**

\[
S^t(w_1, w_2) \simeq \max_{\tilde{\pi}_g, \pi_r} \sum_{\bar{w}_1, \bar{w}_2} \tilde{\pi}_g^t(\bar{w}_1, \bar{w}_2) S_g^t(\bar{w}_1, \bar{w}_2)
\] (37)

\[
+ \sum_{\bar{w}_1, \bar{w}_2} \tilde{\pi}_r^t(\bar{w}_1, \bar{w}_2)(S^t(\bar{w}_1, \bar{w}_2) - k),
\]

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st.

\[ w_i = \sum_{\tilde{w}_1, \tilde{w}_2} \tilde{w}_1 \left( \tilde{\pi}_g^t(\tilde{w}_1, \tilde{w}_2) + \tilde{\pi}_r^t(\tilde{w}_1, \tilde{w}_2) \right), \]

\( i = 1, 2. \)

\[ \sum_{\tilde{w}_1, \tilde{w}_2} (\tilde{\pi}_g^t(\tilde{w}_1, \tilde{w}_2) + \tilde{\pi}_r^t(\tilde{w}_1, \tilde{w}_2)) = 1, \tilde{\pi}_g^t, \tilde{\pi}_r^t \geq 0, \]

where \( S_g^t(\tilde{w}_1, \tilde{w}_2) \) and \( S_r^t(\tilde{w}_1, \tilde{w}_2) \) are defined in problems 3-r and 3-g.

With fine enough grids for \( \tilde{w}_1, \tilde{w}_2, v_1, \) and \( v_2, \) the solution of program 4 is a good approximation to the solution of program 1. The proposed reformulation reduce considerably the dimension of the linear programming problems. Suppose again that \( Q, E, C, W \) and \( M \) have, respectively, 2, 2, 18, 30 and 101 elements, and that we adopt a grid of interim utilities with 26 elements and of \( \tilde{w}_i \) with 30 elements. The decision vector in Program 2 has 291,600 elements and the problem has 3 constraints, Program 3-r has a choice vector with 10,816 elements and has 21 constraints, and Program 3-g has a choice vector with 1,092,416 elements and has 1,821 constraints. Program 4 has a decision vector with 900 elements and 3 constraints. Programs 3-g, 3-r and 4 are solved for a grid with 900 elements. Program 2 for a grid of 676 elements.

Although the dimensions of all these problems are considerably smaller than that of Program 1, Program 3-g is still slow, especially if there is the need to produce successive iterations of a whole grid to obtain the infinite periods solution. In order to obtain a faster approximation of \( S_g^t(\tilde{w}_1, \tilde{w}_2) \), we use the fact that the from (36) any arrangement for a group in the solution must maximize the sum of utilities weighted by \( \mu \) conditional on an incentive constraints that depend on \( \mu \) and a technological constraint and a surplus level. This problem can be solved separately for each \( \mu \), which reduces significantly the dimension of the problem. In our numerical computation, we depart from arrangements obtained this way to compute the solution given by program 4.