Endogenous Supply of Fiat Money

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Abstract

We consider whether reputation concerns can discipline the behavior of a long-lived self-interested agent who has a monopoly over the provision of fiat money. We obtain that when this agent can commit to a choice of money supply, there is a monetary equilibrium where it never overissues. We show, however, that such equilibria do not exist when there is no commitment. This happens because the incentives this agent has to maintain a reputation for providing valuable currency disappear once its reputation is high enough. More generally, we prove that in the absence of commitment overissue happens infinitely often in any monetary equilibrium. We conclude by showing that imperfect memory can restore the positive result with commitment.

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1 Introduction

Frictions in trade are necessary if money is to be valued as a medium of exchange. The standard approach to model this is to assume that trade is anonymous and decentralized and no record keeping is possible. Under these assumptions, a large body of work has shown that money is essential when its supply is exogenous. However, if the amount of money in circulation in an economy is determined by self-interested agents, the absence of record keeping can lead to the so-called “dynamic inconsistency” problem: if money has value, any agent with the ability to print money faces a temptation to overissue, as any deviation from a pre-specified plan of action is likely to go unnoticed. In other words, money may not be feasible if its supply is endogenous.

The way found by the literature to deal with the “dynamic inconsistency” problem, see Berentsen [5], Cavalcanti et. al. [6], Cavalcanti and Wallace [7], and Martin and Schreft [18] for example, is to assume a form of partial record keeping: the behavior of note issuers can be publicly monitored. This solution has two shortcomings. First, it leaves open the question of how this type of partial record keeping arises in the first place. Second, it is not robust to the introduction of monitoring costs, no matter how small, because of a free rider problem. Indeed, the success any technology that allows note issuers to be monitored has in disciplining their behavior depends on how many agents use it. If only a small number of agents do so, the punishment note issuers face if they overissue is insufficient to induce good behavior. However, if a large number of agents uses this technology, its effectiveness is not affected when a single agent stops using it. Therefore, if all agents have to pay a cost, no matter how small, to use it, there can be no situation where a large group of agents uses this technology in the first place. Otherwise, any such agent would have an incentive to stop using it and free ride on the social benefit it brings.

In this article we address the feasibility of fiat money when it is issued by a single self-interested agent, the bank, and its choice of money supply cannot be observed by the other agents in the economy. We do so in an environment where trade is decentralized and agents are anonymous and have heterogeneous preferences, so that there is a natural role for money. The absence of public monitoring means that agents can only learn about the bank’s decisions from their private experience, i.e., information is decentralized. As a consequence, the monitoring of the bank’s behavior is private and imperfect. This assumption about information is a natural one in an economy with decentralized trade.

\footnote{An exception is Monnet [20], who establishes the feasibility of private fiat money without monitoring. His result, however, depends critically on the particular matching process he considers and on the assumption that money is costly to produce.}
The starting point of our analysis is a simple version of the model introduced in Kiyotaki and Wright [14], modified in a number of ways. First, as indicated above, the money supply is privately determined by the bank in each period. Moreover, the bank can either be patient or impatient, and this is also its private information. Second, the other agents in the economy can now decide between staying in autarky or entering the market and transacting with the help of money. The bank’s revenue from money issue in any period is proportional to how much new currency it prints and to the measure of agents who choose the market at that point in time. In particular, holding everything else constant, this revenue is higher if the bank overissues. Finally, autarky is always better than the market if money is always overissued, which happens when the bank is impatient, but the opposite is true when overissue never takes place.

Since the market is always worse than autarky when the bank is impatient, the patient bank faces a trade-off between short-run gains from overissue and long-run losses due to a decrease in its reputation for providing valuable currency. Indeed, if it overissues, the agents in the economy become more convinced that the bank they face is impatient, leading to a smaller revenue from money issue in future periods. The idea that reputation concerns may help solve the “dynamic inconsistency” problem is not new. Klein [15] considers an environment where such trade-off is present. In his model, however, this trade-off is assumed rather than derived, and it turns out that this has important consequences.

Notice that in general the bank’s choice of money supply should affect both the frequency of trade meetings (the extensive margin) and the terms of trade in such meetings (the intensive margin). However, since in our environment money and goods are indivisible and there is an unit upper bound on money holdings, the only margin that is affected by the bank’s decision is the extensive one. This simplifies the analysis considerably, but preserves the trade-off between reputation and short-run gains from overissue.

We first consider the case where the bank’s choice of money supply in the first period is binding. We show that in this case there is an equilibrium where the patient bank never overissues as long as it is sufficiently patient. The intuition for this result is simple. If the patient bank indeed never overissues, its reputation increases over time, which implies a steady stream of revenue from money issue. If, instead, the patient bank deviates and always overissues, its revenue from money issue increases in the short-run as a result of this. However, its reputation for being patient disappears over time, implying that its revenue from money issue decreases to zero in the long-run. If the patient bank cares enough about the future, this is sufficient to discourage it from deviating.

We then consider the no-commitment case, where the bank may change its behavior at any point in time. In this case, a policy for the patient bank where it never overissues is not time-
consistent. Indeed, if the patient bank were never to overissue, its reputation for being patient, and thus providing valuable currency, would increase over time. Eventually a point would be reached where all agents in the market are so convinced that the bank they face is patient that any negative experience is attributed to bad luck. At this stage, the patient bank would rather overissue. The cost of doing so, a reduction in future revenue from money issue due to a decrease in its reputation, is almost zero, while the immediate benefit is substantial. Put differently, the “reputational” cost of overissue eventually becomes negligible, at which point the patient bank has a profitable deviation.

In light of this negative result, a natural question to ask is what type of equilibria are possible in the no-commitment case. For instance, is it possible to have a monetary equilibrium where the net gain of choosing the market is bounded away from zero when the bank is patient? We show that the same logic that rules out the no-overissue equilibrium also rules out these other equilibria. A consequence of this result is that the patient bank must overissue infinitely many times in any monetary equilibrium.

The discussion so far suggests that a monetary equilibrium with no overissue may become feasible in an environment where the “reputational” cost of overissue is bounded away from zero. Motivated by this, we modify the no-commitment case by assuming that in every period a fraction of the population becomes uninformed. We show that with this form of imperfect memory, an equilibrium where the patient bank never overissues is possible. The reason is that now the patient bank has always an incentive to look after its reputation: any time it overissues, the negative impact on its reputation is non-negligible.

Besides the literature on private money, this work also belongs to the literature that looks at reputation as a separation device. A related paper in this literature is Mailath and Samuelson [16], who consider an environment where monitoring is private and imperfect. A crucial difference is that in our environment the agents have an outside option, staying in autarky. In this regard, see Ely and Välimäki [10], who consider a model of reputation where an outside option is present. See also Moav and Neeman [19], who study the interplay between memory and reputation.

The basic setup is developed in the next section and equilibrium is defined in Section 3. The full-commitment case, the case where the bank’s choice of money supply in period one is binding, is considered in Section 4. The no-commitment case is considered in Section 5. The modification of the no-commitment case to include imperfect memory is analyzed in Section 6. Section 7 concludes and several appendices collect details and proofs that are omitted from the main text.

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2Mailath and Samuelson [17] discuss how this approach differs from the more standard approach to reputation that uses “Stackelberg” types (reputation as pooling device).
2 Basic Setup

Time is discrete and indexed by \( t \). The economy has one large infinitely lived agent that we call the bank. Its discount factor \( \delta \) is either zero or \( \delta_p > 0 \). In the first case we say the bank is \textit{impatient}, while in the second one we say the bank is \textit{patient}. The value of \( \delta \), however, is known to the bank only. The economy is also populated by a large number of small infinitely lived agents that we describe in the paragraphs that follow. For simplicity, we refer to these small agents as \textit{agents} only.

The economy starts in \( t = 1 \) with a mass one of agents, all with the same prior belief \( \theta_0 \in (0, 1) \) that the bank is patient. Moreover, in every \( t > 1 \) each agent born in the previous period gives birth to another agent, who inherits his parent’s private history. As we are going to see later on, an agent’s private history determines his belief about the bank’s type. Hence, this last assumption implies that any agent born after \( t = 1 \) starts with the same belief about \( \delta \) as his parent. An agent in his first period of life is said to be \textit{newly born}.

All agents have the same discount factor \( \beta \in (0, 1) \). They also have a type that is determined when they first enter the economy. There are \( K > 2 \) of these types, one for each of the \( K \) types of goods that can be produced in the economy. The probability that a newly born agent is of the type \( k \in \{1, \ldots, K\} \) is the same in every period, \( 1/K \). Agents of type \( k \) can only consume a type \( k \) good, their so-called preferred good.

Production works as follows. All newly born agents receive a non-perishable endowment and make a once and for all decision between moving to autarky or entering the market. In autarky, an agent uses his endowment as an input to a production technology. In each period there is one production possibility, and each good produced yields utility \( a \). In the market, an agent uses his endowment in the production of indivisible and perishable goods. An agent of type \( k \) can only produce, at a cost \( c \) per unit, a good of type \( k + 1 \mod K \), his so-called endowment good. Any agent in the market can hold at most one unit of either goods or money at any point in time.

The bank derives utility from the consumption of all \( K \) goods, but cannot produce any of them. It has, however, the technology to print indivisible units of fiat money. These units provide no direct benefit, but can be offered in exchange for goods. More precisely, each newly born agent who enters the market is approached by the bank with a certain probability \( m \), in which case he receives one unit of fiat money in exchange for one unit of his endowment good. The value of \( m \) is restricted to \( \{m_L, m_H\} \), with \( \frac{1}{2} \leq m_L < m_H < 1 \), and is determined by the bank in each period. No agent observes this choice. If \( \mu \) is the measure of newly born agents who enter the market in a given period, the bank’s flow payoff from choosing \( m \) in this period is \( (\mu + \kappa)m \), where \( \kappa \) is...
infinitesimal. Hence, it is myopically optimal for the bank to choose $m_H$ even when the measure of agents who enter the market is zero. We discuss what happens when $\kappa = 0$ in the next section. From now on we say that the bank overissues when it chooses $m_H$.

The market is organized as follows. There are $K$ sectors, each one specialized in the exchange of one of the $K$ available goods. Agents can identify sectors, but inside each sector they are randomly and anonymously matched in pairs. Since $K > 2$, there are no double coincidence of wants meetings. An agent, however, can trade his endowment good for money and use money to buy his preferred good. If an agent wants money, he goes to the sector that trades his endowment good and searches for an agent with money. If he has money, he goes to the sector that trades his preferred good and searches for an agent who can produce it. When a single coincidence of wants meeting takes place, the buyer transfers his money to the seller, and the latter produces one unit of his endowment good for the buyer, who consumes it to obtain utility $u > c$. Any agent in the market faces one meeting per period.

Notice that we take the behavior of the agents and the bank in the market as given. It is possible, in a natural way, to model the market environment itself as a game involving the agents and the bank. This game has an equilibrium where the agents always exchange their endowment for one unit of money if approached by the bank and, as long as their discount factor is close enough to one, their behavior in the market is as described.

An implicit assumption in the above description of the market is that there is a positive measure of agents in it at any point in time. Since once in the market an agent does not leave it, a necessary and sufficient condition for this is that a positive measure of agents enters the market in period one. When the measure of agents in the market is zero, i.e., the market is “empty”, money does not circulate and the market flow payoff is zero.

Suppose that in period $t$ the choice of $m$ by the bank is $m_t$ and the fraction of agents in the market with money is $\eta_t$. When the market is empty in $t$, $\eta_t = 0$. When the market is not empty in $t$, $\eta_t$ depends on $m_t$ and on the previous choices of $m$ by the bank. In this section, however, we treat the sequences $\{m_t\}$ and $\{\eta_t\}$ as being independent of each other. Notice, though, that if $\eta_t > 0$, then it must be that $\eta_k > 0$ for all $k > t$. Moreover, when $\eta_t > 0$, it must be that $\eta_t \in [m_L, m_H] \subseteq [\frac{1}{2}, 1)$, since the bank’s choice of $m$ is constrained to $\{m_L, m_H\}$.

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3This market structure is adopted for simplicity. It implies that any single coincidence of wants meeting involves a buyer with money and a seller without money, so that trade always occurs at such meetings.

4A more natural assumption to make is that any agent in the market faces $n > 1$ meetings per period, where $n$ is fixed. All the results we obtain in this article remain true under this alternative specification of the market environment. We don’t pursue it since it brings no new insights and adds to the algebra and the notation.
Let $w_{i,t}$ denote the expected lifetime payoff for an agent in the market in period $t$ with $i \in \{0, 1\}$ units of money right before period $t$’s market meeting. Then, if $\eta_t > 0$,

\begin{align*}
w_{1,t} &= \eta_t \beta w_{1,t+1} + (1 - \eta_t)[u + \beta w_{0,t+1}] \\
w_{0,t} &= \eta_t[\beta w_{1,t+1} - c] + (1 - \eta_t)\beta w_{0,t+1}.
\end{align*}

(1)

Observe that an agent with money right before his market meeting in $t$ has probability $\eta_t$ of meeting another agent with money, in which case no trade occurs. With probability $1 - \eta_t$ he meets an agent without money, in which case trade occurs and he obtains utility $u$. A similar interpretation holds for the second equation.\textsuperscript{5}

We can rewrite (1) as

\begin{equation}
w_t = \beta B(\eta_t)w_{t+1} + b(\eta_t) \quad \text{for all} \ t \in \mathbb{N} \ \text{such that} \ \eta_t > 0,
\end{equation}

where

\begin{align*}
B(\eta) &= \begin{pmatrix} \eta & 1 - \eta \\ \eta & 1 - \eta \end{pmatrix}, \ w_t = \begin{pmatrix} w_{1,t} \\ w_{0,t} \end{pmatrix}, \ \text{and} \ b(\eta) &= \begin{pmatrix} (1 - \eta)u \\ -\eta c \end{pmatrix}.
\end{align*}

(2)

Solving this system of equations recursively, we obtain that if the market is not empty in $t$, then

\begin{equation}
w_t = b(\eta_t) + \sum_{\tau = 1}^{\infty} \beta^\tau B(\eta_t) \cdots B(\eta_{t+\tau-1})b(\eta_{t+\tau}).
\end{equation}

(3)

If, on the other hand, the market is empty in $t$, then $w_t = \beta t w_t$, where $t$ is the first period when the market is not empty and $t = +\infty$ if the market is always empty.

Let $v_t$ be the lifetime expected reward from entering the market in period $t$ and $a(m)$ be the $2 \times 1$ row vector $(m \ 1 - m)$. Assume, without loss of generality, that a newly born agent who decides to enter the market does not discount the time between this decision and his first meeting in the market. Then,

\begin{equation}
v_t = m_t(w_{1,t} - c) + (1 - m_t)w_{0,t} = a(m_t)w_t - m_t c.
\end{equation}

(4)

We denote the dependence of $w_{i,t}$, $w_t$, and $v_t$ on the sequence $\{\eta_{t+k-1}\}_{k=1}^{\infty}$ by writing $w_{i,t} = w_{i,t}(\{\eta_{t+k-1}\})$, $w_t = w_t(\{\eta_{t+k-1}\})$, and $v_t = v_t(\{\eta_{t+k-1}\})$.

When the market is not empty in period $t$ and $\eta_k = m_k = \eta \in \{m_L, m_H\}$ for all $k \geq t$, $v_k = v_t$ for all $k \geq t$. In this case, we write $v_t = v(\eta)$.

\textsuperscript{5}Notice that $w_{1,t} - c - w_{0,t} = (1 - \eta_t)(u - c) > 0$. Hence, a newly born agent who enters the market is always willing to accept one unit of money from the bank in exchange for his endowment good if he knows that the market is not empty.
Notice that if $\eta_t > 0$, then $w_{1,t+1} - w_{0,t+1} = (1 - \eta_{t+1})(u - c) + c$, since $\eta_{t+1} > 0$ as well.
Therefore,
\[
\frac{d v_t}{d \eta_t} = \beta(w_{1,t+1}^t - w_{0,t+1}^t) - c - m_t(u - c) < (1 - m_t - \eta_{t+1})(u - c) \leq 0,
\]
as $m_t$ and $\eta_{t+1}$ are bounded below by $1/2$. Consequently, when $\eta_t > 0$, $v_t$ is a strictly decreasing function of $\eta_k$ with $k \geq t$.

Consider now the particular case where $m_t = \eta_t = m \in \{m_L, m_H\}$ for all $t \in \mathbb{N}$ and denote by $w_i(m)$ the expected lifetime utility from entering the market with $i$ units of money. Since $B(\eta)^2 = B(\eta)$, it is straightforward to see from (3) that
\[
\begin{align*}
w_1(m) &= (1 - \beta)^{-1}m(1 - m)(u - c) + (1 - m)u - m(1 - m)(u - c) \\
w_0(m) &= (1 - \beta)^{-1}m(1 - m)(u - c) - mc - m(1 - m)(u - c)
\end{align*}
\]
Hence, $v(m) = (1 - \beta)^{-1}m(1 - m)(u - c) - mc$ is the expected lifetime payoff from choosing the market in this case. Let $v_A = (1 - \beta)^{-1}a$ denote the lifetime expected payoff from choosing autarky.

Assumption 1. $v(m_H) < v_A < v(m_L)$.

Observe that the impatient bank, being myopic, always chooses $m_H$, whether it can commit to its period one choice of $m$ or not. Also observe that if a positive measure of agents enters the market in period 1 and the bank’s choice of money supply is the same in every period, then $\eta_t$ is equal to this choice of $m$ for all $t$. Consequently, the above assumption implies that if a positive measure of agents enters the market in the first period, then: (i) the market is always worse than autarky when the bank is impatient; (ii) the market is always better than autarky when the bank is patient and chooses $m_L$ in every period.

Notice that Assumption 1 also implies that there exists $\theta_M$ in $(0,1)$ such that $\theta_M v(m_L) + (1 - \theta_M)v(m_H) = v_A$. Therefore, it is necessary that $\theta_0 \geq \theta_M$, otherwise no agent would ever enter the market. Indeed, we know from above that $v_t$ is a strictly decreasing function of $\eta_k$ for $k \geq t$ when $\eta_t > 0$. Hence, since the market flow payoff is zero when the market is empty, the highest payoff a newly born agent can obtain if he chooses the market is $v(m_L)$.

Assumption 2. $\theta_0 \geq \theta_M$.

3 Equilibrium

Let $\mathcal{H}_t$ denote the set of possible period $t$ histories for the bank, so that $\mathcal{H}_1 = \{\emptyset\}$ and $\mathcal{H}_t = ([0,1] \times \{m_L, m_H\})^{t-1}$ if $t > 1$. For any $t > 1$, the bank’s history is the sequence of its previous
choices of $m$ together with the list of measures of agents who entered the market in the periods preceding $t$. A strategy for the bank is then a sequence $M = \{M_t\}$ of contingent plans, where $M_t : \{0, \delta_p\} \times \mathcal{H}_t \rightarrow [0, 1]$ is the Borel measurable function mapping the bank's type and period $t$ history into the probability that it chooses $m_L$ in period $t$. Since the impatient bank always overissues, we can restrict attention to strategies $M = \{M_t\}$ such that $M_t(0, \cdot) \equiv 0$ for all $t \in \mathbb{N}$. We refer to the sequence $M(\delta_p) = \{M_t(\delta_p, \cdot)\}$ as the patient bank's strategy.

When making his market-autarky decision, the only piece of information a newly born agent has is the private history he inherits from his parent. Let $H_t$, with typical element $h^t$, denote the set of all possible histories for an agent born in $t$. By assumption, $H_1 = [0, 1]$, the set of possible prior beliefs that the bank is patient. We describe $H_t$ for $t > 1$ in the next two paragraphs. A strategy or decision rule for an agent born in $t$ is then a Borel measurable function $d_t : H_t \rightarrow [0, 1]$, where $d_t(h^t)$ is the probability that he chooses the market given a private history $h^t$.

An agent’s history in his first period of life is his decision together with his subsequent experience in this period. If he chooses autarky, he observes nothing ($\emptyset$). If, instead, he goes to the market, his experience consists of: (i) how many units of money he receives from the bank; (ii) the money holdings of his partner and the terms of trade in his market meeting, if the market is not empty.

Define a family to be the collection of all agents whose genealogy can be traced back to a particular agent born in $t = 1$. Then, the history an agent born in $t > 1$ inherits is the collection of first period histories of the generation $k \leq t - 1$ members of his family.

Now observe that since money and goods are indivisible and there is an unit upper bound on money holdings, trade is always one-to-one when it occurs. Hence, if a newly born agent chooses the market, and the market is not empty, the number $j \in \{0, 1\}$ of market meetings where his partner carries one unit of money summarizes the information about the bank’s type that this agent gathers in his first period of life. Let $\Pi = \{A, \emptyset\} \cup \{M, \{0, 1\} \times \{e, 0, 1\}\}$, where $A$ denotes the event that autarky is chosen, $M$ denotes the event that the market is chosen, and $e$ denotes the event that the market is empty. We can then conclude that $H_t = [0, 1] \times \Pi^{t-1}$ for $t > 1$. In what follows, we denote an arbitrary element of $\Pi$ by $\pi = (d, \omega)$.

Identify the set of families with the unit interval and let $\Delta_t$ denote the set of Borel measurable functions from $H_t$ into $[0, 1]$. Loosely speaking, a strategy profile for the agents is an equivalence class of sequences $\tau = \{\tau_t\}$, where $\tau_t$ maps $[0, 1]$ into $\Delta_t$ and two sequences $\tau_1 = \{\tau_{1,t}\}$ and $\tau_2 = \{\tau_{2,t}\}$ are considered to be the same if for all $t \in \mathbb{N}$ the functions $\tau_{1,t}$ and $\tau_{2,t}$ differ on a set of measure zero in $[0, 1]$. We interpret $\tau_t(i) \in \Delta_t$ as the strategy of the generation $t$ member of the family labeled by $i \in [0, 1]$. The details can be found in Appendix A.
Suppose the bank’s strategy \( M \) is such that \( M(\delta_p) \) is pure and let \( \tau = \{ \tau_t \} \) be a strategy profile for the agents. Since there is no aggregate uncertainty, the measure \( \gamma_1 \) of agents who enter the market in period 1 is deterministic. It is also independent of the bank’s type. Denote by \( m_1 \) the bank’s choice of \( m \) in period 1. It is deterministic by assumption. Together with \( \tau_1 \), it induces a probability measure \( \lambda_2 \) over the Borel sets of \( H_2 \) such that \( \lambda_2(D) \) is the fraction of agents born in \( t = 2 \) with private histories in \( D \subseteq H_2 \). Note that \( \lambda_2 \) is a function of the bank’s type.

Let \( m_2 \) denote the bank’s choice of \( m \) in period 2. If the bank is impatient, \( m_2 = m_H \). If the bank is patient, \( m_2 \) is a deterministic function of \( (\gamma_1, m_1) \). Once more because there is no aggregate uncertainty, the pair \( (\lambda_2, \tau_2) \) completely determines the measure \( \gamma_2 \) of agents who enter the market in \( t = 2 \). Unlike \( \gamma_1, \gamma_2 \) may be a function of the bank’s type. To finish, observe that \( m_2 \) together with \( \lambda_2 \) determine a Borel probability measure \( \lambda_3 \) over \( H_3 \) such that \( \lambda_3(D) \) is the fraction of agents born in \( t = 3 \) that have private histories in \( D \subseteq H_3 \). Like \( \lambda_2, \lambda_3 \) depends on the bank’s type.

Continuing with this process, we obtain sequences \( \{m_t(M, \tau, \delta)\} \) and \( \{\gamma_t(M, \tau, \delta)\} \) such that if \( \delta \) is the bank’s discount factor, then \( m_t(M, \tau, \delta) \) is the bank’s choice of \( m \) in \( t \) and \( \gamma_t(M, \tau, \delta) \) is the fraction of agents born in \( t \) who enter the market. Notice that \( m_t(M, \tau, 0) \equiv m_H \). We also obtain a sequence \( \{\lambda_t(M, \tau, \delta)\} \) such that: (i) \( \lambda_t(M, \tau, \delta) \equiv \delta_{\{0\}} \), the Dirac probability measure over \( H_1 \) with mass one on \( \{0\} \); (ii) \( \lambda_t(M, \tau, \delta) \) is the Borel probability measure over \( H_t \) such that \( \lambda_t(M, \tau, \delta)(D) = \lambda_t(D|M, \tau, \delta) \) is the fraction of agents born in \( t \) with private histories in \( D \subseteq H_t \) when the bank’s discount factor is \( \delta \). The important point is that if the patient bank uses a pure strategy, then both its behavior over time and the aggregate behavior of the agents over time are deterministic.

Let \( m_t \) be the patient bank’s choice of \( m \) in \( t \) and \( \mu_t(\nu_t) \) be the measure of newly born agents who enter the market in \( t \) when the bank is patient (impatient). Notice the change in notation. If the bank is impatient, the fraction of agents in the market in period \( t \) that have money is either zero, when the market is empty, or \( m_H \). On the other hand, if the bank is patient, this fraction, that we denote by \( \alpha_t \), can change over time even when the market is not empty. Precisely, if \( \sum_{\tau=1}^t \mu_\tau = 0 \), then \( \alpha_t = 0 \), while if \( \sum_{\tau=1}^t \mu_\tau > 0 \), then

\[
\alpha_t = \frac{\sum_{\tau=1}^t \mu_\tau m_\tau}{\sum_{\tau=1}^t \mu_\tau}.
\]

Moreover, there may be periods when the market is empty if the bank is of one of type, but not of the other. As a consequence, the belief an agent born in \( t \) has that the bank is patient depends not only on his private history \( h^t \in H_t \), but also on the sequences \( \{\mu_t\}, \{\nu_t\}, \{m_t\} \). Denote this belief by \( \theta(h^t; \{\mu_t\}, \{\nu_t\}, \{m_t\}) \). When there is no risk of confusion, we omit its dependence on the sequences \( \{\mu_t\}, \{\nu_t\}, \{m_t\} \).
Let $\Omega = \{0, 1\} \times \{e, 0, 1\}$ and define $X_t(\delta; \{\mu_t\}, \{\nu_t\}, \{m_t\})$ to be the random variable on $\Omega$ such that if $i, j \in \{0, 1\}$, then:

$$\Pr\{X_t(\delta; \{\mu_t\}, \{\nu_t\}, \{m_t\}) = (i, e)\} = \begin{cases} m_H^i(1 - m_H)^{1-i} & \text{if } \sum_{\tau=1}^t \nu_\tau = 0 \\ 0 & \text{otherwise} \end{cases} ;$$

$$\Pr\{X_t(\delta; \{\mu_t\}, \{\nu_t\}, \{m_t\}) = (i, j)\} = \begin{cases} m_H^{i+j} (1 - m_H)^{2-(i+j)} & \text{if } \sum_{\tau=1}^t \nu_\tau > 0 \\ 0 & \text{otherwise} \end{cases} ;$$

$$\Pr\{X_t(\delta_p; \{\mu_t\}, \{\nu_t\}, \{m_t\}) = (i, e)\} = \begin{cases} m_H^i (1 - m_H)^{1-i} & \text{if } \sum_{\tau=1}^t \mu_\tau = 0 \\ 0 & \text{otherwise} \end{cases} ;$$

$$\Pr\{X_t(\delta_p; \{\mu_t\}, \{\nu_t\}, \{m_t\}) = (i, j)\} = \begin{cases} m_H^i (1 - m_H)^{1-i} \alpha_t^j (1 - \alpha_t)^{1-j} & \text{if } \sum_{\tau=1}^t \mu_\tau > 0 \\ 0 & \text{otherwise} \end{cases} .$$

Observe that $m_H^i(1 - m_H)^{1-i}$ is the probability an agent born in $t$ has of receiving $i$ units of money from the patient bank if he enters the market, and that if $\alpha_t > 0$, then $\alpha_t^j (1 - \alpha_t)^{1-j}$ is the probability he has of meeting $j$ agents with one unit of money in the market in the same period. When the bank is impatient, these probabilities are $m_H^i(1 - m_H)^{1-i}$ and $m_H^j(1 - m_H)^{1-j}$, respectively. We also omit the dependence of $X_t$ on $\{\mu_t\}, \{\nu_t\},$ and $\{m_t\}$ when there is no chance of confusion.

The belief $\theta(h^t)$ can then be computed by the following recursion. In the first period, $\theta(h^1) = h^1$, the prior belief that the bank is patient. Now fix $t \geq 1$ and assume that $\theta(h^t)$ is defined for all $h^t \in H_t$. Moreover, let $h^{t+1} = (h^t, \pi)$, with $\pi = (d, \omega) \in \Pi$, be an element of $H_{t+1}$. If $d = A$, then $\theta(h^t, \pi) = \theta(h^t)$. If $d = M$, so that $\omega \in \Omega$, then

$$\theta(h^t, \pi) = \frac{\theta(h^t) \Pr\{X_t(\delta_p) = \omega\}}{\theta(h^t) \Pr\{X_t(\delta_p) = \omega\} + (1 - \theta(h^t)) \Pr\{X_t(0) = \omega\}},$$

(6)

when the denominator is positive. When the denominator is zero, set $\theta(h^t, \pi)$ equal to $\theta(h^t)$.

It is clear that in order to define an equilibrium where the patient bank uses a pure strategy, we have to take into account that: (i) the agents need the sequences $\{\mu_t\}, \{\nu_t\},$ and $\{m_t\}$ to compute the expected payoff from choosing the market, since this payoff depends on the belief that the bank is patient; (ii) the sequences $\{\mu_t\}, \{\nu_t\},$ and $\{m_t\}$ depend on the aggregate behavior of the agents.
(and the bank). Hence, the requirement that correct expectations about \( \{\mu_t\}, \{\nu_t\}, \text{ and } \{m_t\} \) are held needs to be included in such a definition.

**Definition 1.** Let \( M^* \) be a strategy for the bank and \( \tau^* \) be a strategy profile for the agents. Also, let \( \Theta^*: \bigcup_{t=1}^{\infty} H_t \rightarrow [0,1] \) be a belief updating rule for the agents, i.e., \( \Theta^*(h^t) \) is the belief an agent born in \( t \) with history \( h^t \) has that the bank is patient. The list \( \sigma^* = (M^*, \tau^*, \Theta^*, \{\mu^*_t\}, \{\nu^*_t\}, \{m^*_t\}) \) is a (deterministic) equilibrium if:

(a) \( M^*(\delta_p) \) is pure and \( M^*_t(0, \cdot) \equiv 0 \) for all \( t \in \mathbb{N} \);

(b) \( \Theta^*(h) = \theta(h; \{\mu^*_t\}, \{\nu^*_t\}, \{m^*_t\}) \) for all \( h \in \bigcup_{t=1}^{\infty} H_t \);

(c) The agents hold correct expectations about the sequences \( \{m^*_t\}, \{\mu^*_t\}, \text{ and } \{\nu^*_t\} \). In other words, \( m^*_t = m_t(M^*, \tau^*, \delta_p), \mu^*_t = \gamma_t(M^*, \tau^*, \delta_p), \text{ and } \nu^*_t = \gamma_t(M^*, \tau^*, 0) \) for all \( t \in \mathbb{N} \);

(d) The patient bank’s behavior is sequentially rational. In particular,

\[
\{m_t(M^*, \tau^*, \delta_p)\} \in \text{argmax} \left\{ (1 - \delta_p) \sum_{t=1}^{\infty} \delta_t^{t-1} \gamma_t(M^*, \tau^*, \delta_p) m_t : \{m_t\} \in \{m_L, m_H\}^{\infty} \right\} ;
\]

(e) The decision rules of almost all agents are optimal given the belief updating rule \( \Theta \).

When the patient bank uses a mixed strategy, the evolutions of both \( m_t \) and \( \mu_t \) are no longer necessarily deterministic. If this is the case, the belief a newly born agent has about the bank’s type is not enough to determine his lifetime expected payoff from choosing the market. He also needs a (history dependent) conjecture about how \( m_t \) and \( \mu_t \) are going to behave over time starting with his period of birth. This means that the equilibrium concept just introduced needs to be modified if one wants to consider this more general case. One exception is when the bank's behavior fails to be deterministic only off the equilibrium path, in which case the definition given above is appropriate. We restrict attention to deterministic equilibria in this article.

Suppose \( \sigma^* \) is an equilibrium and let \( \mathbb{N}_1(\sigma^*) = \{t : \mu^*_t > 0, \nu^*_t > 0\} \) be the set of periods where, regardless of the bank’s type, a positive measure agents enters the market in this equilibrium. Lemma 1 implies that \( \mathbb{N} \setminus \mathbb{N}_1(\sigma^*) = \mathbb{N}_0(\sigma) = \{t : \mu^*_t = \nu^*_t = 0\} \). One consequence of this result is that the market is never completely informative about the bank’s type in any equilibrium. Its proof is in Appendix B.
Lemma 1. Suppose $\sigma^*$ is an equilibrium. Then, for all $t \in \mathbb{N}$, $\mu_t^* = 0$ if, and only if, $\nu_t^* = 0$.

If $\sigma^*$ is an equilibrium such that $N_1(\sigma^*)$ is empty, we say that $\sigma^*$ is non-monetary. Such an equilibrium exists. Indeed, let $\tau^* = \{\tau_t^*\}$, $M^* = \{M_t^*\}$, and $\Theta^* : \bigcup_{t=1}^\infty H_t \to [0, 1]$ be such that: (i) $\tau_t^*(i) : H_t \to [0, 1]$ is constant and equal to zero for all $i \in [0, 1]$ and $t \in \mathbb{N}$; (ii) $M_t^*(\cdot, \cdot) \equiv 0$ for all $t \in \mathbb{N}$; and (iii) $\Theta^*(h) = h$ for all private histories $h$. Then, $(M^*, \tau^*, \Theta^*, \{\mu_t^*\}, \{\nu_t^*\}, \{m_t^*\})$ with $\mu_t^* \equiv \nu_t^* \equiv 0$ and $m_t^* \equiv m_H$ is a non-monetary equilibrium.

We finish this section with two other preliminary characterization results. The first one is a straightforward consequence of the assumption that the patient bank has always a myopic incentive to overissue. The second result follows from the first and implies that in any monetary equilibrium $\sigma^*$, the set $N_1(\sigma^*)$ must be infinite. The proofs of both lemmas can be found in Appendix B.

Lemma 2. Let $\sigma^*$ be an equilibrium. Then $m_t^* = m_H$ for all $t \in N_0(\sigma^*)$.

Lemma 3. Suppose $\sigma^*$ is an equilibrium. Then, either $N_1(\sigma^*) = \emptyset$ or $N_1(\sigma^*)$ is infinite.

Without the assumption that the bank prefers to overissue even when the measure of agents entering the market is zero, Lemmas 2 and 3 are not true. In particular, a monetary equilibrium where the measure of agents entering the market is zero after a finite number of periods becomes possible. The proof of Lemma 3 shows that all such equilibria must have the property that if $\bar{t}$ is the last period where a positive measure of agents enters the market, then $m_t^* = m_L$ for all $t > \bar{t}$.

The possibility of this type of monetary equilibrium does not alter the substance of our conclusions in Section 5. Moreover, we do not find these equilibria plausible. In particular, they are not robust to the following perturbation: if in any period after $\bar{t}$ a fraction $\epsilon$ of the newly born agents enters the market, the proof of Theorem 2 shows that the patient bank has a profitable deviation, no matter how small $\epsilon$ is.\footnote{The proof of Theorem 2 does not depend on the assumption that agents start with a common prior.}

4 The Full Commitment Case

In this section we assume that the bank can commit to its period 1 choice of $m$, that is, once it chooses the value of $m$ in the first period, it cannot change it afterwards. This is equivalent to using the equilibrium notion introduced in the previous section, but reducing the set of strategies of the bank to $\{M^{LL}, M^{LH}, M^{HL}, M^{HH}\}$, where $M^{kl} = \{M_t^{kl}\}$, with $k, l \in \{L, H\}$, is such that

$$M_t^{kl}(\delta, \cdot) \equiv \begin{cases} I_{(L)}(k) & \text{if } \delta = 0 \\ I_{(L)}(l) & \text{if } \delta = \delta_p \end{cases}$$
and $I_{\{L\}}$ is the indicator function of $\{L\}$. We show that when $\delta_p$ is close enough to 1, an equilibrium where the bank chooses $M^{HL}$ and money circulates exists. Moreover, the sequence $\{\mu_t\}$ is bounded away from zero in this equilibrium, which implies that the subset of the population that transacts with money when the bank is patient does not die out over time.

For this, let $\Theta^c$ be such that: (i) $\Theta^c(h^t) = h^t$ for all $h^t \in H_1$; (ii) If $h^{t+1} = (h^t, d, \omega)$, then $\Theta^c(h^{t+1}) = \Theta^c(h^t)$ when $d = A$,

$$\Theta^c(h^{t+1}) = \Theta^c(h^t) m^i_j (1 - m_L)^{1-i} \left(1 - \Theta^c(h^t) \right) m^j_i (1 - m_H)^{1-i}$$

when $d = M$ and $\omega \in \{0, 1\} \times \{e\}$, and

$$\Theta^c(h^{t+1}) = \Theta^c(h^t) m^{i+j}_L (1 - m_L)^{2-i-j} \left(1 - \Theta^c(h^t) \right) m^{j+i}_H (1 - m_H)^{2-i-j}$$

when $d = M$ and $\omega = (i, j) \in \{0, 1\} \times \{0, 1\}$. Recall that $e$ denotes the event that the market is empty. Now define $\tau^c = \{\tau^c_t\}$ to be such that $\tau^c_t(\cdot) \equiv d^c_t$, where

$$d^c_t(h^t) = \begin{cases} 1 & \text{if } \Theta^c(h^t) \geq \theta_M \\ 0 & \text{if } \Theta^c(h^t) < \theta_M \end{cases} \quad (7)$$

Recall that $\theta_M$ is the value of $\theta$ for which $v(m_L) + (1 - \theta)v(m_H)$ is equal to $v_A$. To finish, let $\mu^c_t = \gamma_t(M^{HL}, \tau^c, \delta_p), \nu^c_t = \gamma_t(M^{HL}, \tau^c, 0), \text{ and } m^c_t \equiv m_L$. Notice that $\mu^c_t = \nu^c_t > 0$.

**Theorem 1.** There exists $\delta \in (0, 1)$ such that $\sigma^c = (M^{HL}, \tau^c, \Theta^c, \{\mu^c_t\}, \{\nu^c_t\}, \{m^c_t\})$ is an equilibrium if $\delta_p \geq \delta$. Moreover, the sequence $\{\mu^c_t\}$ is bounded away from zero.

The following fact is useful in the proof of Theorem 1. If $\{x_t\}$ is a convergent sequence in the real line with limit $x_\infty$, then

$$\lim_{\delta \rightarrow 1-} (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} x_t = x_\infty. \quad (8)$$

**Proof of Theorem 1:** Suppose the bank follows $M^{HL}$. Since under $\tau^c$ a positive measure of agents enters the market in period 1, the fraction of agents in the market with money is constant over time: equal to $m_L$ when the bank is patient and equal to $m_H$ when the bank is impatient. Hence, $\Theta^c(h^t) = \theta(h^t; \{\mu^c_t\}, \{\nu^c_t\}, \{m^c_t\})$. Moreover, expectations are satisfied by construction. Because $v(m_L)$ is the lifetime expected payoff from entering the market when the bank is patient and $v(m_H)$ is the same payoff when the bank is impatient, $d^c_t$ is, for all $t \in \mathbb{N}$, an optimal decision rule for an agent born in this period. Therefore, we only need to show that $m_L$ is the optimal choice
of $m$ by the patient bank to establish that $\sigma^c$ is an equilibrium. For this notice, by [3, Prop. 2], that $\nu^c_t \downarrow 0$ and there exists $\mu > 0$ such that $\mu^c_t \downarrow \mu > 0$. Since $(1 - \delta_p) \sum_{t=1}^{\infty} \delta_p^{t-1}(\mu^c_t + \kappa)m_L$ is the patient bank’s payoff when it chooses $m_L$ and $(1 - \delta_p) \sum_{t=1}^{\infty} \delta_p^{t-1}(\nu^c_t + \kappa)m_H$ is its payoff when it chooses $m_H$, the desired result is a consequence of (8) together with the assumption that $\kappa$ is infinitesimal.

5 The No Commitment Case

By restricting the bank to make a once and for all decision on the value of $m$ in period 1, we rule out any considerations about the time-consistency of its behavior. In this section we investigate what happens when the bank can change its decision of $m$ at the beginning of every period. In what follows, we refer to the belief that the bank is patient as the belief only and interpret the distribution of these beliefs among the newly born agents as the bank’s reputation.

It turns out that for some purposes it is convenient to reinterpret the decision problems of the successive generations of agents in this economy in the following way. Associate to each family a myopic decision maker, the family lawyer, who is now responsible for the market-autarky decisions of the members of this family; i.e., in each $t$ he decides whether the generation $t$ member of the family he represents enters the market or not. Assume that the period $t$ flow payoff and private history of a family lawyer are, respectively, the expected lifetime payoff and private history of the generation $t$ member of his family. In particular, a strategy profile $\tau = \{\tau_t\}$ for the agents is also a strategy profile for the lawyers (and vice-versa): $\tau(i) = \{\tau_t(i)\}$ is the strategy for the lawyer representing the family indexed by $i$. With these assumptions, a family lawyer behaves over time in exactly the same way as the members of his family would behave if they were to make their market-autarky decisions on their own. Notice that the problem faced by the family lawyers is a two-armed bandit where one arm, the autarky, has known payoffs, and the other arm, the market, has (in principle) non-stationary payoffs.

We start by arguing (somewhat informally) that $\sigma^c$ cannot be an equilibrium in the no commitment case. To see why, notice first that autarky is absorbing in $\sigma^c$. Hence, for the generation $t$ member of a family to enter the market, it must be that all previous generations of his family did the same. Moreover, the market is informative about the bank’s type in this equilibrium. Therefore, if the bank is patient, a fraction close to one of the agents who enter the market in $t$ have

\footnote{The setting in [3] is slightly different from the setting considered in this article. It is straightforward to adapt the proof of their Proposition 2 to our environment.}
beliefs very close to one when \( t \) is sufficiently large. In particular, for each \( k \in \mathbb{N} \) there exists \( \hat{t} \in \mathbb{N} \) such that for a fraction close to one of the families whose generation \( \hat{t} \) members choose the market, their next \( k \) generations do the same whether the patient bank chooses \( m_L \) in \( \hat{t} \) or not. In other words, the patient bank’s reputation is so high at this point that its choice of money supply in \( \hat{t} \) has a negligible impact on its revenue from money issue in the next \( k \) periods. Consequently, by choosing \( k \) large enough, the patient bank has a profitable deviation in \( \hat{t} \).

Notice that the above argument relies on the fact that autarky is absorbing in \( \sigma^c \). It is easy to show that if we modify \( \tau^c \) in \( \sigma^c \) to allow the agents to randomize when indifferent between the market and autarky, then \( \sigma^c \) remains an equilibrium in the full commitment case (as long as the sequences \( \{\mu^*_t\} \) and \( \{\nu^*_t\} \) are changed accordingly), but autarky is no longer absorbing. However, the market is still informative about the bank’s type, and so the measure of newly born agents who are indifferent between the market and autarky must converge to zero over time. Indeed, if this is not the case, then there is a positive measure of family lawyers who send an infinite number of members of their families to the market, but who are indifferent between the market and autarky in infinitely many periods, a contradiction. Hence, the reasoning of the previous paragraph is also valid in this case.

Now observe that if \( \sigma^* = (M^*, \tau^*, \Theta^*, \{\mu^*_t\}, \{\nu^*_t\}, \{m^*_t\}) \) is such that \( m^*_t \equiv m_L \), then it must be that \( \Theta^* = \Theta^c \) and \( \mu^*_t > 0 \) for all \( t \in \mathbb{N} \) in order for \( \sigma^* \) to be an equilibrium. The second fact follows from Lemma 2. Lemma 1 now implies that \( \nu^*_t > 0 \) for all \( t \in \mathbb{N} \) as well, and so the market is never empty under \( \sigma^* \). Therefore, it is optimal for a newly born agent to enter the market if his belief is greater than \( \theta_M \) and to stay in autarky if this belief is smaller than \( \theta_M \). In other words, if \( \sigma^* \) is to be an equilibrium, then the only instance where the decision rules for the agents can differ from the ones of \( \sigma^c \) is when \( \theta = \theta_M \). We are then in the case of the last paragraph, and so \( \sigma^* \) cannot be an equilibrium.

To summarize, there is no equilibrium in the no commitment case where the patient bank never overissues. In face of this negative result, it is natural to ask what type of monetary equilibria are possible in this case. The following theorem is a partial answer to this question. Recall that if \( \sigma^* \) is a monetary equilibrium, then \( N_1(\sigma^*) \) is an infinite set. A consequence of Theorem 2, and the main result of this section, is that in any monetary equilibrium the patient bank overissues infinitely many times in \( N_1(\sigma^*) \).

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8This follows from the consistency of Bayes estimates in the multinomial case.
9More generally, we could let the agents follow asymmetric strategies where their market-autarky decisions differ from the ones of \( \tau^c \) when (and only when) they are indifferent between the market and autarky.
10This is a consequence of [2, Lemma 1] together with the consistency of Bayes estimates in the multinomial case.
Theorem 2. Suppose $\sigma^*$ is a monetary equilibrium and let $v^*_{t}$ be the corresponding expected lifetime payoff of entering the market in period $t$ when the bank is patient. Then, $\liminf_{t \in \mathbb{N}_1(\sigma^*)} v^*_{t} = v_A$.

The proof of Theorem 2 is in Appendix C. A rough sketch of it is as follows. Suppose that $\sigma^*$ is a monetary equilibrium. Lemma 3 implies that $\mathbb{N}_1(\sigma^*)$ is infinite, so that $\liminf_{t \in \mathbb{N}_1(\sigma^*)} v^*_{t} \geq v_A$. Now let $\alpha^*_t$ be the fraction of agents with money in the market in period $t$ when the bank is patient and $\sigma^*$ is the equilibrium. Then, $\{\alpha^*_t\}$ cannot converge to $m_H$, since this would imply that $v^*_t$ eventually becomes smaller than $v_A$. In particular, the market is informative about the bank’s type in $\sigma^*$. Suppose then, by contradiction, that $\liminf_{t \in \mathbb{N}_1(\sigma^*)} v^*_{t} > v_A$. In this case, the sequence $\{\mu^*_t\}_{t \in \mathbb{N}_1(\sigma^*)}$ is bounded away from zero. A consequence of this fact is that $m^*_t = m_L$ infinitely many times in $\mathbb{N}_1(\sigma^*)$, otherwise $\{\alpha^*_t\}$ converges to $m_H$. This, in turn implies that there is a profitable deviation for the patient bank. The existence of such a deviation follows from a reasoning similar to the one used to argue that the patient bank has a profitable deviation in $\sigma^*$.

The lack of commitment implies that in any monetary equilibrium, the gain $v^*_t - v_A$ from choosing the market cannot be bounded away from zero in $\mathbb{N}_1(\sigma^*)$. This does not mean that this gain necessarily converges to zero in this set, which would be the case if $\limsup_{t \in \mathbb{N}_1(\sigma^*)} v^*_t = v_A$. What prevents this more negative result is the existence of an outside option for the agents. In periods where the gain from entering the market is small, even small changes in the patient bank’s reputation can have relatively large effects on its revenue from money issue, as the market-autarky decision is very knife-edge. This has the potential of disciplining the patient bank’s behavior to a certain extent. Not enough, however, to prevent the fraction of agents with money in the market from eventually becoming greater than $m_L$ in any monetary equilibrium, which is a consequence of the result that follows.

Corollary 1. Let $\mathbb{N}_1(\sigma^*, m_k) = \{t \in \mathbb{N}_1(\sigma^*) : m^*_t = m_k\}$. Then, $\mathbb{N}_1(\sigma^*, m_H)$ is infinite if $\sigma^*$ is monetary.

Proof: Let $\alpha^*_t$ be the fraction of agents in the market in period $t$ with money when the bank is patient. Since $a(m)B(\eta) = a(\eta)$ for all $m, \eta \in (0, 1)$,

$$v^*_t = a(m^*_t)b(\alpha^*_t) - m^*_tc + \sum_{k=1}^{\infty} \beta^k a(\alpha^*_{t+k})b(\alpha^*_{t+k}).$$

Suppose then, by contradiction, that $\mathbb{N}_1(\sigma^*, m_H)$ is finite. Since $\alpha^*_t$ can only change values when $t \in \mathbb{N}_1(\sigma^*)$, there exists $\alpha \in [m_L, m_H]$ such that $\{\alpha^*_t\}$ converges from below to $\alpha$; i.e., $\alpha^*_t \rightarrow \alpha$ and

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11This goes in the opposite direction of Ely and Välimäki [10], who provide an instance where that the combination of reputation concerns with the presence of an outside option has very negative effects.
there exists \( t_0 \in \mathbb{N} \) such that \( \alpha^*_t \leq \alpha \) for \( t \geq t_0 \). If \( \mathbb{N}_1(\sigma^*, m_H) \) is empty, \( \alpha^*_t \equiv m_L \). In this case, \( v^*_t = v(m_L) > v_A \) for all \( t \in \mathbb{N}_1(\sigma^*) \), a contradiction. Hence, \( \{\alpha^*_t\} \) is non-constant. Let

\[
v^*(\alpha, m_L) = a(m_L)b(\alpha) - m_Lc + \sum_{k=1}^{\infty} \beta_k a(\alpha)b(m_L).
\]

Since \( v^*_t \) is strictly decreasing in \( \alpha^*_t \) for \( k \geq t \), the sequence \( \{v^*_t\}_{t \in \mathbb{N}_1(\sigma^*)} \) is non-constant and converges to \( v^*(\alpha, m_L) \) from below. Moreover, \( v^*_t > v_A \) for all \( t \in \mathbb{N}_1(\sigma^*) \), otherwise \( \mu^*_t = 0 \). Therefore, \( v^*(\alpha, m_L) = \lim_{t \in \mathbb{N}_1(\sigma^*)} v^*_t > v_A \), a contradiction. \qed

## 6 Imperfect Memory

The analysis so far suggests that in order to have an equilibrium where the patient bank never overissues there must be something that prevents its reputation among the newly born agents from increasing too much when it always chooses \( m_L \). Put differently, we need a mechanism that provides the patient bank with the incentive to always invest in its reputation by never choosing \( m_H \).

With this in mind, we modify our environment by assuming that in any period there is a probability \( \rho > 0 \) that a newly born agent does not inherit his parent’s private history. Instead, this agent’s market-autarky decision is based on his prior belief that the bank is patient, that we assume to be \( \theta_0 \). In this way, regardless of what the patient bank does, there is always a positive measure of newly born agents for whom its reputation is not high. The equilibrium notion we use is still the one introduced in Section 3. We only have to change the definition of \( H_t \) for \( t > 1 \) to incorporate the fact that memory is imperfect. Now, when \( t > 1 \), \( H_t = ([0,1] \times \Pi^{t-1}) \cup \{N\} \), where \( N \) denotes the event that no private history is inherited.

Let \( \Theta^*: \bigcup_{t=1}^{\infty} H_t \to [0,1] \) be the belief updating rule given by

\[
\Theta^*(h) = \begin{cases} 
\theta_0 & \text{if } h = N \\
\Theta^c(h) & \text{otherwise}
\end{cases}
\]

where \( \Theta^c \) is the belief updating rule of Section 4, and define \( \tau^* = \{\tau^*_t\} \) to be such that \( \tau^*(\cdot) \equiv d_t^c \), where \( d_t^c \) is given by (7) with \( \Theta^* \) in the place of \( \Theta^c \). Now let \( \{\mu^*_t\}, \{\nu^*_t\}, \) and \( \{m^*_t\} \) be such that \( \mu^*_t = \gamma_t(M^{HL}, \tau^*, \delta_p) \), \( \nu^*_t = \gamma_t(M^{HL}, \tau^*, 0) \), and \( m^*_t \equiv m_L \), where \( M^{HL} \) is the strategy profile for the bank defined in Section 4. The next theorem shows that the introduction of imperfect memory works as desired: it restores the good equilibrium outcome obtained in Section 4 under the assumption of full commitment.
Theorem 3. Suppose that:

(a) \((1 - \rho)m_L(m_H + m_L) > 1;\)
(b) \(\theta_0 m_L^2/(\theta_0 m_L^2 + (1 - \theta_0)m_H^2) < \theta_M.\)

Then, there exists \(\delta \in (0, 1)\) such that \(\sigma^* = (M^{HL}, \tau^*, \Theta^*, \{\mu^*_t\}, \{\nu^*_t\}, \{m^*_t\})\) is an equilibrium in the no commitment case if \(\delta_p \geq \delta.\)

Notice that since \(m_L^2 < m_H^2,\) condition (b) above is satisfied if \(\theta_0\) is sufficiently close to \(\theta_M.\) Also notice that there are values of \(m_L\) and \(m_H, m_L = 2/3\) and \(m_H = 7/8\) for example, for which condition (a) is satisfied. This condition cannot be satisfied when \(m_L = 1/2,\) though. At this stage it is worthwhile to mention that if we were to assume that each agent in the market faces a fixed number \(n > 1\) of meetings per period, see footnote 4, then less stringent conditions on \(m_L\) and \(m_H,\) as well as on the prior \(\theta_0,\) are necessary in order for Theorem 3 to hold. This is intuitive, given that the greater is the number of per-period meetings in the market, the larger is the negative impact on the patient bank’s reputation after a deviation.

Proof of Theorem 3: We know, from Theorem 1, that (i) \(\Theta^*(\cdot) \equiv \theta(\cdot; \{\mu^*_t\}, \{\nu^*_t\}, \{m^*_t\}),\) (ii) expectations are (trivially) satisfied, and (iii) agents behave optimally given the bank’s behavior and the behavior of the other agents. Hence, we just need to check that the patient bank has no profitable one-shot deviation if \(\delta_p\) is sufficiently close to one. We ignore \(\kappa,\) as it is infinitesimal.

Fix a period \(k\) and suppose the bank is patient. Given any history \(h \in \mathcal{H}_k\) for the bank, we can compute the distribution of beliefs across the agents born in \(k\) induced by \(h.\) Denote this distribution by \(\lambda_k(h)\) and let \(D_k(\theta)\) be the bank’s gain from a one-shot deviation in \(k\) if all agents born in this period have the same belief \(\theta.\) The bank’s gain from a one-shot deviation in \(k,\) as a function of \(h,\) is then given by

\[
D_k(h) = \int D_k(\theta)\lambda_k(d\theta|h).
\]

Because memory is imperfect, for each \(h \in \mathcal{H}_k\) there exists a distribution \(\hat{\lambda}_k(h)\) such that

\[
D_k(h) = (1 - \rho) \int D_k(\theta)\hat{\lambda}_k(d\theta|h) + \rho D_k(\theta_0).
\]

Consider an agent born in \(k\) with belief \(\theta.\) For each \(t \geq 1,\) denote by \(\xi_{t,k}(m, \theta)\) the probability that the generation \(k + t\) member of this agent’s family enters the market in \(k + t\) if (i) the patient bank chooses \(m \in \{m_L, m_H\}\) in \(k\) and \(m_L\) from \(k + 1\) on, and (ii) private histories are always transmitted from one generation to the next. Now let \(\phi_{t,k}(m, \theta)\) be the same probability when
private histories can fail to be transmitted. Then,

$$\phi_{t,k}(m, \theta) = (1 - \rho)^t \xi_{t,k}(m, \theta) + \sum_{\tau=1}^{t-1} \rho(1 - \rho)^\tau \xi_{t,k+(\tau-1)-\tau}(m_L, \theta_0) + \rho. \quad (10)$$

Indeed, with probability $(1 - \rho)^t$ private histories are always passed from one generation to the next, while with probability $\rho(1 - \rho)^\tau$ the last period before $k + t$ where private histories are not passed from one generation to the following is $k + (t - 1) - \tau$, with $\tau \in \{0, \ldots, t - 1\}$.

Observe $\xi_{t,k}$ is independent of $k$, since under $\sigma^*$ all newly born agents follow the same cutoff belief strategy and the market environment is stationary. Therefore, the same is true for the probabilities $\phi_{t,k}$. Because of this, we omit $k$ from both $\xi_{t,k}$ and $\phi_{t,k}$ in what follows. Let $\zeta$ be the indicator function of $[\theta_M, 1]$. Then, for all $k \in \mathbb{N}$,

$$D_k(\theta) = (m_H - m_L)\zeta(\theta) + m_L \sum_{t=1}^{\infty} \delta^t_p [\phi_t(m_H, \theta) - \phi_t(m_L, \theta)]$$

$$= (m_H - m_L)\zeta(\theta) + m_L \sum_{t=1}^{\infty} \delta^t_p (1 - \rho)^t [\xi_t(m_H, \theta) - \xi_t(m_L, \theta)],$$

where the second equality follows from (10). Now observe that for each $t \in \mathbb{N}$, $\xi_t(m_L, \theta) \geq \xi_t(m_H, \theta)$ for all $\theta \in [0, 1)$ and $\xi_t(m_L, 1) = \xi_t(m_H, 1)$. Hence, $D_k(\theta) \leq D_k(1)$ for all $k \in \mathbb{N}$ and $\theta \in [0, 1)$, which implies that

$$D_k(h) \leq (1 - \rho)D_k(1) + \rho D_k(\theta_0) = D_k$$

for each $k \in \mathbb{N}$ and all $h \in \mathcal{H}_k$. It is then enough to show that $D_k$ is negative for all $k \in \mathbb{N}$.

In Appendix D we prove that $\xi_t(m_L, \theta_0) - \xi_t(m_H, \theta_0) \leq \xi_{t+1}(m_L, \theta_0) - \xi_{t+1}(m_H, \theta_0)$ for all $t \in \mathbb{N}$; i.e., the patient bank’s loss of revenue from money issue due to a worse reputation increases over time. Hence,

$$D_k = \rho \left( (m_H - m_L) + m_L \sum_{t=1}^{\infty} \delta^t_p (1 - \rho)^t [\xi_t(m_H, \theta_0) - \xi_t(m_L, \theta_0)] \right) + (1 - \rho)(m_H - m_L)$$

$$\leq (m_H - m_L) - m_L \frac{\delta_p \rho (1 - \rho)}{1 - \delta_p (1 - \rho)} \Delta \xi_1,$$

where $\Delta \xi_1 = \xi_1(m_L, \theta_0) - \xi_1(m_H, \theta_0)$. Since $m_L(1 - m_L) > m_H(1 - m_H)$, condition (b) implies that a newly born agent who enters the market with belief $\theta_0$ ends his first period of life with an updated belief less than $\theta_M$ only when he receives money from the bank and faces an agent with money in his first market meeting. Therefore, $\Delta \xi_1 = m^2_H - m^2_L = (m_H - m_L)(m_H + m_L)$, and so

$$D_k < (m_H - m_L) \left[ 1 - \frac{\delta_p \rho}{1 - \delta_p (1 - \rho)} \right].$$
The desired result now holds since the left-hand side of the above inequality converges to zero when \( \delta_p \) converges to one.\(^{12}\)

\[ \sum_{t=1}^{\infty} \rho^t (1 - \rho)^t \]

\[ \sum_{t=1}^{\infty} \rho^t (1 - \rho)^t \converges to 1 \text{ when } \rho \converges to zero \text{ from above.} \]

\[^{12} \text{Notice that } \rho \sum_{t=1}^{\infty} (1 - \rho)^t \text{ converges to 1 when } \rho \text{ converges to zero from above.} \]

\[^{13} \text{The reason why convertibility may work together with reputation is that its failure can act as a signal that the bank is impatient, thus disciplining the behavior of the patient bank.} \]

7 Conclusion

This work contributes to the literature on endogenous money. It addresses the feasibility of fiat money when its supply is determined by a single self-interested agent (the bank). This is done in an environment where trade is decentralized and agents are anonymous and have heterogeneous preferences, so that money is essential. We depart from previous work by assuming that: (i) there is uncertainty about the bank’s preferences, so that there is a role for reputation; (ii) there is no technology that allows the bank to be publicly monitored, so that information is decentralized and its flow is constrained by the same technology that hinders trade. The main feature of our model is that the bank faces a trade-off between short-run gains from overissue and long-run losses due to a decrease in its reputation for providing valuable currency.

We show that if the patient bank can commit to a choice of money supply, then a monetary equilibrium where it does not overissue exists as long as it is sufficiently patient. This equilibrium, however, is not time-consistent when the patient bank cannot commit to a plan of action. The reason for this is that the patient bank’s incentives to maintain a good reputation by not overissuing disappear once this reputation becomes too high. In other words, the trade-off between gains from overissue and reputation is only significant for moderate reputations. Following this insight, we show that if memory is imperfect, the patient bank’s incentive to maintain a good reputation never disappears, and so a monetary equilibrium with no overissue is possible. The shortcoming of this approach is that even though reasonable, it is somewhat ad-hoc.

An interesting question is whether there are other mechanisms that can discipline the bank’s behavior. In this regard, we believe that there are two possible directions for study. First, it may be that inconvertibility is at the root of the overissue problem. As Friedman and Schwartz point out, “historically, producers of money have established confidence by promising convertibility into some dominant money, generally, specie. Many examples can be cited of fairly long-continued and successful producers of private moneys convertible into specie” [11, p. 45].\(^{13}\) The second alternative, which we are currently investigating, is to introduce competition among money issuers. According to von Hayek, one of its main advocates, “convertibility is a safeguard necessary to impose upon
a monopolist, but unnecessary with competing suppliers who cannot maintain themselves in the business unless they provide money at least as advantageous to the user as anybody else” [13, p. 111].

References


Appendix A

Let $\Omega_t$ be the vector space of bounded Borel measurable functions from $H_t$ into $\mathbb{R}$ endowed with the supremum norm and denote its dual by $\Omega^*_t$. Now let $ba(H_t)$ be the vector space of all signed charges of bounded variation defined over the Borel sets of $H_t$ and endow this set with the total variation norm.\(^{14}\) A well-known result, see [1, Theorem 13.4], is that $\Omega^*_t = ba(H_t)$.

A function $\tau_t : [0, 1] \to \Omega_t$ is weakly Borel measurable if for all $\omega^* \in \Omega^*_t$, the function $\omega^* \tau_t : [0, 1] \to \mathbb{R}$ given by $\omega^* \tau_t(i) = \omega^*(\tau_t(i))$ is Borel measurable. When $\tau_t(i) \in \Delta_t$ for almost all $i \in [0, 1]$, we write $\tau_t : [0, 1] \to \Delta_t$. If $\tau^1_t, \tau^2_t : [0, 1] \to \Omega_t$ are two weakly Borel measurable functions such that $\{i : \tau^1_t(i) \neq \tau^2_t(i)\}$ has measure zero, we say that $\tau^1_t$ and $\tau^2_t$ are equivalent.

Definition: A strategy profile $\tau$ for the agents is an equivalence class of infinite sequences $\{\tau_t\}$, where $\tau_t : [0, 1] \to \Delta_t$ is weakly Borel measurable for all $t \in \mathbb{N}$ and two sequences $\tau$ and $\tau'$ are equivalent if their corresponding elements are equivalent.

Notice that if $\{d_t\}$ is such that $d_t \in \Delta_t$ for all $t \in \mathbb{N}$, then $\tau = \{\tau_t\}$ with $\tau_t \equiv d_t$ is weakly Borel measurable, and hence a possible strategy profile. Indeed, for each $\omega^* \in \Omega_t$, $\omega^* \tau_t : [0, 1] \to [0, 1]$ is a constant, and thus measurable, function.

Appendix B

Lemma 1. Suppose $\sigma^*$ is an equilibrium. Then, for all $t \in \mathbb{N}$, $\mu^*_t = 0$ if, and only if, $\nu^*_t = 0$.

Proof: Let $\underline{t} = \inf\{t \in \mathbb{N} : \mu^*_t > 0 \text{ or } \nu^*_t > 0\}$, where the infimum of an empty set is taken to be plus infinity. If $\underline{t} = +\infty$, we are done. Suppose then that $\underline{t} < +\infty$. First observe that both $\mu^*_\underline{t}$ and $\nu^*_\underline{t}$ must be positive. Indeed, since $\mu^*_t = \nu^*_t = 0$ for $t < \underline{t}$, $\lambda(D[M^*, \tau^*, \delta] > 0$ if, and only if, $D = \{\theta_0, A, \emptyset, A, \emptyset, \ldots, A, \emptyset\}$. Hence, $\mu^*_\underline{t} = \nu^*_\underline{t}$, and so both must be positive by the definition of $\underline{t}$. Consequently, for all $t > \underline{t}$, the measure of agents in the market is greater than zero regardless of the bank’s type. Because $m_L, m_H \in (0, 1)$, if $t > \underline{t}$, then any event in $H_t$ happens with positive probability when the bank is patient if, and only if, it happens with positive probability when the bank is impatient. In particular, this conclusion holds for the set of all private histories that lead an agent born in $t > \underline{t}$ to enter the market.

\(^{14}\)A charge is a finitely additive set function. See [1] for the definition of the (total) variation of a charge. It is straightforward to show that convergence in the total variation norm is equivalent to convergence in the norm $|| \mu || = \sup\{|\mu(D)| : D \text{ is Borel}\}$.
Lemma 2. Let $\sigma^*$ be an equilibrium. Then $m^*_t = m_H$ for all $t \in \mathbb{N}_0(\sigma^*)$.

Proof: Suppose $t \in \mathbb{N}_0(\sigma^*)$. Notice that the patient bank’s flow payoff in this period is $\kappa m_t$. Moreover, its choice of $m$ in $t$ does not affect future payoffs. Indeed, since the measure of agents who enter the market in $t$ is zero, the behavior of the patient bank in this period does not affect the distribution of private histories across the agents in future periods. Hence, a profitable deviation for the patient bank is possible if $m^*_t = m_L$. □

Lemma 3. Suppose $\sigma^*$ is an equilibrium. Then, either $\mathbb{N}_1(\sigma^*) = \emptyset$ or $\mathbb{N}_1(\sigma^*)$ is infinite.

Proof: Suppose $\mathbb{N}_1(\sigma^*)$ is non-empty, but finite; i.e., suppose there is $\bar{t} \in \mathbb{N}$ such that both $\mu^*_t$ and $\nu^*_t$ are positive, but $\mu^*_t = \nu^*_t = 0$ for $t > \bar{t}$. Recall that $w_{i,t} = w_{i,t}(\{\alpha^*_t\})$ is the lifetime expected reward from entering the market in period $t$ with $i$ units of money when the bank is patient. Hence, if an agent born in $t \geq \bar{t}$ with belief $\theta$ enters the market, his lifetime expected payoff is

$$v_i(\theta) = \theta[m^*_t(w_{i,t} - c) + (1 - m^*_t)v_{0,t}] + (1 - \theta)v(m_H). \quad (B.1)$$

Now observe that: (a) $\alpha^*_t = \alpha^*_i$ for all $t > \bar{t}$; (b) $m^*_t = m_H$, otherwise the patient bank would have a profitable deviation in $\bar{t}$; (c) the measure of agents born in $\bar{t}$ with belief in $\{\theta : v_\bar{t}(\theta) \geq v_A\}$ is positive regardless of the bank’s type. These facts imply that: (d) $v_{\bar{t}+1}(\theta) = v_\bar{t}(\theta)$ for all $\theta \in [0,1]$; and (e) $\alpha^*_\bar{t} > m_H$. Indeed, $w_{i,\bar{t}+1} = w_{i,\bar{t}}$ by (a) and $m^*_{\bar{t}+1} = m_H$ by Lemma 2, so that (d) holds by (B.1) and (b). To see why (e) is also true, notice that if $\alpha^*_\bar{t} = m_H$, then (a) and (b) imply that $v_\bar{t}(\theta) = v(m_H) < v_A$, in which case $\{\theta : v_\bar{t}(\theta) \geq v_A\}$ is empty. A consequence of (e) is that the market is informative about the bank’s type in period $\bar{t}$. Therefore, there is a positive probability that any agent who enters the market in $\bar{t}$ ends this period with a strictly higher belief about the bank. Since (e) also implies that $V^*_\bar{t}$ is strictly increasing in $\theta$, the measure of agents who are born in $\bar{t} + 1$ with belief in $\{\theta : v_{\bar{t}+1}(\theta) > v_A\}$ is positive when the bank is patient by (c) and (d). This, however, implies that $\mu^*_{\bar{t}+1} > 0$, a contradiction. □

Appendix C

Theorem 2. Suppose $\sigma^*$ is a monetary equilibrium and let $v^*_t$ be the corresponding expected lifetime payoff of entering the market in period $t$ when the bank is patient. Then, $\liminf_{t \in \mathbb{N}_1(\sigma^*)} v^*_t = v_A$.

Proof: Suppose $\mathbb{N}_1(\sigma^*)$ is infinite. Notice that in order for $t$ to be an element of $\mathbb{N}_1(\sigma^*)$ it is necessary that $v^*_t \geq v_A$, from which we can conclude that $\liminf_{t \in \mathbb{N}_1(\sigma^*)} v^*_t \geq v_A$. Suppose then, by contradiction, that $\liminf_{t \in \mathbb{N}_1(\sigma^*)} v^*_t > v_A$. We want to show that in this case a profitable deviation
is possible for the patient bank. We divide the argument in three parts.

Step 1: We first show that \( \{\alpha_t^*\} \) cannot converge to \( m_H \). Recall that

\[
v_t^* = a(m_t^*)b(\alpha_t^*) - m_t^*c + \sum_{k=1}^{\infty} \beta^k a(\alpha_{t+k}^*)b(\alpha_{t+k}^*).
\]  

(C.2)

Suppose then, by contradiction, that \( \{\alpha_t^*\} \) converges to \( m_H \). Since \( m_t^* = m_H \) for all \( t \in \mathbb{N}_0(\sigma^*) \), there exists \( t_1 \in \mathbb{N} \) such that \( m_t^* = m_H \) if \( t \geq t_1 \). Equation (C.2) then implies that \( \{v_t^*\} \) converges to \( v(m_H) \), and so there exists \( t_2 \in \mathbb{N} \) such that \( v_t^* < v_A \) if \( t \geq t_2 \). This, however, implies that \( \mathbb{N}_1(\sigma^*) \) is finite, a contradiction. Consequently, there exists \( m \in [m_L, m_H] \) and a subsequence \( \{\alpha_{t_j}^*\} \) of \( \{\alpha_t^*\} \) that converges to \( m \). Let \( k(t) = \max\{t' \leq t : t' \in \mathbb{N}_1(\sigma^*)\} \). Since \( \alpha_t^* = \alpha_{k(t)}^* \) for all \( t \in \mathbb{N} \), \( \alpha_{k(t_j)}^* \rightarrow m \) as well. Assume, without loss, that

\[
\{\alpha_t^*\}_{t \in \mathbb{N}_1(\sigma^*)} \rightarrow m.
\]  

(C.3)

Step 2: Now observe that \( \lim \inf_{t \in \mathbb{N}_1(\sigma^*)} v_t^* < v_A \) implies that there exist \( v > v_A \) and \( t_1 \in \mathbb{N} \) such that if \( t \in \mathbb{N}_1(\sigma^*) \) and \( t \geq t_1 \), then \( v_t^* \geq v \). Assume, without loss, that \( v_t^* \geq v \) for all \( t \in \mathbb{N}_1(\sigma^*) \) and let \( \bar{\theta} \in (0, 1) \) be such that \( \bar{\theta}v/2 + (1 - \bar{\theta})v(m_H) = v_A \). Then, any agent born in \( t \in \mathbb{N}_1(\sigma^*) \) with belief in \( [\bar{\theta}, 1] \) enters the market. Alternatively, a family lawyer with belief \( \theta \in [\bar{\theta}, 1] \) in \( t \in \mathbb{N}_1(\sigma^*) \) sends the generation \( t \) member of his family to the market. A straightforward modification of the proof of Theorem 5.1 on survival probabilities in multi-armed bandits in [4] shows that a least one of the two subsequences \( \{\mu_t^*\}_{t \in \mathbb{N}_1(\sigma^*)} \) and \( \{v_t^*\}_{t \in \mathbb{N}_1(\sigma^*)} \) is bounded away from zero.

In what follows we prove that \( \{\mu_t^*\}_{t \in \mathbb{N}_1(\sigma^*)} \) is bounded away from zero. For this, let \( H_\infty = [0, 1] \times \Pi^\infty \) and use \( H_\infty \) to denote a typical element of this set. A standard argument shows that given the bank’s type \( \delta \), the strategy \( \tau^*(i) \) of family \( i \)’s lawyer induces a Borel probability measure \( \lambda^*_\infty(i, \delta) \) on \( H_\infty \). The monotone class lemma, see [1, Theorem 4.12], and the assumption that the functions \( \tau^*_t \) are weakly Borel measurable together imply that for each Borel subset \( D \) of \( H_\infty \)

\[
\lambda^*_\infty(i, \delta)(D) = \lambda^*_\infty(D|i, \delta) \text{ is a measurable function of } i.
\]

Now let \( d_t(h_\infty) \) be the list of market-autarky decisions in \( h_\infty \) up to period \( t \) and \( d_\infty(h_\infty) \) be the list of all market-autarky decisions in \( h_\infty \). Moreover, let \( \#_Md_t(h_\infty) \), with \( t \in \mathbb{N} = \mathbb{N} \cup \{\infty\} \), be the number of elements of \( d_t(h_\infty) \) that are equal to \( M \), \( E_{kt} = \{h_\infty \in H_\infty : \#_Md_t(h_\infty) \geq k - 1\} \), and \( E_{k\infty} = \{h_\infty \in H_\infty : \#_Md_\infty(h_\infty) \geq k - 1\} \), where \( k \in \mathbb{N} \). Notice that \( E_{k\infty} = \bigcup_{k=1}^{\infty} E_{kt} \). To finish, let \( p_t(k|i, \delta) = \lambda^*_\infty(E_{kt}|i, \delta) \) and \( n_t(k|i, \delta) = p_t(k+1|i, \delta) - p_t(k+2|i, \delta) \), where once again \( t \in \mathbb{N} \) and \( k \in \mathbb{N} \).

Suppose then, by contradiction, that \( \{\mu_t^*\}_{t \in \mathbb{N}_1(\sigma^*)} \) is not bounded away from zero, so that \( \{v_t^*\}_{t \in \mathbb{N}_1(\sigma^*)} \) is. By definition, if the bank’s discount factor is \( \delta \), \( n_t(k|i, \delta) = \int n_t(k|i, \delta)di \), with
$t \in \mathbb{N}$, is the measure of families for which $k$ of its members choose the market up to period $t$ and $n_\infty(k|\delta) = \int n_\infty(k|i, \delta)di$ is the measure of families for which $k$ of its members enter the market. Notice that for each $k, \delta$, and $i$, $p_t(k|i, \delta) \rightarrow p_\infty(k|i, \delta)$. Hence, $n_t(k|\delta) \rightarrow n_\infty(k|\delta)$ by the dominated convergence theorem. Since $\nu^*_t = \sum_{k=1}^{\infty}[n_t(k|0) - n_{t-1}(k-1|0)]$, where $n_0(k|\delta) \equiv 0$ by definition, it must be that $\sum_{k=1}^{\infty} n_\infty(k)$, the measure of families for which only a finite number of their members enter the market, is less than one. Otherwise, $\{\nu^*_t\}$ converges to zero. Denote this set of families by $\mathcal{F}$.

Now let $\theta_t(i)$ be the belief of the member of $i$ that is born in $t$ and suppose that all members of $i$ who enter the market do not use their initial money holdings – whether they receive money from the bank upon entering the market or not – to update beliefs. If $i \in \mathcal{F}$, then (C.3) together with an application of Kolmogorov’s Strong Law of Large Numbers implies that $\{\theta_t(i)\}$ converges to zero almost surely.\(^{15}\) Therefore, by [2, Lemma 1], $\{\theta_t(i)\}$ converges to zero almost surely even when the members of $i$ who enter the market use their initial money holdings to update beliefs. We know, however, that it is optimal to choose autarky if $\theta_t(i) < \theta_M$, since $\nu(m_L)$ is the highest payoff possible from choosing the market when the bank is patient. Hence, $\mathcal{F}$ cannot have a positive mass, a contradiction.

To finish this step, notice that $\mathbb{N}_1(\sigma^*, m_L)$ must be an infinite set. Suppose not. Because $\{\mu^*_t\}_{t \in \mathbb{N}_1(\sigma^*)}$ is bounded away from zero, $\lim_{t \rightarrow \infty} \sum_{k \in \mathbb{N}_1(\sigma^*) \cap \{1, \ldots, t\}} \mu^*_k = +\infty$. Hence,

$$\alpha^*_t = \frac{\sum_{k \in \mathbb{N}_1(\sigma^*, m_L) \cap \{1, \ldots, t\}} \mu^*_k}{\sum_{k \in \mathbb{N}_1(\sigma^*) \cap \{1, \ldots, t\}} \mu^*_k} m_L + \frac{\sum_{k \in \mathbb{N}_1(\sigma^*, m_H) \cap \{1, \ldots, t\}} \mu^*_k}{\sum_{k \in \mathbb{N}_1(\sigma^*) \cap \{1, \ldots, t\}} \mu^*_k} m_H \rightarrow m_H,$$

since $A_t$ converges to zero. This, however, contradicts (C.3).

Step 3: Let $ca(\mathbb{N})$ be the Banach space of all signed measures endowed with the total variation norm $|| \cdot ||$ and define $\chi_t(i, \delta) \in ca(\mathbb{N})$, with $t \in \mathbb{N}$, to be such that if $B \subseteq \mathbb{N}$, then

$$\chi_t(i, \delta)(B) = \chi_t(B|i, \delta) = \begin{cases} 0 & \text{if } B = \emptyset \\ \sum_{k \in B} n_t(k|i, \delta) & \text{otherwise} \end{cases}.$$

Now let $B \in 2^\mathbb{N}$ be such that if $B \in \mathcal{B}$, then either $B$ is finite or $B = \{k, k+1, \ldots\}$ for some $k \in \mathbb{N}$. It is obvious that $\{\chi_t(B|i, \delta)\}$ converges to $\chi_\infty(B|i, \delta)$ if $B$ is finite. If $B = \{k, k+1, \ldots\}$ for some $k \in \mathbb{N}$, then $\chi_t(B|i, \delta) = p_t(k+1|i, \delta)$ for all $t \in \mathbb{N}$, and so $\{\chi_t(B|i, \delta)\}$ also converges to 

\(^{15}\)An intuition for this result is as follows. Consider an individual who tosses a coin infinitely many times, and this coin is either fair or biased towards heads. Even if this bias changes over time, as long as the limiting frequency of heads in the biased coin is greater than 1/2, this individual will still learn the coins’ true type.
\[ \chi_\infty(B|i, \delta). \] Hence, \( \{\chi_t(B|i, \delta)\} \) converges to \( \chi_\infty(B|i, \delta) \) for all \( B \in \mathcal{C} \), the algebra generated by \( \mathcal{B} \).

By [12, Theorem 3.1], if \( \Lambda_t(B|i, \delta) = \chi_t(B|i, \delta) - \chi_\infty(B|i, \delta) \), with \( t \in \mathbb{N} \), then \( \Lambda_t(B|i, \delta) \to 0 \) for all \( B \subseteq \mathbb{N} \). By Phillips’ lemma, see [8, p. 83], we can then conclude that \( ||\chi_t(i, \delta)|| \to 0 \). Egoroff’s Theorem, see [9], now tell us that for all \( \epsilon > 0 \) there exists a set \( \mathcal{J}_\epsilon \) of families of measure at least \( 1 - \epsilon \) such that \( \{\chi_t(i, \delta)\} \) converges to \( \chi_\infty(i, \delta) \) uniformly on \( \mathcal{J}_\epsilon \). For our purposes, we can then assume that \( \{\chi_t(i, \delta)\} \) converges uniformly to \( \chi_\infty(i, \delta) \) on \( [0, 1] \).

Let \( \mu \) be a lower bound for \( \{\mu^*_t\}_{t \in \mathbb{N}_1(\sigma^*)} \) and choose \( K \in \mathbb{N} \) such that \( \sum_{t=K+1}^{\infty} \frac{\delta^t_p}{\mu(m_H - m_L)/2} \). Moreover, let \( \epsilon \in (0, 1 - \overline{\theta}) \) be such that if an agent is born in \( t \in \mathbb{N}_1(\sigma^*) \) with belief in \( [1 - \epsilon, 1] \), then the next \( K \) generations of his family always enter the market as long as their date of birth also lies in \( \mathbb{N}_1(\sigma^*) \). Equation (C.3) together with [2, Lemma 1] imply that if an infinite number of members of family \( i \) enter the market, \( \{\theta_t(i)\} \) converges to one almost surely. Consequently, there exists \( \overline{N} \) such that if \( \#Md_t(h^\infty) \geq \overline{N} \), then the probability that \( \theta_t(i) \geq 1 - \epsilon \) is at least \( 1 - \mu(1 - \delta_p)(m_H - m_L)/2 \), as almost sure convergence implies convergence in measure.

Now let \( t_1 \in \mathbb{N} \) be such that \( |\mathbb{N}_1(\sigma^*) \cap \{1, \ldots, t_1\}| = \overline{N} \), where \( |B| \) denotes the cardinality of \( B \). By the results obtained so far, for all \( \epsilon > 0 \) there exists \( t_2 \in \mathbb{N} \) such that if \( t \geq t_2 \), then \( |\chi_t(B|i, \delta_p) - \chi_\infty(B|i, \delta_p)| \leq \epsilon \) for all \( B \in \mathcal{B} = \{\{1\}, \{2\}, \ldots, \{\overline{N} - 1\}, \{\overline{N}, \ldots, \infty\}\} \) and \( i \in [0, 1] \). Assume, without loss, that there exists \( t_2 \) such that \( \chi_t(B|i, \delta_p) = \chi_\infty(B|i, \delta_p) \) for all \( B \in \mathcal{B} \) and \( i \in [0, 1] \). Hence, if \( t \geq \overline{t} = \max\{t_1, t_2\} \), then for almost all families \( i \) either: (i) \( \#Md_t(h^\infty) \) is constant in \( t \), so that no members of \( i \) enter the market after \( t \); or (ii) \( \#Md_t(h^\infty) \geq \overline{N} \). In other words, if \( t \in \mathbb{N}_1(\sigma^*) \cap \{\overline{t} + 1, \ldots\} \), then almost all agents who enter the market in \( t \) have at least \( \overline{N} \) previous members of his family who did the same.

To finish, let \( t \in \mathbb{N}_1(\sigma^*, m_L) \) be such that \( t > \overline{t} \) and set \( \kappa \) equal to zero (we can do so since \( \kappa \) is infinitesimal). The patient bank’s lifetime payoff from sticking to its prescribed strategy from period \( t \) on is

\[ \sum_{k=1}^{\infty} \delta^k_p \mu^*_t m^*_t + \mu^*_t m_L. \]

If, instead, it does a one-shot deviation in \( t \), its lifetime payoff is at least

\[ \mu^*_t m_H + \left( 1 - \frac{1}{2} \mu(1 - \delta_p)(m_H - m_L) \right) \sum_{k=1}^{K} \delta^k_p \mu^*_t m^*_t. \]

\[ \text{For completeness, we state a weaker version of Theorem 3.1 in [12] below.} \]
Hence, the patient bank’s payoff gain from a one-shot deviation in period $t$ is no less than

$$
\mu^*_t(m_H - m_L) - \frac{1}{2} \mu(1 - \delta_p)(m_H - m_L) \sum_{k=1}^{K} \delta_p^k \mu_{t+k}^* m_{t+k}^* - \sum_{k=K+1}^{\infty} \delta_p^k \mu_{t+k}^* m_{t+k}^*
$$

$$
\geq \mu(m_H - m_L) - \frac{1}{2} \mu(1 - \delta_p)(m_H - m_L) \sum_{k=1}^{K} \delta_p^k - \sum_{k=K+1}^{\infty} \delta_p^k > 0,
$$

since $\mu^*_t m^*_t < 1$ for all $t \in \mathbb{N}$. We can then conclude that the patient bank has a profitable deviation, the desired result. \qed

**Theorem.** Let $X$ be a metric space and suppose $C$ is a family of open sets in $X$ where:

1. If $C_1, C_2 \in C$, then $C_1 \cap C_2 \in C$;
2. If $C_1, C_2 \in C$ and $C_1^c \cap C_2^c = \emptyset$, then $C_1 \cup C_2 \in C$, where $C^c$ denotes the closure of $C$;
3. If $K, U \in X$ are such that $K$ is compact, $U$ is open, and $K \subset U$, then there exist $C_1, C_2 \in C$ such that $K \subset C_1 \subset X \setminus C_2 \subset U$;
4. If $\{C_n\}$ and $\{D_n\}$ are two sequences of elements of $C$ such that $\{C_n\}$ is increasing, $\{D_n\}$ is decreasing, and $C_n \subset D_n$ for all $n \in \mathbb{N}$, then there exists $C_0 \in C$ such that $C_n \subset C_0 \subset D_n$ for all $n \in \mathbb{N}$.

Now let $\{\mu_n\}$ be a sequence of Borel probability measures on $X$ such that if $U$ is open, then $\mu_n(U) = \sup\{\mu_n(K) : K \subset U$ is compact $\}$. If $\{\mu_n(C)\}$ converges for all $C \in C$, then $\{\mu_n(B)\}$ converges for all Borel subsets $B$ of $X$.

**Appendix D**

**Lemma 4.** $\xi_t(m_L, \theta_0) - \xi_t(m_H, \theta_0) \leq \xi_{t+1}(m_L, \theta_0) - \xi_{t+1}(m_H, \theta_0)$ for all $t \in \mathbb{N}$

**Proof:** Let $c(h^t)$ denote the number of meetings with money in $h^t \in H_t$. It is straightforward to show that when private histories are always transmitted from one generation to the next, an agent born in $t$ with history $h^t$ chooses the market if, and only if, $c(h^t) \leq \alpha 2t + \gamma$, where $\alpha > 0$ depends on $m_L$ and $m_H$ and $\gamma \geq 0$ depends on $\theta_0$. Therefore,

$$
\xi_t(m, \theta_0) = \sum_{(c_1, \ldots, c_t) \in C_t} \left( \begin{array}{c} 2 \\ c_1 \end{array} \right) \cdots \left( \begin{array}{c} 2 \\ c_n \end{array} \right) m^{c_1(1-m)} \cdots \sum_{B_t(c_1, \ldots, c_t)} 2^{c_1-1} m_L^{c_2} \cdots (1-m_L)^{2(\gamma-2c_2-\cdots-c_t)},
$$
where \( C_t = \{(c_1, \ldots, c_t) : c_{\tau} \leq \lfloor \alpha 2^{\tau} + \gamma \rfloor - c_1 - \cdots - c_{\tau-1}, \, \tau = 1, \ldots, t-1\} \). Here \( \lfloor x \rfloor \) is, by definition, the greatest integer smaller than \( x \). Now let \( L_t(m) = \xi_t(m, \theta_0) - \xi_{t+1}(m, \theta_0) \). Then,

\[
L_t(m) = \sum_{(c_1, \ldots, c_{t+1}) \in D_{t+1}} B_t(c_1, \ldots, c_t) m^{c_1} (1 - m)^{2-c_1} \frac{m_1^{c_2+\cdots+c_{t+1}} (1 - m_L)^{2-t-c_2-\cdots-c_{t+1}}}{m_L^{c_2, \ldots, c_{t+1}}},
\]

where \( D_{t+1} = \{(c_1, \ldots, c_{t+1}) : (c_1, \ldots, c_t) \in C_t, \, c_{t+1} \geq \lfloor \alpha 2(t+1) + \gamma \rfloor - c_1 - \cdots - c_t + 1\} \). Hence, \( L_t(m_L) - L_t(m_H) \) is equal to

\[
\sum_{(c_1, \ldots, c_{t+1}) \in D_{t+1}} B_{t+1}(c_1, \ldots, c_{t+1}) m_L(c_2, \ldots, c_{t+1}) \left[ m_L^{c_1} (1 - m_L)^{2-c_1} - m_H^{c_1} (1 - m_H)^{2-c_1} \right].
\]

To finish observe that the term in brackets in the above expression is non-positive if, and only if, \( c_1 \leq 2\alpha \). Therefore \( L_t(m_L) \leq L_t(m_H) \), as desired. \( \square \)

The proof of Lemma 4 can be easily modified to cover the more general case where each agent in the market faces \( n \) meetings per period.