

Common Belief Foundations of Global Games*

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Abstract

Uniqueness in global games turns on the nature of the residual uncertainty conditional on public information. When there is enough correlation between types relative to dispersion, there is a unique equilibrium. Endogenous public signals do not always eliminate correlation across types, and so uniqueness can persist even with informative public signals. In the type space, correlation ensures the key property of the *stationarity of beliefs* across player characteristics. This property underpins rationalizability. We derive a set of conditions on beliefs that are necessary and sufficient for one or other action to be rationalizable in terms of operators on sets of types that resemble standard belief operators.

*Preliminary

1 Introduction

Games often have many equilibria. Even when they have a single equilibrium, they often have many actions that are rationalizable, and are therefore consistent with common knowledge of rationality. Yet a pathbreaking paper by Carlsson and van Damme (1993) suggested a natural perturbation of complete information that gives rise to a unique rationalizable equilibrium for each player. They introduced the idea of “global games” - where any payoffs of the game are possible and each player observes the true payoffs of the game with a small amount of noise. They showed - for the case of two player two action games - that as the noise about payoffs become small, there is a unique equilibrium; the equilibrium strategies played also constitute the unique rationalizable strategies. This result has since been generalized in a number of directions and used in a number of applications.¹ When the global game approach can be applied to more general games, it can be used to derive unique predictions in settings where the underlying complete information game has multiple equilibria, making it possible to carry out comparative static and policy analysis.

However, a number of recent papers have raised questions both on the basic theoretical rationale for global games and the applicability of the framework for the analysis of real world problems. Three strands of the argument from the literature are particularly worthy of note.

1. In most economic environments where coordination is important, interactions endogenously generate informative public information that might be used as a coordination device. An especially important source

¹Morris and Shin (1998) analyzed a global game with a continuum of players making binary choices, and this case has been studied in a number of later applications. Morris and Shin (2003) survey some theory and applications of global games.

of endogenous public information are market prices (see Atkeson (2001), Tarashev (2003), Hellwig, Mukherji and Tsyvinski (2006) and Angeletos and Werning (2006))². When prices convey information, increased precision of private information will feed increased accuracy of (endogenous) public signals. Thus uniqueness conditions will fail if private signals are sufficiently accurate.

2. While asymmetric information may exist in a large variety of economic settings, it does not always conform to the global game notion of “noisy signals”. Global game results turn on the relative precision of private and public signals, but if we do not know what these noisy signals are in real life, debates about relative precisions have no conceptual basis (see, for example, Kurz (2006), Sims (2005a, 2005b), Svensson (2005), Woodford (2005)).
3. While common knowledge of payoffs is relaxed in global games, there is still assumed to be common knowledge of the information structure, which is surely a no more realistic assumption. A recent paper by Yildiz (2006) shows that the exact form of the perturbation away from common knowledge of payoffs is crucial in determining the rationalizable outcome. The global game prediction is not the only possible perturbation that yields unique rationalizable outcomes. What claim does the global game approach have for being a “natural” or “reasonable” perturbation?

The objective of our paper is to evaluate these arguments and questions concerning the global game methodology, and to provide a framework that

²Angeletos, Hellwig and Pavan (2006a, 2006b) note (inter alia) how other sources of endogenous public information may lead to multiplicity in such coordination games.

can both deepen our understanding of the theoretical basis for global games and to provide guidance for applied researchers on the scope (and limitations) of the global game approach.

The canonical information structure associated with the global game approach is one where players observe some underlying fundamental variable with some noise. This is for the historical reason that the early papers (Carlsson and van Damme (1993), Morris and Shin (1998)) adopted this formalism. The noise is a convenient way to relax common knowledge of the fundamentals, but in subsequent applications of global games the noisy information structure has been taken more literally - as players failing (literally) to observe the true fundamentals perfectly. Many of the criticisms of the global game approach presumes such a literal interpretation of the global games approach.

However, there are pitfalls in taking the noisy information structure too literally, as the underlying logic of the argument becomes identified with a particular formalism, and the general scope of the approach becomes obscured by debates surrounding the merits or otherwise of the particular formalism. The logic underlying the global game approach turns out to be more robust, and is not tied to taking noisy signals literally.

In this paper, we do three things. First, we will present a global game analysis where we dispense with talk of “noisy signals” altogether, and instead deal with heterogeneity of players. Borrowing the distinction used in auction theory, we will deal with “private values” problems (or more accurately, “affiliated values” problems), rather than with “common value” problems.

Second, with our framework, we will address the issue of whether endogeneity of public information brings about multiplicity of equilibrium. Our

answer is “not always”. Equilibrium uniqueness turns on the nature of the residual uncertainty conditional on all available public information. If there is correlation across types conditional on public information (or equivalently, if it is common knowledge that types are correlated), then uniqueness is the norm. Multiplicity of equilibria come about when public information removes correlation across types. In many settings, correlation would be the norm, rather than the exception. In particular, we note some important differences between prices and truthful polls as public signals.

Finally, we will characterize the higher-order beliefs that allow the global game argument to “work”. Essentially, the property that matters is the stationarity of beliefs with respect to the ordering of types. Global game arguments work because the beliefs that player types have over their neighboring types do not change abruptly as we consider types along the ordering. Beliefs are stationary, in this sense. We derive conditions on the beliefs and higher-order beliefs of types that are necessary and sufficient for rationalizability. In this way, we show precisely what deviations from common knowledge are implied by the global game apparatus.

2 Contribution Game

In light of the concerns raised by some authors on the exact interpretation of “noisy signals” of fundamental variables, we will dispense with talk of noisy signals altogether, but instead consider a problem where players’ payoffs depend directly on some characteristic of the player, and the focus is on the uncertainty concerning other players’ characteristics. Using the distinction often used in auction theory, we will consider “private values” problems rather than “common value” problems.

As a vehicle for our arguments, let us consider a voluntary contribution

game for a public good, where each individual decides whether to contribute toward the public good, or to opt out. Contributing toward the public good is a risky action, as the successful provision of the public good requires contribution from a known, critical mass of the players. If the contribution falls short of this critical mass, the provision of the public good fails, resulting in poor payoffs for those who decided to contribute. Opting out is a safe action for an individual agent, but leads to a socially sub-optimal outcome.³

There is a continuum of players with unit mass, indexed by $i \in [0, 1]$. Each player has to choose between two actions - to contribute to the public good, or to opt out. Denote by κ the proportion of agents who contribute. The public good is successfully provided when κ is larger than some critical threshold $\hat{\kappa}$. The consumption value of the public good is 1, but player i faces a cost x_i in contributing to the provision of the public good. Thus, the payoff to player i from contributing to the public good is

$$\begin{cases} 1 - x_i & \text{if } \kappa \geq \hat{\kappa} \\ -x_i & \text{if } \kappa < \hat{\kappa} \end{cases} \quad (1)$$

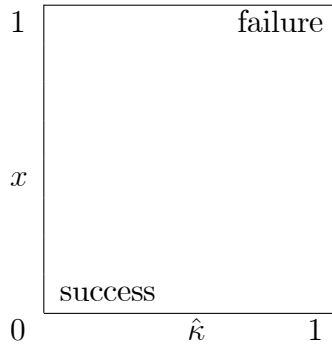
The payoff to opting out is zero, and does not depend on κ . If $x_i < 0$, then the decision is trivial, since contributing is a dominant action. Similarly, if $x_i > 1$, then opting out is dominant. However, for x_i between zero and one, the optimal choice depends on the probability that player i attaches to κ exceeding the threshold $\hat{\kappa}$. We will be concerned with how this belief is determined in the game.

The payoffs of the game could be interpreted in terms of a speculative attacks game, where $\hat{\kappa}$ is the incidence of attack that is sufficient to break a peg, and x_i is player i 's cost of attack, interpreted as the opportunity cost of

³Similar games have been examined in laboratory experiments by Van Huyck, Battalio and Beil (1990, 1991). Crawford (1991) gives an evolutionary rationalization of their results.

joining in the speculative attack. Here, the “public good” is only a public good from the perspective of the speculators.

Before getting into any detailed analysis, it is worthwhile taking a step back and considering what a reasonable outcome would be in such a game. In experimental settings⁴, coordination is more demanding when the critical threshold $\hat{\kappa}$ is high. In terms of the actions of the players, they will contribute if their cost is low relative to the critical threshold $\hat{\kappa}$. Conversely, they will opt out if their cost is high relative to $\hat{\kappa}$.



We may conjecture that the $(\hat{\kappa}, x)$ -space can be divided into a region where the public good is provided successfully, and a region where it is not. In general, we may expect that successful provision is more likely in the bottom left hand corner, where the threshold is low and costs are also low. Conversely, in the opposite corner, we may expect provision to fail.

The evidence from experiments, such as from Heinemann, Nagel and Ockenfels (2004) suggest that such threshold strategies arise fairly commonly in games of this kind even when the fundamentals are ostensibly common knowledge among the players. We say “ostensibly”, since the cautious actions of the players betray a lack of confidence in the proposition that x is common knowledge.

⁴e.g. Van Huyck et al. (1990, 1991)

The question is whether there is a single dividing line that separates the two regions, or whether the dividing line is “fuzzy” and we have a region of indeterminacy where both outcomes are possible in equilibrium. We explore this question more systematically in what follows.

2.1 Inference Problem

The joint density over the players’ costs of contribution is determined as follows. Player i ’s cost is given by

$$x_i = \theta + \varepsilon_i$$

where θ is a Gaussian random variable with mean y and variance $1/\alpha$, and ε_i is Gaussian with mean zero and variance $1/\beta$. The random variables $\{\varepsilon_i\}$ are mutually independent, and independent of θ .

Here, the term θ represents the common element in the players’ costs of contribution, and ε_i is the idiosyncratic element. However, the player knows only the total cost, rather than each component individually. The players therefore face an inference problem on what the distribution of costs are in the population, conditional on his own cost. Thus, when player i observes a high cost of contribution x_i , he has to weigh the possibility that his own high cost reflects the generally high costs prevailing in the population against the possibility that his own high cost is mostly idiosyncratic, and the costs are generally low in the population. His incentives to contribute would be very different in the two circumstances.

Another example of an inference problem of this type is involved in assessing my relative ranking after taking an exam when I only know my own score and not the distribution of scores as a whole. When I do well in an exam, should I take this to be a signal that I am ranked high in the class,

or should I take it as a signal that the exam was easy, and that others have done equally well? In other words, how typical am I?

Voting in political contests is another example. When my political views change after experiencing a major national event, how much should I believe this change to be an idiosyncratic shift on my part, and how much should I attribute my shift of views as being a reflection of a change in the “national mood” as a whole?

In the speculative attack interpretation of our game, the cost of attacking a peg is determined relative to his own portfolio of assets. Although each speculator has some information about the costs of others, there are small differences in the costs due to differences in the initial portfolios.

Denote by $\lambda(x)$ the proportion of players whose cost of contribution is x or less. The “ λ ” stands for “lower”. Then, $\lambda(x)$ is a random variable with realizations in the unit interval, and which is a function of the random variables $\{\theta, \varepsilon_i\}_{i \in [0,1]}$ and the threshold x . We will derive the density function of $\lambda(x_i)$ conditional on player i 's cost being x_i . Let

$$G(z|x_i) \tag{2}$$

be the cumulative distribution function (c.d.f.) of $\lambda(x_i)$ conditional on x_i , evaluated at z . In other words,

$$G(z|x_i) = \Pr(\lambda(x_i) \leq z|x_i) \tag{3}$$

In words, $G(z|x_i)$ is the probability that the proportion of players with cost lower than x_i is z or less, conditional on cost x_i . Figure 1 illustrates the derivation of $G(z|x_i)$.

Given the common cost element θ , the proportion of players who have cost below x_i is

$$\Phi\left(\sqrt{\beta}(x_i - \theta)\right) \tag{4}$$

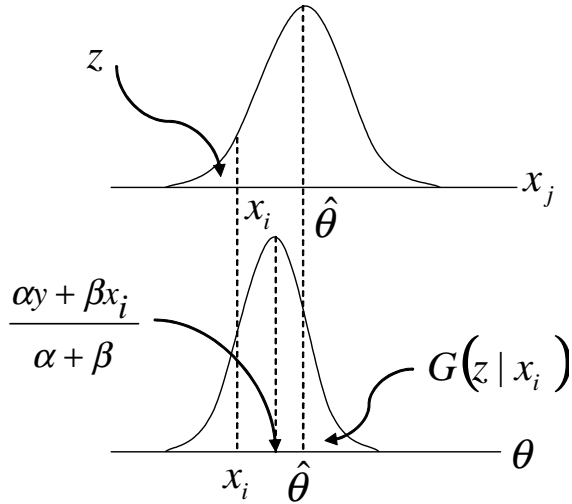


Figure 1: Deriving $G(z|x_i)$

where $\Phi(\cdot)$ is the cumulative distribution function for the standard normal. Let $\hat{\theta}$ be the realization of θ at which this proportion is exactly z . In other words,

$$\hat{\theta} = x_i - \frac{\Phi^{-1}(z)}{\sqrt{\beta}} \quad (5)$$

The top panel of figure 1 illustrates $\hat{\theta}$. When $\theta \geq \hat{\theta}$, the proportion of players that have cost less than x_i is z or less. In other words, $\lambda(x_i) \leq z$ whenever $\theta \geq \hat{\theta}$. Hence, $G(z|x_i)$ is the probability of $\{\theta | \theta \geq \hat{\theta}\}$ conditional on x_i . The bottom panel of figure 1 illustrates the argument. Conditional on x_i , the density over θ is normal with mean

$$\frac{\alpha y + \beta x_i}{\alpha + \beta} \quad (6)$$

and precision $\alpha + \beta$. The probability that $\theta \geq \hat{\theta}$ is the area under this density to the right of $\hat{\theta}$, namely

$$1 - \Phi\left(\sqrt{\alpha + \beta}\left(\hat{\theta} - \frac{\alpha y + \beta x_i}{\alpha + \beta}\right)\right) \quad (7)$$

This expression gives $G(z|x)$. Substituting out $\hat{\theta}$ by using (5) and rearranging, we can re-write (7) to give:

$$G(z|x_i) = \Phi\left(\frac{\alpha}{\sqrt{\alpha+\beta}}(y-x_i) + \sqrt{\frac{\alpha+\beta}{\beta}}\Phi^{-1}(z)\right) \quad (8)$$

Differentiation of this expression with respect to z will give us the density of $\lambda(x_i)$ conditional on x_i .

There is one special case where the density is invariant to both x_i and y , and this is the limiting case when $\beta \rightarrow \infty$. In this when the idiosyncratic component of cost shrinks to zero, so that everyone ends up with the same cost, given by θ . The limiting case of (8) as $\beta \rightarrow \infty$ is given by:

$$G(z|x) \rightarrow \Phi(\Phi^{-1}(z)) = z$$

so that G is the identity function. In other words, the c.d.f. of $\lambda(x_i)$ is the 45 degree line, and hence the density over $\lambda(x_i)$ is uniform. Thus, in this limit, player i believes that he is “typical” in quite a strong sense, in that he puts equal weight on every realization of $\lambda(x_i)$.

Figure 2 illustrates the derivation of $G(z|x_i)$ in the limiting case. As β becomes large, the posterior density over θ is centered around x_i itself, and is identical in shape to the density over costs conditional on θ . Then $G(z|x_i)$ is the area under the posterior density to the right of $\hat{\theta}$. But since the densities in the two panel are of identical shape, the area to the right of $\hat{\theta}$ in the bottom panel is the same as the area to the left of x_i in the top panel. Hence, $G(z|x_i) = z$.

3 Monotone Equilibrium

If we were interested only in solving for equilibrium in monotone strategies - that is strategies where players contribute when their cost falls below some

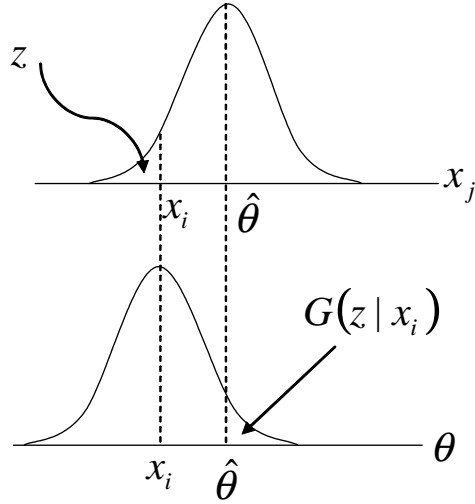


Figure 2: Limiting case of $G(z|x_i)$ as $\beta \rightarrow \infty$

threshold x and opts out when it lies above x - we need only solve for the switching point x . When everyone follows the switching strategy around x , then the proportion of players who contribute (denoted by κ) is the proportion of players whose cost is below x . Hence

$$\kappa = \lambda(x) \tag{9}$$

The indifference condition at the switching point x equates the probability of successful provision of the public good with the cost of contribution, given by x . The probability of successful coordination conditional on x is

$$\begin{aligned} \Pr(\kappa \geq \hat{\kappa}|x) &= \Pr(\lambda(x) \geq \hat{\kappa}|x) \\ &= 1 - G(\hat{\kappa}|x) \\ &= \Phi\left(\frac{\alpha}{\sqrt{\alpha+\beta}}(x-y) - \sqrt{\frac{\alpha+\beta}{\beta}}\Phi^{-1}(\hat{\kappa})\right) \end{aligned} \tag{10}$$

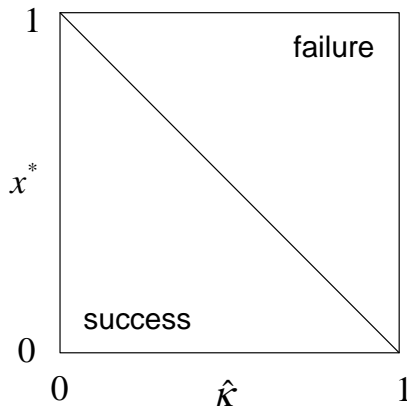


Figure 3: Equilibrium in limit as $\beta \rightarrow \infty$

Hence, the set of equilibrium thresholds x is given by the solutions of the equation:

$$x = \Phi \left(\frac{\alpha}{\sqrt{\alpha+\beta}} (x - y) - \sqrt{\frac{\alpha+\beta}{\beta}} \Phi^{-1}(\hat{\kappa}) \right) \quad (11)$$

If there is a unique solution to (11), there is a unique switching equilibrium. In fact, as we will show later in the paper, a unique solution to (11) is necessary and sufficient for there to be a unique rationalizable outcome.

Drawing on our discussion from the last section, a sufficient condition for uniqueness is when $\beta \rightarrow \infty$, since the right hand side of (11) tends to the constant function $1 - \hat{\kappa}$. In this case, the unique equilibrium is the switching equilibrium around the point x where

$$x = 1 - \hat{\kappa} \quad (12)$$

The dividing line between the region of successful provision and failure of provision is just the diagonal straight line as illustrated in figure 3.

In general, a sufficient condition for there to be a unique solution to (11) is that the slope of the right hand side is at most 1 everywhere, namely

$$\phi(\dots) \cdot \frac{\alpha}{\sqrt{\alpha+\beta}} \leq 1$$

where ϕ is the normal density evaluated at the expression inside the brackets in (10). Since $\phi \leq 1/\sqrt{2\pi}$, a sufficient condition for uniqueness is

$$\frac{\alpha}{\sqrt{\alpha+\beta}} \leq \sqrt{2\pi} \quad (13)$$

This condition is also necessary for uniqueness for values of y and $\hat{\kappa}$ such that the right hand side of (11) is exactly 1/2. In this sense the bound identified in (13) is tight.

3.1 Correlation and Dispersion

To understand better the meaning of (13) let us re-express the condition in more familiar terms. Let ρ be the correlation between the costs across players conditional on all public information (in our case, the public signal y). Let σ^2 be the variance of players' costs in the population conditional on public information. Hence,

$$\begin{aligned} \rho &= \frac{\text{Cov}(x_i, x_j|y)}{\sqrt{\text{Var}(x_i|y)}\sqrt{\text{Var}(x_j|y)}} \\ &= \frac{\frac{1}{\alpha}}{\frac{1}{\alpha} + \frac{1}{\beta}} \\ &= \frac{\beta}{\alpha + \beta} \end{aligned} \quad (14)$$

and

$$\begin{aligned} \sigma^2 &= \text{Var}(x_i|y) \\ &= \frac{1}{\alpha} + \frac{1}{\beta} \end{aligned} \quad (15)$$

Using (14) and (15), we can express the bound (13) as a lower bound on the correlation across types, given by:

$$\rho \geq \frac{1}{1 + 2\pi\sigma^2} \quad (16)$$

Uniqueness is guaranteed provided that the correlation across types is high relative to the dispersion of costs in the population. In the limiting case where $\beta \rightarrow \infty$, the correlation tends to 1, while the dispersion in costs is prevented from shrinking to zero by the variability of the common component θ around y . The bound (16) also suggests another way that uniqueness results - namely, when the dispersion becomes very large, and there is enough heterogeneity in players' costs.

The general property that underlies the bound (16) is the *stationarity of beliefs* to shifts in one's own cost x_i . The probability that the proportion of players who have cost lower than oneself should not vary "too much" to one's own cost. Both higher correlation across types or very large dispersion in costs will result in such insensitivity of beliefs to one's cost.

Thus, when there is a failure to achieve a unique monotone equilibrium, the diagnosis lies in the correlation across players' costs to fall short of the threshold. We now examine the issue of endogenous public signals in this light.

3.2 Endogenous Public Signals

Angeletos and Werning (2006) and Hellwig, Mukherji and Tsyvinski (2006) have argued that when the precision of the public signal is endogenous, and reflects the information flow from the private information of the players, then the precision of the public signal (α in our model) is increasing as β becomes large. When α increases in response to an increase in β , then (10) will not be a constant function in the limit as $\beta \rightarrow \infty$. In particular, when α increases at a sufficiently fast rate relative to β , then (10) (seen as a function of x) will converge to a step function that jumps from zero to one. We consider two scenarios.

Scenario 1: Informative Poll

Let us first consider the scenario where θ is drawn from an improper uniform density, and costs of contribution $\{x_i\}$ are distributed in the population with the normal density around θ with variance $1/\beta$. Then, a poll is conducted by selecting n players, and each player in the sample reveals his cost truthfully to the pollster. The average of the private signals is then disclosed truthfully as the public signal y . Given normality and conditional independence of the costs, the precision of the public signal y will then be:

$$\alpha = n\beta \tag{17}$$

Let us now see what happens as n becomes large, so that the poll becomes an increasingly accurate signal of the common component θ of costs. From (10), the probability of successful provision conditional on being at the switching point x is given by

$$\Pr(\kappa \geq \hat{\kappa}|x) = \Phi\left(\sqrt{n+1}\left(\frac{n\sqrt{\beta}}{n+1}(x-y) - \Phi^{-1}(\hat{\kappa})\right)\right) \tag{18}$$

As $n \rightarrow \infty$, this expression (seen as a function of x) converges to the step function where the probability of successful provision jumps from 0 to 1 at the point:

$$x = y + \frac{\Phi^{-1}(\hat{\kappa})}{\sqrt{\beta}} \tag{19}$$

To understand what is going on here, consider figure 1, and how the probability of successful provision is determined. The probability of successful provision is the area to the left of $\hat{\theta}$ in the bottom panel of figure 1. As $\alpha \rightarrow \infty$, the precision of the posterior beliefs increases, making the density more concentrated around the posterior mean. Thus, the probability of successful provision will be either zero or one depending on whether y is to the

right, or left of $\hat{\theta}$. Thus, the jumping point will happen when $y = \hat{\theta}$. We know from (5) that $\hat{\theta} = x - \Phi^{-1}(\hat{\kappa})/\sqrt{\beta}$. Hence, the outcome depends on:

$$x \geq y + \frac{\Phi^{-1}(\hat{\kappa})}{\sqrt{\beta}} \quad (20)$$

In the limit as the size of the poll becomes large, we have $y = \theta$. Provided that the proportion of players with dominant actions is not so large as to make the problem trivial, the jumping point (19) lies in the unit interval. Then, there are two solutions to (11), given by $x = 0$ and $x = 1$. Both the switching strategy around 0 and the switching strategy around 1 are equilibria, and so we have multiplicity.

Referring back to the bound (16), we can diagnose the failure of uniqueness to the fall in the correlation of costs across players conditional on the public signal y . As $n \rightarrow \infty$, we have $\alpha \rightarrow \infty$ so that

$$\rho = \frac{\beta}{\alpha + \beta} \rightarrow 0$$

Even as correlation disappears, the dispersion of costs σ^2 does not shrink to zero due to the idiosyncratic element in costs, and so the bound (16) is violated.

In interpreting the significance of the re-emergence of multiplicity with the announcement of the informative poll, we have to ask how reasonable it is that the public signal will remove correlation across types. To some extent, this question is an empirical matter. However, if the poll fails to elicit the common component θ perfectly (for instance, because it is contaminated with a systematic bias), then the correlation across types will persist due to the bias.

Scenario 2: Price

Angeletos and Werning (2006) also consider a case where the public signal of θ is given by the price determined in a noisy rational expectations equilibrium where investors have exponential utility. Their discussion formalizes an argument by Atkeson (2001) in his discussion of an earlier paper of ours (Morris and Shin (2001)), where he argues that price will serve as an informative public signal that can coordinate players' actions.

In such an equilibrium, the precision α of the price function is of the order of β^2 . If we then take limits as $\beta \rightarrow \infty$, we again lose the correlation across types conditional on the price, and we end up with multiple switching equilibria as in the case of the informative poll.

To see this, consider the set of switching equilibria when

$$\alpha = \beta^2 \tag{21}$$

The probability of successful provision is then given by

$$\Pr(\kappa \geq \hat{\kappa}|x) = \Phi\left(\sqrt{\beta^2 + \beta}\left(\frac{\beta^2}{\beta^2 + \beta}(x - y) - \frac{1}{\sqrt{\beta}}\Phi^{-1}(\hat{\kappa})\right)\right) \tag{22}$$

The probability of successful provision is exactly $\frac{1}{2}$ when the expression inside the first set of brackets is zero - that is, when

$$\frac{\beta^2}{\beta^2 + \beta}(x - y) - \frac{1}{\sqrt{\beta}}\Phi^{-1}(\hat{\kappa}) = 0 \tag{23}$$

As $\beta \rightarrow \infty$, the solution of this equation is $x = y$. If y lies in the undominated region of costs, then (22) converges to the step function that jumps from 0 to 1 at $x = y$. Then, there are two solutions to (11), given by $x = 0$ and $x = 1$. We thus have multiplicity.

In assessing the significance of this result, we again need to ask how realistic it is to suppose that the price is a public signal that removes correlation

across types. Thus, the question concerns not so much the informativeness of the price, as such, but rather the nature of the *residual uncertainty* that remains once the price is observed. If the residual uncertainty is such that there is enough correlation between player types, then there will be uniqueness in the global game.

We illustrate this point with an example that Angeletos and Werning (2006) examined in the appendix to their paper. This is the case where

$$\theta = v + \eta \tag{24}$$

where v is the liquidation value of an asset that is drawn from the improper uniform prior, and η is a normal random variable, independent of v with mean 0 and variance $1/\gamma$. As before, player i 's cost of contribution to the public good is

$$\begin{aligned} x_i &= \theta + \varepsilon_i \\ &= v + \eta + \varepsilon_i \end{aligned} \tag{25}$$

In effect, player i 's cost x_i serves as a “noisy signal” of the liquidation value of the asset, but where the signals have both an idiosyncratic noise term ε_i , but also a common noise term η . The idea is that, even if (hypothetically) we can gather all the private information in some way, there is still some residual uncertainty about the liquidation value v .

For uniqueness of the global game, the common noise term η would be irrelevant if we could gather all the private information through a poll. The common noise term η would not contaminate the poll of the sum $v + \eta$, and we would again lose uniqueness, as seen under scenario 1.

However, the question is whether the price will do as well as the informative poll in ascertaining the average cost $v + \eta$. The answer is no. In

the appendix we solve for the noisy rational expectations equilibrium of the asset market with traders who have the exponential utility

$$\exp\left\{-\frac{c}{\tau}\right\} \quad (26)$$

where c is consumption, and $\tau > 0$ is the risk tolerance of the traders. In the limit as $\beta \rightarrow \infty$ the equilibrium price p is given by

$$p = v + \eta - \frac{1}{\tau\gamma}s \quad (27)$$

where s is the Gaussian noisy supply term, and γ is the precision of the common noise term η . Provided that traders are risk averse (i.e. τ is finite) and the common noise term persists (γ is finite), the price is a noisy signal of the average cost $v + \eta$. As $\beta \rightarrow \infty$, the correlation ρ of players' costs conditional on price goes to 1. However, since price is a noisy signal of the average cost $v + \eta$, the variance of costs σ^2 conditional on the price does not shrink to zero. Thus, we satisfy the uniqueness condition:

$$\rho \geq \frac{1}{1 + 2\pi\sigma^2} \quad (28)$$

The main lesson from this example is that there are important differences between the *price* and the *informative poll* in terms of their role as the public signal. The informative poll will always lead to the elimination of the correlation across types once the sample size of the poll becomes large. This leads to multiplicity of the global game. However, price may not lead to the elimination of the correlation across types, even as the idiosyncratic component in the “noisy signal” of the traders vanishes, as seen above.

Price can only be as informative as the informative poll when the demands of the traders become so sensitive to the price that the demand curves become vertical. In effect, risk-averse traders behave like risk-neutral traders, and

submit infinitely aggressive demand and supply curves. It is for this reason that the noise in the price function disappears in the limit as $\beta \rightarrow \infty$. As long as the traders have some reason to exercise caution, they will not submit vertical demand curves, and the price will fail to be fully informative of the average signal.

Thus, our bottom line is that endogenous public signals need not imply multiplicity of equilibrium in the global game. The crucial issue is the nature of the residual uncertainty conditional on public information. If the public signal is such that all correlation is lost across types conditional on the public signal, then multiplicity results. As long as some correlation remains, then sufficiently large heterogeneity among players will ensure uniqueness of equilibrium.

4 Optimists and Pessimists

Drawing on our observation that what matters for equilibrium uniqueness is the *stationarity of beliefs*, it is intuitively plausible that nothing in our results depends on the common prior assumption. As long as the stationarity of beliefs is maintained, we can accommodate heterogeneity of beliefs.⁵

To support our claim, consider a variation of the public goods contribution game with heterogenous priors. Keeping all features of our game as before, we make one change in the assumption on players' beliefs. We will suppose that the players have different beliefs about the prior mean of θ , and (crucially) their beliefs about the prior depends on their realized cost level x_i .

There are two groups - optimists and pessimists. They have equal mass

⁵Izmalkov and Yildiz (2006) have also studied global games with heterogeneous priors. They studied environments where uniqueness was maintained, but the nature of the heterogeneity - common knowledge of optimism or pessimism - leads to different outcomes.

in the population. For optimists, when their own cost is x_i , they believe that the prior mean of θ is

$$y_i^O(x_i) = x_i + \frac{\phi^O}{\sqrt{\alpha}} \quad (29)$$

for some number ϕ^O . For pessimists, when their own cost is x_i , they believe that the prior mean of θ is

$$y_i^P(x_i) = x_i + \frac{\phi^P}{\sqrt{\alpha}} \quad (30)$$

for some ϕ^P . For our labelling to be consistent with our interpretation in terms of optimists and pessimists, we have

$$\phi^O < 0 < \phi^P$$

In other words, optimists believe that the population as a whole has lower costs than himself, while pessimists believe that the population as a whole has higher costs than himself. Conditional beliefs of the optimists and pessimists therefore differ. We have

$$\begin{aligned} E_i^O(\theta) &= \frac{\alpha y_i^O + \beta x_i}{\alpha + \beta} \\ &= x_i + \frac{\sqrt{\alpha}}{\alpha + \beta} \phi^O \end{aligned} \quad (31)$$

and analogously for pessimists. Note that conditional beliefs are “stationary” in that there is a one-for-one shifts in beliefs about θ with shift in one’s own cost x_i .

Suppose that all players use switching strategies, but we allow the possibility that optimists and pessimists use different switching points. Let \hat{x}^O be the switching point for optimists and \hat{x}^P the switching point for pessimists. Then, the total mass of players contributing to the public good given the common cost component θ is

$$\frac{1}{2} \Phi\left(\sqrt{\beta}(\hat{x}^O - \theta)\right) + \frac{1}{2} \Phi\left(\sqrt{\beta}(\hat{x}^P - \theta)\right)$$

Define the value $\hat{\theta}$ as the value of θ that is just low enough to make successful provision possible. That is

$$\frac{1}{2}\Phi\left(\sqrt{\beta}\left(\hat{x}^O - \hat{\theta}\right)\right) + \frac{1}{2}\Phi\left(\sqrt{\beta}\left(\hat{x}^P - \hat{\theta}\right)\right) = \hat{\kappa} \quad (32)$$

Let us restrict our attention to the special case where the threshold value is $\hat{\kappa} = 0.5$. Then, we have from (32),

$$\hat{\theta} = \frac{\hat{x}^O + \hat{x}^P}{2}$$

The equilibrium condition for the switching point is that any player must be indifferent between contributing and not contributing. In other words, the cost of contribution must be equal to the probability of successful provision.

$$\begin{aligned} \hat{x}^O &= \Phi\left(\sqrt{\alpha}\left(\hat{\theta} - E^O(\theta)\right)\right) \\ \hat{x}^P &= \Phi\left(\sqrt{\alpha}\left(\hat{\theta} - E^P(\theta)\right)\right) \end{aligned} \quad (33)$$

Meanwhile

$$\begin{aligned} \hat{\theta} - E^O(\theta) &= \frac{\hat{x}^P - \hat{x}^O}{2} - \frac{\sqrt{\alpha}}{\alpha + \beta}\phi^O \\ \hat{\theta} - E^P(\theta) &= \frac{\hat{x}^O - \hat{x}^P}{2} - \frac{\sqrt{\alpha}}{\alpha + \beta}\phi^P \end{aligned} \quad (34)$$

Substituting (34) into (33) and re-arranging, we obtain two equations for our two unknowns - \hat{x}^O and \hat{x}^P

$$\begin{aligned} \hat{x}^P &= \hat{x}^O + \frac{2}{\sqrt{\alpha}}\Phi^{-1}\left(\hat{x}^O\right) + \frac{2\sqrt{\alpha}}{\alpha + \beta}\phi^O \\ \hat{x}^O &= \hat{x}^P + \frac{2}{\sqrt{\alpha}}\Phi^{-1}\left(\hat{x}^P\right) + \frac{2\sqrt{\alpha}}{\alpha + \beta}\phi^P \end{aligned}$$

Solving for \hat{x}^P ,

$$\hat{x}^P = 1 - \Phi\left(\frac{\alpha}{\alpha + \beta}\left(\phi^O + \phi^P\right) + \Phi^{-1}\left(\hat{x}^P + \frac{2}{\sqrt{\alpha}}\Phi^{-1}\left(\hat{x}^P\right) + \frac{2\sqrt{\alpha}}{\alpha + \beta}\phi^P\right)\right)$$

There is a unique solution for the switching point, since the right hand side is decreasing in \hat{x}^P . Now, take the limit as $\alpha \rightarrow \infty$. In this limit, the solution is

$$\hat{x}^P = 1 - \Phi(\phi^O + \phi^P + \Phi^{-1}(\hat{x}^P))$$

For the special case where $\phi^O + \phi^P = 0$, we have

$$\begin{aligned} \hat{x}^P &= 1 - \Phi(\Phi^{-1}(\hat{x}^P)) \\ &= 1 - \hat{x}^P \end{aligned}$$

so that $\hat{x}^P = \hat{x}^O = 0.5$. The fact that the switching points of the optimists and pessimists are the same is because we have taken the limit as $\alpha \rightarrow \infty$. In general, they will differ.

5 Common Belief and Rationalizability

We now turn to an analysis of rationalizability in the game, and the implications for common belief among the players. The role of higher order beliefs in global games has been much discussed. Carlsson and van Damme (1993) noted the relation of their result to Rubinstein (1989) and the lack of high levels of knowledge. We emphasized the importance of removing common knowledge in Morris and Shin (1995), the working paper version of Morris and Shin (1998). Morris and Shin (2003) discusses in detail the general connection between global game results and the abstract game theoretic literature on higher order beliefs. Hellwig (2002) reports results on the relation between approximate common knowledge and the multiplicity of equilibria in the model of Morris and Shin (2004).

As discussed already, Yildiz (2006) notes that for any type with multiple rationalizable actions and any of those rationalizable actions, there is always a way of finding a nearby type in the product topology. Thus he highlights

the fact that some common knowledge properties - i.e., properties of the tails of higher order beliefs - matter in generating multiplicity.

Our objective in this section is to characterize the higher-order beliefs that make one or other action rationalizable in the public good contribution game. In this way, we will accomplish two goals.

- We prove that whenever there is a unique equilibrium in monotone strategies, there is a unique rationalizable outcome for all types of all players.
- We provide a necessary and sufficient condition on the beliefs of a type that make a particular action rationalizable. In this way, we make explicit the higher order beliefs that makes the global game argument “work”.

A *type* of player i is the ordered pair that consists of his own cost of contribution and his beliefs over other players’ types. In order to avoid the notational clutter and technical issues in defining the full type space with a continuum of players, we will adopt a reduced form method of argument in this section. In a later section, we will conduct the analysis of rationalizability more rigorously in the context of a simpler, two-player global game, where the type space and belief operators are defined more formally.

Thus, for the discussion in this section, we will exploit the symmetry and anonymity of the game to label a type of player i with his contribution cost x_i , with the interpretation that this type’s beliefs over other types are given by the joint density over costs generated by our assumptions.

We also introduce the following notation. Denote by L_x the set of all types across all players whose cost of contribution is *strictly less than* x , and denote by U_x the set of all types across all players whose cost of contribution

is *strictly greater than* x . Each L_x and U_x is an event in the sense of being a subset of the universal type space. We call the collection of events $\{L_x\}_{x \in \mathbb{R}}$ the *lower simple events*, and call the collection $\{U_x\}_{x \in \mathbb{R}}$ the *upper simple events*. As a convention, we allow $x = -\infty$ and $x = \infty$ so that the empty event \emptyset and the universal type space Ω qualify both as lower simple events and upper simple events.

Define the operator $S_C(\cdot)$ on lower simple events $\{L_x\}$ so that $S_C(L_x)$ is the set of all types who strictly prefer to contribute when all types in L_x contribute. Analogously, we define the operator $S_N(\cdot)$ on upper simple events such that $S_C(U_x)$ is the set of all types who strictly prefer not to contribute when all types in U_x do not contribute. We first prove the preliminary result that the lower simple events are closed under $S_C(\cdot)$, and upper simple events are closed under $S_N(\cdot)$.

Lemma 1 *If L_x is a lower simple event, then so is $S_C(L_x)$. Similarly, if U_x is an upper simple event then so is $S_N(U_x)$.*

To prove this result, we begin by defining a function $u(x, \hat{x})$ that gives the payoff to contributing to the public good when the cost of contribution is x and all others use a switching strategy around \hat{x} . In our case, the probability of successful provision of the public good given cost x when all others follow the switching strategy around \hat{x} is

$$\Pr(\lambda(\hat{x}) \geq \hat{\kappa}|x) \tag{35}$$

By using a similar argument to that used to derive equation (8), we can show that

$$\Pr(\lambda(\hat{x}) \geq \hat{\kappa}|x) = \Phi\left(\sqrt{\alpha + \beta}\left(\hat{x} - \frac{\alpha y + \beta x}{\alpha + \beta} - \frac{\Phi^{-1}(\hat{\kappa})}{\sqrt{\beta}}\right)\right) \tag{36}$$

Hence we can define $u(x, \hat{x})$ as

$$u(x, \hat{x}) \equiv \Phi\left(\sqrt{\alpha + \beta}\left(\hat{x} - \frac{\alpha y + \beta x}{\alpha + \beta} - \frac{\Phi^{-1}(\hat{\kappa})}{\sqrt{\beta}}\right)\right) - x \tag{37}$$

Suppose x' is the value of x that solves

$$u(x, \hat{x}) = 0$$

Thus, if all types whose cost is strictly below \hat{x} contribute, then irrespective of the actions of types whose cost is \hat{x} or higher, types whose cost is strictly below x' strictly prefer to contribute. Note the importance of strategic complementarity in reaching this conclusion. Hence, we have

$$S_C(L_{\hat{x}}) = L_{x'}$$

so that the lower simple events are closed under $S_C(\cdot)$. The argument for the closure of the upper simple events under $S_N(\cdot)$ is analogous.

From (37) and (36), we can characterize the event $S_C(L_{\hat{x}})$ as the set of types whose cost is strictly lower than its probability that the mass of players whose types are in $L_{\hat{x}}$ is larger than $\hat{\kappa}$.

Define the sequence $\underline{x}_1, \underline{x}_2, \underline{x}_3, \dots$ in terms of the following equations

$$\begin{aligned} \underline{x}_1 &= 0 \\ u(\underline{x}_2, \underline{x}_1) &= 0 \\ u(\underline{x}_3, \underline{x}_2) &= 0 \\ &\vdots \end{aligned}$$

Since $u(x, \hat{x})$ is decreasing in the first argument and increasing in the second, the sequence $\underline{x}_1, \underline{x}_2, \underline{x}_3, \dots$ is an increasing sequence that is bounded by the smallest solution \underline{x} to $u(x, x) = 0$. Hence, the sequence $\underline{x}_1, \underline{x}_2, \underline{x}_3, \dots$ converges to \underline{x} . Similarly, define the decreasing sequence $\bar{x}_1, \bar{x}_2, \bar{x}_3, \dots$ in

terms of the equations

$$\begin{aligned}\bar{x}_1 &= 1 \\ u(\bar{x}_2, \bar{x}_1) &= 0 \\ u(\bar{x}_3, \bar{x}_2) &= 0 \\ &\vdots\end{aligned}$$

The sequence converges from above to the largest solution \bar{x} to $u(x, x) = 0$.

From the definition of the operators $S_C(\cdot)$ and $S_N(\cdot)$, we have

$$S_C^k(\emptyset) = L_{\underline{x}_k} \quad \text{and} \quad S_N^k(\emptyset) = U_{\bar{x}_k} \quad (38)$$

Since the sequence $\underline{x}_1, \underline{x}_2, \underline{x}_3, \dots$ is strictly increasing and the sequence $\bar{x}_1, \bar{x}_2, \bar{x}_3, \dots$ is strictly decreasing, we thus have the following set inclusion properties of the $S_C(\cdot)$ and $S_N(\cdot)$ operators.

$$\emptyset \subsetneq S_C(\emptyset) \subsetneq S_C^2(\emptyset) \subsetneq \dots \quad (39)$$

$$\emptyset \supsetneq S_N(\emptyset) \supsetneq S_N^2(\emptyset) \supsetneq \dots \quad (40)$$

where $S_C^k(\cdot)$ denotes the k -fold application of the $S_C(\cdot)$ operator. The event $S_C^k(\emptyset)$ consists of all types for whom not contributing is dominated after k rounds of deletions of strictly dominated strategies. We can define converse operators $B_N(\cdot)$ and $B_C(\cdot)$ defined on the complements of simple events such that

$$B_N(\sim L_x) \equiv \sim S_C(L_x) \quad \text{and} \quad B_C(\sim U_x) \equiv \sim S_N(U_x) \quad (41)$$

where $\sim E$ denotes the complement of the event E in the universal type space Ω . The event $B_N(\sim L_x)$ consists of all types for whom not contributing is undominated when every type in L_x contributes. From (41), the collection of complements of simple events are closed under the operators $B_C(\cdot)$ and

$B_N(\cdot)$. By taking the complements of (39) and (40), we have

$$\begin{aligned}\Omega &\supsetneq B_N(\Omega) \supsetneq B_N^2(\Omega) \supsetneq \dots \\ \Omega &\supsetneq B_C(\Omega) \supsetneq B_C^2(\Omega) \supsetneq \dots\end{aligned}\tag{42}$$

The event $B_N^k(\Omega)$ is the set of all types for whom not contributing is undominated after k rounds of deletion of strictly dominated strategies. The event

$$\bigcap_{k \geq 1} B_N^k(\Omega)\tag{43}$$

consists of all types for whom not contributing is rationalizable. The event (43) has the feature that it is the largest fixed point of the $B_N(\cdot)$ operator, since

$$B_N\left(\bigcap_{k \geq 1} B_N^k(\Omega)\right) = \bigcap_{k \geq 1} B_N^k(\Omega)\tag{44}$$

The set inclusion relations (42) and the fixed point property above shares many similarities with belief operators, and the common belief operator associated with evident events. We will show in a later section that this is not a coincidence, and for a simpler two-player game, we show that the B_N operator has the interpretation as a belief operator on the type space.

From (41), and our earlier characterization of events in $S_C^k(\emptyset)$ we can characterize the types that belong to the event $B_N^k(\Omega)$ in terms of their higher order beliefs. For any statement P , let us use the notation $\mu\{P\}$ to denote the proportion of players for whom P is true. We have

Theorem 2 *Not contributing is rationalizable for the type of player i with cost x_i if and only if all of the following sequence of statements are true:*

$$\begin{aligned}
(S1) \quad x_i &\geq 0 \\
(S2) \quad x_i &\geq \Pr(\mu\{S1\} \geq \hat{\kappa} | x_i) \\
(S3) \quad x_i &\geq \Pr(\mu\{S2\} \geq \hat{\kappa} | x_i) \\
&\vdots
\end{aligned}$$

In other words, not contributing is rationalizable for player i if and only if

- player i 's cost is larger than 0, and
- player i 's cost is larger than his probability that the mass of players with cost larger than 0 is larger than the critical threshold $\hat{\kappa}$, and
- player i 's cost is larger than his probability that the mass of players whose cost is larger than their respective probability of the [mass of players having cost larger than 0 is larger than $\hat{\kappa}$] is larger than $\hat{\kappa}$,
- and so on...

The ‘if’ part of the theorem states that if a type belongs to $\bigcap_{k \geq 1} B_N^k(\Omega)$, then not contributing is rationalizable. This part is true by construction. The ‘only if’ part of the theorem states that any type for whom not contributing is rationalizable must belong to $\bigcap_{k \geq 1} B_N^k(\Omega)$. This part follows from the fact that $\bigcap_{k \geq 1} B_N^k(\Omega)$ is the largest fixed point of $B_N(\cdot)$, and hence is the largest set of types for whom not contributing is undominated after the iteration deletion of dominated strategies. In this way, we can characterize the higher-order beliefs of those types for whom contributing is rationalizable, and thus characterize the allowable perturbations of higher order beliefs

from the common knowledge case, and links the discussion of global games with that of higher-order beliefs in Yildiz (2006).

The set of types for whom not contributing is rationalizable can be derived analogously. The event

$$\bigcap_{k \geq 1} B_C^k(\Omega) \cap \bigcap_{k \geq 1} B_N^k(\Omega) \quad (45)$$

consists of all types for whom both ‘contribute’ and ‘not contribute’ are rationalizable actions. This set of types can be characterized in terms of the limits \underline{x} and \bar{x} of the sequences of threshold points defined above. Both \underline{x} and \bar{x} are solutions to the equation

$$\Pr(\lambda(x) \geq \hat{\kappa}|x) = x \quad (46)$$

and hence, we can characterize the uniqueness of the rationalizable outcome in terms of the sensitivity of beliefs about other types as one’s own cost changes.

Theorem 3 *There is a unique rationalizable outcome if and only if there are no distinct x and x' such that*

$$\frac{\Pr(\lambda(x) \geq \hat{\kappa}|x) - \Pr(\lambda(x') \geq \hat{\kappa}|x')}{x - x'} = 1 \quad (47)$$

The ‘only if’ part follows from the fact that, if there *were* such distinct x and x' , then they would be distinct solutions to (46), and so the smallest solution will differ from the largest solution. That is, $\underline{x} < \bar{x}$, so that for all types with cost in $[\underline{x}, \bar{x}]$, both actions are rationalizable. The ‘if’ part follows from the fact that when there is a unique solution to (46), then we have $\underline{x} = \bar{x}$.

Theorem 3 states that a necessary and sufficient condition for dominance solvability is that players’ beliefs about their relative position in the population is not too sensitive to their own realized costs in the relevant region

where $x \in [0, 1]$. The perceived probability of being above the $\hat{\kappa}$ -th quantile does not increase “too fast” with one’s own realized cost when neither action is dominant.

6 Common p -Belief

The “belief” operators $B_C(\cdot)$ and $B_N(\cdot)$ that we used in the previous section did not correspond exactly to the conventional belief operators that describe the probability weights that players assign to events. In this section, we will show that the rationalizability argument developed above for the public good contribution game can be conducted using conventional belief operators.

We lay out the construction in terms of the following two player game. Each player has two possible actions, "Invest" (I) or "Not Invest" (N). The payoff to not investing is always 0. Each player’s payoff to investing is equal to θ if the other player invests, where θ is an unknown parameter of the game. The payoff to investing is $\theta - 1$ if the other player does not invest, so there are strategic complementarities in investment. Payoffs are summarized in the following table:

	Invest	Not Invest	
Invest	θ, θ	$\theta - 1, 0$	(48)
Not Invest	$0, \theta - 1$	$0, 0$	

Now suppose that θ is normally distributed with mean y and precision α . Also assume that each player i observes a normal signal $x_i = \theta + \varepsilon_i$ of the true parameter θ , where each ε_i is independently normally distributed with mean 0 and precision β . If β is sufficiently large - specifically if

$$\frac{\alpha^2}{\beta} \left(\frac{\alpha + \beta}{\alpha + 2\beta} \right) \leq 2\pi. \quad (49)$$

then this game has a unique rationalizable outcome. As $\beta \rightarrow \infty$, the unique

rationalizable outcome has each player choosing the risk dominant action in the game: action I if $\theta > \frac{1}{2}$ and action N if $\theta < \frac{1}{2}$.⁶

We develop a framework that deals explicitly with higher order beliefs.

6.1 Finite Type Spaces

Let T_i be a finite set types of player i . Type t_i of player i has beliefs $\pi_i(t_i) \in \Delta(T_j \times \mathbb{R})$ over the types of the other player T_j and the state $\theta \in \mathbb{R}$. We write $\tilde{\theta}_i(t_i)$ for type t_i 's expectation of θ and $\pi_i(t_j|t_i)$ for the probability that he assigns to player $j \neq i$ being of type t_j .

While the whole domain of relevant uncertainty is the space $T_1 \times T_2 \times \mathbb{R}$, we find it convenient to focus analysis on the type space along, $T_1 \times T_2$. Now fix an event $E \subseteq T_1 \times T_2$ and let $\pi_i(E|t_i)$ be the probability that type t_i assigns to the event E :

$$\pi_i(E|t_i) = \sum_{\{t_j:(t_i,t_j) \in E\}} \pi_i(t_j|t_i).$$

6.2 Belief Operators

Monderer and Samet (1989) introduced a useful language for discussing higher order beliefs. For any event E in the type space, let $B_i^p(E)$ be the set of states where player i assigns probability at least p to the event E :

$$B_i^p(E) = \{(t_i, t_j) : \pi_i(E|t_i) \geq p\}.$$

Let $B_*^p(E)$ be the set of states where both players believe event E with probability at least p :

$$B_*^p(E) = B_1^p(E) \cap B_2^p(E).$$

⁶See Morris and Shin (2003) for a complete analysis of this example.

Let $C^p(E)$ be the set of states where both players believe event E with probability at least p , both players believe with probability p that both believe it with probability p , and so on:

$$C^p(E) = \bigcap_{k \geq 1} [B_*^p]^k(E).$$

Monderer and Samet (1989) proved a crucial characterization of common p -belief: say that an event E is p -evident if, whenever it is true, everyone believes it with probability p , or

$$E \subseteq B_*^p(E).$$

Monderer and Samet (1989) showed:

Theorem 4 *Event E is common p -belief at type profile (t_1, t_2) (i.e., $(t_1, t_2) \in C^p(E)$) if and only if there exists a p -evident event F such that $(t_1, t_2) \in F \subseteq B_*^p(E)$.*

This theorem generalizes the classic result of Aumann (1976) relating iterated and fixed point definitions of common knowledge.

It will also be useful to work with strict versions of belief operators. Let $S_i^p(E)$ be the set of states where player i assigns probability strictly greater than p to the event E :

$$S_i^p(E) = \{(t_i, t_j) : \pi_i(E|t_i) > p\}.$$

A useful result connects strict and weak versions of the belief operator. Say that $E \subseteq T_1 \times T_2$ is a "simple" event if $E = E_1 \times E_2$, where each $E_i \subseteq T_i$. Let $S_+^p(E)$ be set of states where one or the other player assigns probability strictly greater than p to the event E ,

$$S_+^p(E) = S_1^p(E) \cup S_2^p(E).$$

Proposition 5 For any simple event E and $k \geq 1$,

$$[B_*^p]^k (E) = \sim [S_+^{1-p}]^k (\sim E)$$

and thus

$$C^p (E) = \sim \bigcup_{k \geq 1} [S_+^{1-p}]^k (\sim E).$$

Observations like this were used in Monderer and Samet (1996) and Morris, Rob and Shin (1993).

6.3 Functional Belief Operators

For any function $f_i : T_i \rightarrow \mathbb{R}$, let $B_i^{f_i} (E)$ be the set of states where player i assigns probability at least f_i to the event E , where f_i is a random variable. Thus

$$B_i^{f_i} (E) = \{(t_i, t_j) : \pi_i (E|t_i) \geq f_i (t_i)\}.$$

We will also write $f = (f_1, f_2)$ and

$$\begin{aligned} B_*^f (E) &= B_*^{f_1, f_2} (E) = B_1^{f_1} (E) \cap B_2^{f_2} (E) \\ C^f (E) &= C^{f_1, f_2} (E) = \bigcap_{k \geq 1} [B_*^{f_1, f_2}]^k (E). \end{aligned}$$

Say that event E is common f -belief at (t_1, t_2) if $(t_1, t_2) \in C^f (E)$. Say that an event E is f -evident if $E \subseteq B_*^f (E)$. The characterization of Monderer and Samet (1989) will immediately generalize to functions:

Theorem 6 Event E is common f -belief at type profile (t_1, t_2) if and only if there exists a f -evident event F such that $(t_1, t_2) \in F \subseteq B_*^f (E)$.

Let $S_i^{f_i} (E)$ be the set of states where player i assigns probability strictly greater than f_i to the event E :

$$S_i^{f_i} (E) = \{(t_i, t_j) : \pi_i (E|t_i) > f_i (t_i)\}.$$

Let $S_+^f(E)$ be set of states where one or the other player assigns probability strictly greater than p to the event E :

$$S_+^f(E) = S_1^{f_1}(E) \cup S_2^{f_2}(E).$$

We adopt the convention that if $f = (f_1, f_2)$, then $1 - f = (f'_1, f'_2)$, where $f'_i(t_i) = 1 - f_i(t_i)$.

Proposition 7 *For any simple event E and $k \geq 1$,*

$$[B_*^f]^k(E) = \sim \left[S_+^{1-f} \right]^k(\sim E)$$

and thus

$$C^f(E) = \sim \bigcup_{k \geq 1} \left[S_+^{1-f} \right]^k(\sim E).$$

6.4 Characterizing Rationalizable Actions

Now we know that action N must be the unique rationalizable action for player i if $\tilde{\theta}_i(t_i) < 0$. But since $\pi_i(\emptyset|t_i) = 0$, we can re-write this as $\tilde{\theta}_i(t_i) < \pi_i(\emptyset|t_i)$. Thus N is the unique rationalizable action of at least one player if

$$(t_1, t_2) \in S_+^{\tilde{\theta}}(\emptyset).$$

But observe that if a player i assigns probability more than $\tilde{\theta}_i(t_i)$ to his opponent not investing for sure, then he does not want to invest for sure. Then N is the unique rationalizable action of at least one player if

$$(t_1, t_2) \in S_+^{\tilde{\theta}}\left(S_+^{\tilde{\theta}}(\emptyset)\right).$$

Iterating this logic, and applying a similar logic to the action I , we have:

Lemma 8 *Action N is the unique rationalizable action for player i if*

$$(t_i, t_j) \in S_i^{\tilde{\theta}_i} \left(\sim \bigcup_{k \geq 1} [S_+^{\tilde{\theta}}]^k (\emptyset) \right)$$

Action I is the unique rationalizable action for player i if

$$(t_i, t_j) \in S_i^{1-\tilde{\theta}_i} \left(\sim \bigcup_{k \geq 1} [S_+^{1-\tilde{\theta}}]^k (\emptyset) \right)$$

In fact, we can generalize this result by giving an exact characterization of when each action is rationalizable:

Proposition 9 *Action N is rationalizable if and only if*

$$(t_i, t_j) \in B_i^{\tilde{\theta}_i} C^{\tilde{\theta}} (\Omega).$$

Action I is rationalizable for player i if and only if

$$(t_i, t_j) \in B_i^{1-\tilde{\theta}_i} C^{1-\tilde{\theta}} (\Omega).$$

We can reconcile lemma 8 and proposition 9 by observing that, by proposition 7,

$$B_i^{\tilde{\theta}_i} C^{\tilde{\theta}} (\Omega) = \sim S_i^{1-\tilde{\theta}_i} \left(\sim \bigcup_{k \geq 1} [S_+^{1-\tilde{\theta}}]^k (\emptyset) \right)$$

and

$$B_i^{1-\tilde{\theta}_i} C^{1-\tilde{\theta}} (\Omega) = \sim S_i^{\tilde{\theta}_i} \left(\sim \bigcup_{k \geq 1} [S_+^{\tilde{\theta}}]^k (\emptyset) \right).$$

6.5 Sufficient Conditions for Uniqueness: Common Knowledge of Typicalness

Lemma 8 and Proposition 9 give an exact characterization of when there is a unique rationalizable outcome: there is a unique rationalizable outcome for player i if and only if

$$(t_i, t_j) \in S_i^{\tilde{\theta}_i} \left(\sim \bigcup_{k \geq 1} [S_+^{\tilde{\theta}}]^k (\emptyset) \right) \cup S_i^{1-\tilde{\theta}_i} \left(\sim \bigcup_{k \geq 1} [S_+^{1-\tilde{\theta}}]^k (\emptyset) \right).$$

We would like to understand what makes this condition hold better. One way to do this is to identify sufficient conditions for this to hold. Let

$$\begin{aligned} p_i(t_i) &= \pi_i \left(\left\{ (t_i, t_j) : \tilde{\theta}_j(t_j) > \tilde{\theta}_i(t_i) \right\} \middle| t_i \right) \\ q_i(t_i) &= \pi_i \left(\left\{ (t_i, t_j) : \tilde{\theta}_j(t_j) = \tilde{\theta}_i(t_i) \right\} \middle| t_i \right) \end{aligned}$$

We will maintain the assumption that there is common knowledge that $q_i(t_i) = 0$ and $\tilde{\theta}_i(t_i) \neq \frac{1}{2}$, for all i and t_i . If $p_i(t_i) = \frac{1}{2}$, player i considers himself to be typical, i.e., he has no reason to believe that his opponent's expectation is either higher or lower than his. We have:

Proposition 10 *If there is common knowledge that $p_1 = p_2 = \frac{1}{2}$, then there is a unique rationalizable outcome.*

If $p_i(t_i) = \frac{1}{2}$, player i considers himself to be typical, i.e., he has no reason to believe that his opponent's expectation is either higher or lower than his. Thus we will refer to "common knowledge that $p_1 = p_2 = \frac{1}{2}$ " as common knowledge of typicalness. Biais and Boesaerts (1998) made the primitive assumption that a player's expectation of an other player's expectation is equal to his own expectation. This assumption plays a very similar role.

6.6 Sufficient Conditions for Multiplicity: Almost Public Events

The existence of multiple rationalizable actions for some types is associated with common knowledge or approximate common knowledge of some events that can coordinate behavior. Thus we have:

Proposition 11 *If, for some p , it is common p -belief that both players' expectations of θ are greater than $1 - p$ and, for some q , it is common q -belief*

that both players' expectations of θ are less than q , i.e.,

$$t \in \left(\bigcup_{p \in [0,1]} C^p \left(\left\{ t | \tilde{\theta}(t) \geq (1-p, 1-p) \right\} \right) \right) \cap \left(\bigcup_{q \in [0,1]} C^q \left(\left\{ t | \tilde{\theta}(t) \leq (q, q) \right\} \right) \right)$$

then both actions are rationalizable for both players.

7 Asymmetric Information Revisited

With our preliminary results on the relation between higher order beliefs and global games, we can review some of the standard results in global games through the lens of common belief of typicalness.

7.1 Limit Uniqueness

The uniqueness condition (49) on the precision of public and private information seemed quite mysterious. Common knowledge of typicalness allows us to understand this result much better. In fact, as $\beta \rightarrow \infty$, we attain common knowledge of typicalness. Above, we saw that a sufficient condition for multiplicity was

$$\left(\bigcup_{p \in [0,1]} C^p \left(\left\{ t | \tilde{\theta}(t) \geq (1-p, 1-p) \right\} \right) \right) \cap \left(\bigcup_{q \in [0,1]} C^q \left(\left\{ t | \tilde{\theta}(t) \leq (q, q) \right\} \right) \right) \neq \emptyset$$

This condition is satisfied if there exist $\underline{x} < x < \bar{x}$, p and q such that

$$\frac{\alpha y + \beta \underline{x}}{\alpha + \beta} \geq 1 - p$$

$$\Pr_i(x_j \geq \underline{x} | x_i = \underline{x}) \geq p$$

$$\frac{\alpha y + \beta \bar{x}}{\alpha + \beta} \leq q$$

$$\Pr_i(x_j \leq \bar{x} | x_i = \bar{x}) \geq q$$

These conditions can be satisfied only if there exist $\underline{x} < \bar{x}$ such that

$$\Pr_i(x_j \leq \bar{x} | x_i = \bar{x}) \geq \frac{\alpha y + \beta \bar{x}}{\alpha + \beta}$$

and

$$\Pr_i(x_j \leq \underline{x} | x_i = \underline{x}) \leq \frac{\alpha y + \beta \underline{x}}{\alpha + \beta}.$$

Thus the higher order beliefs sufficient conditions for multiplicity are tight in this example.

7.2 A non-Normal Example

The uniqueness condition comes for normality assumptions about public and private information. What happens if we relax normality? Morris and Shin (2005b) have some preliminary results on this question. Here we just report a striking example and again relate it back to our results on higher order beliefs.

We will report an example with a continuous but non-differentiable prior where there are multiple equilibria no matter how accurate the private signals. Suppose that $\theta = \frac{1}{2} + \tau\eta$, where η is distributed according to density g with $g(\eta) = \frac{1}{2}e^{-|\eta|}$ and each player observes a signal $x_i = \theta + \sigma\varepsilon_i$, where the ε_i are independently and uniformly distributed on the interval $[-\frac{1}{2}, \frac{1}{2}]$. We can examine when there are unique or multiple equilibria as a function of the parameters σ and τ . We show in Morris and Shin (2005b) that there are multiple equilibria for all σ if $\tau > \frac{1}{4}$, but if $\tau < \frac{1}{4}$, there is a unique equilibrium for sufficiently small σ . Rather than report the algebra underlying the result, it is useful to understand the result in terms of approximate common knowledge events.

For small τ , the prior has a sharp spike at $\theta = \frac{1}{2}$. In this case, for every $\sigma > 0$, it will be possible to find x such that the events (1) that each player's

private signal is above $\frac{1}{2} - x$; and (2) that each player's private signal is below $\frac{1}{2} + x$ is common p -belief for $p > \frac{1}{2}$. This allows the construction of multiple equilibria.

8 Concluding Remarks

This paper has aimed at achieving three objectives.

- First, we have presented a global game analysis where we dispense with talk of “noisy signals”, and instead deal with heterogeneity of players.
- Second, with our framework, we have been able to address the issue of whether endogeneity of public information brings about multiplicity of equilibrium. Our answer is “not always”. The crucial issue is the nature of the residual uncertainty conditional on the public signal. If there is enough correlation across types, uniqueness is the norm. In particular, prices are different from informative polls.
- Finally, we have characterized the higher-order beliefs that allow the global game argument to “work”. Essentially, the property that matters is the stationarity of beliefs with respect to the ordering of types in the region where the players do not have dominant actions.

Appendix

This appendix solves for a rational expectations equilibrium in an asset market where players' signals have a common noise term. The liquidation value of an asset v is distributed normally with mean y and variance $1/\alpha$. Trader i receives a noisy signal

$$x_i = v + \eta + \varepsilon_i,$$

with $\eta \sim N\left(0, \frac{1}{\gamma}\right)$, $\varepsilon_i \sim N\left(0, \frac{1}{\beta}\right)$. Conjecture that price p takes the form

$$p = c(v + \eta) + (1 - c)y - ds,$$

with the noisy supply term $s \sim N\left(0, \frac{1}{\delta}\right)$. The covariance matrix for (x_i, p) is

$$V = \begin{bmatrix} \frac{1}{\alpha} + \frac{1}{\gamma} + \frac{1}{\beta} & \frac{c}{\alpha} + \frac{c}{\gamma} \\ \frac{c}{\alpha} + \frac{c}{\gamma} & \frac{c^2}{\alpha} + \frac{c^2}{\gamma} + \frac{d^2}{\delta} \end{bmatrix}$$

The determinant is $D = \frac{c^2\alpha\delta + c^2\gamma\delta + d^2\alpha\beta + d^2\alpha\gamma + d^2\beta\gamma}{\alpha\beta\gamma\delta}$, and the inverse is

$$V^{-1} = \frac{1}{D} \begin{bmatrix} \frac{c^2}{\alpha} + \frac{c^2}{\gamma} + \frac{d^2}{\delta} & -\frac{c}{\alpha} - \frac{c}{\gamma} \\ -\frac{c}{\alpha} - \frac{c}{\gamma} & \frac{1}{\alpha} + \frac{1}{\gamma} + \frac{1}{\beta} \end{bmatrix}$$

Hence, the conditional mean of v based on trader i 's information is

$$\begin{aligned} E(v|x_i, p) &= y + \frac{1}{\alpha} \begin{bmatrix} 1 & c \end{bmatrix} V^{-1} \begin{bmatrix} x_i - y \\ p - y \end{bmatrix} \\ &= \frac{d^2\beta\gamma}{c^2\alpha\delta + c^2\gamma\delta + d^2\alpha\beta + d^2\alpha\gamma + d^2\beta\gamma} x_i \\ &\quad + \frac{c\gamma\delta}{c^2\alpha\delta + c^2\gamma\delta + d^2\alpha\beta + d^2\alpha\gamma + d^2\beta\gamma} p \\ &\quad + \frac{c^2\alpha\delta - c\gamma\delta + c^2\gamma\delta + d^2\alpha\beta + d^2\alpha\gamma}{c^2\alpha\delta + c^2\gamma\delta + d^2\alpha\beta + d^2\alpha\gamma + d^2\beta\gamma} y \end{aligned}$$

The conditional variance is

$$\begin{aligned} Var(v|x_i, p) &= \frac{1}{\alpha} - \frac{1}{\alpha^2} \begin{bmatrix} 1 & c \end{bmatrix} V^{-1} \begin{bmatrix} 1 \\ c \end{bmatrix} \\ &= \frac{c^2\delta + d^2\beta + d^2\gamma}{c^2\alpha\delta + c^2\gamma\delta + d^2\alpha\beta + d^2\alpha\gamma + d^2\beta\gamma} \end{aligned}$$

Given exponential utility with risk tolerance τ , trader i 's demand is

$$\frac{\tau (E_i(v) - p)}{\text{Var}(v|x_i, p)}$$

Hence, aggregate demand is

$$\frac{\tau (\bar{E}(v) - p)}{\text{Var}(v|x_i, p)}$$

where $\bar{E}(v|x_i, p)$ is obtained by replacing x_i with $v + \eta$ in $E_i(v)$. Setting aggregate demand equal to supply s ,

$$p = \bar{E}(v) - \frac{s}{\tau} \text{Var}(v|x_i, p)$$

We solve for a linear equilibrium by substituting in the conjectured form of the price $p = c(v + \eta) + (1 - c)y - ds$. We have

$$\begin{aligned} p &= \bar{E}(v) - \frac{s}{\tau} \text{Var}(v|x_i, p) \\ &= \frac{\gamma\delta c^2 + \beta\gamma d^2}{c^2\alpha\delta + c^2\gamma\delta + d^2\alpha\beta + d^2\alpha\gamma + d^2\beta\gamma} (v + \eta) \\ &\quad + \frac{c^2\alpha\delta + d^2\alpha\beta + d^2\alpha\gamma}{c^2\alpha\delta + c^2\gamma\delta + d^2\alpha\beta + d^2\alpha\gamma + d^2\beta\gamma} y \\ &\quad - \frac{1}{\tau} \frac{c^2\delta + d^2\beta + d^2\gamma + cd\tau\gamma\delta}{c^2\alpha\delta + c^2\gamma\delta + d^2\alpha\beta + d^2\alpha\gamma + d^2\beta\gamma} s \end{aligned}$$

Now, take the limit as $\alpha \rightarrow 0$, in which the prior mean on v is uninformative. In this limit, the coefficient on y goes to zero, so that $c = 1$. The coefficient on s in the limit as $\alpha \rightarrow 0$ is given by

$$-\frac{1}{\tau} \frac{\delta + d^2\beta + d^2\gamma + d\tau\gamma\delta}{\gamma\delta + d^2\beta\gamma}$$

Next, take the limit as $\beta \rightarrow \infty$. The equilibrium price p in this limit is

$$p = v + \eta - \frac{1}{\tau\gamma} s$$

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