The Allocation of Indivisible Objects via Rounding

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Abstract

The problem of allocating indivisible objects arises in the allocation courses, spectrum licenses, landing slots at airports and assigning students to schools. This paper proposes a technique for making such allocations that is based on rounding a fractional allocation. Under the assumption that no agent wants to consume more than $k$ items, the rounding technique can be interpreted as giving agents lotteries over approximately feasible integral allocations that preserve the ex-ante efficiency and fairness properties of the initial fractional allocation. The integral allocations are only approximately feasible in the sense that up to $k - 1$ more units than the available supply of any good is allocated.

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1 Introduction

Among the important contributions to the problem of allocating indivisible objects are Shapley and Shubik (1972) and Scarf and Shapley (1974). Both papers are concerned with the allocation of indivisible objects under the constraint that no agent consumes more than one object (unit demand). The first paper allows transfers, the second does not. When the unit demand assumption holds, the literature has identified a host of mechanisms for different settings with attractive properties, see for example, Demange, Gale and Sotomayor (1986), Hylland and Zeckhauser (1979) or Bogomolnaia and Moulin (2001). An important reason for the surfeit of positive results is that in this environment any fractional assignment can be expressed as a convex combination of feasible integral assignments (this is a consequence of the Birkhoff-von Neuman Theorem). Thus, in the quasi-linear case, efficient allocations can be supported with Walrasian prices. In the non-transferable case, fractional assignments can be interpreted as lotteries over feasible integral allocations.

When additional constraints are imposed on the set of allocations or when the unit demand assumption is relaxed, it is no longer the case that fractional assignments lie in the convex hull of feasible integral allocations.\(^1\) Thus, in more general settings to obtain allocations with attractive properties one is forced to yield on feasibility. The question is, how much? In this paper we describe a technique, the iterative rounding method (IRM), for producing mechanisms for allocating indivisible goods when no agent wishes to consume more than \(k\) goods (\(k\)-demand).\(^2\) This can be modeled by assuming the utility assigned to bundles of size \(k+1\) or larger is zero. If one wishes to preserve monotonicity of valuations, this can be captured by setting the utility of any bundle equal to the utility of the highest value \(k\)–subset of that bundle. Formally, if \(u(\cdot)\) is an agent’s utility function, then,

\[
u(S) = \max_{A \subseteq S: |A| \leq k} \{ u(A) \}.
\]

for any bundle \(S\).

These mechanisms will possess a number of attractive features but violate the resource constraints of the problem. Under the \(k\)-demand assumption these violations will be of size \(k-1\). In the special case when \(k=1\) we recover the Birkhoff-von Neumann Theorem. In many settings \(k-1\) will be small relative to the available supply of the good. One can interpret this to mean that by withholding a relatively small amount of each good it will

\(^1\)Budish et al. (2012) uses standard results about the integrality of polyhedra to identify cases where fractional assignments can be interpreted as lotteries over feasible integral allocations.

\(^2\)This technique is discussed in detail in Lau, Ravi and Singh (2011).
be possible to obtain allocations with desirable properties.

To illustrate the application of the technique we give two examples. The environment for both examples is the same. In each case, let $G$ be a set of distinct goods and $s_j$ the (integral) supply of good $j \in G$, i.e., the number of copies of good $j$. For convenience, assume no agent wishes to consume more than one copy of any $j \in G$.

In the first example, agents in $N$ have quasi-linear preferences. In this case it is well known that Walrasian prices that clear the market do not exist. An application of the IRM shows that under the $k$-demand assumption there exist Walrasian prices where the excess demand for any good is at most $k - 1$. This result formalizes the intuition that in the absence of strong complementarities in preferences, one should be able to clear the market using Walrasian prices. This also provides support for the use of clock auctions before a package bidding phase for combinatorial auctions under the $k$-demand assumption. If the clock auction reaches the right Walrasian prices, because the excess demand on any item is small, most of the price discovery needed to support an efficient allocation has been made.

The second example does not permit transfers. In this case, under the $k$-demand assumption, the IRM returns a lottery over assignments of bundles to agents that is

1. approximately efficient, and, ex-ante envy-free.
2. The mechanism is asymptotically strategy-proof.
3. The allocation consumes no more than $s_j + k - 1$ units of good $j \in G$.

An instance of where $k$ in item 3 would be ‘small’ is in course allocation. Here, each element of $N$ is a student, each element of $G$ a class and $s_j$ the number of available seats in class $j \in G$. In this context, $k$ would be an upper limit on the number of distinct classes a student can take during a term. At Northwestern University, for example, $k$ is typically no more than 4 or 5 and $s_j$ a factor of 10 larger than that.

The outcomes of the IRM described here can be contrasted with those of a mechanism proposed by Budish (2011) for the same setting. Budish’s mechanism is based on computing an approximate competitive equilibrium from equal incomes. That mechanism returns an allocation that is approximately efficient, approximately envy-free in an ex-post sense, asymptotically strategy-proof and violates the resource constraints, but by $O(\sqrt{\min\{2k, |G|\} |G|})$. This scales with $|G|$ and can be significantly larger than $k - 1$.

Unlike Budish (2011), the IRM does not require the computation of a fixed point. Instead, it solves a sequence of linear programs and iteratively rounds the corresponding
fractional solution into an integer solution. An implication of the rounding method used is that it allows one to interpret a fractional assignments of courses to students as a lottery over integral allocations that violate, in a limited way, the resource constraints. Thus, it is possible to obtain randomized mechanisms that are ex-ante efficient and fair.

In the next section of this paper we describe the IRM as applied to the allocation of indivisible goods when agents have $k$-unit demands. A consequence of the IRM is that fractional allocations can be interpreted as lotteries of integer allocations that are approximately feasible. We then apply the IRM in two settings: one with quasi-linear preferences and one without transfers.

The subsequent section, 3, applies the IRM to combinatorial exchanges. In the final section, 4, we restrict attention to assignment settings where agents have unit demands. However, there are side constraints on the mix of agents that consume each type of good. Such problems arise in school choice settings. For example, when $G$ is a set of schools and $N$ a set of students, one might impose constraints on the ethnic or gender composition of each school.

## 2 The Iterative Rounding Mechanism

Let $N$ be a set of agents, $G$ be a set of distinct goods and $s_j$ the (integral) supply of good $j \in G$, i.e., the number of copies of good $j$. For convenience, assume no agent wishes to consume more than one copy of any $j \in G$ (but they are willing to consume up to $k$ distinct goods). To describe the set of feasible allocations introduce decision variables $x_i(S)$ which have value 1 if the bundle $S \subseteq G$ is assigned to agent $i \in N$ and zero otherwise. An assignment $\{x_i(S)\}_{i \in N, S \subseteq G}$ is feasible if the following inequalities are satisfied:

\[ \sum_{S \subseteq G} x_i(S) \leq 1 \quad \forall i \in N \]  
\[ \sum_{i \in N} \sum_{S \ni j} x_i(S) \leq s_j \quad \forall j \in G \]

Inequality (2) ensures that each agent receives at most one bundle. Inequality (3) ensures no good $j \in G$ is allocated more than $s_j$ times. Let $P$ be the set of non-negative solutions to (2-3).

The IRM takes as input an extreme point, $x^* \in \arg \max \{u \cdot x : x \in P\}$ where $u \geq 0$ and $u_i(S) = 0$ for all $i \in N$ and $S \subseteq G$ such that $|S| > k$. It then rounds $x^*$ into a 0-1
vector $\bar{x}$ that satisfies (2) and is such that
\[
\sum_{i \in N} \sum_{S \ni j} \bar{x}_i(S) \leq s_j + k - 1 \quad \forall j \in G.
\] (4)

Beginning with $x^*$, we remove from (2-3) all variables $x_i(S)$ for which $x^*_i(S) = 0$. In other words, a variable that is zero in $x^*$ will be rounded down to zero and fixed at that value in all subsequent iterations. Similarly, remove from (2-3) all variables $x_i(S)$ for which $x^*_i(S) = 1$ and adjust the right hand sides of (3) accordingly. In other words, a variable set to 1 by $x^*$ is fixed at 1 in all subsequent iterations. In the system that remains pick a non-negative extreme point (fractional or otherwise) that optimizes the vector $u$ and repeat. At some iteration, we obtain an extreme point with no variable set to 1. Call it $y$. As we show below, there must exist a $j \in G$ such that
\[
|\{i \in N : y_i(S) > 0, S \ni j\}| \leq s_j + k - 1.
\]
For each such $j$, remove the corresponding constraint (3) and in the relaxed system find an extreme point that optimizes $u$ and repeat. Stop once all variables have been fixed at either 0 or 1 and denote the resulting 0-1 vector by $\bar{x}$.

There are three observations to be made about $\bar{x}$.

1. At each iteration, inequality (2) holds. Thus, $\bar{x}$ satisfies (2).

2. At each iteration, the original program is (possibly) relaxed. Thus, $u \cdot \bar{x} \geq u \cdot x^*$.

3. Because $\bar{x}_i(S) = 1$ only if $x^*_i(S) > 0$, it follows that for the inequalities in (3) thrown away, $\sum_{i \in N} \sum_{S \ni j} \bar{x}_i(S) \leq s_j + k - 1$.

The key lemma is the following:\footnote{Actually it is a consequence of Király, Lau and Singh (2008), but for completeness we include a proof.}

**Lemma 2.1** Let $u_i(S) \geq 0$ and $u_i(S) = 0$ for all $|S| > k$. Let $x^*$ be an extreme point of $P$ in $\arg \max \{u \cdot x : x \in P\}$ such that $x^*_i(S) < 1$ for all $i \in N$ and $S \subseteq G$. Then, there exists a $j \in G$ such that
\[
|\{i \in N : x^*_i(S) > 0, S \ni j\}| \leq s_j + k - 1.
\]

**Proof:** First we show that $|N| < |G|$. Observe that an extreme point of (2-3) can have at most $|N| + |G|$ non-zero variables. As $x^*_i(S) < 1$ for all $i \in N$ and $S \subseteq G$, it follows...
that for each $i$, $|\{S : x^*_i(S) > 0\}| \geq 2$. Hence, the number of non-zero variables in $x^*$ is at least $2|N|$. Therefore, $|N| \leq |G|$. 

Let $z_i(S) = 1$ if $x^*_i(S) > 0$ and zero otherwise. Suppose, for a contradiction that the conclusion of the lemma is false. Then,

$$\sum_{i \in N} \max_{S \subseteq G} \{z_i(S)\} > s_j + k - 1 \quad \forall j \in G.$$  

Adding inequality (5) up over $j \in G$ and using the fact that $z_i(S) = 1$ implies $|S| \leq k$ gives:

$$\sum_{j \in G} s_j + |G|(k - 1) < \sum_{j \in G} \sum_{i \in N} \max_{S \subseteq G} \{z_i(S)\} \leq \sum_{i \in N} k \max_{S \subseteq G} z_i(S) \leq k|N|$$

As the left hand side of the above is bounded below by $k|G|$ and $|N| < |G|$ we get a contradiction.

The proof of Theorem 2.2 is nonconstructive. In the appendix, we provide a polynomial time algorithm using iterative rounding to compute a lottery over integral solutions of $E_k$ whose expectation is arbitrarily close to the given fractional solution in $Q_k$.

**Theorem 2.2** $Q_k$ is in the convex hull of $E_k$.

**Proof:** Suppose not. Then, there is a $z \in Q_k$ not in $E_k$. Hence, there must be a hyperplane that separates $z$ from $E_k$. Let $u$ be the vector of coefficients of that hyperplane. Choose it so that $ux < uz$ for all $x \in E_k$. Let $x^*$ be an extreme point solution to $\max \{ux : x \in Q_k\}$. By Lemma 2.1, there is a $\hat{x} \in E_k$ such that $u\hat{x} \geq ux^* \geq uz$, a contradiction.

The proof of Theorem 2.2 is nonconstructive. In the appendix, we provide a polynomial time algorithm using iterative rounding to compute a lottery over integral solutions of $E_k$ whose expectation is arbitrarily close to the given fractional solution in $Q_k$.

The IRM provides a great deal of flexibility in the choice of the initial extreme point, $x^*$. We give examples to illustrate the applications.

### 2.1 Quasi-linear Preferences

Suppose agents have quasi-linear preferences and let $u_i(S)$ be the monetary value that agent $i \in N$ assigns to bundle $S \subseteq G$. We assume that each $u_i$ is monotone non-decreasing...
and that the $k$-demand assumption holds. In this setting it is well known that without further restrictions on the preferences of the agents, Walrasian prices that clear the market need not exist.

To find an efficient allocation we solve $\max\{u \cdot x : x \in P\}$. In this program set $u_i(S) = 0$ for all $S$ such that $|S| > k$. This does not change the value of the efficient allocation but does shrink the set of efficient allocations.

Let $p^*$ be the optimal dual variables associated with (3) in the program $\max\{u \cdot x : x \in P\}$. We can interpret $p^*$ as a Walrasian price vector. For each $i \in N$ let $D_i(p^*) = \arg \max_{S \subseteq G}[u_i(S) - \sum_{j \in S} p_j^*]$, i.e., agent $i$’s demand correspondence. Lemma 2.1 implies that we can give to each agent an element of their demand correspondence at price $p^*$ such that (4) is satisfied. Formally, under (1) there exist Walrasian prices where the excess demand for any object will be at most $k - 1$. Informally, as long as agents do not demand bundles that are very large, there exist Walrasian prices that approximately clear the market.

As a second example, consider the problem of designing an individually rational, weakly dominant strategy, efficient mechanism in this setting. It is well known that the Vickrey-Clarke-Groves mechanism will do the trick. However, computing the efficient allocation is NP-hard.

The IRM can be used to construct a polynomial time mechanism that individually rational, weakly dominant strategy but approximately but approximately efficient. To see this, let $x^* \in \arg \max\{u \cdot x : x \in P\}$. As before we may assume that $u_i(S) = 0$ whenever $|S| > k$. Observe that $x^*_i(T^i)$ is monotone in $u_i(T^i)$. By Theorem 2.2 we can represent $x^*$ as a lottery over $E_k$. Thus, in expectation, the allocation rule defined by this lottery is exactly the solution to the problem of finding the efficient allocation. Hence, this lottery can be implemented in a way that is incentive compatible in expectation (the expectation is with respect to the randomization induced by the lottery). Furthermore, in the worst case, our mechanism over allocates at most $k - 1$ copies of each good. Thus, if each good has a large number of copies, our mechanism will yield an almost feasible allocation.

### 2.2 Non-transferable Utility

Here we describes a choice of $x^*$ that ensures the rounded solution $\bar{x}$ gives to each agent a bundle they prefer at least as much as their maxi-min share. As before we assume the $k$-demand assumption holds. For each $h$ and $i \in N$, in (2-3) set $x_i(S) = 0$ for any bundle

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4In the last decade there has been an extensive literature on designing polynomial time mechanisms for combinatorial auctions. See for example L. Blumrosen and N. Nisan (2007) for a survey.
S that \( i \) ranks below \( h^{th} \) place in her preference ordering. Denote the corresponding restriction of \( P \) by \( P_h \). Let \( h^* \) be the smallest index such that \( P_{h^*} \) is non-empty. Choose any \( u \) that satisfies the conditions of the lemma and let \( x^* \) be an extreme point solution of \( \max \{ u \cdot x : x \in P_{h^*} \} \). Consider now the corresponding \( \bar{x} \). As \( \bar{x}_i(S) = 1 \) only if \( x^*_i(S) > 0 \), it follows that each agent receives a bundle that she ranks in place \( h^* \) or higher.

As another illustration, let \( u_i(S) \) be the von-Neumann Morgenstern utility that agent \( i \) assigns to bundle \( S \) (not necessarily quasi-linear). A fractional allocation \( x \in P \) is ex-ante envy-free if

\[
\sum_{S \subseteq G} u_i(S)x_i(S) \geq \sum_{S \subseteq G} u_i(S)x_j(S) \quad \forall i \neq j
\]  

Choose an \( x^* \) in \( Q_k \) that satisfies (6). Assuming one exists,\(^5\) Theorem 2.2 tells us it can be expressed as a lottery over the elements in \( E_k \).

### 2.2.1 Strategy-Proofness

In this section we show how to determine an ex-ante envy-free allocation in a way that is asymptotically strategy proof. Specifically, as the number of agents and goods scales with \( m \), no agent can gain by misreporting their preferences.

Scale the number of agents and supply of each good by \( m \). Let \( N^* \) be the set of \( m|N| \) agents. As there are a finite number of objects there can only be a finite number of orderings of these objects. Let \( T \) be the set of distinct orderings which we will call types. In this expanded economy an agent is defined by a pair \((i, t)\). The first term in the pair is the name of the agent and the second is her type. Let \( u^t_i(S) \) be the utility that agent \( i \) of type \( t \) assigns to bundle \( S \). Let \( m_t \) be the number of agents of type \( t \). Consider the following program for finding a utilitarian allocation that is envy-free:

\[
\max \sum_{t \in T} \sum_{i \in N^*} \sum_{S \subseteq G} u^t_i(S)x^t_i(S)
\]  

\[
\sum_{S \subseteq G} x^t_i(S) \leq 1 \quad \forall i \in N \quad \forall t \in T \quad \forall i \in N^*
\]  

\[
\sum_{t \in T} \sum_{i \in N^*} \sum_{S \ni j} x^t_i(S) \leq ms_j \quad \forall j \in G
\]  

\[
\sum_{S \subseteq G} u^t_i(S)x^t_i(S) \geq \sum_{S \subseteq G} u^t_j(S)x^t_j(S) \quad \forall i \neq j \quad \forall r \neq t
\]

\(^5\) It does because of Hylland and Zeckhauser (1979).
Suppose the optimal solution is unique. This can be guaranteed by invoking a tie breaking rule or perturbing the objective function slightly. Recall, that we can set $u_i^t(S) = 0$ for all $(i, t)$ and $|S| > k$. Call (7-10) the disaggregate formulation.

Now consider the following ‘aggregate’ formulation:

\[
\max \sum_{t \in T} \sum_{S \subseteq G} m_t u_i^t(S) y^t(S) \tag{11}
\]

\[
\sum_{S \subseteq G} y^t(S) \leq 1 \forall t \in T \tag{12}
\]

\[
\sum_{t \in T} \sum_{S \ni j} m_t y^t(S) \leq m s_j \forall j \in G \tag{13}
\]

\[
\sum_{S \subseteq G} u_i^t(S) y^t(S) \geq \sum_{S \subseteq G} u_i^r(S) y^r(S) \forall t, r \in T \tag{14}
\]

We can recover the optimal solution for the disaggregate formulation from the aggregate formulation by setting $x_t^i(S) = y^t(S)$ for all $i$ of type $t$.

Suppose agent $i$ of type $p$ pretends to be of type $q$. We will show that the impact on the allocations of the other agents from this misreport can be computed by solving (11-14) with a perturbed right hand side.

If agent $i$ of type $p$ pretends to be of type $q$ the aggregate formulation becomes:

\[
\max \sum_{t \in T} \sum_{S \subseteq G} m_t u_i^t(S) y^t(S) - \sum_{S \subseteq G} u_i^p(S) y^p(S) + \sum_{S \subseteq G} u_i^q(S) y^q(S) \tag{15}
\]

\[
\sum_{S \subseteq G} y^t(S) \leq 1 \forall t \in T \tag{16}
\]

\[
\sum_{t \in T} \sum_{S \ni j} m_t y^t(S) - \sum_{S \ni j} y^p(S) + \sum_{S \ni j} y^q(S) \leq m s_j \forall j \in G \tag{17}
\]

\[
\sum_{S \subseteq G} u_i^t(S) y^t(S) \geq \sum_{S \subseteq G} u_i^r(S) y^r(S) \forall t, r \in T \tag{18}
\]

Let $z$ be the optimal solution to (15)-(18). To determine the effect of agent $(i, p)$’s misreport on the other agents adjust the right hand sides of the aggregate formulation by $z^p - z^q$. This gives the following formulation:

\[
\max \sum_{t \in T} \sum_{S \subseteq G} m_t u_i^t(S) y^t(S) - \sum_{S \subseteq G} u_i^p(S) z^p(S) + \sum_{S \subseteq G} u_i^q(S) z^q(S) \tag{19}
\]
\[ \sum_{S \subseteq G} y^t(S) \leq 1 \forall t \in T \]  
(20)

\[ \sum_{t \in T} \sum_{S \ni j} m_i y^t(S) \leq m s_j + \sum_{S \ni j} z^p(S) - \sum_{S \ni j} z^q(S) \forall j \in G \]  
(21)

\[ \sum_{S \subseteq G} u^1_i(S) y^t(S) \geq \sum_{S \subseteq G} u^1_r(S) y^r(S) \forall t, r \in T \]  
(22)

Compare program (19)-(22) to program (11)-(14). Their constraints differ only in the right hand sides. If we scale the last constraint in each by \( m \), then, the right hand sides of both differ by \( \epsilon = O(1/m) \). Therefore, for \( \epsilon \) small enough, the optimal basis of the two programs will coincide. Hence, the difference in optimal solution to the two programs will be \( O(\epsilon) \). In other words, the agent who misreports their type can only change their allocation by \( O(\epsilon) \). Thus, by the envy-free constraint, their utility changes by at most \( O(\epsilon) \). In fact, we can say more. As \( m \) the number of agents grows and the optimal basis remains unchanged and is unique, the utility of an agent can only decrease from misreporting their type. When agent \( i \) of type \( p \) pretends to be of type \( q \), he does not change the proportion of each resource that goes to agents of type \( q \) (this is from the uniqueness of the optimal basis) and only increases the number of agents vying for those resources.

3 Combinatorial Exchanges

In this section we consider a setting motivated by an FCC proposal (see Hoffman and Menon (2010)) to set up a centralized combinatorial exchange to enable efficient restructuring of spectrum holdings by allowing agents to submit package bids. Package bids are motivated by complementarities in spectrum. Possession of spectrum in neighboring cells will reduce the cost of interference and thus is more valuable than having them separately.

In the presence of complementarities, Walrasian prices that support a surplus maximizing allocation need not exist. If the complementarities are not excessive we show there exist Walrasian prices where the excess demand or supply is not to large.

Let \( N \) be the set of buyers, \( M \) the set of sellers and \( G \) the set of all goods. For each \( i \in N \) denote by \( u_i(S) \) the monetary value that buyer \( i \) assigns to bundle \( S \subseteq G \). For each \( j \in M \) let \( B^j \) be the set of goods that seller \( j \) possesses. Note \( G = \bigcup_{j \in M} B^j \). Denote by \( c_j(S) \) the opportunity cost that seller \( j \) assigns to bundle \( S \subseteq B^j \).

To formulate the problem of finding a surplus maximizing allocation of goods from sellers to buyers as an integer program, let \( x_i(S) = 1 \) if buyer \( i \in N \) is assigned bundle
S ⊆ G and zero otherwise. Similarly, let yj(S) = 1 if seller j ∈ M gives up bundle S ⊆ Bj and zero otherwise. Then, the problem of finding a surplus maximizing allocation is

\[
\max \sum_{i \in N} \sum_{S \subseteq G} u_i(S)x_i(S) - \sum_{j \in M} \sum_{S \subseteq B_j} c_j(S)y_j(S) \tag{23}
\]

\[
\sum_{S \subseteq G} x_i(S) \leq 1 \ \forall i \in N \tag{24}
\]

\[
\sum_{S \subseteq B_j} y_j(S) \leq 1 \ \forall j \in M \tag{25}
\]

\[
\sum_{i \in N} \sum_{S \ni g} x_i(S) - \sum_{j \in M} \sum_{S \ni g} y_j(S) \leq 0 \ \forall g \in G \tag{26}
\]

The objective function in (23) measures the surplus generated from trade. Constraint (24) ensures no buyer receives more than one bundle. Constraint (25) ensures that each seller parts with no more than one bundle. This is without loss of generality given the subadditivity assumption.

Note that surplus maximization is not the only objective one could choose. For example, one could replace (23) with

\[
\max \sum_{i \in N} \sum_{S \subseteq G} u_i(S)x_i(S) + \sum_{j \in M} \sum_{S \subseteq B_j} c_j(B_j \setminus S)y_j(S) \tag{27}
\]

Our results will apply to this choice of objective function as well.

Our goal is to establish the existence of an integer solution with high surplus that satisfies (24-25) without violating (26) to much. We consider a setting where complementarities are not excessive for both sellers and buyers. In particular, for each Bj there is a partition Pj1, ..., Pjt such that |Pjr| ≤ k for all r = 1, ..., t. Also

\[
c_j(S) = \sum_{r=1}^{t} c_j(S \cap Pjr). \]

Hence cj(·) is additive across the elements of the partition but permits complementarities or substitutes within the elements of the partition.

For each buyer i there is a partition P1i, ..., Pti such that |Pri| ≤ k for all r = 1, ..., t. Also

\[
u_i(S) = \sum_{r=1}^{t} u_i(S \cap Pri).\]
This is similar to the assumption introduced on the opportunity cost of the sellers. We believe it to be relevant to spectrum settings because interfering spectrum assets have similar frequency and are located close to one another and so can be categorized in groups of small size.\footnote{We could assume the $k$-demand condition instead.}

Under the assumption of seller and buyer preferences stated above, we can reformulate the program (23-26). Inequality (24) can be replaced by a collection of inequalities

$$\sum_{S \subset P^{i}_{r}} x_{i}(S) \leq 1 \forall i \in N \forall r = 1, \ldots, t (28)$$

Similarly, (25) can be replaced by

$$\sum_{S \subset P^{j}_{r}} y_{j}(S) \leq 1 \forall j \in M \forall r = 1, \ldots, t (29)$$

Consider now the following variation of the IRM. First, solve the linear programming relaxation of (23, 28, 29, 26). Observe that under this reformulation we can always assume that $x_{i}(S), y_{j}(S) = 0$ for any $S$ such that $|S| > k$. If a variable in the linear programming solution has an integer value, fix that variable and change the corresponding constraints. In particular, in every step we will maintain the following balance condition for a active subset $G^{*} \subset G$ of the goods

$$\sum_{i \in N} \sum_{S \ni g} x_{i}(S) + \sum_{i \in N} \sum_{S \ni g} \bar{x}_{i}(S) - \sum_{j \in M} \sum_{S \ni g} y_{j}(S) - \sum_{j \in M} \sum_{S \ni g} \bar{y}_{j}(S) = 0 \forall g \in G^{*}.$$  

Here $\bar{x}, \bar{y}$ denote the variables that were fixed to integer values in earlier iterations.

If the optimal fractional solution has no integral variables, we show that there exists $g \in G^{*}$ such that the number of non-zero variables (both $x_{i}(S)$ and $y_{j}(S)$) where $g \in S$ is at most $2k - 1$. Given this fact we drop the balance condition for this item $g$, and resolve the LP. Thus, when the algorithm terminates, we obtain an integral solution that violated the balance condition at most $2k - 1$.

The following result also follows from Király, Lau and Singh (2008).

**Theorem 3.1** The IRM returns an integral solution that violates (26) by at most $2k - 1$.

**Proof:** If the optimal fractional solution $(x^{*}, y^{*})$ has no integral variables, we show that there exists $g \in G^{*}$ such that the number of non-zero variables (both $x_{i}^{*}(S)$ and $y_{j}^{*}(S)$) where $g \in S$ is at most $2k - 1$. 

\[\sum_{i \in N} \sum_{S \ni g} x_{i}(S) + \sum_{i \in N} \sum_{S \ni g} \bar{x}_{i}(S) - \sum_{j \in M} \sum_{S \ni g} y_{j}(S) - \sum_{j \in M} \sum_{S \ni g} \bar{y}_{j}(S) = 0 \forall g \in G^{*}.\]
To each non-zero variable \( x_i^*(S) \), or \( y_j^*(S) \) assign 1 token. We distribute the total number of tokens in the following way. Without loss of generality, consider the token assigned to \( x_i(S) \). Allocate 1/2 of the token to the constraint corresponding to agent \( i \); give \( \frac{1}{2k} \) tokens to the constraints corresponding to every good \( g \in G \).

Clearly, if there is no good \( g \) where we can drop the balance constraint, then every constraint get at least 1 token. Furthermore, equality occurs only when every constraint binds, every \( S \) has size \( k \), and in this case the set of binding constraints is not linearly independent.

One of the design proposals being considered for the upcoming FCC incentive auction involves a clock auction (an ascending price auction on each item separately) followed by a package bidding phase (Ausubel, Cramton and Milgrom (2006)). The clock phase it is argued is both simple and facilitates price discovery. The package bidding phase limits collusion and enhances efficiency. One can interpret the clock auction as attempting to find Walrasian prices. Theorem 3.1 suggests that with the right Walrasian prices, the clock phase should terminate in an allocation with high surplus and relatively small excess demand or supply. In this sense the clock auction does help with price discovery.

4 Assignment Settings

In this section we consider an environment that satisfies the unit demand constraint. The assignment of goods to agents, however, must satisfy additional constraints. Such problems are motivated by school assignment problems where students must be assigned to schools in a way to ensure diversity on various dimensions (see Ehlers et al. (2011) for example). Budish et al. (2012) identified a class of side constraints that could be incorporated without violating the integrality property of the unit demand model. The constraints we consider do not belong to this class.

As before let \( G \) be a set of goods and \( N \) a set of agents but in this case each agent wishes to consume at most one good. Let \( x_{ij} = 1 \) if agent \( i \in N \) is assigned good \( j \in G \) and zero otherwise. Each agent is also endowed with a 0-1 vector of dimension \( k \) that records which of \( k \) binary characteristics she possesses. Examples of characteristics are gender, race or citizenship. Denote by \( C^r \) the set of agents with characteristic \( r = 1, \ldots, k \). An assignment of goods to agents is feasible if it satisfies the following

\[
\sum_{j \in G} x_{ij} = 1 \quad \forall i \in N \tag{30}
\]
\[ \sum_{i \in N} x_{ij} \leq s_j \quad \forall j \in G \quad (31) \]
\[ q^r_j \leq \sum_{i \in C^r} x_{ij} \leq Q^r_j \quad \forall r = 1, \ldots, k, \quad \forall j \in G \quad (32) \]

Let \( A \) denote the set of non-negative solutions of (30-32). Inequality (32) arises, for example, in controlled school choice problems. Here \( N \) is a set of students, \( G \) a set of schools.

Let \( A^\beta \) denote the set of non-negative integer solutions to (30) and (33, 34) below:

\[ \sum_{i \in N} x_{ij} \leq s_j + \beta \quad \forall j \in G \quad (33) \]
\[ q^r_j - \beta \leq \sum_{i \in C^r} x_{ij} \leq Q^r_j + \beta \quad \forall r = 1, \ldots, k, \quad \forall j \in G \quad (34) \]

**Theorem 4.1** \( A \) is in the convex hull of \( A^{2k+1} \).

**Proof:** Similar to the proof of Theorem 3.1, we can show that starting from any fractional solution in \( A \), the IRM returns an integral solution in \( A^{2k+1} \). From this, we can use an analog argument as in Theorem 2.2 to obtain the desired theorem. \( \blacksquare \)

Suppose agents have von-Neumann Morgenstern utilities, in particular, agent \( i \) gets utility \( u_{ij} \) from good \( j \). Mimicking section 2.2 we might seek to design an asymptotically strategy-proof mechanism that is ex-ante envy-free, efficient and approximately feasible. However, the quota constraints present in this application (but not in section 2.2) may render this impossible. If there are no lower quota constraints, using a general equilibrium approach, one can derive an ex-ante envy-free and feasible fractional solution. The linear program approach used also allows us to find an envy-free fractional solution that minimizes the degree of in-feasibility. For example, one can consider the following program\(^7\)

\[ \max \sum_i u_{ij} x_{ij} - \alpha \]
\[ \sum_{j \in G} x_{ij} = 1 \quad \forall i \in N \quad (35) \]
\[ \sum_{i \in N} x_{ij} \leq s_j \quad \forall j \in G \quad (36) \]

\(^7\)One also can put weights on the objective function.
\[ q_j^r - \alpha \leq \sum_{i \in C^r} x_{ij} \leq Q_j^r + \alpha \forall r = 1, \ldots, k, \forall j \in G \]  \hspace{1cm} (37)

\[ \sum_j u_{ij}x_{ij} \geq \sum_j u_{ij}x_{kj} \forall i, k \in N \]  \hspace{1cm} (38)

Given the solution of this LP, we can find a corresponding lottery over approximately feasible integer solutions, which is ex-ante envy free, asymptotic strategy-proof and optimizes the trade-off between efficiency and feasibility. The approximately feasible integer solutions will violate the supply and quota constraints by at most \(2k+1\). For example, in the school choice context the quotas \((s_j, q_j^r, Q_j^r)\) are large compared with \(k\), which is the number of student characteristics taken into account.

5 Conclusion

This paper has examined some of the consequences of relaxing the unit demand assumption in the allocation of indivisible goods to \(k\)-unit demand. Provided \(k\) is small relative to the supply of goods, one can obtain allocations with a number of attractive features by discarding a relatively small number of the goods.

References


Appendix

Recall that Theorem 2.2 shows that any $x \in Q_k$ can be expressed as a convex combination of points in $E_k$. In this section we show how to (approximately) decompose any $x \in Q_k$ into a convex combination of points in $E_k$.

In the following we assume $Q_k$ is full dimensional (dimension $d$): that is $Q_k$ has an interior point. Assume $E_k$ is bounded with diameter $D$. Denote by $|x - y|$ the Euclidean distance between $x, y$. Recall that we have a subroutine that will for any fractional $x \in Q_k$ and any cost vector $c$, return an integral $\bar{x} \in E_k$ such that $c \bar{x} \geq c x$.

Given this subroutine, we exhibit a polynomial time algorithm that for a given point $x \in Q_k$, finds at most $d + 1$ integral points in $E_k$ whose convex hull is arbitrarily close to $x$. The algorithm also returns a lottery over these $d + 1$ integral vectors whose expectation is close to $x$.

Given a fractional solution $x \in Q_k$. We assume there exists $\delta > 0$ such that $B(x, \delta) \in Q_k$. Here $B(x, \delta)$ denotes the ball of radius $\delta$ at $x$. This is with loss of generality, because otherwise we can always choose $x'$ in the interior of $Q_k$ close to $x$.

Given an allowable error $\epsilon > 0$, the algorithm is the following.

Algorithm In each step maintain a subset $S$ of points in $E_k$. Each iteration consists of the following steps.

1. Compute $y \in \text{conv}(S)$ that is closest to $x$. If $|y - x| < \epsilon$, the algorithm terminates.
2. Otherwise, because \( y \) is the closest point to \( x \) in \( S \), \( y \) lies in a hyperplane of \( \text{conv}(S) \). Thus, there exists a subset \( S' \subset S \) of size at most \( d \) such that \( y \in \text{conv}(S') \). (Recall \( d \) is the dimension).

Consider \( z = x + \delta \frac{x-y}{|x-y|} \). Notice, \( z \in Q_k \) because \( B(x, \delta) \in Q_k \). Use IRM to find an integral \( z' \in E_k \), such that

\[
<z, x - y> \leq <z', x - y>.
\]

3. Update \( S := S' \cup \{z'\} \); and repeat.

To show that the algorithm terminates in polynomial time, we show that after each iteration, the distance \(|x - y|\) is reduced by at least a constant factor. To prove this, let \( y' \) be the point in the interval \([z', y]\) that is closest to \( x \). We will prove the following.

\[\text{Claim 5.1} \quad \text{There exists } 0 < \gamma < 1 \text{ that depends on } D \text{ and } \delta \text{ such that } |x - y'| < (1 - \gamma)|x - y|\]

**Proof:** Let \( t \) be the point in the interval \((z', y)\) such that \(<t - z, x - y> = 0\). Because \(<z, x - y> \leq <z', x - y>\), such a \( t \) exists. See Figure 1.

Now,

\[
\frac{|x - y|^2}{|x - y'|^2} = \frac{|t - y|^2}{|t - z|^2} = \frac{|t - z|^2 + (|x - z| + |x - y|)^2}{|t - z|^2} \geq \frac{|t - z|^2 + \delta^2}{|t - z|^2}
\]

We have \(|t - z| \leq |z' - z|\). Furthermore, because the diameter of \( E_k \) is \( D \), \(|z' - z| \leq D\). Thus, \(|t - z| \leq D\).

Hence, we obtain

\[
\frac{|x - y|^2}{|x - y'|^2} \geq \frac{D^2 + \delta^2}{D^2}
\]

Thus, there exists \( 0 < \gamma < 1 \), depending on \( D \) and \( \delta \) such that

\[
|x - y'| < (1 - \gamma)|x - y|,
\]
which is what we need to prove.

The claim above shows that after each iteration the distance between $x$ and $y$ is reduced by at least a factor of $(1 - \gamma)$. Consider $K = \frac{\ln(D/\epsilon)}{\gamma}$, we have

$$D(1 - \gamma)^K \leq \epsilon,$$

Thus, after at most $K$ iterations, the algorithm will terminate.