

Games of Commitment: Independence & Unanimity

F. Nava

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Abstract. *This paper investigates the effects of commitments to rule out strategies on the analysis of a strategic form game. Two extreme classes are discussed in detail: in one commitments arise from unilateral pledges, while in the other a unanimous agreement is required for a commitment to be enforced. The set of payoffs implemented by structures belonging to either class is identified. If players have to commit at once, though pledges are generally beneficial, there is no guarantee that any pledge in either class be efficient. Conditions for commitments to implement efficient allocations are presented. But if players have several rounds in which to commit, it is shown that there always exist unanimous commitments and a sufficiently large number of contracting rounds to approximately implement any allocation exceeding the worst Nash payoff of all players. Thus, even in non-convex games, efficient rationalizable allocations can be implemented by pure commitments if players have several instances in which to agree.*

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1. Introduction and Related Literature

Intrinsic to the definition of game is the ability of players to fully commit to any of their available actions at any choice instance.¹ Such definition takes action sets as exogenous elements of the problem and presumes that players can under no circumstance rule out any of the available strategies before the game takes place. If, indeed, such description is appropriate for many strategic environments, it does abstract from the institutions determining which choices are available to players. In fact, if actions sets were determined by players' commitments to rule out strategies, such abstraction would focus the analysis on the interim payoffs of the game. An alternative approach consists of, explicitly, taking into account such options to rule out strategies before the game takes place, thereby making choice sets endogenous. Actions chosen would then necessarily depend on commitments previously undertaken.

Different institutions enabling players to rule out strategies can be envisioned. Some have commitments arising from unilateral pledges, as in the case of a producer setting its capacity, others from joint agreements amongst players, as in the case of a traditional marriage. While either type of institution can enhance cooperation amongst players by eliminating tempting, but costly strategies, commitments undertaken and actions played in equilibrium may significantly differ.

The analysis characterizes the set of allocations implemented by different institutions that can be used to commit the players' course of action. The methodology adopted extends the original game by modeling explicitly how commitments arise before actions are chosen. Any extension in which all players' choices that precede the game only determine the set of feasible actions in the game is referred to as commitment game. The main conclusion of study is that as the number of instances in which to pledge increases, the need for random pledges vanishes. In proving such result, properties of different commitment institutions will be discussed.

When commitments arise from unilateral pledges, an extension of the game only specifies the set of commitments to which each player can pledge. The characterization of the payoffs implemented by mechanisms belonging to this class, shows how unilateral commitments can, partially, internalize the order of move in a game. As shown in [22], conditions on the original game can be provided that guarantee that unilateral pledges implement equilibria of the original game modified so that all but one player move first. In such extensions a trade-off between the

¹Possibly conditioning choice of commitment to the outcome of some randomization device.

benefits of moving early and the loss in the flexibility of punishing opponents can arise. Such trade-off may cause pledges that do not entirely limit one's ability to act to be optimal. Though, unilateral pledges can lead to mutually beneficial outcomes, they do not, necessarily, implement any efficient outcome. Exemplary cases show how such pledges can implement efficient payoff profiles that are not self-enforcing.

A small literature, discussed in detail below, investigates the consequences of unilateral commitments on strategic behavior. The analysis of this class of extensions generalizes results, concurrently developed by Renou in [22], by making the commitment devices present in the economy arbitrary and by allowing players to randomize upon their choice of commitment.

Unilateral pledges are seldom efficient. Intuition dictates that more stringent requirements on the consensus necessary to rule out strategies may warrant the implementation of efficient outcomes, since any player's refusal to commit would carry bigger consequences on which actions are available to others.

When commitments are jointly determined, an extension of the game will need to specify not only a set of feasible contracts, but also the nature of the agreement amongst players necessary to enforce each of those contract. The most cooperative class of commitment extensions requires a unanimous agreement in order to rule out any strategies. In such environments, any player can object to any commitment that modifies the structure of original game.

The characterization of the payoff implemented by mechanisms belonging to this class, shows that, if players have a single period in which to commit and if pledges can only deterministically rule out strategies, not necessarily can any efficient allocation be implemented. The concavity of the payoff hull of the original game can, indeed, lead to such negative conclusion. It may appear that in order to overcome such shortcoming in the implementation result lotteries on commitments ought to belong to the set of feasible contracts. In fact, such assumption would result in the implementation of all allocations exceeding each players worst Nash outcome, since any deviation from such contracts could be discouraged by appropriate beliefs about equilibrium play in the subgame.

Surprisingly however, such random contract may never be necessary to implement the very same payoff hull if players have several stages in which to unanimously commit. Any such commitment extension has a specified number of rounds in which players can agree to rule out strategies, not previously eliminated.² The central result to the analysis of unanimous commitment games, asserts that, under weak dimensionality conditions on the original game, any payoff profile that exceeds the worst Nash outcome can be approximately implemented as the number of rounds in which to commit increases. Moreover only contracts that fully commit players to a single pure strategy profile of the original game are necessary to attain such result. The claim hinges on the existence of continuation values inducing each player to occasionally concede to unfavorable contracts at various stages, in order to support concession by the others' to contracts favoring him at later stages. Hence, time suffices in forming beliefs about sequences of concessions to deterministic contracts that implement any payoff profile that can be obtained in a single instance with random contracts. Any such sequence of beliefs can be interpreted as a description of the bargaining process amongst players trying to sign one of such deterministic contracts. An implication of such claim is that even when a single round of deterministic commitment is inefficient, not observing a random contract being signed should not, *per se*, be interpreted as a signal for inefficiency.

²In these extensions time is immaterial. It merely defines the number of rounds left to negotiate and the set of available contracts conditional on agreements previously signed.

Though increasing either the set of feasible contracts, or the number contracting rounds implements the same outcome set, the contracts signed in either scenario do significantly differ. Indeed, increasing the number of rounds often results in players signing with positive probability contracts that would never be individually rational to them with a single commitment instance. The simplicity of a stochastic extension is counterbalanced by the considerable number of contracts required in the implementation and by the empirical counterpart of such contracts that often involve lottery-contracts being signed. A multi-round extension, though technically more involved, requires far less contracts and all of them deterministic to realize the same objective.

The study, also, shows that the very same result obtains without any dimensionality condition on the original game if players can at each round commit to commitments signed at the next round. Thus, a richer hierarchical commitment structure enhances the generality of the implementation claim.

1.1. Underlying Questions

The analysis addresses the following questions: (1) what payoff profiles can be implemented if pledges are unilateral and what profiles if a unanimous agreement is required to commit? (2) for which conditions on the original game will commitments never lead to efficiency gains? and what conditions would grant the implementation of all efficient payoff profiles? (3) can increasing the number of rounds in which players commit, lead to the implementation of all efficient payoff profiles, when such allocations cannot be implemented in a single instance? (4) what pledges should be observed and would random contracts ever be necessary to implement any payoff profile?

1.2. Related Literature

The idea that unilaterally limiting one's flexibility at the choice instance can be beneficial is deeply rooted in the game theoretic literature. Many examples of the advantages and limitations of commitment can be found in Schelling's classic [23]. A part of the literature on the topic investigates how bargaining impasses can arise from unilateral and uncertain commitments to favorable positions. Crawford [7] shows why rational players, when bargaining, may elect to commit, even when doing so leads with some positive probability to an inefficient impasse. Powell [21] shows how such conflicting commitments arise when compromise solutions are not enforceable. Ellingsen and Miettinen [10] extend results by making commitments certain introducing small costs to commit. Their analysis shows that costs reduce the multiplicity of equilibria and that equilibria in which the probability of an impasse tends to one can arise, as costs tend to zero. Hart and Moore [15] discuss the role of unilateral pledges in an environment with one sided incomplete information and bargaining. They show how a trade-off between commitment and flexibility could arise in the choice of the optimal pledge.

More recently, a literature has developed relating unilateral commitments to the endogenous timing of moves in a game. In the context of duopoly game Hamilton and Slutsky [14] show how the ability to unilaterally commit to any single action, by endogenizing the timing play, may implement the leader-follower equilibria of the game. Van Damme and Hurkens [8] provide conditions under which the equilibria of the original game are robust to unilateral commitments to single pure strategies. Renou [22] extends results further allowing all players to unilaterally commit to any subsets of their action set. For such technology, necessary and sufficient conditions are provided for implementation of any pure strategy profile in the commitment expansion of the

original game. Therein, it is shown that commitments to single pure strategies may not suffice for the implementation of an efficient allocation, even when unilateral pledges to subsets of strategies do. Again, a trade-off between commitment and flexibility can arise, which in turns crates a rationale for the optimality of limited commitment. The results presented section 5 generalize the implications of such models. A related paper by Bade, Haeringer and Renou [3] extends results to well-behaved two players continuous games.

1.3. Roadmap

The paper proceeds as follows: section 2 presents a Cournot duopoly example motivating the analysis, section 3 introduces structure and properties of the commitments studied; section 4 defines the equilibria of the game extended by commitments, argues existence, and provides simplifying results; section 5 characterizes payoffs implemented by unilateral commitments, and provides conditions for efficiency and examples; section 6 characterizes the set of payoffs implemented by unanimous agreements, gives sufficient conditions for efficiency, shows that no deterministic commitment necessarily supports any efficient profile and discusses the benefits of joint pledges; section 7 argues that whenever agents have arbitrarily many stages in which to commit, all allocations exceeding each player's Nash threat, including the efficient ones, are implementable. Therein, it is shown how harmful concession at each commitments stage may be motivated by beliefs about favorable concessions at later stages. Section 8 strengthens such result by providing conditions for which the implementation claim holds true even when player are bound to use only commitments that tie them to a single pure strategy profile of the original game. Section 9 reports examples motivating theory developed. Section 10 concludes and discusses related projects. In appendix (section 11) it is possible to find proofs omitted from the main text. An extended web-appendix with more results and examples will be posted soon.

2. A Duopoly Example

Consider a scenario in which two duopolists compete on the quantity of output to supply to a market. Suppose that the two duopolist make use of the same technology, implicitly defined by the quadratic total cost function, $c(q) = 6q - 3q^2$ for $q \in [0, 1]$.³ Assume that the price at which goods are demanded in the market is linear in the aggregate quantity produced, $Q = q_1 + q_2$. Specifically, let it be defined by $d(Q) = 10 - 3Q$. Therefore, for any possible profile of outputs, $(q_i, q_j) \in [0, 1]^2$, profits for producer $i \neq j \in \{1, 2\}$ would reduce to:

$$u_i(q_i, q_j) = d(q_i + q_j)q_i - c(q_i) = 4q_i - 3q_iq_j$$

The demand and the marginal cost curve for this market are plotted in of figure 1A.

If the two producers must choose output independently, no course of action can lead to profit profiles outside:⁴

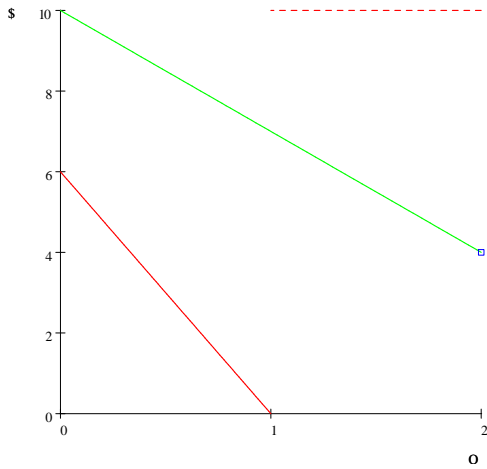
$$\mathcal{I} = \left\{ (u_1, u_2) \in \mathbb{R}^2 \mid u_2 \leq 16/3 + u_1 - 8\sqrt{u_1/3}, u_1 \geq 0 \ \& \ u_2 \geq 0 \right\}$$

The concavity of \mathcal{I} implies that some feasible allocations can only be obtained by coordinating the output choices by the two producers. In fact, the set of feasible ex-ante payoff profiles, denoted by \mathcal{U} , consists of all payoffs in $\text{co}(\mathcal{I})$. The unique Nash equilibrium for this, quantity

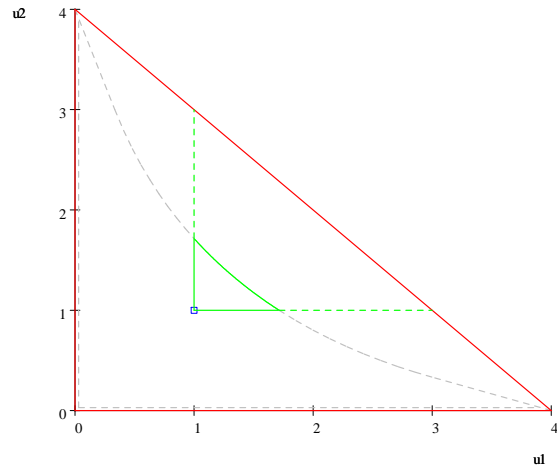
³Marginal costs are decreasing on $[0, 1]$, as it is immediate to check.

⁴By linearity of each producer's profit function in its own choice, any independent randomization is payoff equivalent to some deterministic choice of output.

competition, duopoly model, requires both firms to produce at full capacity, $q_1 = q_2 = 1$ and make unit profits.⁵ Figure 1B depicts the aforementioned payoff sets.



1A: green demand, red a firm's marginal costs, blue the Nash equilibrium



1B: red \mathcal{U} , gray \mathcal{I} , blue \mathcal{N} , green dashed $\underline{\mathcal{N}}$ & solid \mathcal{K}_{UP}

In such duopoly game, because any action that does not involve producing at full capacity is strictly dominated, no player would benefit from limiting unilaterally his output capacity. In fact, no capacity chosen by a firm could ever induce the other producer to reduce output below capacity. Thus, no such commitment would ever be undertaken in equilibrium.

If, instead, producers had the ability to limit their choice set by jointly committing to deterministically reduced output capacities, it would be possible to increase the set of allocations implemented as a equilibria of the enlarged game to include all payoff profiles in \mathcal{I} that exceed the Nash outcome of either player. For instance consider how payoff profile $(4/3, 4/3)$ could be implemented by having both producers jointly pledge to limit their capacity to $2/3$. For each producer, signing such contract would yield additional profits, given that any deviation at the contract signing stage would necessarily lead to the unique Nash outcome of the original game and unit profits.

It could appear that the implementation of the efficient payoff profiles of the game requires the use of a stochastic contracts that make either producers the monopolist with a probability in $[1/4, 3/4]$. For instance, a probability of $1/2$ would yield to both producers ex-ante expected profits of 2.

The analysis presented shows that any such allocation can, also, be implemented by deterministic pledges to rule out strategies, if producers have sufficiently many rounds in which to jointly pledge to reduced capacities. Thus, in this setup having multiple stages in which to commit defies the need for randomness in contracts. Indeed, consider how the duopolists could with two rounds of deterministic contracting implement the profit allocation $(1, 2)$, that lies outside \mathcal{I} . With a single stage of contracting, commitments to capacities $(0.5512, 0.4262)$ would support profits $(1.5, 1)$ as an equilibrium, because both producers would benefit. If producers could commit to capacity contracts at two stages, even payoff profile $(1, 2)$ could be supported. In fact, suppose that at the second stage there be at least three different contracts that, if signed, commit the players to make the second producer the monopolist by shutting down the first firm. If both players have to agree for such contract to be enforced, there is an equilibrium with the desired payoffs independently of the other devices present in this economy. In such equilibrium both produces

⁵The unique Nash equilibrium is, also, the unique correlated equilibrium.

sign up to three of such devices with equal probability, π , and miscoordination on one of these devices leads players to sign the contract (1.5, 1) at the latter round. If any deviation from such strategy is punished by the other producer with the refusal to contract any further, the described strategies constitute an equilibrium of any voluntary commitment expansion of the game which includes capacity profiles, (0.5512, 0.4262) and (0, 1). Since:

$$\begin{aligned} u_1(\pi_1, \pi_2) &= (1/3)0 + (2/3)1.5 \geq 1 \\ u_2(\pi_1, \pi_2) &= (1/3)4 + (2/3)1 \geq 1 \end{aligned}$$

Such strategy profiles, thus, implement an ex-ante profits did not belong to the equilibrium set of any single stage commitment expansion. The analysis will show how, with arbitrary many rounds of deterministic contracting, it is possible to implement any profit profile exceeding each player's worst Nash threat, as a subgame perfect equilibrium of the enlarged contracting game.

This modeling may be used to rationalize observing producers voluntarily committing to sign with positive probability contracts that are never beneficial to them, such as one making the other producer the monopolist. Such harmful commitments arise from the necessity by any producer to concede in some event to the other being the monopolist in order to induce the rival to concede to him monopoly power in some other events at latter contracting stages, if an agreement has not materialized at the current instance. Indeed, having the multiple rounds of commitment results in an implicit equilibrium theory of producers bargain upon which contract to sign.

Thus, in the duopoly example, the empirical counterpart of a collusive outcome, with ex-ante profits of 2 for each producer, may involve observing contracts that shut down one of the two firms with certainty, being signed. Rather than observing contracts that randomize upon which firm to shut down.

3. Structure

3.1. Definitions: Payoff Hulls & Equilibria

Consider a strategic form game $\Gamma = \{N, \{A_i, u_i\}_N\}$, for N the set of players, A_i the set of actions and $u_i : A \rightarrow \mathbb{R}$ the utility map, for any player $i \in N$.⁶ Throughout the paper, $\Delta(\cdot)$ denotes the map from a set to its simplex.⁷ For any player $i \in N$, let $\Delta_i = \Delta(A_i)$ and let $\Delta_N = \times_{i \in N} \Delta(A_i)$.

Define set of ex-ante and ex-post feasible payoff profiles, respectively, by:

$$\mathcal{U}(\Gamma) = \{u(\mu) \in \mathbb{R}^N \mid \mu \in \Delta(A)\} \quad \& \quad \mathcal{A}(\Gamma) = \{u(a) \in \mathbb{R}^N \mid a \in A\}$$

The set of payoff profiles belonging to the weak Pareto frontier of the game is, then, denoted by $\mathcal{P}(\Gamma)$.⁸ Also, let $\mathcal{I}(\Gamma) = \{u(\sigma) \in \mathbb{R}^N \mid \sigma \in \Delta_N\}$ denote the set of feasible payoffs if players are bound not correlate their strategies. By construction any payoff profile that belongs to Nash equilibrium payoff hull, $\mathcal{N}(\Gamma)$, must also belong to the independent hull, $\mathcal{N}(\Gamma) \subseteq \mathcal{I}(\Gamma)$. The Nash equilibrium strategy set is denoted by $\mathcal{E}^{\mathcal{N}}(\Gamma) \subseteq \Delta_N$.⁹

⁶We adopt the following common conventions: $A = \times_{i \in N} A_i$, $A_{-j} = \times_{i \in N \setminus j} A_i$ & $A_j = \times_{i \in j} A_i$ for $\forall j \in 2^N$. Where by 2^N we denote the power set of the set N . Similar conventions are adopted for elements.

⁷For X a set, $\Delta(X) = \{\mu \in \mathbb{R}_+^X \mid \sum_X \mu(x) = 1\}$.

⁸Specifically, define:

$$\mathcal{P}(\Gamma) = \{u \in \mathcal{U}(\Gamma) \mid \nexists u' \in \mathcal{U}(\Gamma) : u' - u > \mathbf{0}\}$$

⁹Specifically $\mathcal{N}(\Gamma) = \{u(\sigma) \in \mathcal{U}(\Gamma) \mid \sigma \in \mathcal{E}^{\mathcal{N}}(\Gamma)\}$, for:

$$\mathcal{E}^{\mathcal{N}}(\Gamma) = \{\sigma \in \Delta_N \mid u_i(a_i, \sigma_{-i}) - u_i(\sigma) \leq 0 \text{ for } \forall a_i \in A_i \text{ \& } i \in N\}$$

For latter reference, define the correlated equilibrium payoff polytope by $\mathcal{C}(\Gamma) \subseteq \mathcal{U}(\Gamma)$ and the correlated equilibrium strategy set by $\mathcal{E}^{\mathcal{C}}(\Gamma) \subseteq \Delta(A)$.¹⁰ The correlated equilibrium payoff polytope is known to be a non-empty convex subset of the payoff hull [by Aumann [2] & Nau McCardle [19]], containing all Nash equilibria on the facets of the polytope [by Nau Canovas Hansen [20]].¹¹ Also, by the *revelation principle* for strategic-form games, any equilibrium of the game expanded by means of a system of pre-play communication is equivalent to a some the correlated equilibrium outcome [by Forges [11] & Myerson [16]].¹²

Finally, let *Nash rational* payoff hull consists of all ex-ante feasible payoffs that exceed the worst Nash equilibrium outcome of each player. That is:

$$\underline{\mathcal{N}}(\Gamma) = \left\{ u \in \mathcal{U}(\Gamma) \mid u_i \geq \min_{u' \in \mathcal{N}(\Gamma)} u'_i, \forall i \in N \right\} \supseteq \mathcal{N}(\Gamma)$$

The notation \underline{u}^i is often used to denote a payoff profile that yields worst Nash outcome to player i . That is: $\underline{u}^i \in \arg \min_{u \in \mathcal{N}(\Gamma)} u_i$. The term *Nash threat* is used throughout to refer to such payoff profile.

3.2. Independent and Joint Commitment

Deterministic Commitments. This section describes how commitment expansions of a strategic form game are modeled throughout the paper. First, the framework is introduced, then properties of such expansions are discussed.

In the context of the paper, the term commitment is used to refer to voluntary and fully enforceable pledges to rule out actions before the choice stage. Therefore, commitments do, in general, rule out, but do not rule in strategies at the choice stage. In this setup, each player's pledge can, in principle, depend on the pledges made by others, but will never depend on the actions chosen by others.¹³

For any game, Γ , a commitment structure, $(M, k) = \{M_i, k_i\}_{i \in N}$, consists of a collection of message spaces, M_i , and of a collection of correspondences mapping from message profiles to the action simplexes: $k_i : M \rightrightarrows \Delta_i$.¹⁴ Throughout the paper, to guarantee existence and to avoid mixing on mixtures, assume that commitment correspondences are closed and convex valued.¹⁵ Additionally, to ease the analysis, only consider commitment structures with a finite messaging spaces. For a given commitment structure (M, k) , the extended game by $\{N, \{M_i, k_i, A_i, u_i\}_N\}$ consists of a two stage simultaneous move game. At first, players choose which messages to send, then commitments are realized based on reports and actions are chosen within the, consequently, restricted strategy sets. Assume that all player at the choice stage know all messages sent at the commitment stage. *Perfect monitoring* of the messages makes the extended game one of complete information and enhances the ability to punish deviating players.

¹⁰That is, $\mathcal{C}(\Gamma) = \{u(\mu) \in \mathcal{U}(\Gamma) \mid \mu \in \mathcal{E}^{\mathcal{C}}(\Gamma)\}$, for:

$$\mathcal{E}^{\mathcal{C}}(\Gamma) = \left\{ \mu \in \Delta(A) \mid \sum_{a_{-i}} \mu(a) (u_i(a'_i, a_{-i}) - u_i(a)) \leq 0 \text{ for } \forall a_i, a'_i \in A_i \text{ \& } i \in N \right\}$$

¹¹That is: $\emptyset \subset \mathcal{N}(\Gamma) \subseteq \partial \mathcal{C}(\Gamma) \subseteq \mathcal{C}(\Gamma)$. Whenever ∂X denotes the boundary of set X .

¹²In fact, $\mathcal{N}(\Gamma(\mathcal{C})) = \mathcal{C}(\Gamma)$ for a large enough class of communication mechanisms, \mathcal{C} .

¹³This assumption is invoked in order to guarantee existence of equilibria at the action stage. In fact, if such assumption were not invoked, it could be that the extent of each player's commitment be unknown to him, when choosing which action to play [if others randomize]. This would undermine assumptions necessary that grant existence.

¹⁴That is $k_i : M \rightarrow \mathcal{F}(\Delta_i)$. Where $\mathcal{F}(\Delta_i)$ denotes the power set of the action space simplex of agent i .

¹⁵So long as agent can to randomize conditional commitments, there would be no need of assuming the commitment correspondence to be convex valued. But for notational ease, such assumption is maintained.

$$\begin{array}{ccc} t = 0 & t = 1 & t = 2 \\ \hline M & k(m) \subseteq A & U \end{array} \xrightarrow{\text{blue arrow}}$$

A commitment structure is *voluntary*: if for any agent $i \in N$, there exists message $m_i \in M_i$ such that $k_i(m) = \Delta_i$ for any profile of messages of the others $m_{-i} \in M_{-i}$. In any such structure, if an agent ever commits, he must do so voluntarily, because he could have prevented any commitment on his part by sending the no commitment message, m_i . The analysis focuses on voluntary structures, because they entail an inalienable right for players to control actions and restraints that affect their own choice set in the original game.

A commitment structure is *with veto* whenever for any agent $i \in N$, there exists message $m_i \in M_i$ such that $k(m) = \Delta_N$ for any profile of messages of the others $m_{-i} \in M_{-i}$.¹⁶ Such a commitment structure requires all to agree for anyone to commit.

A commitment structure is said to be *independent* (or unilateral) if it is voluntary and if for any player $i \in N$ and message profile $m \in M$:

$$k_i(m) = k_i(m_i, m'_{-i}) \quad \text{for } \forall m'_{-i} \in M_{-i}$$

Such a mechanisms are without veto, since no player can ever block unilateral commitments undertaken by others. Indeed, independence implies that $k_i : M_i \rightrightarrows \Delta_i$ for any $i \in N$. A particular type of independent commitment structure is discussed in detail by Renou in [22].

A commitment structure is said to be *unanimous* if it is with veto and if $k_i(m) \neq \Delta_i$ for some $m \in M$ and $i \in N$, implies that, for $\forall j, l \in N$:

$$k_l(m'_j, m_{-j}) = \Delta_l \quad \text{for } \forall m'_j \in M_j \setminus m_j$$

A unanimous structure requires an agreement amongst all players to enforce any given pledge, because any player can block any commitment. For convenience, define a unanimous commitment structure to be in *canonical form* if $M_i = M_*$ for any $i \in N$ and that $k_i(m) \neq \Delta_i$ for some $i \in N$ implies that $m = m_i^N \in M$. The only restriction upon simple unanimity is that all players have a common messaging space. Pledges in canonical form unanimous supergame can, thus, arise only on the main diagonal of the game. Focusing on such subclass simplifies the analysis, while posing no limitation on implementation of any particular allocation. For instance:

$1 \setminus 2$	m_1	m_2	m_3	...
m_1	$k(m_1^2)$	Δ_N	Δ_N	...
m_2	Δ_N	$k(m_2^2)$	Δ_N	...
m_3	Δ_N	Δ_N	$k(m_3^2)$	Δ_N
...	Δ_N	...

The analysis, currently, investigates these two extreme classes of commitment devices. The study of structures requiring agreement only amongst those committed, also, lies within the goals of the theory presented.¹⁷ Define the set of closed and convex valued commitments structures by:

$$K(A) = \{(k, M) \mid k : M \rightrightarrows \Delta_N, \text{ closed convex valued}\}$$

¹⁶Any commitment structure with veto is voluntary, but the converse is not generally true.

¹⁷A commitment structure is *direct* whenever $M_j = k(M) \subset \Delta_N$, for any agent $j \in N$.

An intermediate form of commitment may require agreement only by those committed. Indeed a direct commitment structure is said to be *consensual*, if it is direct and if for any agent $i \in N$ it satisfies:

$$k_i(m) = \begin{cases} \text{proj}_i m_i & \text{if } m_i = m_j \text{ for } \forall j \in N(m_i) \\ \Delta_i & \text{if } \text{otw} \end{cases}$$

for $N(m_i) \equiv \{j \in N : \text{proj}_j m_i \neq \Delta_j\}$. Consensual commitment structure is voluntary, but without veto.

Similarly, let $U(A)$ and $I(A)$, respectively, define the sets of possible unanimous and independent commitment structures. That is, $U(A) = \{(k, M) \in K(A) | k \text{ unanimous}\}$. When clarity is not compromised, only the map k may be used to denote a commitment structure (k, M) .

Digression on Pure. A relevant class of commitment mechanisms, $K_P(A) \subset K(A)$, allows only pledges that rule out pure strategies of the original game. Mechanisms belonging to the such class are called *pure commitments* and require agents to commit play within some subset of their action set. That is, $k \in K_P(A)$ if $k_i(m) = \Delta(\bar{A}_i)$ for some $\bar{A}_i \in 2^{A_i} \setminus \emptyset$, for any profile of messages $m \in M$ and for any player $i \in N$.¹⁸ For notational convenience let $U_P = K_P \cap U$ and $I_P = K_P \cap I$.

Stochastic Commitments. All the mechanisms described up to this subsection were deterministic. Because the analysis shows when focusing on deterministic devices entails no loss, consider a richer class of mechanisms that includes random commitment devices. In such scenario, players can pledge to lotteries that, if agreed upon, randomly enforce profiles of commitments. Again, as a benchmark, assume that after the lottery is performed all information about messages sent and commitments realized is released to all agents; perfect monitoring.

Within such class only lotteries on single pure actions and no commitment need to be considered, because no additional allocation can ever be implemented by having stochastic devices that commit to subsets of strategies. Thus for simplicity, let stochastic commitment structure consists of a map from messages to lotteries on profiles of commitments in $\times_{i \in N} [A_i \cup \{\Delta_i\}]$. That is $k : M \rightarrow \Delta(\times_{i \in N} [A_i \cup \{\Delta_i\}])$.¹⁹ Consequently, denote the set of stochastic structures by:

$$K^S(A) = \{(k, M) | k : M \rightarrow \Delta(\times_{i \in N} [A_i \cup \{\Delta_i\}])\}$$

Equilibrium outcomes of a stochastic commitment structure cannot, in principle, be implemented by any deterministic commitment device. Indeed, mechanisms in this class presume that players have some ability to forcefully condition their choice of how to rule out actions upon the outcome of a randomization device. Such delegation ability requires an stronger technological extension of the game, because it can sustain in equilibrium lotteries on commitments that are not self-enforcing for any given deterministic device.

Notation $k(am)$ will be used to denote the probability that players are committed to profile $\bar{a} \in \times_{i \in N} [A_i \cup \{\Delta_i\}]$ given that they signed up to lottery $k(m)$. As, usual messaging spaces will be assumed finite. Note that stochastic independent commitment structure consist of maps $k_i : M_i \rightarrow \Delta(A_i \cup \{\Delta_i\})$ for any player $i \in N$. In this scenario $k(m) = \prod_{i \in N} k_i(m_i)$, for any $m \in M$.

4. Subgame Perfect Equilibria of a Commitment Game

All propositions and remarks in the following sections should be completed with the, systematically, omitted quantifier: “For any complete information strategic form game, Γ, \dots ”. Also, all sets, should be taken to depend on the original game, Γ , as previously outlined. Such dependence is omitted for clarity alone. Also, recall that by construction all claims are for closed and convex valued commitments with a Cartesian product structure.

¹⁸An alternative definition says that $k \in K_P(A)$ if $k_i(m) = 2^{A_i} \setminus \emptyset$ for any profile of messages $m \in M$ and for any player $i \in N$.

¹⁹In general, for \mathcal{F}_i being the power set of Δ_i , a stochastic device consist of a map $k : M \rightarrow \Delta(\times_N \mathcal{F}_i)$.

4.1. Equilibria of the Committed Subgame

This section discusses equilibrium behavior in subgames in which strategies are chosen from convex and compact subsets of the independent strategy simplexes, $k_i(m) \subseteq \Delta_i$ for any $i \in N$. The perfect monitoring assumption focuses the analysis on deterministic subgames.²⁰ The subgame induced by pledges, $k(m)$, is denoted by $\langle u, k(m) \rangle = \{N, \{k_i(m), u_i\}_N\}$. The set of Nash equilibrium strategies and payoffs for such a subgame are, respectively, defined by:

$$\begin{aligned} \mathcal{E}^{\mathcal{N}} \langle u, k(m) \rangle &= \{ \sigma \in k(m) \mid u_i(\sigma_{-i}, \sigma'_i) - u_i(\sigma) \leq 0 \text{ for } \forall \sigma'_i \in k_i(m) \ \& \ i \in N \} \\ \mathcal{N} \langle u, k(m) \rangle &= \{ u(\sigma) \in \mathcal{U} \mid \sigma \in \mathcal{E}^{\mathcal{N}} \langle u, k(m) \rangle \} \end{aligned}$$

Nash equilibria in such subgames exist, because expected utility is linear in the probabilities and because domains are compact and convex. Therefore:

LEMMA 1. *For any convex & compact commitment, $k(m) \subseteq \Delta_N$, in the independent action simplex, $\mathcal{E}^{\mathcal{N}} \langle u, k(m) \rangle \neq \emptyset$.*

If players can credibly rule out mixed strategies, any payoff in the independent hull can be implemented as a subgame equilibrium for an appropriate commitments.

REMARK 2. *For any $\sigma \in \Delta_N$ there exists a $k \subseteq \Delta_N$ such that $\sigma \in \mathcal{E}^{\mathcal{N}} \langle u, k \rangle$.*

Indeed when all players are fully committed to an independent strategy, there is no way to deviate from it in the subgame. More interestingly, for any two commitment subgames that can be ranked by inclusion, if an equilibrium strategy of the larger subgame is not ruled by the restrictions in the smaller subgame, then that strategy profile must, also, be an equilibrium of the smaller subgame. Formally:

REMARK 3. *For any $k \subseteq k' \subseteq \Delta_N$, if $\sigma \in \mathcal{E}^{\mathcal{N}} \langle u, k' \rangle$ and $\sigma \in k$, then $\sigma \in \mathcal{E}^{\mathcal{N}} \langle u, k \rangle$.*

In fact, whenever an equilibrium strategy of the remains available, conditional on the other players behaving according to it, it is, still, optimal for any agent to follow such strategy, because the set of possible deviations from it is smaller than in the larger subgame. Therefore, any equilibrium of the original game remains an equilibrium of any subgame in which the corresponding strategy is not ruled out by the choice of commitments.

4.2. Equilibria of the Commitment Supergame

This section defines the equilibria of the messaging supergame and completes the description of subgame perfect behavior in commitment games.

Deterministic Commitment. For any closed and convex valued commitment structure $(k, M) \in \mathbf{K}(A)$, denoted only by k in this section, define the set of feasible maps from message profiles to Nash continuation payoffs of the committed subgame by:²¹

$$V(k) = \{ v \in \mathcal{U}^M \mid v(m) \in \mathcal{N} \langle u, k(m) \rangle \text{ for } \forall m \in M \}$$

²⁰Thus, in the case of commitments stochastic, one should take, in this section, $k(m)$ to represent the realization of the commitment lottery.

²¹Such object is well defined because of the non-emptiness of equilibrium correspondence for compact convex commitment.

Implicit in such definition of continuation values is the ability of agents to condition their course of action not only upon the commitments realized, but also upon messages sent by other players at the contracting stage. Therefore, equilibrium action profiles may differ in the same restricted action domain $k(m) = k(m')$, whenever such commitments originate from different message profiles, $m \neq m'$. This is a direct consequence of the perfect monitoring assumption discussed and invoked previously. Weaker monitoring structures limiting the punishment ability of the players could reduce the set of implementable payoffs.

For any feasible profile of continuation payoffs, v , denote the simultaneous move commitment supergame by: $\langle v, M \rangle = \{N, \{M_i, v_i\}_N\}$. The set of equilibrium payoffs of a game, Γ , expanded by means of commitment structure, (k, M) , thus, comprises all Nash equilibrium payoff profiles of any the feasible game $\langle v, M \rangle$. That is to say:

$$\mathcal{K}(k) = \cup_{v \in V(k)} \mathcal{N} \langle v, M \rangle$$

Such definition characterizes the subgame perfect equilibria of expanded two stage game. Behavioral strategy profiles of a commitment game are denoted by (π, σ) and satisfy $\pi \in \times_{i \in N} \Delta(M_i)$ and $\sigma(m) \in k(m)$, for any $m \in M$. Occasionally, a continuation maps in $V(k)$ is denoted by $u(\cdot|\sigma) = u(\sigma(\cdot))$, for $\sigma(m) \in \mathcal{E}^N \langle u, k(m) \rangle$ the behavioral strategy following a message profile $m \in M$.²²

An alternative definition, $\mathcal{K}^*(k) = \cup_{v \in V(k)} \mathcal{C} \langle v, M \rangle$, identifies payoffs implemented by commitments if communication can precede the contracting stage. Since $\mathcal{N} \langle v, M \rangle \subseteq \mathcal{C} \langle v, M \rangle$, it follows, immediately, that:

$$\mathcal{K}(k) \subseteq \mathcal{K}^*(k)$$

Stochastic Commitment. The definition of stochastic device adopted implies that players are free to choose amongst their actions only if not committed, since otherwise their choices are completely determined. For any $m \in M$, $k(m)$ consists of a distribution on $\times_{i \in N} [A_i \cup \{\Delta_i\}]$. Thus, if $k \in K^S(A)$, the requirement on the continuation payoffs becomes that:²³

$$V(k) = \{v \in \mathcal{U}^M | v(m) = z(k(m)|m) \text{ for } [z(\bar{a}|m) \in \mathcal{N} \langle u, \bar{a} \rangle \text{ for } \forall \bar{a} \in \times_{i \in N} [A_i \cup \{\Delta_i\}]] \text{ for } \forall m \in M\}$$

In this context, the equilibrium definition for the commitment supergame, $\mathcal{K}(k)$, remains unchanged with respect to the deterministic case, but for replacing the deterministic continuation values with the stochastic ones.

5. Implementation & Properties for the Independent Class

This section presents the characterization of the set of implementable allocations for independent commitment extensions of a complete information games. Though developed independently, the results presented generalize a claim established by Renou, in [22]. That work focuses on pure strategy implementation of pure action profiles by means of a specific unilateral commitment structure. His results are strengthened, first by looking at the pure strategy implementation of mixed action profiles of arbitrary structures and then by focusing on the general implementation

²²Thus, for any behavioral strategy (π, σ) :

$$u_i(\pi|\sigma) = \sum_M \pi(m) u_i(m|\sigma) = \sum_M \pi(m) \sum_A \sigma(a|m) u_i(a) = \sum_A u_i(a) \sum_M \sigma(a|m) \pi(m)$$

²³For $z(k(m)|m) = \sum_{\times_{i \in N} [A_i \cup \{\Delta_i\}]} k(\bar{a}|m) z(\bar{a}|m)$.

result.²⁴ Because the more novel contributions of the paper are for the unanimous class, the section will be kept concise.²⁵

For $(k, M) \in \mathbf{I}$, let $\underline{u}^i \langle k(m) \rangle$ denote a utility profile associated to a Nash threat of player $i \in N$ in the committed subgame $\langle u, k(m) \rangle$. Also, let $w_i(m_{-i}|k)$ denote the highest payoff that player i could obtain knowing the other players will be committed to $k_{-i}(m_{-i})$ and that they will play Nash threat against him at any latter stage. Specifically:

$$\underline{u}^i \langle k(m) \rangle \in \arg \min_{u' \in \mathcal{N} \langle u, k(m) \rangle} u'_i \quad \& \quad w_i(m_{-i}|k) = \max_{m_i \in M_i} \underline{u}^i \langle k(m) \rangle$$

Then the first proposition characterizes what payoffs can be implemented when players never randomize on how to commit.

PROPOSITION 4. *For any $(k, M) \in \mathbf{I}$, strategy profile $\sigma \in \Delta_N$ is implemented in pure strategies by commitment $k(m)$ if and only if $\sigma \in \mathcal{E}^N \langle u, k(m) \rangle$ and $u_i(\sigma) \geq w_i(m_{-i}|k)$, for $\forall i \in N$.*

Where “is implemented in pure strategies” means that commitment strategies, π_i , are pure for any player $i \in N$. The claim fully characterizes the strategy profiles that can be implemented in pure strategies by any independent commitment structure.²⁶ Any allocation implemented in pure strategies, $u(m|\sigma) \in \mathcal{K}(k)$, necessarily belongs to the independent hull of the original game, \mathcal{I} , since $u(m|\sigma) = u(\sigma)$. The above proposition holds true, even, for pure strategy commitment structures, $(k, M) \in \mathbf{I}_P$.²⁷ The original result by Renou focused on the implementation of pure strategy profiles, $\sigma(a) = 1$ some $a \in A$, through pure commitment structures.

It is, always, possible to construct an independent commitment structure that implements in pure strategies any Nash equilibrium of the original game. But, such structure may require commitments to mixed strategies if equilibrium is not pure. Indeed, if all players can and do commit to a Nash equilibrium strategy, nobody can benefit from a deviation at the commitment stage. But unless the original equilibrium is in pure strategies, such commitment structure cannot be pure, in \mathbf{I}_P . Moreover, the only strategies that can be implemented by having all players commit with certainty to a single mixed strategy of the original game are the Nash equilibria of that game.

Another property of such game extensions is that, whenever all actions in the original game can be ranked by all players by strict dominance, making use of independent commitments cannot enlarge the set of equilibrium payoffs. Thus, whenever this is the case and the Nash equilibria are inefficient, unilateral pledges never implement any efficient allocation. A typical example of such strategic environment is the prisoner’s dilemma.

For $M_i(\pi) = \{m_i | \pi_i(m_i) > 0\}$, the general implementation result for the independent class, asserts that:

PROPOSITION 5. *For any $(k, M) \in \mathbf{I}$, a strategy profile (π, σ) of the game extension is implemented if and only if $\sigma(m) \in \mathcal{E}^N \langle u, k(m) \rangle$ for any $m \in M(\pi)$ and $u_i(\pi|\sigma) = u_i(m_i, \pi_{-i}|\sigma) \geq w_i(\pi_{-i}|k)$, for any $m_i \in M_i(\pi)$ and $i \in N$.*

Such ability to randomize amongst commitment chosen may result in the implementation of payoff profiles outside of the independent hull of the original game, \mathcal{I} . In the next section an

²⁴The first generalization is a straightforward extension of theorem 2 in [22], while the latter departs more significantly from such claim.

²⁵A web-appendix with additional results and clarifications will be posted soon.

²⁶In general, the existence of strategy profile, σ , implemented in pure strategies cannot be guaranteed, since existence may require mixing even at the messaging stage.

²⁷That is, whenever, for any $i \in N$, any message $m_i \in M_i$ leads to a commitment $k_i(m_i) \subseteq A_i \setminus \emptyset$.

example shows how preventing players from mixing at the commitment stage limits the scope of the implementation.

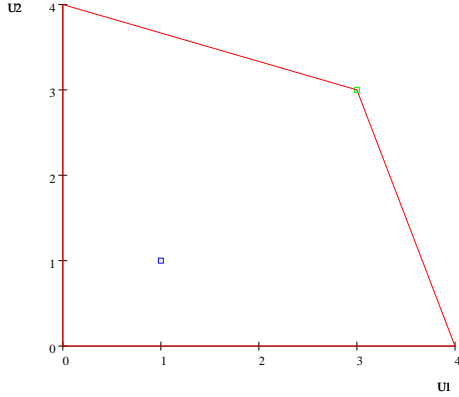
5.1. Examples: Independent Commitments & Efficiency

The first game, due to Renou and depicted in 2A, illustrates the trade-off between commitment and flexibility to punish other players. For $\delta < 1$, consider the game:

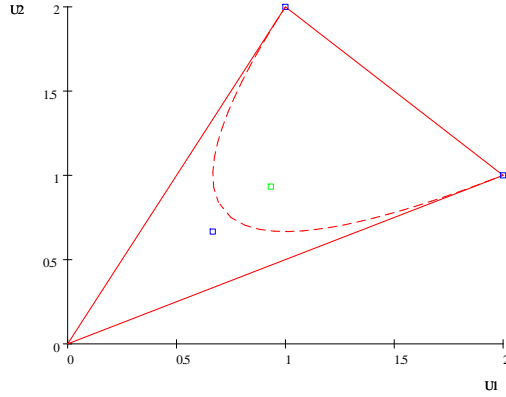
$1 \setminus 2$	a	b	c
a	3, 3	1, 2	0, 4
b	2, 1	0, 0	$\delta, 2$
c	4, 0	2, δ	1, 1

Independent commitments implement the efficient outcome profile (3, 3) only if $\delta \geq 0$. In fact, consider the pure independent structure that has $M_i = 2^{A_i} \setminus \emptyset$ and $k_i(m_i) = m_i$ for any $m_i \in M_i$, for any player $i \in N$. For such structure $m = \{a, b\}^2 \in M$ implements (a, a) as an equilibrium so long as $\delta \geq 0$, because $(a, a) \in \mathcal{E}^N \langle m \rangle$ and because $w_i(\{a, b\}) = 3$ for $\delta \geq 0$. If $\delta < 0$ instead, $w_i(\{a, b\}) = 4$ and a deviation to $\{c\}$ becomes profitable in such structure. Indeed, it is possible to show that no independent structure implements (3, 3) in this case.

For δ positive, both commitment and flexibility are required to implement the efficient outcome, because a full commitment to strategy $\{a\}$ would limit the ability to punish the opponent if it deviates to commitment $\{c\}$, while no commitment would prevent coordination.



2A: $\delta = 0$, red \mathcal{U} , blue \mathcal{N} & green \mathcal{K}_1



2B: red solid \mathcal{U} , dashed \mathcal{I} , blue \mathcal{N} & green \mathcal{K}_1

In the next example, depicted in 2B, randomizing upon commitments increases the set of implementable allocations. Consider the coordination game depicted in the left table:

$1 \setminus 2$	a	b
a	2, 1	0, 0
b	0, 0	1, 2

$1 \setminus 2$	$\{a\}$	$\{b\}$	$\{ab\}$
$\{a\}$	2, 1	0, 0	2, 1
$\{b\}$	0, 0	1, 2	1, 2
$\{ab\}$	2, 1	1, 2	2/3, 2/3

The Nash equilibria of such battle of the sexes have either both players coordinating on one of the two actions or both players mixing and receiving a payoff of 2/3. Again, consider the pure independent commitment structure that has $M_i = 2^{A_i} \setminus \emptyset$ and $k_i(m_i) = m_i$ for any $m_i \in M_i$, for any player $i \in N$. Feasible continuation values, $v \in V(k)$, for this game extension are depicted in the right table above. The game $\langle M, v \rangle$ then has a fully mixed equilibrium in which both players

receive a payoff of $14/15$.²⁸ Clearly no such such payoff could be implemented, were players never to randomize at the commitment stage, because $(14/15, 14/15) \notin \mathcal{I}$.

5.2. Considerations on Efficiency, Correlation & Endogenous Order

Though the ability to unilaterally rule actions generally benefits players, allocations in which a player is harmed beyond his worst Nash outcome can in principle be implemented. A lower-bound for a player's equilibrium payoff in any unilateral extension is the independent min-max value of the original game, since any player by refusing to commit could, always, guarantee himself such outcome. Formally, for $\mathcal{K}_I = \cup_{k \in I} \mathcal{K}(k)$ it must be that:²⁹

$$\mathcal{K}_I \subseteq \mathcal{R} = \left\{ u \in \mathcal{U} \mid u_i \geq \min_{\sigma_{-i} \in \Delta_{-i}} \max_{\sigma_i \in \Delta_i} u(\sigma), \forall i \in N \right\}$$

The examples in the previous section meant to display how such structures could lead to the implementation of mutually beneficial outcomes and ultimately to efficiency. But any prisoner's dilemma shows that this need not to be the case. More surprisingly, the implementation of the efficient profiles may require players to maintain some flexibility in order to discourage deviant behaviors. Thus within such class limiting attention to commitments to single pure strategy profiles may entail a loss in generality. A necessary condition for the implementation of an efficient allocation is that the set of individually rational, independent strategy efficient allocation profiles be non-empty. Indeed:

PROPOSITION 6. *If $\mathcal{R} \cap \mathcal{I} \cap \mathcal{P} = \emptyset$, then $\mathcal{K}_I \cap \mathcal{P} = \emptyset$.*

Such assumption prevent the implementation of any efficient outcome because the probability of pledging to an inefficient subgame is never zero.

Perfect monitoring of commitments allows payers to correlate their strategy of play. It may appear that coordinating strategies be the only advantage that could derive from unilateral pledges. But, in general, equilibrium outcomes of independent commitment structures need not to be *self-enforcing*.³⁰ Indeed, the first example of the previous section presented such an instance, since by strict domination the efficient allocation $(3, 3)$ would never obtain as the outcome of pre-play communication.

Some results in the literature relate the equilibria of independent commitment structures, to equilibria of the original game modified in the order of play. In fact, consider the extensive form games in which subsets of players move at different stages in a specified order and receive payoffs according to the original utility functions. Proposition 3 in [22], conditions on the original game that guarantee that any equilibrium of the extensive form game, in which all but one player move first, be an equilibrium of a unilateral commitment extension of the game. The implementation of arbitrary orders of play, though, requires players to have more rounds in which to commit as well as a different set of dimensionality assumptions on the game. Therefore, such ability to pledge can in some circumstances generate beliefs that rationalize different orders of play.³¹

²⁸In such equilibrium, $\pi_1 \{a\} = \pi_2 \{b\} = 8/15$ and $\pi_2 \{a\} = \pi_1 \{b\} = 1/15$.

²⁹This holds because a player can chose not to commit and play his min-max strategy in each subgame. Thereby achieving the desired payoff in any committed subgame.

³⁰Define an allocation to be self-enforcing, if it belongs to the correlated equilibrium payoff hull, \mathcal{C} .

³¹I defer to the web-appendix has a formal treatment of some endogenous order claims.

6. Implementation & Properties for the Unanimous Class

This section characterizes payoffs implemented when a unanimous agreement is required to commit. If deterministic, such contracts generally lead to efficiency gains, but not necessarily Pareto efficiency. Indeed, in such contracting environment random pledges may be needed to implement any efficient allocation.

First, properties of unanimous structures are investigated, then the set payoffs implemented by the entire class \mathcal{U} are characterized and, finally, efficiency is discussed. All proofs of the claims to follow only make use of unanimous structures that are in canonical form.

6.1. Deterministic Commitment

The initial four remarks outline elementary properties of the payoff sets that can be implemented by unanimous commitment structures. First, notice that any payoff that exceeds the worst Nash threat for all players and that belongs to the independent payoff hull of the original game can be supported as an equilibrium for an appropriate commitment structure. That is, for $\mathcal{K}_{\mathcal{U}} = \cup_{k \in \mathcal{U}} \mathcal{K}(k)$:

REMARK 7. *For any $u \in \mathcal{I} \cap \underline{\mathcal{N}}$ there exists a unanimous commitment structure, $k \in \mathcal{U}$, such that $u \in \mathcal{K}(k)$. Hence, $\mathcal{K}_{\mathcal{U}} \supseteq \mathcal{I} \cap \underline{\mathcal{N}}$.*

Because any player would have a weak incentive to commit to such a payoff, whenever all the others agreed to it, if threatened by his worst equilibrium outcome. As a consequence, all Nash equilibria of the original game belong to the set of commitment payoffs since any Nash strategy is both independent and in $\underline{\mathcal{N}}$. That is, $\mathcal{K}_{\mathcal{U}} \supseteq \mathcal{I} \cap \underline{\mathcal{N}} \supseteq \underline{\mathcal{N}}$. Also, observe that unanimous pledges never implement a payoff outside the Nash rational hull, because some player would have an incentive to veto such pledge:

REMARK 8. *For any unanimous commitment structure, $k \in \mathcal{U}$, and any $u \in \mathcal{K}(k)$ it must be that $u \in \underline{\mathcal{N}}$. Hence, $\underline{\mathcal{N}} \supseteq \mathcal{K}_{\mathcal{U}}$.*

The third observation points out that allocations implemented in pure strategies by unanimous pledges must belong to the intersection between the Nash rational and the independent payoff hull of the original game, since otherwise some agent could again profitably veto.

REMARK 9. *There exists $k \in \mathcal{U}$ and $v \in V(k)$ such that $\pi(m) = 1$ for some $m \in M$ & $v(\pi) = u \in \mathcal{K}(k)$ if and only if $u \in \mathcal{I} \cap \underline{\mathcal{N}}$.*

The last and more useful remark states that enlarging the messaging space of a given unanimous commitment structure cannot reduce the set of payoffs that can be supported as unanimous commitment equilibria. More precisely:

REMARK 10. *For $(M, k), (\overline{M}, \overline{k}) \in \mathcal{U}$, if $\overline{M} \supseteq M$ and if $k(m) = \overline{k}(m)$ for any $m \in M$, then $\mathcal{K}(\overline{k}) \supseteq \mathcal{K}(k)$.*

This observation hinges on the cooperative nature of the commitment games originating from unanimous structures. The remark, significantly, simplifies the analysis of any unanimous structure.

Whenever the game possesses a convex payoff hull $\mathcal{I} = \mathcal{U}$, the set of payoffs implemented by unanimous commitment devices, $\mathcal{K}_{\mathcal{U}}$, is completely determined by the preceding remarks. In such scenario, any feasible payoff profile that exceeds each player's Nash threat is in equilibrium of some mechanism $k \in \mathcal{U}$. More generally, it must be that:

COROLLARY 11. *If $\mathcal{I} \cap \underline{\mathcal{N}} = \underline{\mathcal{N}}$, then $\mathcal{K}_U = \underline{\mathcal{N}}$.*

In fact, whenever the incentive structure in the original game has independent randomizations spanning all possible payoff allocations exceeding the worst uncommitted threat, then any such allocation can be implemented as an equilibrium of some unanimous commitment structure.

Before extending results to non-convex games, notice that again perfect monitoring can be exploited by players to correlate their strategies. Indeed, any payoff in the convex hull of the Nash equilibria of the original game can be approximately implemented by some unanimous commitment structure. Formally, denoting by $\text{co}(\cdot)$ the convex hull operator, the following holds true:

PROPOSITION 12. *For any $u \in \text{co}(\mathcal{N})$ and any $\varepsilon > 0$ there exists a $k \in U$ and a $\hat{u} \in \mathcal{K}_{\mathcal{N}}(k)$ such that $\|u - \hat{u}\| < \varepsilon$. Moreover, it is possible to find one such k that never restricts any player's strategy.*

This proposition asserts that $\text{co}(\mathcal{N}) \subseteq \text{cl}(\mathcal{K}_U)$, for $\text{cl}(\cdot)$ the closure of a set. Therefore, a unanimous commitment mechanism can be found that mimics any direct communication. Its is possible to do so, because the perfect monitoring allows player to condition their actions upon the message previously sent and publicly observed. The proposition, also, asserts that any such payoff can be implemented by a structure in which commitments never materialize in equilibrium. As discussed later, this result makes the analysis of structures in which players have several rounds in which to commit more tractable.

The next proposition generalizes the results of proposition 11 to non-convex games that possess a unique Nash equilibrium outcome.

PROPOSITION 13. *If $|\mathcal{N}| = 1$, then: (1) $\mathcal{K}_U^* = \text{co}(\mathcal{K}_U) = \text{co}(\mathcal{I} \cap \underline{\mathcal{N}})$.
(2) $\mathcal{K}_U \subseteq \overline{\mathcal{I} \cap \underline{\mathcal{N}}}$.*

For $\overline{\mathcal{I} \cap \underline{\mathcal{N}}} \equiv \{u \in \underline{\mathcal{N}} \mid \exists u' \in \mathcal{I} \cap \underline{\mathcal{N}} : u' \geq u\}$ denoting the set of payoffs exceeding the Nash threat, but dominated by some element in $\mathcal{I} \cap \underline{\mathcal{N}}$. The first part of proposition states that in unique NE games, whenever communication, precedes commitment, there is no benefit to mediation. Additionally, it identifies the set of payoffs to be attained for such communication structure, to consist only of mixtures of pure strategy equilibria of the Nash commitment game. The second part, instead, identifies an upper-bound for the payoffs attainable without communication. The constructed bound is contained in the convex hull of pure Nash equilibria, that is $\overline{\mathcal{I} \cap \underline{\mathcal{N}}} \subseteq \text{co}(\mathcal{I} \cap \underline{\mathcal{N}})$. Indeed, the smallest convex set containing the Nash payoff set will, by (1), be $\text{co}(\mathcal{I} \cap \underline{\mathcal{N}})$. The upper-bound in (2) is not, in general, convex. The most interesting consequence of part (2) of the claim is that all allocations that are efficient within the commitment payoff set, \mathcal{K}_U , always, belong to the set $\mathcal{I} \cap \underline{\mathcal{N}}$. Hence, for games with a unique equilibrium payoff, when no communication is permitted, the pure strategy equilibria of the messaging stage suffice in implementing all efficient allocations within \mathcal{K}_U . Another consequence of the claim is that, in such games, unanimous commitments do not, in general, lead to Pareto efficiency. Examples, in section 9, show that not all payoffs in $\overline{\mathcal{I} \cap \underline{\mathcal{N}}}$ are necessarily implemented and that any tighter characterization depends on the specific incentive structure of the original game.

The next proposition provides an upper-bound on the set of commitment equilibria for the general case. Let $\mathcal{U}_{\mathcal{N}} = \text{cc}(\mathcal{I}, \text{co}(\mathcal{N}))$, for $\text{cc}(\cdot, \cdot)$ the convex combination operator³² and $\underline{\mathcal{N}}_i = \{u \in \mathcal{U} \mid u_i \geq \min_{\mathcal{N}(\Gamma)} u_i\}$, then it must be that:

³²For any two sets, X & Y , denote $\text{cc}(X, Y) \equiv \{\alpha x + (1 - \alpha)y \mid x \in X, y \in Y \text{ \& } \alpha \in [0, 1]\}$.

PROPOSITION 14. $\mathcal{K}_U \subseteq \mathcal{K}_U^* \subseteq \bigcap_{i \in N} \text{co}(\underline{\mathcal{N}}_i \cap \mathcal{U}_N)$.

Notice that such upper-bound set is convex, because any intersection of convex sets is itself convex. Also, such bound is weakly bigger than any bound in the preceding propositions, since:

$$\text{co}(\mathcal{I} \cap \underline{\mathcal{N}}) \subseteq \text{co}(\mathcal{U}_N \cap \underline{\mathcal{N}}) \subseteq \bigcap_{i \in N} \text{co}(\mathcal{U}_N \cap \underline{\mathcal{N}}_i)$$

Though it may, still, be that $[\bigcap_{i \in N} \text{co}(\mathcal{U}_N \cap \underline{\mathcal{N}}_i)] \cap \mathcal{P} = \emptyset$, in which case efficiency could never be attained. Therefore, this bound provides a necessary, but not sufficient, condition to test efficiency, both with and without communication. The proof of this claim depends almost entirely on individual rationality and does not make use of all equilibrium implications. Indeed, the fact that any player must be indifferent amongst all messages sent with positive probability was never used in the proof.³³ This observation can potentially tighten the result. Currently, the paper refrains from doing so, because gains derived from such tightening are limited, while characterization of the bounding set becomes harder.

Efficiency in Unanimous Commitment Structures. This subsection highlights the implications on efficiency of the results presented in the previous section. An immediate consequence of proposition 11 is that, whenever the original game displays a convex independent payoff hull above the worst Nash threats, any allocation that is efficient and Nash rational can be implementable. That is, if $\mathcal{I} \cap \underline{\mathcal{N}} = \underline{\mathcal{N}}$, then $\mathcal{P} \cap \underline{\mathcal{N}} \subseteq \mathcal{K}_U$. Thus, in the convex case a single round of deterministic commitment can attain any feasible efficiency gain.

Efficient allocations, however, are not necessarily implemented whenever the game is not convex. Indeed, it is possible to show that whenever no efficient payoff belongs to the set $\mathcal{I} \cap \underline{\mathcal{N}}$, then no efficient allocation will ever belong to the commitment equilibrium payoff hull. Specifically:

PROPOSITION 15. *If $\mathcal{I} \cap \underline{\mathcal{N}} \cap \mathcal{P} = \emptyset$, then $\mathcal{K}_U \cap \mathcal{P} = \emptyset$.*

Indeed, this is the case for a non-convex prisoner's dilemma, as discussed in the examples section. But, even in games meeting such assumption, hostile to coordination, unanimous pledges generally achieve efficiency gains on the equilibrium set of the original game. As will be the case, whenever $\mathcal{I} \cap \underline{\mathcal{N}} \neq \emptyset$. The next sections show how either the ability to delegate the choice of commitment or the ability to commit in different round always lead to efficiency.

6.2. Stochastic Commitment

In this scenario, all agents are allowed to delegate their choice of commitment to a third party that is able to forcefully impose the choice of commitment conditional on the realization of some lottery. Such powerful technology helps agents to discipline their incentives problems ultimately leading to Pareto efficiency. In fact, because any payoff in \mathcal{A} may be attained as a payoff of a subgame for some fully restrictive profile of commitments, it must be that any payoff in \mathcal{U} be attained by some lottery on such commitment profiles. But this observation reduces the problem to the one of deterministic commitments on convex games. Consequently, the following holds true, for $\mathcal{K}_U^S = \bigcup_{k \in U \cap K^S} \mathcal{K}(k)$:

LEMMA 16. $\mathcal{K}_U^S = \underline{\mathcal{N}}$.

³³For any two messages sent with positive probability $m', m'' \in \{m | \pi_i(m) > 0\}$ by player $i \in N$:

$$u_i(\pi | \sigma) = u_i(m', \pi_{-i} | \sigma) = u_i(m'', \pi_{-i} | \sigma)$$

Even though it may appear that random contracts are necessary to implement the entire Nash rational hull, the rest of analysis shows that this is not the case. Indeed, if players were given several rounds in which to sign pure deterministic contracts, there would be no need for randomness in contracts in order to implement the aforementioned payoff hull. Though outcomes in the two scenarios may coincide, contracts signed in any given equilibrium could significantly differ. In fact, increasing the number of rounds will make up for reducing the number of contracts.

7. Layered Unanimous Commitments & Efficiency

The initial subsection introduces the structure. The one following first presents claims on the evolution of the equilibrium set and then states the layered implementation theorem.

7.1. Structure

In an environment in which agents are allowed to commit to subsets of actions, it may seem natural to allow agents to commit to their choice of commitment as well. This section discusses the effects of such commitments to commit. It may appear that such ability to iteratively commit enhances the chances of coordination, thereby increasing the set of equilibrium payoffs. Indeed, after notation is appropriately extended, it is claimed not only that an additional stage of commitment can, always, be used to directly communicate upon equilibria of the latter stage, but also that as the number of stages increases one may get arbitrarily close to the implementation of the entire Nash rational hull, \mathcal{N} .

Define a game to have $t + 1$ layers of commitment if there is a sequence of commitment structure, $\{k_0, \dots, k_t\} = k^t$, that for any layer $s \in \{1, \dots, t\}$ and any player $i \in N$ satisfies:

$$k_{i,s} : M_s \rightrightarrows \Delta(M_{i,s-1}) \quad \& \quad k_{i,0} : M_0 \rightrightarrows \Delta_i$$

Therefore in such commitment games, conditional on pledges previously agreed upon, players can at every stage stipulate contracts on how to restrict their ability to choose at the next stage. The figure below is meant to illustrate the timing of play.

$$\begin{array}{ccccccc} t & t-1 & \dots & -1 & -2 & & \\ \hline M_t & k_t(m_t) \subseteq M_{t-1} & \dots & k_0(m_0) \subseteq A & U & & \end{array}$$

Continuation maps for the stage $s > 0$ subgame are defined recursively. Because of the perfect monitoring assumption, strategies at any stage can depend upon all messages previously sent. As discussed earlier, the set of continuation values for the commitment instance that precedes the game must belong to:

$$V(k_0) = \{v \in \mathcal{U}^{M_0} \mid v(m_0) \in \mathcal{N} \langle u, k_0(m_0) \rangle \text{ for } \forall m_0 \in M_0\}$$

for $\langle u, k_0(m_0) \rangle$ denoting the original game restricted by $k_0(m_0)$. The recursive definition of the collection of equilibrium continuation values for any layer $s > 0$ states:

$$V(k^s) = \{v \in \mathcal{U}^{M_s} \mid [v(m_s) \in \mathcal{N} \langle v_{s-1}, k_s(m_s) \rangle \text{ for some } v_{s-1} \in V(k^{s-1})] \text{ for } \forall m_s \in M_s\}$$

Such definition entails the perfect monitoring assumption, because the lower layer continuation values v_{s-1} can depend upon the message profile m_s . Consequently, the collection of subgame perfect payoff profiles of the game extended by a commitment structure with $t + 1$ layers, k^t , can be defined as:

$$\mathcal{K}(k^t) = \cup_{v \in V_t(k^t)} \mathcal{N} \langle v, M_t \rangle$$

Denote the class of $t + 1$ layered commitment mechanisms, that are unanimous at every stage and that originate from action set A , by $U^t(A)$. For $M^{t,s} = \times_{r=s}^t M_r$ and $M^{t,t+1} = \emptyset$, a strategy profile for the players consists of a maps (π^t, σ) , with domains such that $\pi_s : M^{t,s+1} \rightarrow \times_{i \in N} \Delta(M_{i,s})$ for any $s \in \{0, \dots, t\}$ and $\sigma : M^t \rightarrow \Delta_N$, satisfying the following conditions:

$$\begin{aligned} \sigma(m^t) &\in k_0(m_0) \text{ for } \forall m^t \in M^t \\ \pi_s(m^{t,s+1}) &\in k_{s+1}(m_{s+1}) \text{ for } \forall m^{t,s+1} \in M^{t,s+1} \ \& \ s \in \{0, \dots, t\} \end{aligned}$$

7.2. Implementation of the Nash Rational Hull

An immediate consequence of commitments to commit being voluntary is that the equilibrium payoff hull remains, for any number of stages stage, a subset of set of Nash rational hull. In fact, for $\mathcal{K}_{\cup}^t = \cup_{k^t \in U^t} \mathcal{K}(k^t)$:

REMARK 17. $\mathcal{K}_{\cup}^t \subseteq \underline{\mathcal{N}}$, for any $1 \leq t < \infty$.

Since any player receiving a payoff below his worst Nash equilibrium outcome could always deviate to vetoing any commitment at all stages and guarantee himself a Nash outcome by subgame perfection.

As for one stage mechanisms, a richer contracting structure implements a bigger set of payoffs. In fact, any equilibrium of a t layered unanimous extension is for appropriately chosen beliefs, also, an equilibrium of an extension with more contracts at each stage. Therefore:

PROPOSITION 18. For $(M^t, k^t), (\bar{M}^t, \bar{k}^t) \in U^t$, if $\bar{M}^t \supseteq M^t$ and if $k_s(m_s) = \bar{k}_s(m)$ for any $m \in M_s$ and $s \leq t$, then $\mathcal{K}(\bar{k}^t) \supseteq \mathcal{K}(k^t)$.

Since players can threaten punish at any stage t a deviation from M_t with the worst Nash outcome.

Another consequence commitment being voluntary, that an additional stage can never reduce the set of equilibrium payoffs. In fact:

COROLLARY 19. $\mathcal{K}(k^t) \supseteq \mathcal{K}(k^{t-1})$, for any $k^t \in U^t$.

This inclusion follows directly from corollary 7. Indeed, any payoff $u \in \mathcal{K}(k^{t-1})$ can be implemented in pure strategies at the $t + 1^{\text{st}}$ commitment stage, since by veto property for any choice of k_t there exists $\bar{m}_t \in M_t$ such that $k_t(\bar{m}_{i,t}, m_{-i,t}) = \times_{j \in N} \Delta(M_{j,t})$ for any $i \in N$, and there exists $v \in V(k^t)$ such that:

$$v(\bar{m}_{i,t}, m_{-i,t}) = u \text{ for } \forall m_{-i,t} \in M_{-i,t} \ \& \ i \in N$$

The previous comments imply that, when a game is convex, increasing the number of layers of commitment serves no purpose, by remark 17, corollary 19 and the observation that all allocations above the Nash threat were already implemented by single layer unanimous commitment, $\underline{\mathcal{N}} \subseteq \mathcal{K}_{\cup}^0$. Indeed, if $\mathcal{I} \cap \underline{\mathcal{N}} = \underline{\mathcal{N}}$, then $\mathcal{K}_{\cup}^t = \underline{\mathcal{N}}$ for any $0 \leq t < \infty$.

As in the single layer case, because perfect monitoring structures can mimic direct communication an additional layer of commitment implements any payoff in the convex hull of equilibrium payoffs of the lower layer. More precisely, the generalization of proposition 12 states that:

PROPOSITION 20. $\text{cl}(\mathcal{K}_{\cup}^t) \supseteq \text{co}(\mathcal{K}_{\cup}^{t-1})$, for any $1 \leq t < \infty$.

Therefore an additional layer of commitment can certainly enlarge the equilibrium payoff hull whenever \mathcal{K}_U^{t-1} is not convex.³⁴

The proof of the main theorem included of the section draws heavily on the next observation, a generalization of remark 7. It states that any payoff that, at t^{th} stage, belongs to both the Nash rational hull and independent payoff hull of a game $\langle v_t, M_t \rangle$, for some equilibrium continuation value $v_t \in V(k^t)$, can be implemented at the $t + 1^{\text{st}}$ stage. That is:

PROPOSITION 21. *For any $(M^t, k^t) \in U^t$ and $v_t \in V(k^t)$ it must be that $\mathcal{I}\langle v_t, M_t \rangle \cap \underline{\mathcal{N}} \subseteq \mathcal{K}_U^{t+1}$.*

Again deviations from such contracts can be made unappealing by threatening each player with his worst Nash outcome.

That several stages of unanimous commitments can enhance coordination amongst players may not be too surprising. However, the more remarkable property of such class of mechanisms is that it suffices to approximately implement all Nash rational payoffs. Thus, even in games that are not convex, a mechanism can be found in such class that approximately implements any efficient payoff exceeding the Nash threats as a subgame perfect equilibrium of the layered commitment extension of original game. Indeed, the limit of the sequence of closures of the commitment equilibrium set always converges to a payoff set whose closure is the Nash rational hull.

THEOREM 22. *Let $C = \lim_{t \rightarrow \infty} \text{cl}(\mathcal{K}_U^t)$, then $\text{cl}(C) = \underline{\mathcal{N}}$.*

This result hinges on the ability of players to trade at any stage, occasional concessions on unfavorable commitments in exchange of a more favorable treatment from others whenever the concession does not occur. Such trades would, then, be self-enforcing given the nature of the chosen commitment structure and the knowledge of the continuation values. A possible interpretation of this claim is that forming an agreement that supports an efficient allocation is an activity that may take some time (i.e. several stages) even in contracting environments designed to voluntarily enhance cooperation.

Hence, for any outcome implemented by a stochastic mechanism, there exists a deterministic unanimous structure with multiple layers implementing the same allocation. The model presented delivers a rationale for observing several stages of contracting even in environments in complete information, since, as shown, this may help to discipline the incentive problems, in games that are not convex if deterministic contracts are to be signed. The intuition for underlying the proof of this theorem will be made clear from the analysis of the examples. The next section shows that under weak dimensionality conditions the implementation of the Nash rational set only requires pure commitments.

8. Multiple Round Pure Implementation

It may appear from the previous analysis the implementation Nash rational set requires layers of unanimous mixed strategy pledges. But whenever the original game satisfies appropriate dimensionality conditions, discussed in what follows, the implementation result holds even if players can only commit to pure strategy profiles of the original game, but at several instances. Recall that a single stage commitment structure was said to be pure whenever $k_i(m) \in 2^{A_i} \setminus \emptyset$ for any profile of messages $m \in M$ and all players $i \in N$.

³⁴A consequence of such claim and of proposition 13 is that, if $\mathcal{I} \cap \underline{\mathcal{N}} \neq \underline{\mathcal{N}}$ and $|\mathcal{N}| = 1$, then it must be that $\text{cl}(\mathcal{K}_U^1) \supseteq \mathcal{K}_U^*$.

In this section define a commitment structure with $t + 1$ rounds to be pure if there exists a sequence of unanimous commitment structures, $\{k_0, \dots, k_t\} = k^t$, that at any instance $s \in \{0, \dots, t\}$ satisfies:

$$k_s : M_s \rightarrow A \cup \{\Delta_N\}$$

In such structures, if there is at any round an agreement on a pure strategy profile of the original game pledges end, because all are fully committed. If instead no agreement is found, $k_s(m_s) = \Delta_N$, players can again try to commit at the next round, $s - 1$. Indeed, any such structure, k^t , is strategically equivalent to layered unanimous commitment structures, \bar{k}^t , satisfying:

$$\bar{k}_s : M_s \rightarrow M_{s-1} \cup \{\times_{i \in N} \Delta(M_{i,s-1})\} \quad \& \quad \bar{k}_0 : M_0 \rightarrow A \cup \{\Delta_N\}$$

Note that any stage of such layered structure may posses contracts that fully commit to pure strategies in the final game because a full commitment to fully commit completely determines actions at all latter stages.

Denote the class of commitment mechanisms with $t + 1$ rounds, that are unanimous, pure and that originate from action set A , by $U_P^t(A)$. Also, let $\mathcal{K}_{U_P}^t = \cup_{k^t \in U_P^t} \mathcal{K}(k^t)$. Before proceeding to the pure strategy implementation result, three claims ancillary to its prove are outlined. The first two are corollaries to propositions 19 and 20. They claim that, even in pure commitment environments, does the set of equilibrium payoff profiles weakly increase with the number of rounds and that players may exploit the contracting expansion in order to communicate. Let $\underline{A} = \mathcal{A} \cap \underline{\mathcal{N}}$, for \mathcal{A} the pure strategy payoff hull.

COROLLARY 23. $\mathcal{K}_{U_P}^t \supseteq \mathcal{K}_{U_P}^{t-1} \supseteq \mathcal{N} \cup \underline{A}$, for any $1 \leq t < \infty$.

COROLLARY 24. $\text{cl}(\mathcal{K}_{U_P}^t) \supseteq \text{co}(\mathcal{K}_{U_P}^{t-1})$, for any $1 \leq t < \infty$.

The third claim, instead, provides a technique to enlarge the set of implementable payoffs at each round of commitment. On the latter result hinges much of the proof of the pure strategy implementation result.

PROPOSITION 25. For $u_t \in \mathcal{K}_{U_P}^t$, $u \in \mathcal{A}$ and $q \in \mathbb{N}_+$, if $(1/q^{N-1})u + (1 - 1/q^{N-1})u_t \in \underline{\mathcal{N}}$ then:

$$(1/q^{N-1})u + (1 - 1/q^{N-1})u_t \in \mathcal{K}_{U_P}^{t+1}$$

The last and central result of the section provides conditions for which the use of multiple round pure unanimous structures suffices for the implementation of the Nash rational payoff hull. Contracts required for such claim are very simple and involve at each layer only commitments to pure strategies of the original game. For $\underline{N} = \{i \in N \mid \exists u \in \underline{\mathcal{N}} \text{ s.t. } u_i > \underline{u}_i^i\}$, $\text{aff}(\cdot)$ the affine hull operator and $\text{vert}(\cdot)$ the vertices of a polytope, the statement of the claim is as follows:³⁵

THEOREM 26. If $\text{vert}(\mathcal{U} \cap \text{aff}(\underline{\mathcal{N}})) \subseteq \mathcal{A} \cup \underline{\mathcal{N}}$ and if there exist $u \in \text{co}(\mathcal{N} \cup \underline{A})$, $u' \in \mathcal{A}$, $\alpha \in [0, 1]$ and $j \in \underline{N}$ such that:

$$\begin{aligned} \alpha u'_i + (1 - \alpha)u_i &> \underline{u}_i^i \quad \text{for } \forall i \in \underline{N} \setminus j \\ \alpha u'_i + (1 - \alpha)u_i &\geq \underline{u}_i^i \quad \text{for } \forall i \in N \setminus (\underline{N} \setminus j) \end{aligned}$$

then for $C_P = \lim_{t \rightarrow \infty} \text{cl}(\mathcal{K}_{U_P}^t)$, it must be that $\text{cl}(C_P) = \underline{\mathcal{N}}$.

³⁵For $x, y \in \mathbb{R}^n$ let $L(x, y) = \{(1 - \lambda)x + \lambda y \mid \lambda \in \mathbb{R}\}$, the line. Then, a set H is a flat if $x, y \in H$ implies $L(x, y) \subseteq H$. The affine hull of X , denoted $\text{aff}(X)$, consists of the intersection of all flats that contain X .

If X is a convex polytope, then $x \in \text{vert}(X)$ if $y, z \in X$, $\lambda \in (0, 1)$ and $x = (1 - \lambda)z + \lambda y$ implies $x = y = z$.

The first is a mere dimensionality assumption. It guarantees that the Nash rational set can be spanned by commitments to pure strategies in addition to the Nash equilibria of the original game. Such assumption is without loss when there are only two players, because $\mathcal{U} \cap \text{aff}(\underline{\mathcal{N}})$ is either \mathcal{U} or \mathcal{N} or a facet of \mathcal{U} . When there are more than two players, even though the assumption holds in great generality, cases in which it does not can be constructed. The second assumption, instead, guarantees that it be possible to implement at some round a payoff in the relative interior of the Nash rational hull. This assumption is needed for the implementation result, because it makes sure that at some layer there be equilibria in which all relevant players, \underline{N} , receive a payoff above the Nash threat. This condition is in the spirit similar to the multiplicity assumption required to implement payoffs in a finitely repeated game in [5]. An example in which the latter assumption fails in a two player game will be reported in the next section.

In the proof of this result, all binding contracts signed at any stage by the players with strictly positive probability entail a full commitment to a strategy profile in A that can never be renegotiated. Thus, empirical counterpart of contracting simply results in a pure commitment contract being signed sometime before the game plays out.

In such a contracting environments joint randomizations on action profiles are realized by voluntary probabilistic concessions to unfavorable contracts at earlier stages in order to encourage coordination on favorable contracts at latter stages if an agreement does not materialize. In fact, even contracts that yield payoffs lower than the Nash threat for some player may occasionally be observed, because of beliefs about future concessions. Thus, the theorem, to the extent of its assumptions, makes the point that even in non-convex games there is non need to observe stochastic contracts for an efficient allocation to be implemented. The model implicitly generates a non-cooperative theory for how players bargaining upon deterministic contracts can implement efficient outcomes.

The generality of the second assumption on the original game depends on the class games to which Γ belongs. The implementation is easier for continuous compact games. The web-appendix discusses such matters in detail. The next section presents examples that clarify the dynamics underlying the proof of the implementation theorem.

9. Examples: 2 Player Unanimous Case

For graphical clarity, this section focuses on two player games. First, examples of games in which a single stage of deterministic commitment suffices in implementing any efficient Nash rational allocation are presented. Then, examples of games in which several stages are required are described. The latter should clarify how layers of commitment can be generate concessions amongst players.³⁶

9.1. Single Stage Efficiency

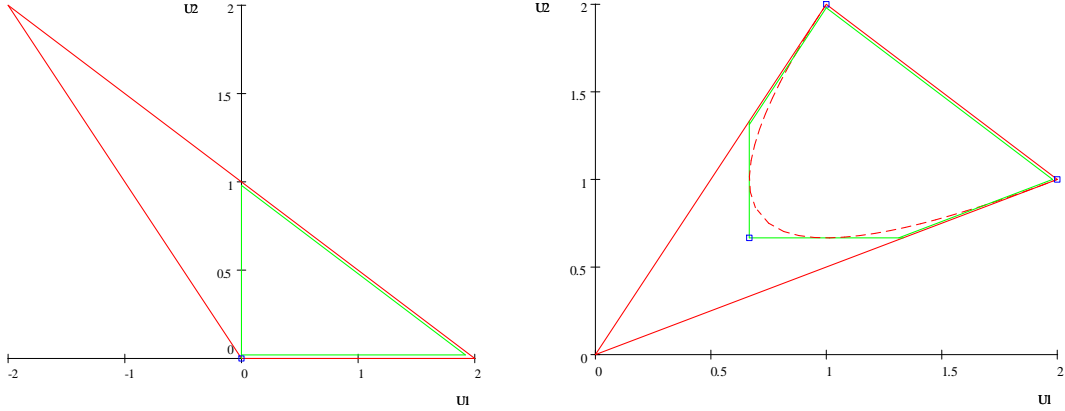
The first example consists of a convex, $\mathcal{I} = \mathcal{U}$, game in which a borrower, player 1, and a lender, player 2, choose simultaneously which loan to offer and whether to take the new loan. Specifically:

$1 \setminus 2$	n	c
n	$0, 0$	$0, 0$
c	$-2, 2$	$2, 0$

The relevant payoff hulls of the game are depicted in figure 3A. In all the Nash equilibria the borrower will chose not to take the loan, because of the fear of the lender offering the bad loan.

³⁶More examples can be found in web-appendix.

Thus, $\mathcal{N} = (0, 0)$. If the lender could unilaterally commit to offering the good loan, the borrower would accept such offer. Thus, $(2, 0) = \mathcal{K}_{\mathcal{I}_P} \cap \mathcal{P} \cap \underline{\mathcal{N}}$.³⁷ By convexity of the game it also follows, trivially, that $(2, 0) = \mathcal{K}_{\mathcal{U}_P} \cap \mathcal{P} \cap \underline{\mathcal{N}}$ and that $\mathcal{K}_{\mathcal{U}} = \underline{\mathcal{N}}$. More interestingly, by theorem 26, it is also clear that $\lim_{t \rightarrow \infty} \mathcal{K}_{\mathcal{U}_P}^t \simeq \underline{\mathcal{N}}$ ³⁸ Therefore, there must be contracting rounds in which the borrower agrees with positive probability to pledge to the unfavorable loan contract, in order to motivate the lender to offer the good contract at following rounds, if an agreement is not reached.



3A: in red \mathcal{I} , in blue \mathcal{N} & in green $\mathcal{K}_{\mathcal{U}}$

3B: red solid \mathcal{U} , dashed \mathcal{I} , blue \mathcal{N} & green $\mathcal{K}_{\mathcal{U}}^*$

A typical example of non-convex game, which requires no pledges to be made in equilibrium, for efficiency to be attained in a single stage, is the battle of the sexes:

$1 \setminus 2$	n	c
n	2, 1	0, 0
c	0, 0	1, 2

Indeed, no restriction is needed for efficiency either because $\mathcal{P} \subset \text{co}(\mathcal{N}) \subseteq \mathcal{K}^*(k)$ for a commitment structure $k \in \mathcal{U}$. Or because, by proposition 12, any payoff in the convex hull of Nash equilibria of the original game can be arbitrarily closely approximated by an equilibrium of a pure unanimous structure, $\text{co}(\mathcal{N}) \subseteq \text{cl}(\mathcal{K}_{\mathcal{U}_P})$. In any such equilibrium, the commitment structure is used as a direct communication device. Also, $\text{cl}(\mathcal{K}_{\mathcal{U}_P}^1) = \underline{\mathcal{N}}$, since two stages may be needed to approximately implement payoffs in $\underline{\mathcal{N}} \setminus [\mathcal{I} \cup \text{co}(\mathcal{N})]$. The commitment equilibrium set is depicted in figure 3B.

9.2. When a Single Round is not Efficient, but Several are

The first two examples of the subsection required infinitely many contracting stages to implement the Nash rational hull, while the third and final only requires two stages. Consider a prisoners' dilemma type game, in which the hull of feasible payoffs, \mathcal{U} , cannot be spanned by independent randomizations, $\mathcal{I} \subsetneq \mathcal{U}$. Indeed, consider the game:

$1 \setminus 2$	n	c
n	1, 1	6, 0
c	0, 6	2, 2

³⁷Note that the game satisfies $\mathcal{K}_{\mathcal{I}} \supseteq \mathcal{P} \cap \underline{\mathcal{N}}$. Meaning that it is possible to construct mixed independent commitment structures implementing such profiles. For structures allowing player to commit to all strategy profiles only outcome $(2, 0)$ could be implemented.

³⁸Indeed, $\mathcal{K}_{\mathcal{U}_P}^1 \simeq \underline{\mathcal{N}}$.

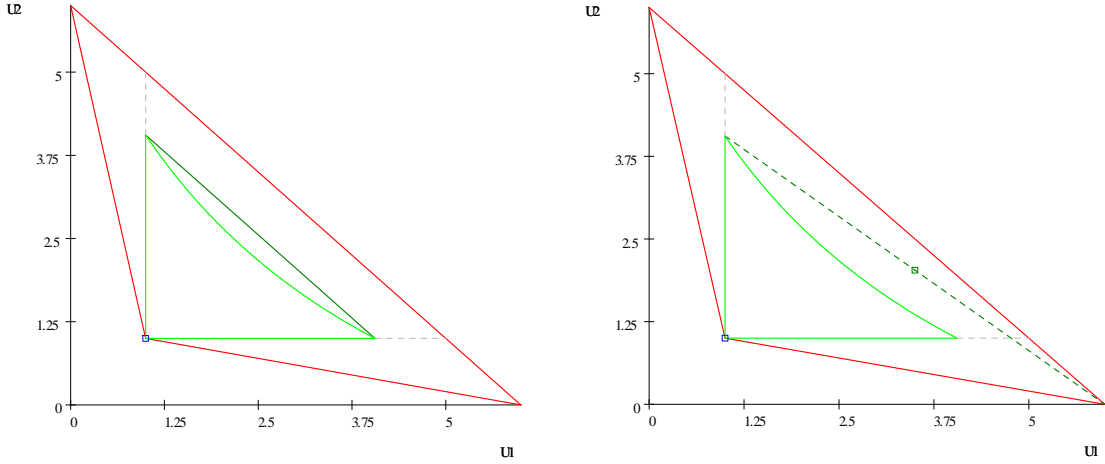
In such context, any efficient outcome corresponds to a specific randomization on action profiles (n, c) and (c, n) . The payoff hulls for this game is depicted in figure 4A. In every committed subgame of this game, players attach to cooperation, c , the lowest chance possible in their committed strategy set, because such strategy is always strictly dominated. The set $\mathcal{I} \cap \underline{\mathcal{N}}$ of this game is completely characterized by three non-linear constraints in the utility space:³⁹

$$u_2 \leq u_1 - 4\sqrt{2 + 3u_1} + 12, \quad u_1 \geq 1 \quad \& \quad u_2 \geq 1$$

For instance, for $u_1 = 1$, the highest payoff for the opponent that still belongs to $\mathcal{I} \cap \underline{\mathcal{N}}$ is $u_2 \approx 4.056$. Such values, belonging $\partial\mathcal{I}$, could be rationalized by the commitment profile, $\sigma_1(n) \leq .079$ & $\sigma_2(n) \leq 0.588$. No deterministic single stage commitment, if accepted, can ever yield to any player a payoff greater than 4.056. Indeed, by remark 7, proposition 13 and the observation that any $u \in \underline{\mathcal{N}}$ such that $u \leq v$ for some $v \in \mathcal{I} \cap \underline{\mathcal{N}}$ satisfies $u \in \mathcal{I} \cap \underline{\mathcal{N}}$, it follows that:

$$\mathcal{K}_U = \overline{\mathcal{I} \cap \underline{\mathcal{N}}} = \mathcal{I} \cap \underline{\mathcal{N}}$$

Thus, allocation $(2, 2)$ in which both players cooperate is implemented is implemented by a pure deterministic contract. Additionally, proposition 13 and the previous observation imply that $\mathcal{K}_U^* = \text{co}(\mathcal{I} \cap \underline{\mathcal{N}})$. Thus, even if communication were to precede a single stage of deterministic commitment, efficiency could not be attained.⁴⁰



4A: In red \mathcal{U} , in light green \mathcal{K}_U & in dark green the boundary of \mathcal{K}_U^* 4B: boxed in green a point in \mathcal{K}_U^* , dashed the lottery supporting it

This game, though, satisfies the assumptions for the pure implementation of the Nash rational hull. Indeed, if players can repeatedly commit efficiency gains can be attained with each additional stage of contracting. Potential gains, though, will diminish as the number of stages increases. In fact, at any stage t it is possible to implement payoffs, $u_t \in \mathcal{K}_{U_P}^t$, that are no further than $1/2^{t-1}$ utils from the closest efficient payoff, $\|u_t, \mathcal{P}\|_+ \leq 1/2^{t-1}$. The following sequence of games $\{\langle v_s, M_s \rangle\}_{s=0}^\infty$ for continuation payoffs $v_s \in V(k^s)$ of a canonical pure unanimous structure can

³⁹To characterize $\partial[\mathcal{I} \cap \underline{\mathcal{N}}]$, the following problem is solved, for any $i, j \in N$, for $u_j \geq 1$ and $\sigma_i = \sigma_i(n)$:

$$\begin{aligned} & \max_{\sigma_i, \sigma_j} [\sigma_i \sigma_j + 6\sigma_i(1 - \sigma_j) + 2(1 - \sigma_i)(1 - \sigma_j)] \\ \text{s.t. } & \sigma_i \sigma_j + 6\sigma_j(1 - \sigma_i) + 2(1 - \sigma_i)(1 - \sigma_j) \geq u_j \end{aligned}$$

This program attains maximal value $u_i = u_j - 4\sqrt{2 + 3u_j} + 12$ for $u_j \geq 1$.

⁴⁰Since, $\text{co}(\mathcal{I} \cap \underline{\mathcal{N}}) = \text{co}((1, 1), (4.056, 1), (1, 4.056))$.

be used to verify the validity of the claim:

$t = 1$	m	m'	...	$t = 2$	m	m'	...	$t = 3$	m	m'	...
m	0,6	2,2	...	m	6,0	1,4	...	m	6,0	3.5,2	...
m'	2,2	0,6	...	m'	1,4	6,0	...	m'	3.5,2	6,0	...
...
$t = 4$	m	m'	...	$t = 5$	m	m'	...				
m	0,6	4.75,1	...	m	0,6	2.375,3.5	...				
m'	4.75,1	0,6	...	m'	2.375,3.5	0,6
...

These mechanisms are unanimous and pure because $(6, 0), (0, 6), (2, 2) \in \mathcal{A}$ and because for $u_t = v_{t+1}(m, m') \in \mathcal{K}(k^t)$, for any $t > 0$.⁴¹ The sequence $u_t = v_{t+1}(m, m')$ of equilibrium payoffs possesses the desired property, $\|u_t, \mathcal{P}\|_+ \leq 1/2^{t-1}$. The same technique may be used to produce: $u_6 = (1.1875, 4.75)$, $u_7 = (3.59375, 2.375)$ and so on. By remark 18, the symmetric payoff profiles, $\bar{u}_t = (u_{2,t}, u_{1,t})$, also, belong to the equilibrium set of a structure with the symmetric commitments. Hence by remark 24, the convex hull of such payoff profiles belongs to the closure of equilibrium payoff hull of the preceding commitment stage. That is, $\text{co}((1, 1), u_t, \bar{u}_t) \subset \text{cl}(\mathcal{K}(k^{t+1}))$. Equilibrium payoff sets can, similarly, be expanded in any direction, not only the efficient one, so long as there are payoffs in that direction belonging to $\underline{\mathcal{N}}$. For instance, k^7 so that:

$t = 7$	m''	m'''	...
m''	0,6	1.1875,4.75	...	1.1875,4.75	...
...	1.1875,4.75	0,6
...	0,6	1.1875,4.75	...
m'''	1.1875,4.75	...	1.1875,4.75	0,6	...
...

For an appropriate number of such messages, call it $n \in \mathbb{N}$, the following allocation can, also, be implemented: $([1.1875 - 1.1875/n], [4.75 + 1.25/n]) \in \mathcal{K}_{\mathcal{N}}(k^7)$. Indeed, for $n = 7$, payoff profile $(1.018, 4.929) \in \mathcal{K}_{\mathcal{N}}(k^7)$.

The last two observations applied iteratively, guarantee approximate implementation of any payoff profile belonging to the Nash rational hull in a finite number of stages. Efficiency gains at each stage are driven by the fear of the other player abandoning any further negotiation at all latter stages, if no probabilistic concession to unfavorable contracts is ever made. Therefore, contracts that yield to some player a payoff below his min-max value could be observed with positive probability in equilibrium. Such phenomenon cannot be avoided, since all efficient payoff profiles of the game involve randomizations on action profiles that are dominated for one of the two players.

It is trivial to verify that had the agents the ability to stochastically delegate their choice of commitment, any Nash rational allocation could be implemented with a single stage commitment extension. Specifically, any individually rational efficient allocation could be implemented by a stochastic commitment: $p(6, 0) \oplus (1 - p)(0, 6)$ for $p \in [1/6, 5/6]$.

In the next example, if pledges either are to be made at single instance, or need to be pure at each instance no deterministic commitment can ever enlarge the set of equilibrium payoffs.

⁴¹If any player randomizes uniformly on described messages, the other is indifferent amongst all such messages.

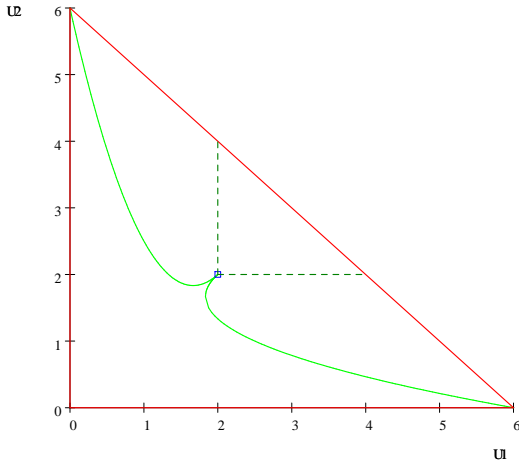
Consider the game:

$1 \backslash 2$	l	c	r
t	2, 2	1, 1	1, 1
m	1, 1	0, 0	6, 0
d	1, 1	0, 6	0, 0

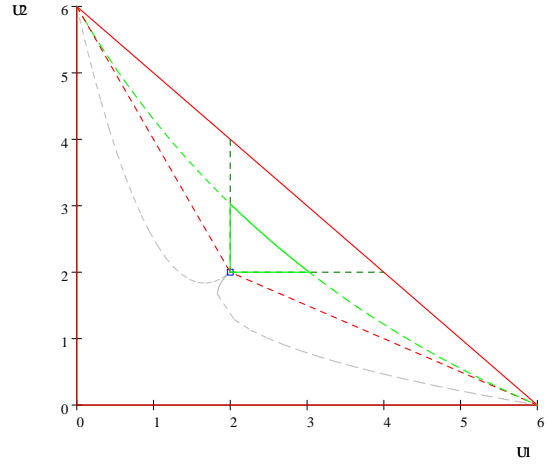
Such game possesses a unique Nash equilibrium outcome, in which the first agent plays t and the second l . Uniqueness can be shown by iterative elimination of strictly dominated strategies. This equilibrium lies on the boundary of the independent utility hull. Additionally, $\mathcal{N} = \mathcal{I} \cap \underline{\mathcal{N}}$, because any payoff $u \in \mathcal{I}$ must satisfy the following two inequalities, for $i \neq j \in \{1, 2\}$:

$$u_i \leq u_j + 1.5(2 - u_j)^2$$

In this game, the upper-bound on the commitment equilibrium payoff set, derived in proposition 14, requires: $\mathcal{K}_U \subseteq \text{co}(\mathcal{I} \cap \underline{\mathcal{N}}) = \mathcal{N}$. Therefore, a single layer of commitment cannot enlarge the set of equilibrium payoffs in such environment.



5A: in red \mathcal{U} , in green \mathcal{I} & in blue \mathcal{K}_U



5B: dashed in green $\partial \mathcal{I}_0$ & solid in green \mathcal{K}_U^1

Increasing the number layers, though, helps even in such negative scenario. Indeed, consider the canonical commitment structure $k_0 \in \mathcal{U}$ characterized by messaging space $M_{*0} = A \cup \{\Delta_N\}$ and by commitment map:

$$k_0(m) = \begin{cases} a & \text{if } m = a^N \cap a \in A \\ \Delta_N & \text{if } \text{otw} \end{cases}$$

Notice that continuation payoffs, v , are unique, $|V(k_0)| = 1$ and that $\mathcal{I}(v, M_{*0}) \cap \underline{\mathcal{N}} \supseteq \mathcal{N}$. Such payoff set is, in fact, defined by:

$$u_2 \leq 18 + u_1 - 12(1 + (u_1/2))^{-5}, \quad u_1 \geq 2 \quad \& \quad u_2 \geq 2$$

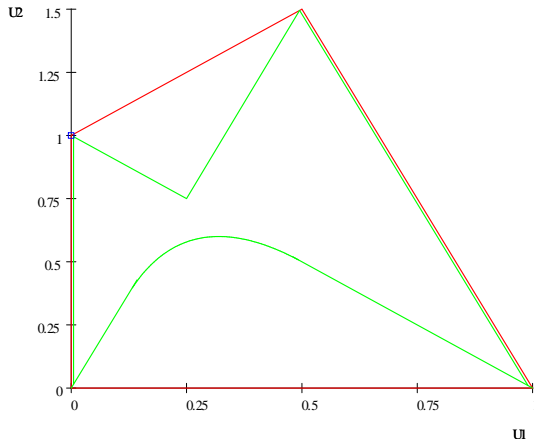
The last remark in turn implies that $\mathcal{K}_U^1 \supseteq \mathcal{I}(v, M_{*0}) \cap \underline{\mathcal{N}}$ by proposition 21. Thus, a second layer of commitment increases the equilibrium payoff set even when a layer alone could not. Further expansions are trivial and follow the scheme presented in the previous example.

In this game, pure layered contracts never enlarge the equilibrium set. Since any pure commitment would be vetoed at the first commitment instance, such commitments would also be vetoed at earlier instances given that any disagreement can only lead to the Nash outcome. Indeed, because no player can ever stand to lose, there is no room for beneficial concessions when bargaining upon pure contracts.

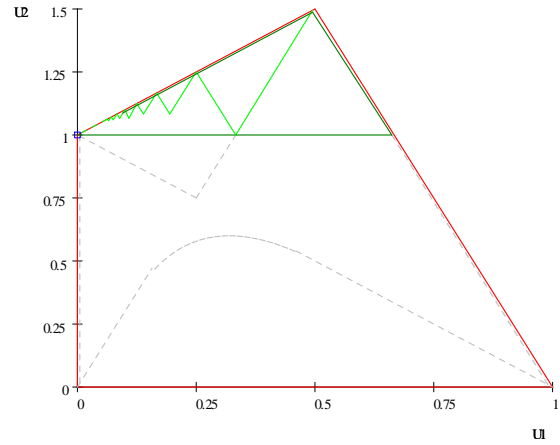
The last example consists of game in which the set $\mathcal{I} \cap \underline{\mathcal{N}}$ is not connected and the implementation of the Nash rational set occurs in a finite number of stages.

$1 \setminus 2$	n	c
n	0, 0	1/2, 3/2
c	0, 1	1, 0

In this game, $\mathcal{I} \cap \underline{\mathcal{N}} = (0, 1) \cup \text{co}((1/2, 3/2), (2/3, 1), (1/3, 1))$. Therefore, because all vertices of $\underline{\mathcal{N}}$ belong to $\mathcal{I} \cap \underline{\mathcal{N}} \subset \mathcal{K}(k^0)$ for an appropriately chosen $k^0 \in U^0$, by proposition 20 it follows that $\text{cl}(\mathcal{K}_U^1) = \mathcal{K}_U^* = \underline{\mathcal{N}}$. So that two layers of commitment suffice to implement the entire Nash rationalizable set. The set \mathcal{K}_U^0 is harder to characterize tightly and its exact computation is, for brevity, omitted. However, it is easy to verify that $\mathcal{K}_U^0 \subsetneq \underline{\mathcal{N}}$, since not all payoff in $\text{co}\{(1/2, 3/2), (0, 1)\}$ are attained at the first stage of commitment. Indeed, a single layer of commitment only implements payoffs in $\{(1 + 1/2p, 1/2p) \mid p \in \mathbb{N}_+\}$ in such subspace. The set \mathcal{K}_U^0 is depicted in figure 6B. One additional layer is need if one wants the implementation to be in pure commitments, $\text{cl}(\mathcal{K}_{U_p}^2) = \underline{\mathcal{N}}$.



6A: in red \mathcal{U} , in green \mathcal{I} & in blue \mathcal{N}



6B: in light green \mathcal{K}_U^0 & in dark green \mathcal{K}_U^1

10. Conclusions & Perspectives

The analysis discussed how the availability of commitment devices can lead to the implementation of allocations that are not self-enforcing in the original game. In the unilateral case, the informational content implicit in any commitment made this possible. The theorems presented in the independent section characterized the payoffs that obtained and the pledges signed with a single instance of commitment. The main contribution in the analysis of mechanisms in this class, was of the characterization of the equilibria in which players randomize amongst pledges. A trade-off between the flexibility to punish opponents and a commitment to favorable strategies arose, when pledges were made independently. Therefore in such environments, restricting attention to commitments to single pure strategies could limit the scope of the implementation.

The second and central part of analysis focused on the most cooperative form commitment devices. The analysis showed that endowing players with the ability to unanimously pledge to rule out strategies did not necessarily lead to full cooperation. Indeed, even though efficiency gains were often obtained, without further assumptions on the original game, there would be no guarantee of ever implementing an efficient payoff profile. In non-convex games, conditions were found preventing any efficient allocation from ever being implemented through a single stage of

contracting. It was, also, observed that perfect monitoring of pledges does provide players with a communication channel and that the use of such channel did not require any pledge to be enforced.

The main contribution in the analysis of such devices was the approximate implementation of the Nash rational hull in extensions with multiple rounds of commitment. Indeed increasing the number of commitment rounds was shown to enlarge the set of implementable payoffs, because at any instance occasional concessions to unfavorable pledges could be exchanged with beliefs in a more favorable treatment at later instances, were an agreement not to be found. Therefore, in such cooperative commitment environment, layered contracts can, for appropriately chosen beliefs, implement any allocation that a stochastic device would implement. When players made use of such hierarchies of contracts, the model could be interpreted as generating a non-cooperative theory of bargaining on contracts involving at each layer threats to dismiss further negotiations and occasional concessions to unfavorable outcomes to sustain cooperation.

Dimensionality conditions on the original game were established for which such implementation result would hold true even if at every stage the only available contracts involved pledges to single pure strategy profiles of the initial game. Thus, the empirical counterpart of such bargaining would merely result in the observation of some pure strategy contract being signed by all sometime before the game takes place. The contracts signed in this process and, thus, observed, are not necessarily implemented with a single layer, since concession may be required to support them in equilibrium.

The unanimity assumption did significantly increase the set of implementable payoffs, when compared to unilateral pledges, because such technology enhanced cooperation. The consensual technology, requiring agreement just amongst those committed, even though more flexible and, arguably, more empirically relevant, does not lead to efficiency gains on the unanimous case. The analysis of such devices is forthcoming.

Quite different dynamics arise if the original game is one with incomplete information. In such case, pledges do not only discipline incentives, but they also disclose information about the types of agents playing the game. In such scenario even unanimous pledges may not suffice in implementing efficient profiles. The commitment analysis of such games lies in the scopes of this theory.

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11. Appendix

Because such notation will be used extensively throughout the section, recall that $\underline{u}^i \in \arg \min_{u \in \mathcal{N}(\Gamma)} u_i$ and $\underline{u}_i^j = \arg \min_{u \in \mathcal{N}(\Gamma)} u_i$. The joint probability of an action profile $a \in A$ induced by the players’ independent randomizations, σ_i , often denoted by $\sigma(a) = \prod_{i \in N} \sigma_i(a_i)$.⁴² The dependence on the original game Γ is generally omitted for sake of clarity.

⁴²Similar conventions will be used for other independent randomizations.

11.1. Subgame Existence and Remarks

To prove Nash equilibrium existence in any committed subgame, one may, just, proceed by Kakutani fixed point theorem noting that domains are compact and convex and objective functions are weakly concave. Instead, a proof is provided that relies on the Brouwer fixed point theorem alone and that can be found in more generality in Geanakoplos [12].

LEMMA 1. *For any convex \mathcal{E} compact commitment profile, $k(m) \subseteq \Delta_N$, in the independent action simplex $\mathcal{E}^N \langle u, k(m) \rangle \neq \emptyset$.*

Proof. Define for any $i \in N$ the map:

$$\begin{aligned} f_i & : k(m) \rightarrow k_i(m) \\ f_i(\sigma) & \equiv \arg \max_{\sigma'_i \in k_i(m)} \left[u_i(\sigma'_i, \sigma_{-i}) - \|\sigma'_i - \sigma_i\|^2 \right] \end{aligned}$$

The maximand of any map f_i is continuous and strictly concave in σ'_i . Strictly concave because sum of a linear function and negative quadratic function in σ'_i . Continuous because sum of two continuous functions in σ'_i . Thus, for any convex compact commitment $k_i(m)$ the map f_i is well defined. Additionally, by the maximum principle, f_i is a continuous map in σ , because its maximand is continuous in σ . Let $f : k(m) \rightarrow k(m)$ denote $f = \{f_i\}_{i \in N}$. Brouwer theorem can be applied to the map f in order to prove existence of a fixed point, $\sigma = f(\sigma)$, because the map is continuous in σ and its domain is compact and convex. It must still be shown that any fixed point of the map f corresponds to Nash equilibrium of the committed subgame, $u_i(\sigma) \geq u_i(\sigma'_i, \sigma_{-i})$ for any $\sigma'_i \in k_i(m)$ and $i \in N$. For any σ_{-i} , the maximand of f_i is known to be strictly concave in σ'_i and, hence, with a unique maximizer. Let $\sigma_i = \arg \max_{\sigma'_i \in k_i(m)} \left[u_i(\sigma'_i, \sigma_{-i}) - \|\sigma'_i - \sigma_i\|^2 \right]$, by convexity of $k_i(m)$ and linearity of u for any $\varepsilon \in (0, 1)$ it must be that:

$$\begin{aligned} 0 & \geq u_i(((1-\varepsilon)\sigma_i + \varepsilon\sigma'_i), \sigma_{-i}) - \|((1-\varepsilon)\sigma_i + \varepsilon\sigma'_i) - \sigma_i\|^2 - \left[u_i(\sigma_i, \sigma_{-i}) - \|\sigma_i - \sigma_i\|^2 \right] = \\ & = (1-\varepsilon)u_i(\sigma_i, \sigma_{-i}) + \varepsilon u_i(\sigma'_i, \sigma_{-i}) - \varepsilon^2 \|\sigma'_i - \sigma_i\|^2 - u_i(\sigma_i, \sigma_{-i}) = \\ & = \varepsilon(u_i(\sigma'_i, \sigma_{-i}) - u_i(\sigma_i, \sigma_{-i})) - \varepsilon^2 \|\sigma'_i - \sigma_i\|^2 \\ & \Rightarrow u_i(\sigma'_i, \sigma_{-i}) - u_i(\sigma_i, \sigma_{-i}) \leq \varepsilon \|\sigma'_i - \sigma_i\|^2 \quad \text{for } \forall \varepsilon \in (0, 1) \\ & \Leftrightarrow u_i(\sigma'_i, \sigma_{-i}) - u_i(\sigma_i, \sigma_{-i}) \leq 0 \end{aligned}$$

Thus $\sigma_i \in \arg \max_{\sigma'_i \in k_i(m)} u_i(\sigma'_i, \sigma_{-i})$. Which proves the claim. ■

The prove of the two remarks about subgame behavior follows:

REMARK 2. *For any $\sigma \in \Delta_N$ there exists an $k \subseteq \Delta_N$ such that $\sigma \in \mathcal{E}^N \langle u, k \rangle$.*

Proof. Namely, for any $\sigma \in \Delta_N$, the fully committed subgame must satisfy $\sigma \in \mathcal{E}^N \langle u, \sigma \rangle$. ■

REMARK 3. *For any $k \subseteq k' \subseteq \Delta_N$, if $\sigma \in \mathcal{E}^N \langle u, k' \rangle$ and $\sigma \in k$, then $\sigma \in \mathcal{E}^N \langle u, k \rangle$.*

Proof. Indeed, for $\forall i \in N$, if $\sigma_i \in \arg \max_{\sigma'_i \in k'_i} u_i(\sigma'_i, \sigma_{-i})$ it must be that $\sigma_i \in \arg \max_{\sigma'_i \in k_i} u_i(\sigma'_i, \sigma_{-i})$, since $k_i \subseteq k'_i$. ■

11.2. Attainable Payoffs in Independent Game Extensions

The pure strategy independent implementation result generalizes a claim by Renou in [22]:

PROPOSITION 4. *For any $(k, M) \in \mathbf{I}$, strategy profile $\sigma \in \Delta_N$ is implemented in pure strategies by commitment $k(m)$ if and only if $\sigma \in \mathcal{E}^N \langle u, k(m) \rangle$ and $u_i(\sigma) \geq w_i(m_{-i}|k)$, for $\forall i \in N$.*

Proof. \Rightarrow : If strategy profile σ is implemented in pure strategies by $k(m)$, then by subgame perfection $\sigma \in \mathcal{E}^N \langle u, k(m) \rangle$. And by optimality in the commitment supergame it follows that

$u_i(\sigma) \geq w_i(m_{-i}|k)$, for $\forall i \in N$. Because if for some player $u_i(\sigma) < w_i(m_{-i}|k)$, that player should profitably deviate to $\max_{m_i \in M_i} \underline{u}_i^i(k(m))$.

⇐: Since $\sigma \in \mathcal{E}^{\mathcal{N}}(u, k(m))$, σ is an equilibrium of the subgame following commitment profiles m . Then, choosing continuation values, $v \in V(k)$, appropriately:

$$v(\bar{m}) = \begin{cases} \underline{u}^i(\bar{m}) \in \mathcal{N}(u, k(\bar{m})) & \text{if } \bar{m}_i \neq m_i \cap \bar{m}_{-i} = m_{-i} \\ u(\sigma) & \text{if } \bar{m} = m \\ ? & \text{if } \text{otw} \end{cases}$$

one can verify that, whenever $u_i(\sigma) \geq w_i(m_{-i}|k) = \max_{m_i \in M_i(k)} \underline{u}_i^i(k(m))$, the choice of commitment m is a pure strategy equilibrium of commitment game $\langle v, M \rangle$. ■

For $M_i(\pi) = \{m_i | \pi_i(m_i) > 0\}$, the general implementation result assets:

PROPOSITION 5. *For any $(k, M) \in \mathbb{I}$, a strategy profile (π, σ) of the game extension is implemented if and only if $\sigma(m) \in \mathcal{E}^{\mathcal{N}}(u, k(m))$ for any $m \in M(\pi)$ and $u_i(\pi|\sigma) = u_i(m_i, \pi_{-i}|\sigma) \geq w_i(\pi_{-i}|k)$, for any $m_i \in M_i(\pi)$ and $i \in N$.*

Proof. This is an immediate consequence of the previous claim and the observation that players when mixing at the messaging stage must be indifferent amongst the messages sent with positive probability and prefer them to possible deviations for some equilibrium continuation values. ■

PROPOSITION 6. *If $\mathcal{R} \cap \mathcal{I} \cap \mathcal{P} = \emptyset$, then $\mathcal{K}_{\mathbb{I}} \cap \mathcal{P} = \emptyset$.*

Proof. Clearly, if $\mathcal{R} \cap \mathcal{I} \cap \mathcal{P} = \emptyset$, no strategy σ implemented in pure strategies by a commitment $k(m)$ can be efficient, since $u(\sigma) \in \mathcal{R} \cap \mathcal{I}$ by arguments made in the text. To show that this must be the case even for mixed equilibria of all structures in \mathbb{I} ... ■

11.3. Attainable Payoffs in Unanimous Game Extensions

The proofs of the four remarks on unanimous commitment structures follow.

REMARK 7. *For any $u \in \mathcal{I} \cap \underline{\mathcal{N}}$ there exists a unanimous commitment structure, $k \in \mathbb{U}$, such that $u \in \mathcal{K}(k)$. Hence, $\mathcal{K}_{\mathbb{U}} \supseteq \mathcal{I} \cap \underline{\mathcal{N}}$.*

Proof. Since $u \in \mathcal{I}$, there exists $\sigma \in \Delta_N$ such that $u(\sigma) = u$. Then, consider the canonical form unanimous commitment structure with messaging spaces $M_* = \{1, 2\}$ and such that

$$k(m) = \begin{cases} \sigma & \text{if } m = (1, \dots, 1) \\ \Delta_N & \text{if } \text{otw} \end{cases}$$

Such structure implements u as a pure strategy Nash equilibrium, so long as $u \in \underline{\mathcal{N}}$. In fact, the following subgame perfect strategy result in the desired payoff u , for any $m \in M$ & $i \in N$:

$$\begin{aligned} \pi_i(1) &= 1 \\ \sigma_i(m) &= \begin{cases} \sigma_i & \text{if } m_j = 1 \text{ for } \forall j \in N \\ \in \arg \min_{\bar{\sigma} \in \mathcal{E}^{\mathcal{N}}} u_k(\bar{\sigma}) & \text{if } m_j = 1 \text{ for } \forall j \in N \setminus k \\ \in \mathcal{E}^{\mathcal{N}} & \text{otw} \end{cases} \end{aligned}$$

Providing the desired result. ■

REMARK 8. *For any unanimous commitment structure, $k \in \mathbb{U}$, and any $u \in \mathcal{K}(k)$ it must be that $u \in \underline{\mathcal{N}}$. Hence, $\underline{\mathcal{N}} \supseteq \mathcal{K}_{\mathbb{U}}$.*

Proof. Suppose, by contradiction that $\bar{u} \in \mathcal{K}(k)$ for some $k \in \mathbb{U}$ and that $\bar{u} \notin \underline{\mathcal{N}}$. By assumption $\exists i \in N$ such that $\underline{u}_i^i > \bar{u}_i$. But were this the case, agent i would profit by vetoing the commitment.

In fact, denoting player i 's veto message by message $\bar{m}_i \in M_i$, it must be that for any $v \in V(k)$ and $\pi_{-i} \in \times_{j \in -i} \Delta(M_j)$:⁴³

$$v_i(\bar{m}_i, \pi_{-i}) \geq \underline{u}_i^i > \bar{u}_i$$

Because when not committed at the action stage subgame perfection will require $\sigma(\bar{m}_i, m_{-i}) \in \mathcal{N}$, which would contradict $\bar{u} \in \mathcal{K}(k)$. ■

REMARK 9. *There exists $k \in \mathbf{U}$ and $v \in V(k)$ such that $\pi(m) = 1$ for some $m \in M$ & $v(\pi) = u \in \mathcal{K}(k)$ if and only if $u \in \mathcal{I} \cap \underline{\mathcal{N}}$.*

Proof. By the proof of remark 7 it must be that if $u \in \mathcal{I} \cap \underline{\mathcal{N}}$, then u can be implemented by $k \in \mathbf{U}$ and $v \in V(k)$ such that $\pi(m) = 1$ for some $m \in M$. Thus only the converse is to be shown. If $k \in \mathbf{U}$ and $v \in V(k)$ such that $\pi(m) = 1$ for some $m \in M$ & $v(\pi) = u \in \mathcal{K}(k)$, then:

$$v(\pi) = v(m) \in \mathcal{N} \langle u, k(m) \rangle \subseteq \mathcal{I} \langle u, k(m) \rangle \subset \mathcal{I}$$

Also, $v(\pi) \in \underline{\mathcal{N}}$ by remark 8 Thus, the two observations prove the claim. ■

REMARK 10. *For $(M, k), (\bar{M}, \bar{k}) \in \mathbf{U}$, if $\bar{M} \supseteq M$ and if $k(m) = \bar{k}(m)$ for any $m \in M$, then $\mathcal{K}(\bar{k}) \supseteq \mathcal{K}(k)$.*

Proof. To show this first note that if $u \in \mathcal{K}(k)$ then there exists $v \in V(k)$ such that $u = v(\pi)$ and $v(\pi) \in \mathcal{N} \langle v, M \rangle$. Given such map v construct a continuation value \bar{v} on $\mathcal{U}^{\bar{M}}$ such that

$$\bar{v}(m) = \begin{cases} v(m) & \text{if } m \in M \\ \underline{u}^i & \text{if } m_i \notin M_i \text{ \& } m_{-i} \in M_{-i} \\ ? & \text{if } \text{otw} \end{cases}$$

The so defined map \bar{v} must belong to $V(\bar{k})$ since $\bar{v}(m) \in \mathcal{N} \langle u, k(m) \rangle$ for any $m \in M$ and $\bar{v}(m) \in \mathcal{N}$ when $m_i \notin M_i$ and $m_{-i} \in M_{-i}$ for some $i \in N$. Additionally, it must be that $\bar{v}(\pi) = v(\pi)$ and $\bar{v}(\pi) \in \mathcal{N} \langle \bar{v}, \bar{M} \rangle$, because all Nash requirements hold for any $i \in N$:

$$\bar{v}_i(\pi) \geq \bar{v}_i(m_i, \pi_{-i}) \text{ for } \forall m_i \in \bar{M}_i$$

Thus, $u \in \mathcal{K}(\bar{k})$ since, for so chosen continuation values, no benefit can ever be derived from deviating to one of the additional strategies. ■

Now, it is shown that, whenever \mathcal{I} is convex, any payoff rationalized by the Nash threat can be implemented by a unanimous commitment structure.

COROLLARY 11. *If $\mathcal{I} \cap \underline{\mathcal{N}} = \underline{\mathcal{N}}$, then $\mathcal{K}_{\mathbf{U}} = \mathcal{K}_{\mathbf{U}}^* = \underline{\mathcal{N}}$.*

Proof. (1) $\mathcal{K}_{\mathbf{U}} \subseteq \underline{\mathcal{N}}$ by remark 8. Also, $\mathcal{K}_{\mathbf{U}} \supseteq \mathcal{I} \cap \underline{\mathcal{N}} = \underline{\mathcal{N}}$ by remark 7. But since $\mathcal{K}_{\mathbf{U}}^* \supseteq \mathcal{K}_{\mathbf{U}}$ and because by the same logic of remark 8 one obtains that $\underline{\mathcal{N}} \supseteq \mathcal{K}_{\mathbf{U}}^*$, it is also the case that: $\mathcal{K}_{\mathbf{U}}^* = \underline{\mathcal{N}}$. ■

The next proposition shows that unanimous structure can be exploited by players in order to directly communicate.

PROPOSITION 12. *For any $u \in \text{co}(\mathcal{N})$ and any $\varepsilon > 0$ there exists a $k \in \mathbf{U}$ and a $\hat{u} \in \mathcal{K}(k)$ such that $\|u - \hat{u}\| < \varepsilon$. Moreover, it is possible to find one such k that never restricts any player's strategy.*

Proof. By Caratheodory's theorem (reported below) it is known that, for any $u \in \text{co}(\mathcal{N})$, there exists a $\mathcal{N}' \subseteq \mathcal{N}$ with $|\mathcal{N}'| \leq N + 1$ such that one may express:

$$u = \sum_{\mathcal{N}'} u' \theta(u')$$

⁴³Recall that \bar{m}_i is such that $k(\bar{m}_i, m_{-i}) = \times_N \Delta_i$ for any profile of messages of the others $m_{-i} \in M_{-i}$.

For some probability measure θ on a subset of Nash equilibrium strategy profiles, \mathcal{N}' . The prove intends to construct a unanimous commitment structure approximating any such payoff as a Nash equilibrium.

For any canonical form unanimous commitment structure (M_*, k) consider the set $\Theta_{M_*} = \{t \in \mathbb{N}^{|\mathcal{N}'|} \mid \sum_{\mathcal{N}'} t(u') / |M_*| = 1\}$ of probability distributions on the Nash outcomes. Let θ_{M_*} denote the closest element (in the sup-norm) to θ in this set:

$$\theta_{M_*} = \arg \min_{t \in \Theta_{M_*}} \|t - \theta\|_+$$

For any order on the Nash payoffs, define:

$$u[\theta_{M_*}] \equiv \underbrace{[u', \dots, u']}_{|M_*| \theta_{M_*}(u')\text{-times}}, \dots, \underbrace{[u'', \dots, u'']}_{|M_*| \theta_{M_*}(u'')\text{-times}}, \dots, \underbrace{[u''', \dots, u''']}_{|M_*| \theta_{M_*}(u''')\text{-times}}$$

Then, construct a canonical commitment structure, k , in which continuation values $u(\cdot|\sigma) \in V(k)$ can satisfy:

$$\begin{aligned} u(m|\sigma) &= u(m_{1,2}, m'_{-1,2}|\sigma) \text{ for any } m \in M \text{ and } m'_{-1,2} \in M_{-1,2} \\ u(\cdot, m_{-i}|\sigma) &= \text{per}[u[\theta_{M_*}]] \text{ for any } m_{-i} \in M_{-i} \text{ and } i \in \{1, 2\} \end{aligned}$$

where $\text{per}[\cdot]$ denotes of the permutation operator. Such structure obtains, for instance, by labeling $M_* = \{1, 2, \dots, |M_*|\}$ and requiring payoff profiles to be cyclic permutations, denote $\text{cp}[\cdot]$, of the same payoff vector $\text{per}[u[\theta_{M_*}]]$. That is for any $m_1 = l \in M_*$:

$$u(l, \cdot, m_{-1,2}|\sigma) = \text{cp}^l[u(1, \cdot, m_{-1,2}|\sigma)]$$

For clarification, the table below depicts for $N = 2$ the commitment supergame associated to such mechanism. Label elements in the Nash subset by: $\mathcal{N}' = \{u^1, \dots, u^{|\mathcal{N}'|}\}$.

$1 \setminus 2$	1	2	...	$\theta_{M_*}(1)$	$\theta_{M_*}(1) + 1$...	$ M_* - \theta_{M_*}(\mathcal{N}') + 1$...	$ M_* $
1	u^1	u^1	...	u^1	u^2	...	$u^{ \mathcal{N}' }$...	$u^{ \mathcal{N}' }$
2	$u^{ \mathcal{M}_\theta }$	u^1	u^1	$u^{ \mathcal{N}' }$	$u^{ \mathcal{N}' }$
...	...	$u^{ \mathcal{M}_\theta }$	$u^{ \mathcal{N}' }$
$\theta_{M_*}(1)$	$u^{ \mathcal{M}_\theta }$	u^1
...
$ M_* $	u^1	...	u^1	u^2	$u^{ \mathcal{N}' }$	u^1

It needs to be shown that such a canonical commitment structure exists, is compatible with unanimity and possesses a Nash equilibrium with payoffs:

$$u_{M_*} = \sum_{\mathcal{N}'} u' \theta_{M_*}(u')$$

By remark 8, such continuation values can originate from a canonical unanimous structure that satisfies $\sigma(m_*^N) \in k(m_*^N)$ for any $m_* \in M_*$, for σ defined by $u(m_*^N|\sigma) = u(\sigma(m_*^N))$. Since for any $m_* \in M_*$:

$$u(m_*^N|\sigma) \in \mathcal{N} \langle u, k(m_*^N) \rangle$$

Such restriction can, always, met by choosing a commitment structure that satisfies $k(m_*^N) = \Delta_N$ for any $m_* \in M_*$. Additionally, because any other payoff profile belongs to $\mathcal{N}' \subseteq \mathcal{N}$ and therefore satisfies all unanimity requirements. Thus, there is no need to restrict any players strategy.

To then show that there is a NE of the commitment supergame with the desired payoff, notice that any player $i \in N \setminus \{1, 2\}$ is indifferent amongst all of his messages game, because by construction they do not affect payoffs. Also, remark that any player $i \in \{1, 2\}$ is indifferent between all of his messages when the remaining players randomize uniformly on their messaging spaces, $\pi_l(m_l) = 1/|M_*|$ for any $m_l \in M_l$ & $l \in N \setminus i$. In fact, for $j \neq i \in \{1, 2\}$ and $m_i \in M_*$:

$$\begin{aligned} u_i(m_i, \pi_{-i}|\sigma) &= \sum_{M_{-i}} u_i(m|\sigma)\pi_{-i}(m_{-i}) = \sum_{M_{-i}} u_i(m|\sigma)/|M_*|^{N-1} = \\ &= |M_*|^{N-2} \left[\sum_{m_j \in M_*} u_i(m|\sigma) \right] / |M_*|^{N-1} = \\ &= \left[\sum_{\mathcal{N}'} u'_i |M_*| \theta_{M_*}(u') \right] / |M_*| = \sum_{\mathcal{N}'} u'_i \theta_{M_*}(u') \end{aligned}$$

Hence, for anyone to randomize uniformly in the supergame would constitute a subgame perfect equilibrium of the commitment expansion of the original game yielding the desired payoff. Consequently, $u_{M_*} \in \mathcal{K}(k)$ for the above described unanimous commitment structure.

It still needs to be shown that for $\forall \varepsilon > 0$ there $\exists \delta < \infty$ such that $\|u - u_{M_*}\| \leq \varepsilon$, whenever $|M_*| > \delta$. Because of continuity of u with respect to θ , it suffices to show that $\|\theta - \theta_{M_*}\|_+ \leq \zeta(\varepsilon)$. This obtains, for instance, by letting $|M_*| > \delta = 2/\zeta(\varepsilon)$. By way of contradiction suppose that $|M_*| > 2/\zeta(\varepsilon)$ & $\|\theta - \theta_{M_*}\|_+ > \zeta(\varepsilon)$. For the moment, also, assume that $\arg \max_{u'} |\theta(u') - \theta_{M_*}(u')|$ is unique. By hypothesis there exist a u' such that $|\theta(u') - \theta_{M_*}(u')| > \zeta(\varepsilon)$. Because both measures integrate to unity, if $\theta(u') > \theta_{M_*}(u')$, there also exists u'' such that $\theta(u'') < \theta_{M_*}(u'')$. But then, it is possible to construct $\theta'_{M_*} \in \Theta_{M_*}$ that is closer to θ , than θ'_{M_*} . Indeed, let:

$$\theta'_{M_*}(u) = \begin{cases} \theta_{M_*}(u) + 1/|M_*| & \text{if } u = u' \\ \theta_{M_*}(u) - 1/|M_*| & \text{if } u = u'' \\ \theta_{M_*}(u) & \text{if } \text{otw} \end{cases}$$

In fact, $|\theta(u') - \theta_{M_*}(u')| > |\theta(u') - \theta'_{M_*}(u')|$ since $\zeta(\varepsilon) > 2/|M_*| > 1/|M_*|$. Also:

$$|\theta(u'') - \theta'_{M_*}(u'')| < \max \{1/|M_*|, |\theta(u'') - \theta'_{M_*}(u'')|\} < |\theta(u') - \theta_{M_*}(u')|$$

But this would contradict $\theta_{M_*} \in \arg \min_{t \in \Theta_{M_*}} \|t - \theta\|_+$, since $\|\theta - \theta_{M_*}\|_+ > \|\theta - \theta'_{M_*}\|_+$ and $\theta_{M_*} \in \Theta_{M_*}$. If the $\arg \max_{u'} |\theta(u') - \theta_{M_*}(u')|$ is not unique applying the described procedure to one of the maxima produces a vector with equal sup-norm distance from θ , but with one less component attaining such maximal distance. Iterating on the procedure until the maximizer is unique provides the result. ■

In the proof of the above result it was asserted that any point in the convex hull of a the Nash equilibria could be represented by a randomization on no more than $N + 1$ elements of Nash equilibrium payoff set. Such result is a straightforward application of Caratheodory's theorem, reported below. The exact proof of this theorem can be found in [4] and is, therefore, omitted.

CARATHEODORY THEOREM *If $X \subset \mathbb{R}^n$, then $\forall x \in \text{co}(X)$, there $\exists X' \subseteq X$ and a probability measure μ on X' such that $|X'| \leq n + 1$ & $x = \sum_{X'} x' \mu(x')$.*

The characterization of the implementable payoffs for unique Nash equilibrium games follows.

PROPOSITION 13. *If $|\mathcal{N}| = 1$, then: (1) $\mathcal{K}_U^* = \text{co}(\mathcal{K}_U) = \text{co}(\mathcal{I} \cap \mathcal{N})$.*

$$(2) \mathcal{K}_U \subseteq \overline{\mathcal{I} \cap \mathcal{N}}.$$

Proof. When $|\mathcal{N}| = 1$, for any $k \in U$ and $m \in M$ such that $k(m) \neq \Delta_N$, if $\pi(m) > 0$ in some Nash equilibrium of the commitment supergame, it must be that $u(m|\sigma) \in \mathcal{N}$. Indeed, were this not the case, there would exist an agent $j \in N$ for which $u_j(m|\sigma) < \underline{u}_j$. But for such player j the strategy of sending message m_j is weakly dominated by the option of vetoing any commitment.

In particular, in any such equilibrium equilibrium of the messaging game agent j would choose $\pi_j(m_j) = 0$ whenever $\pi_{-j}(m_{-j}) > 0$. Thus, $u(\pi|\sigma) \in \mathcal{K}_U$ and $\pi(m) > 0$ implies $u(m|\sigma) \in \underline{\mathcal{N}}$.

Then, to prove part (1) note that any payoff in $\mathcal{I} \cap \underline{\mathcal{N}}$ can be implemented as a pure strategy NE by remark 9. Given this observation it is immediate to notice that $\mathcal{K}_U^* \supseteq \text{co}(\mathcal{K}_U) = \text{co}(\mathcal{I} \cap \underline{\mathcal{N}})$, since $\mathcal{K}^*(k) \supseteq \text{co}(\mathcal{K}(k))$ for any $k \in U$.⁴⁴ Because $v(m) \in \mathcal{I} \cap \underline{\mathcal{N}}$ for any $m \in M$, $v \in V(k)$ and $k \in U$, we also have $\mathcal{K}_U^* \subseteq \text{co}(\mathcal{I} \cap \underline{\mathcal{N}})$.

The latter remark also implies that $\mathcal{K}_U \subseteq \text{co}(\mathcal{I} \cap \underline{\mathcal{N}})$. This being the smallest convex set containing \mathcal{K}_U . For part (2) of the claim, note that if $u = v(\pi) \in \mathcal{K}_U$ for some $v \in V(k)$, since $v(m) \in \mathcal{I} \cap \underline{\mathcal{N}}$ for any $m \in M$ such that $\pi(m) > 0$, it must be that for any message profile m such that $k(m) \neq \Delta_N$ and any player $i \in N$:

$$v_i(\pi) = \pi_{-i}(m_{-i})v_i(m) + (1 - \pi_{-i}(m_{-i}))u_i^i \leq v_i(m)$$

proving the desired result. ■

The next proposition derives an upper-bound for the set of commitment equilibria. There is a potential for tighten such bound since not all restrictions are taken into account.

PROPOSITION 14. $\mathcal{K}_U \subseteq \mathcal{K}_U^* \subseteq \bigcap_{i \in N} \text{co}(\mathcal{U}_N \cap \underline{\mathcal{N}}_i)$.

Proof. First, note that if for any $k \in U$ and $v \in V(k)$ the equilibrium is in pure strategies it must belong to the desired set, since any such equilibrium payoff belongs to $\mathcal{I} \cap \underline{\mathcal{N}}$ by remark 9 and because:

$$\mathcal{I} \cap \underline{\mathcal{N}} \subseteq \text{co}(\mathcal{I} \cap \underline{\mathcal{N}}) \subseteq \text{co}(\mathcal{U}_N \cap \underline{\mathcal{N}}) \subseteq \bigcap_{i \in N} \text{co}(\mathcal{U}_N \cap \underline{\mathcal{N}}_i)$$

For any $k \in U$ and $v \in V(k)$, any mixed equilibrium payoff profile can be written as $v(\pi) = \sum_{M_i} \pi(m_i)v(m_i, \pi_{-i})$. By unanimity for any $m_i \in M_i$ there exist at most one profile $m_{-i} \in M_{-i}$ such that $k(m) \neq \Delta_N$. Therefore for any m_i such that $\pi_i(m_i) > 0$:

$$v(m_i, \pi_{-i}) = \pi_{-i}(m_{-i})v(m) + (1 - \pi_{-i}(m_{-i}))u^\pi(m_i) \in \mathcal{U}_N$$

for $u^\pi(m_i) = \sum_{\bar{m}_{-i} \in M_{-i} \setminus m_{-i}} \pi_{-i}(\bar{m}_{-i}) / (1 - \pi_{-i}(m_{-i})) v(m_i, \bar{m}_{-i}) \in \text{co}(\mathcal{N})$. Since any player's choice of commitment is optimal, it must also be that $v(m_i, \pi_{-i}) \in \underline{\mathcal{N}}_i$, because otherwise agent i could benefit by deviating to the veto message. Collecting the last few observations, implies that $v(\pi) \in \text{co}(\mathcal{U}_N \cap \underline{\mathcal{N}}_i)$, for any $i \in N$. Hence, the promised result, that $v(\pi) \in \mathcal{K}(k)$ implies $v(\pi) \in \bigcap_{i \in N} \text{co}(\mathcal{U}_N \cap \underline{\mathcal{N}}_i)$, always holds. The same holds true whenever players can communicate before committing. Since, for any $i \in N$ and $m_i \in M_i$:

$$v(m_i, \mu_{-i}|m_i) = \mu_{-i}(m_{-i}|m_i)v(m) + (1 - \mu_{-i}(m_{-i}|m_i))u^\mu(m_i) \in \mathcal{U}_N \cap \underline{\mathcal{N}}_i$$

Which proves the claim. ■

Next, it is shown that one stage of unanimous commitment does not, generally suffice to implement any efficient allocation.

PROPOSITION 15. *If $\mathcal{I} \cap \underline{\mathcal{N}} \cap \mathcal{P} = \emptyset$, then $\mathcal{K}_U \cap \mathcal{P} = \emptyset$.*

Proof. The stated assumption, immediately, implies that $\mathcal{N} \cap \mathcal{P} = \emptyset$. It needs to be shown that for any game satisfying this assumption and for any continuation payoff $v \in V(k)$, generated from some unanimous commitment structure $k \in U$, it must be that:

$$\mathcal{N}(v, M_*) \cap \underline{\mathcal{N}} \cap \mathcal{P} = \emptyset$$

⁴⁴The correlated equilibrium set of any game contains the convex hull of the set of Nash equilibria.

In fact, any payoff in $\langle v, M_* \rangle$ that does not lead to a commitment must be inefficient, since $\mathcal{N} \cap \mathcal{P} = \emptyset$. It follows that any mixed equilibrium of $\mathcal{N} \langle v, M_* \rangle$ is inefficient, because it must put positive probability off diagonal payoffs. But any pure equilibrium must be inefficient as well, because if no agreement is reached its payoffs belong to \mathcal{N} and if an agreement is reached payoffs must belong to $\mathcal{I} \cap \underline{\mathcal{N}}$ by remark 9. Thus, for such a game no efficient allocation can ever be implemented with a single instance of commitment. ■

The proof that stochastic commitments implement the entire Nash rational sets follows:

LEMMA 16. $\mathcal{K}_U^S = \mathcal{K}_U^{*S} = \underline{\mathcal{N}}$.

Proof. Consider $u \in \underline{\mathcal{N}}$. Such payoff profile can be represented as $u = u(\mu)$, for $\mu \in \Delta(A)$. Then consider the canonical unanimous commitment structure (M_*, k) satisfying $M_* = \{1, 2\}$ and:

$$k(m) = \begin{cases} \mu & \text{if } m = (1, \dots, 1) \\ \Delta_N & \text{if } \text{otw} \end{cases}$$

This commitment structure is well defined since $\mu \in \Delta(\times_{i \in N} [A_i \cup \{\Delta_i\}])$. If the contract is ever implemented no options will be left to any player at the action stage. Thus, one must only check that signing the contract more beneficial than vetoing it for any player. But choosing continuation values $v \in V(k)$ so that $v(m_i = 2, 1, \dots, 1) = \underline{u}^i$, verifies such claim, since for any $i \in N$:

$$v_i(1, \dots, 1) = u_i(\mu) \geq \underline{u}_i^i$$

As before, no payoff below the worst threat would ever be accepted because of the voluntary nature of the mechanism. ■

11.4. Commitments to Commit

In this section the proofs of results on multiple stages of commitment are reported. First, it is shown that no payoff outside the Nash rational payoff hull can ever be implemented as an equilibrium independently of the stages of commitment structure. For convenience denote the veto message by m_v . That is, $k_t(m_v, m_{-i,t}) = \times_{i \in N} \Delta(M_{i,t-1})$, for any $m_{-i,t} \in M_{-i,t}$ and $i \in N$.

REMARK 17. $\mathcal{K}_U^t \subseteq \underline{\mathcal{N}}$, for any $1 \leq t < \infty$.

Proof. Suppose, by contradiction that $\bar{u} = u_i(\pi_t | \pi^{t-1}, \sigma) \in \mathcal{K}(k^t)$ for some $k^t \in \mathcal{U}^t$ and that $\bar{u} \notin \underline{\mathcal{N}}$. Then, $\exists i \in N$ such that $\underline{u}_i^i > \bar{u}_i$. But were this the case agent i could profit by vetoing at any stage. Indeed, switching to $\bar{\pi}_{i,s}(m_v | m^{t,s+1}) = 1$, when $m_i^{t,s+1} = (m_v, \dots, m_v)$ for any $m_{-i}^{t,s+1} \in M_{-i}^{t,s+1}$ at each stage $s \in \{0, \dots, t\}$, is beneficial since:

$$u_i(\bar{\pi}_{i,t}, \pi_{-i,t} | \bar{\pi}_i^{t-1}, \pi_{-i}^{t-1}, \sigma) \geq \underline{u}_i^i > \bar{u}_i$$

Because when not committed at the action stage subgame perfection will require $\sigma(m_v^t, m_{-i}^t) \in \mathcal{N}$, contradicting $\bar{u} \in \mathcal{K}(k^t)$. ■

Again enlarging the set of feasible commitments cannot reduce the set of equilibrium payoffs.

PROPOSITION 18. For $(M^t, k^t), (\bar{M}^t, \bar{k}^t) \in \mathcal{U}^t$, if $\bar{M}^t \supseteq M^t$ and if $k_s(m_s) = \bar{k}_s(m)$ for any $m \in M_s$ and $s \leq t$, then $\mathcal{K}(\bar{k}^t) \supseteq \mathcal{K}(k^t)$.

Proof. Let (σ, π^t) be an equilibrium strategy of k^t , supported by continuation values v^t , so that $v_t(\pi_t) \in \mathcal{K}(k^t)$. Then, choosing continuation values for the larger commitment structure \bar{k}^t so that at each information stage $m^{t,s+1} \in M^{t,s+1}$, for any $s \leq t$:

$$\bar{v}_s(m | m^{t,s+1}) = \begin{cases} v_s(m_s | m^{t,s+1}) & \text{if } m_s \in M_s \\ \underline{u}^i & \text{if } m_{i,s} \in \bar{M}_{i,s} \setminus M_{i,s} \ \& \ m_{-i,s} \in M_{-i,s} \\ ? & \text{if } \text{otw} \end{cases}$$

would support strategy $(\bar{\sigma}, \bar{\pi}^t)[M^t] = (\sigma, \pi^t)[M^t]$ as an equilibrium of \bar{k}^t . Thus, for off-equilibrium strategies matching the described continuation values, $\bar{v}_t(\bar{\pi}_t) = v_t(\pi_t) \in \mathcal{K}(k^t)$. ■

Adding stages of commitment cannot decrease the set of attainable payoffs.

COROLLARY 19. $\mathcal{K}(k^t) \supseteq \mathcal{K}(k^{t-1})$, for any $k^t \in \mathcal{U}^t$.

Proof. It must be shown that if $u \in \mathcal{K}(k^{t-1})$, then $u \in \mathcal{K}(k^t)$. Because unanimous mechanisms are with veto, the message $m_{i,t} = m_v$ can be sent by any agent $i \in N$ at the $t + 1^{\text{st}}$ layer. Then, for any payoff $u \in \mathcal{K}(k^{t-1})$, there exists $v_t \in V(k^t)$ such that:

$$v_t(m_v, m_{-i,t}) = u \quad \text{for } \forall m_{-i,t} \in M_{-i,t} \text{ \& } i \in N$$

Since $u \in \mathcal{K}(k^{t-1})$, implies that $u \in \mathcal{N}\langle v_{t-1}, M_{t-1} \rangle$ for some $v_{t-1} \in V(k^{t-1})$ and because $k_t(m_v, m_{-i,t}) = \times_{i \in N} \Delta(M_{i,t-1})$ for $\forall m_{-i,t} \in M_{-i,t}$ & $i \in N$. Therefore such continuation values are obtained by choosing $v_{t-1}(\cdot | m_v, m_{-i,t}) = v_{t-1} \in V(k^{t-1})$ for $\forall m_{-i,t} \in M_{-i,t}$ & $i \in N$.

Given the so constructed continuation values, for any agent $i \in N$ it would be optimal to veto the $t + 1^{\text{st}}$ layer when all the other players are vetoing themselves, since:

$$v_{i,t}(m_v^N) = v_i(m_{i,t}, m_v^{N-1}) \quad \text{for any } m_{i,t} \in M_{i,t} \text{ \& } i \in N$$

Therefore such strategies would constitute a Nash equilibrium of the $t + 1$ stage game, implying that $u \in \mathcal{K}(k^t)$. ■

Next, it is proven that an additional layer of commitment allows to approximately implement any convex combination of the lower layer equilibrium payoffs.

PROPOSITION 20. $\text{cl}(\mathcal{K}_{\mathcal{U}}^t) \supseteq \text{co}(\mathcal{K}_{\mathcal{U}}^{t-1})$, for any $1 \leq t < \infty$.

Proof. Also, this claim is a straightforward application of propositions 12 and 18. Because applying the technique provided in that proof it is possible to, approximately, attain as an equilibrium of the t^{th} stage of commitment any payoff in the convex hull of the Nash equilibria of the $(t + 1)^{\text{st}}$ stage. That is, for any $u \in \text{co}(\mathcal{K}(k^{t-1}))$ for some $k^{t-1} \in \mathcal{K}_{\mathcal{U}}^{t-1}$ it is possible to construct $k^t \in \mathcal{K}_{\mathcal{U}}^t$ such that $u \in \text{cl}(\mathcal{K}(k^t))$. This is accomplished just as in proposition 12. ■

Payoffs in the independent hull of a commitment game at stage t belong for some commitment structure, k_{t+1} , to the equilibrium set of stage $t + 1$.

PROPOSITION 21..For any $(M^t, k^t) \in \mathcal{U}^t$ and $v_t \in V(k^t)$ it must be that $\mathcal{I}\langle v_t, M_t \rangle \cap \underline{\mathcal{N}} \subseteq \mathcal{K}_{\mathcal{U}}^{t+1}$.

Proof. It is shown that for any $u \in \mathcal{I}\langle v_t, M_t \rangle \cap \underline{\mathcal{N}}$ there exists a unanimous commitment structure $k^{t+1} \in \mathcal{U}^{t+1}$ such that $u \in \mathcal{K}_{\mathcal{U}}(k^{t+1})$. Specifically, set $k^{t+1} = (k_{t+1}, k^t)$ and note that by definition because $u \in \mathcal{I}\langle v_t, M_t \rangle$ there exists $\pi_t \in \times_{i \in N} \Delta(M_{i,t})$ such that $u = v_t(\pi_t)$. Let k_{t+1} be a canonical form unanimous structure with messaging space $M_{*t+1} = \{1, m_v\}$ and such that $k_{t+1}(1, \dots, 1) = \pi_t$. For such a commitment structure u can be supported as an equilibrium payoff choosing continuation payoffs $v_{t+1} \in V(k^{t+1})$ so that:

$$v_{t+1}(m_{t+1}) = \begin{cases} u & \text{if } m_{t+1} = 1^N \\ \underline{u}^i & \text{if } m_{i,t+1} = m_v \quad \cap \quad m_{-i,t+1} = 1^{N-1} \\ ? & \text{if } \text{otw} \end{cases}$$

Such continuation maps are well defined since $u \in \mathcal{N}\langle v_t, \pi_t \rangle$ and because $\underline{u}^i \in \mathcal{N} \subseteq \cup_{v \in V_t(k^t)} \mathcal{N}\langle v, M_t \rangle$. Hence, when all others' commit every player would weakly prefer to commit that to be awarded his worst equilibrium payoff, which belongs to the equilibrium set of any commitment structure at any layer, by the veto property. ■

Let $\partial(\cdot)$ denote the boundary of a set. For notational simplicity, for any $a \in A$ let $[a]^t$ denote a commitment to fully commit to fully (...) commit to action profile a . That is:

$$[a]^{t+1} \equiv [\pi_t([a]^t) = 1] \quad \& \quad [a]^0 \equiv [\sigma(a) = 1]$$

Similarly let $A^t \equiv \{[a]^t | a \in A\}$. The next theorem states that any Nash rationalizable allocation may be approximately attained as an equilibrium of a game with t layers of commitment, for t large enough.

THEOREM 22. *Let $C \equiv \lim_{t \rightarrow \infty} \text{cl}(\mathcal{K}_U^t)$, then $\text{cl}(C) = \underline{\mathcal{N}}$.*

Proof. By corollary 19 and remark 17 it must be that $\underline{\mathcal{N}} \supseteq \text{cl}(\mathcal{K}_U^t) \supseteq \text{cl}(\mathcal{K}_U^{t-1})$ for any $t \geq 1$. Therefore, the sequence $\{\text{cl}(\mathcal{K}_U^t)\}_{t=0}^\infty$ possesses a limit. It needs to be shown that the closure of such limit cannot differ from $\underline{\mathcal{N}}$. By way of contradiction suppose that the limit be a set C such that $\text{cl}(C) \subsetneq \underline{\mathcal{N}}$. It will be shown that in such scenario some stages of commitment always lead to the implementation of payoffs outside of $\text{cl}(C)$ as commitment equilibria. Contradicting C being $\lim_{t \rightarrow \infty} \text{cl}(\mathcal{K}_U^t)$.

The limiting set C must be convex, because $\text{cl}(\mathcal{K}_U^t) \supseteq \text{co}(\text{cl}(\mathcal{K}_U^{t-1}))$, a direct consequence of proposition 20. Consider some number of stages t , a point $u_\mu = \sum \mu(a)u(a) \in \underline{\mathcal{N}} \setminus \text{cl}(C)$ and a point $u_0 \in \partial \text{cl}(\mathcal{K}_U^t)$ for which $\alpha u_\mu + (1 - \alpha)u_0 \in \underline{\mathcal{N}} \setminus \text{cl}(\mathcal{K}_U^t)$ for any $\alpha \in (0, 1]$. Such points exist because $\text{cl}(\mathcal{K}_U^t)$ is a closed subset of the closed and convex set $\underline{\mathcal{N}}$.

The commitment structure, $k^{t+1} \in U^{t+1}$, will be used to construct payoffs that lie outside of the supposed limiting set. Let (\bar{M}^t, \bar{k}^t) be the commitment structure supporting u_0 as an equilibrium payoff and note that by proposition 18 u_0 is, also, supported by the commitment structure $(M^t, k^t) \in U^t$ expanded for $\forall s \leq t$ to $M_{i,s} = \bar{M}_{i,s} \cup A^s$ and:

$$k_s(m) = \begin{cases} \bar{k}_s(m) & \text{if } m \in \bar{M}_{i,s} \\ \pi_s([a]^{s-1}) = 1 & \text{if } m = ([a]^s, \dots, [a]^s) \text{ for } a \in A \\ \times_{i \in N} \Delta(M_{i,s-1}) & \text{if } \text{otw} \end{cases}$$

Then, simply consider an additional stage of canonical unanimous commitment (M_{*t+1}, k_{t+1}) defined by $M_{*t+1} = m_v \cup A^t$ and the map:

$$k_{t+1}(m) = \begin{cases} \pi_{t+1}([a]^t) = 1 & \text{if } m = ([a]^t, \dots, [a]^t) \text{ for } a \in A \\ \times_{i \in N} \Delta(M_{i,t}) & \text{if } \text{otw} \end{cases}$$

There are continuation values, $v_{t+1} \in V_{t+1}(k^{t+1})$, for which the payoff $(1/|A|^{N-1})u_\mu + (1 - 1/|A|^{N-1})u_0 \in \mathcal{I}(v_{t+1}, M_{*t+1})$. Namely, for continuation values:

$$v_{t+1}(m) = \begin{cases} u(a) & \text{if } m = ([a]^{t+1})^N \\ u_0 & \text{if } \text{otw} \end{cases}$$

and for strategy profiles defined by, for $i \in N \setminus 1$:

$$\pi_{i,t+1}(m_{i,t+1}) = \begin{cases} 1/|A| & \text{if } m_{i,t+1} \in A^{t+1} \\ 0 & \text{if } \text{otw} \end{cases}$$

$$\pi_{1,t+1}(m_{1,t+1}) = \begin{cases} \mu(a) & \text{if } m_{1,t+1} \in A^{t+1} \\ 0 & \text{if } \text{otw} \end{cases}$$

it must be that $v_{t+1}(\pi_{t+1}) = (1/|A|^{N-1})u_\mu + (1 - 1/|A|^{N-1})u_0$. Thus, by proposition 21 $u_1 = (1/|A|^{N-1})u_\mu + (1 - 1/|A|^{N-1})u_0 \in \text{cl}(\mathcal{K}_U^{t+2})$. Recall that u_μ and u_0 were chosen so that $\alpha u_\mu + (1 - \alpha)u_0 \in \underline{\mathcal{N}} \setminus \text{cl}(\mathcal{K}_U^t)$ for any $\alpha \in (0, 1]$. Thus, $u_1 \in \underline{\mathcal{N}} \setminus \text{cl}(\mathcal{K}_U^t)$. Iterating such procedure would yield a sequence $u_r = (1/|A|^{N-1})u_\mu + (1 - 1/|A|^{N-1})u_{r-1} \in \text{cl}(\mathcal{K}_U^{t+2r})$ which converges to u_μ as r diverges to infinity. But $u_\mu \notin \text{cl}(C)$, would contradict C being the limit of $\text{cl}(\mathcal{K}_U^t)$. ■

11.5. Pure Unanimous Implementation

The first two claims of this section are trivial extensions to propositions 19 and 20.

COROLLARY 23. $\mathcal{K}_{\mathcal{U}_P}^t \supseteq \mathcal{K}_{\mathcal{U}_P}^{t-1} \supseteq \mathcal{N} \cup \underline{\mathcal{A}}$, for any $1 \leq t < \infty$.

Proof. Since any commitment pure structure $k^t \in \mathcal{U}_P^t$ it must be that any payoff $u \in \mathcal{K}(k^{t-1})$ can be implemented at the t^{th} layer by having all player veto any commitment at that stage. In fact, choosing continuation values that yield payoff u if anyone vetoes, would make the desired strategy an equilibrium. The second part of the claim follows because $\mathcal{K}_{\mathcal{U}_P}^0 \supseteq \mathcal{A} \cap \underline{\mathcal{N}} = \underline{\mathcal{A}}$ and because $\mathcal{K}_{\mathcal{U}_P}^0 \supseteq \mathcal{N}$ by collective veto. ■

COROLLARY 24. $\text{cl}(\mathcal{K}_{\mathcal{U}_P}^t) \supseteq \text{co}(\mathcal{K}_{\mathcal{U}_P}^{t-1})$, for any $1 \leq t < \infty$.

Proof. The technique described in the proof of propositions 12 and 20 remains unaffected by the restriction since next layer's entire messaging space is a pure commitment that sustains all payoffs in $\mathcal{K}_{\mathcal{U}_P}^{t-1}$. ■

The third claim, instead, provides a technique that can be used to enlarge the set of implementable payoffs with each layer of commitment:

PROPOSITION 25. For $u_t \in \mathcal{K}_{\mathcal{U}_P}^t$, $u \in \mathcal{A}$ and $q \in \mathbb{N}_+$, if $(1/q^{N-1})u + (1 - 1/q^{N-1})u_t \in \underline{\mathcal{N}}$ then:

$$(1/q^{N-1})u + (1 - 1/q^{N-1})u_t \in \mathcal{K}_{\mathcal{U}_P}^{t+1}$$

Proof. Under the stated assumptions let $u = u(a)$ for $a \in A$. Then, consider a canonical form commitment structure (M_{*t+1}, k_{t+1}) such that $k_{t+1}(m^N) = [a]^{t+1}$ for any $m \in M_{*t+1}(a) \subset M_{*t+1}$ and such that $|M_{*t+1}(a)| \geq q$. Then, the following continuation values are feasible, $v_{t+1} \in V(k^{t+1})$, for $k^{t+1} \in \mathcal{U}_P^{t+1}$:

$$v_{t+1}(m) = \begin{cases} u & \text{if } m_i \in M_{*t+1}(a) \ \& \ m_i = m_j \ \forall i, j \in N \\ u_t & \text{if } m_i \in M_{*t+1}(a) \ \forall i \in N \ \& \ m_i \neq m_j \ \text{some } i, j \\ \underline{u}^i & \text{if } m_i \in M_{*t+1} \setminus M_{*t+1}(a) \ \cap \ m_{-i} \in M_{-i, t+1}(a) \\ ? & \text{if } \text{otw} \end{cases}$$

For so chosen continuation values for all players to randomize uniformly on $M_{*t+1}(a)$ provides an allocation with the desired payoff, $(1/q^{N-1})u + (1 - 1/q^{N-1})u_t$. Such strategy is part of an equilibrium because any deviation from it by any player would give him is Nash threat value and would, therefore, not be profitable by $(1/q^{N-1})u + (1 - 1/q^{N-1})u_t \in \underline{\mathcal{N}}$. ■

The last formal result of the section provides conditions for which the use of pure unanimous layered commitments suffices for the implementation of the Nash rational payoff hull.

THEOREM 26. If $\text{vert}(\mathcal{U} \cap \text{aff}(\underline{\mathcal{N}})) \subseteq \mathcal{A} \cup \mathcal{N}$ and if there exist $u \in \text{co}(\mathcal{N} \cup \underline{\mathcal{A}})$, $u' \in \mathcal{A}$, $\alpha \in [0, 1]$ and $j \in \underline{N}$ such that:

$$\begin{aligned} \alpha u'_i + (1 - \alpha)u_i &> \underline{u}_i^i \ \text{for } \forall i \in \underline{N} \setminus j \\ \alpha u'_i + (1 - \alpha)u_i &\geq \underline{u}_i^i \ \text{for } \forall i \in N \setminus (\underline{N} \setminus j) \end{aligned}$$

then for $C_P \equiv \lim_{t \rightarrow \infty} \text{cl}(\mathcal{K}_{\mathcal{U}_P}^t)$, it must be that $\text{cl}(C_P) = \underline{\mathcal{N}}$.

Proof. First it is shown that under the stated assumptions it is possible to construct $u_3 \in \mathcal{K}_{\mathcal{U}_P}^3$ such that $u_{3,i} > \underline{u}_i^i$ for $\forall i \in \underline{N}$. Then, it is argued that, whenever such a utility profile exists, it is possible to approximately implement any allocation in the Nash rational set. If $\underline{\mathcal{N}} = \mathcal{N}$, there is nothing to prove so assume $\underline{N} > 0$.

Observe that $\mathcal{N} \cup \underline{\mathcal{A}} \subseteq \mathcal{K}_{\text{UP}}^0$, because allocations in \mathcal{N} are implemented by any commitment structure and because allocations in $\underline{\mathcal{A}}$ are implemented by pure unanimous commitments $[a]^0$ for $a \in \underline{\mathcal{A}}$. Then note that by corollary 24 it must be that $\text{co}(\mathcal{N} \cup \underline{\mathcal{A}}) \subseteq \mathcal{K}_{\text{UP}}^1$. Then by assumption there exist values $u_1 \in \text{co}(\mathcal{N} \cup \underline{\mathcal{A}})$, $u' \in \mathcal{A}$ and $\alpha \in [0, 1]$ for which $\alpha u' + (1 - \alpha)u_1$ satisfies the desired inequalities. But if this is the case, inequalities must, also, be met by any payoff profile $(1/q^{N-1})u' + (1 - 1/q^{N-1})u_1$ for which $(1/q^{N-1}) \leq \alpha$ and $q \in \mathbb{N}_+ \cup \{\infty\}$, since u_1 belongs to $\underline{\mathcal{N}}$ which is convex. Then, for any such q , it is possible to support as a subgame perfect equilibrium of the contracting extension, payoff profile:

$$u_2 = (1/q^{N-1})u' + (1 - 1/q^{N-1})u_1 \in \mathcal{K}_{\text{UP}}^2$$

By having players randomize uniformly on q contracts that either fully commit to strategy profile $a \in A$, if $u' = u(a)$, or that never commit, if $u' \in \mathcal{N}$. In fact, so long as any deviation from such strategy is punished with the Nash threat and if miscoordination leads to u_1 being implemented, no player would ever have an incentive to deviate from such strategy conditional on believing that the others are playing according to it.

Therefore, there exists $u_2 \in \mathcal{K}_{\text{UP}}^2$ such that $u_{2,i} > \underline{u}_i^j$ for $\forall i \in \underline{N} \setminus j$ for some $j \in \underline{N}$. It must be shown that there exists an $\gamma \in (0, 1)$ and $u'' \in \mathcal{A} \cup \mathcal{N}$ such that $\beta u'' + (1 - \beta)u_{2,i} > \underline{u}_i^j$ for $\forall i \in \underline{N}$ and $\beta \in (0, \gamma]$.

For this to be true it would suffice to find $u'' \in \mathcal{A} \cup \mathcal{N}$ such that $u''_j > \underline{u}_j^j$ and $u''_i = \underline{u}_i^i$ for $\forall i \in \underline{N} \setminus j$. Because $\text{vert}(\mathcal{U} \cap \text{aff}(\underline{\mathcal{N}})) \subset \mathcal{A} \cup \mathcal{N}$, such u exists. In fact, note that $\mathcal{U} \cap \text{aff}(\underline{\mathcal{N}}) = \text{co}(\text{vert}(\mathcal{U} \cap \text{aff}(\underline{\mathcal{N}}))) \supseteq \underline{\mathcal{N}}$ implies that for any $u \in \mathcal{U} \cap \text{aff}(\underline{\mathcal{N}})$ it must be that $u_i = \underline{u}_i^i$ for $\forall i \in \underline{N} \setminus j$ and that for $\forall j \in \underline{N}$ there exist $u'' \in \text{vert}(\mathcal{U} \cap \text{aff}(\underline{\mathcal{N}}))$ such that $u''_j > \underline{u}_j^j$, because the vertices could not otherwise span $\mathcal{U} \cap \text{aff}(\underline{\mathcal{N}})$. Thus, the desired payoff profile $u'' \in \text{vert}(\mathcal{U} \cap \text{aff}(\underline{\mathcal{N}}))$ exists.

Again, for $q \in \mathbb{N}_+$, such that $1/q^{N-1} \leq \gamma$, it is possible to support as an of the contracting extension payoff profile:

$$u_3 = (1/q^{N-1})u'' + (1 - 1/q^{N-1})u_2 \in \mathcal{K}_{\text{UP}}^3$$

By having players randomize uniformly on q contracts that either fully commit to strategy profile $a \in A$, if $u'' = u(a)$, or that never commit, if $u \in \mathcal{N}$.

Now, a weakly increasing sequence of sets $\{C_t\}_{t \geq 3}$ such that $C_t \subseteq \mathcal{K}_{\text{UP}}^t$ is constructed. It will then be shown that $\text{cl}(\lim_{t \rightarrow \infty} C_t) = \underline{\mathcal{N}}$. Let $V \equiv \text{vert}(\mathcal{U} \cap \text{aff}(\underline{\mathcal{N}})) \subseteq \mathcal{A} \cup \mathcal{N}$, consider $u_3 \in \mathcal{K}_{\text{UP}}^3$ such that $u_{3,i} > \underline{u}_i^i$ for $\forall i \in \underline{N}$ and define $C_3 = \{u_3\}$. Then for any $u_t \in C_t$ and $v \in V$ define:

$$q(v, u_t) \equiv \inf_{q \in \mathbb{N}_+} q \quad \text{s.t.} \quad (1/q^{N-1})v_i + (1 - 1/q^{N-1})u_{i,t} \geq \underline{u}_i^i, \text{ for } \forall i \in \underline{N}$$

$$U_t \equiv \{(1/q(v, u_t)^{N-1})v + (1 - 1/q(v, u_t)^{N-1})u_t \mid u_t \in C_t \ \& \ v \in V\}$$

For such objects, the recursion equation defining the weakly increasing sequence of equilibrium sets can be stated as:

$$C_{t+1} = \begin{cases} C_t \cup U_t & \text{if } t \geq 2 \text{ even} \\ \text{co}(C_t) & \text{if } t \geq 2 \text{ odd} \end{cases}$$

Any payoff profile $u_{t+1} \in C_{t+1}$ also belongs to the equilibrium set $\mathcal{K}_{\text{UP}}^{t+1}$, if $C_t \subseteq \mathcal{K}_{\text{UP}}^t$. This follows by corollary 24 if t is odd and if t is even, either because $C_t \subseteq \mathcal{K}_{\text{UP}}^t \subseteq \mathcal{K}_{\text{UP}}^{t+1}$ or because for any payoff in U_t there is an equilibrium of some commitment structure that yields that payoff. Namely, for $(1/q(v, u_t)^{N-1})v + (1 - 1/q(v, u_t)^{N-1})u_t$, one in which all players uniformly randomize

on $q(v, u_t)$ deterministic devices⁴⁵ that commit to action profile v if signed by all and in which continuation values are such that miscoordination on such contracts leads payoff profile u_t , while any deviation of an agent $i \in N$ is punished with his Nash threat, \underline{u}^i . Specifically,...

Since the sequence $\{C_t\}_{t \geq 3}$ is weakly increasing and since at any stage it belongs to the Nash rational payoff set $\underline{\mathcal{N}}$, such sequence must converge. Let $C_* \equiv \lim_{t \rightarrow \infty} C_t$. It is immediate that C_* is convex, because all odd steps convexify. Then, suppose by way of contradiction that $\text{cl}(C_*) \subsetneq \underline{\mathcal{N}}$. Since both sets are convex there exists $u_* \in \partial \text{cl}(C_*) \setminus \partial \underline{\mathcal{N}}$. For any payoff $u \in \underline{\mathcal{N}} \setminus \partial \underline{\mathcal{N}}$ define:

$$U(u) \equiv \{(1/q(v, u)^{N-1})v + (1 - 1/q(v, u)^{N-1})u \mid v \in V\}$$

and note that $\text{co}(U(u)) \supseteq \underline{\mathcal{N}} \cap B_\varepsilon(u)$, for some ball of radius ε around u .⁴⁶ Such observation immediately implies a contradiction, if a payoff $u_* \in \partial \text{cl}(C_*) \setminus \partial \underline{\mathcal{N}}$ can be implemented, $u_* \in C_t$, through a finite number of layers t , since $\mathcal{K}_{\text{UP}}^{t+2} \supseteq \text{co}(U(u_*)) \supseteq \underline{\mathcal{N}} \cap B_\varepsilon(u_*)$ would imply that $u_* \notin \partial \text{cl}(C_*)$. Any payoff $u_t \in C_t$ chosen to be arbitrarily close to u_* , will have $\text{co}(U(u_t))$ arbitrarily close to $\text{co}(U(u_*))$, by continuity. Therefore, since $\text{co}(U(u_*)) \setminus C_* \neq \emptyset$ by $u_* \in \partial \text{cl}(C_*)$, for any sequence of equilibrium payoffs $\{u_t\}$ converging to u_* , it will be the case that:

$$\lim_{t \rightarrow \infty} \text{co}(U(u_t)) \setminus C_* = \text{co}(U(u_*)) \setminus C_* \neq \emptyset$$

Implying that for any $\eta > 0$ there exists some number of layers T such that whenever $t > T$ and for some norm $d(\cdot)$:

$$d(\text{co}(U(u_t)), \text{co}(U(u_*))) < \eta$$

But this would lead to a contradiction since it would imply that for that for t big enough $\text{co}(U(u_t)) \setminus C_* \neq \emptyset$, because otherwise the norm could not converge. ■

⁴⁵If $v = u(a)$ for some $a \in A$ let such device be $[a]^t$ and otherwise if $v \in \mathcal{N}$ take $[A]^t$, for $[A]^t \equiv [\pi_t([A]^{t-1}) = 1]$ & $[A]^0 \equiv \times_N \Delta(A)$.

⁴⁶Where the radius ε can potentially depend on u .