**Abstract**

Using the measure of risk aversion suggested by Kihlstrom and Mirman [1974] and [1981], we propose a dynamic consumption-savings-portfolio choice model in which the consumer-investor maximizes the expected value of a non-additively separable utility function of current and future consumption. Preferences for consumption streams are CES and the elasticity of substitution can be chosen independently of the the risk aversion measure. The additively separable case is a special case. Because choices are not dynamically consistent, we follow the "consistent planning" approach of Strotz [1956] and also interpret our analysis from the game theoretic perpective taken by Peleg and Yaari [1973]. The equilibrium of the Lucas asset pricing model with i.i.d. consumption growth is obtained and the equity premium is shown to depend on the elasticity of substitution as well as the risk aversion measure. The nature of the dependence is examined. Our results are contrasted with those of the non-expected utility recursive approach of Epstein-Zin and Weil.

# Risk Aversion and the Elasticity of Substitution in General Dynamic Portfolio Theory: Consistent Planning by Forward Looking, Expected Utility Maximizing Investors

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## 1 Introduction

Until recently the model used to analyze a consumer-investor's dynamic consumption-saving and portfolio choices was the "additively separable" model in which the consumer-investor is assumed to maximize

$$U\left(\{c_t\}_{t=1}^{\infty}\right) = E\left[\sum_{t=1}^{\infty} \beta^t u\left(\tilde{c}_t\right)\right], \tag{1}$$

the expected value of the discounted sum of utilities of per-period random consumption, $\tilde{c}_t$. Early analyses of this model were provided in the papers by Levhari and Srinavasan [1969], Merton [1969], [1971], Samuelson [1969] and Hakansson [1970]. Merton [1973] used this framework to obtain a dynamic asset pricing model. The implications of Merton's asset pricing model were clarified in Breeden [1979]. In his pioneering [1978] paper, Lucas also used the additively separable model to construct a dynamic asset pricing model. The Lucas and Merton models provide the foundation for much of the subsequent work on dynamic asset pricing. In a significant part of this research, the utility function, $u(\cdot)$, of per-period consumption is assumed to be in the constant relative risk averse (CRRA) class. If $u(\cdot)$ is CRRA with relative risk aversion $\alpha$, the utility function (1) of the certain consumption stream $\{c_t\}_{t=1}^{\infty}$ is in the CES class with elasticity of substitution

$$\sigma = \frac{1}{\alpha}.$$

This, of course, means that, when this model is used to analyze asset pricing, the impact of risk aversion, as measured by $\alpha$, cannot be separated from the impact of a change in the elasticity of substitution of the ordinal preferences for consumption streams $\{c_t\}_{t=1}^{\infty}$.

Recent contributions by Epstein [1988], Epstein and Zin [1989], [1990], [1991] and Weil [1989], [1990], have extended the analysis of the Lucas model to particular cases of recursive, nonadditive preferences of the type introduced in Kreps and Porteus [1978], [1979a], [1979b]. The motivation for the Epstein-Zin and Weil (EZW) extensions was the inability of the Lucas model with CRRA preferences to explain the size of the equity premium without assuming values of $\alpha$ that were widely regarded as "too large." This "equity premium puzzle" was pointed out in Prescott and Mehra [1985] and Grossman, Melino and Shiller [1987] which built on the earlier contribution in Grossman and Shiller [1981].

By using recursive, non-additive Kreps-Porteus preferences EZW were able to choose the risk aversion parameter independently of the elasticity of substitution. The additional degree of freedom in the EZW formulation would appear to enhance the ability of the Lucas model to provide an explanation for a wide variety of consumption-saving, portfolio choice and asset pricing phenomena. As noted in the introduction to Epstein and Zin [1991], "the disentangling of risk aversion from the elasticity of substitution is a problem that has been highlighted by the empirical literature on the behavior of asset returns and consumption over time. Representative agent optimizing models have not performed well empirically (see, among others, Hansen and Singleton [1983], Prescott and Mehra [1985] and Grossman, Melino, and Shiller [1987]). One possible reason for this poor performance is that the maintained specification of preferences is too rigid." But as Weil [1989] points out "the solution to the equity premium puzzle documented by Mehra and Prescott [1985] cannot be found by simply separating risk aversion (from) intertemporal substitution. If the dividend growth process is *i.i.d.*, the risk-premium, when appropriately defined, is independent of the intertemporal elasticity of substitution, and thus is the same whether or not the time-additive, expected utility restriction is imposed. When the dividend growth process is non-*i.i.d.*, relaxing the parametric restriction adds, for plausible parameter values, a risk free rate puzzle to Mehra and Prescott's equity premium puzzle." Kocherlakota [1990] demonstrates an even stronger result. In analyzing the Lucas asset pricing model he assumes Epstein-Zin preferences and that "the growth rate of the aggregate endowment is *i.i.d.*" Using this model he demonstrates that "an econometrician with data on asset prices and aggregate consumption cannot separately identify" the elasticity of substitution and the relative risk aversion. He concludes that the Epstein-Zin preferences have "no more explanatory power than" the preferences represented by the utility function in (1). The inability of the EZW framework to add explanatory power beyond that of the additively separable model in the *i.i.d.* case is rooted in the observation made in a number of papers that, when the returns to the risky assets are *i.i.d.*, the EZW optimal portfolio depends only on the risk aversion measure and is independent of the elasticity of substitution. See for example, Swenson [1989] and Bhamra and Uppal [2006].

In the EZW approach, consumer-investors do not maximize expected utility. Epstein and Zin [1989] briefly discussed but rejected an alternative approach, based on expected utility maximization, to the problem of using different parameters to measure risk aversion and the elasticity of substitution. The alternative approach that they argued was inappropriate for use in a dynamic setting is based on the discussion of risk aversion with many commodities in Kihlstrom and Mirman [1974] and [1981]. There is, indeed, a difficulty that arises when one attempts to implement the Kihlstrom-Mirman approach in a non-additively separable dynamic model of the type envisioned by Epstein and Zin. Specifically, dynamic consistency requires that the best current choice depends on choices made in the past. Thus, memory of past choices is a necessary condition for dynamic consistency. It was not simply the fact that the memory of the past was necessary that Epstein and Zin found troubling. Indeed, memory of the past is required even in additively separable dynamic models that assume preferences are influenced by habits; see, for example, Constantinides [1990], Abel [1990] and [1999] and Campbell and Cochrane [1999]. What did prove objectionable and decisively so was the fact that, because of the usual discounting assumptions, memory of the very distant past would become most important. In a recent working paper, Van den Heuvel [2007] has investigated the approach rejected by Epstein and Zin. Under the discounting assumptions of Van den Heuvel's paper, memory of the distant past is shown to become unimportant.

In the current paper, we also reconsider the possibility of implementing an alternative dynamic model based on the Kihlstrom-Mirman expected utility maximizing approach. In our alternative formulation, preferences change over time since the consumer-investor's only concern is with current and future consumption. Because the consumer-investor ignores past consumption, we refer to him as having "forward looking" von Neuman-Morganstern preferences. In the dynamic model considered below, the consumer-investor's current consumption-saving and portfolio choices at each point in time maximize his expected utility of current and future consumption. This implies that the consumer-investor's current choices are independent of past consumption. This independence of past consumption implies a dynamic inconsistency. The current choices are not the ones the consumer-investor would have committed himself to make if such a commitment had been possible at an earlier point in time.

The dynamic inconsistency we are forced to deal with is similar to but distinct from that which arises in models of hyperbolic discounting such as those first considered by Strotz [1956] and recently reconsidered in the work of Laibson [1997], Harris and Laibson [2001] and Luttmer and Marrioti [2003] and [2007]. In analyzing the model we propose, we follow the "consistent planning" approach of Strotz [1956] and assume that, when making his current choice, the consumer-investor will "take account of future disobedience." This consistent planning approach was also used by Pollak [1968] and by Phelps and Pollak [1968]. In the dynamic consumption-savings,portfolio model we consider, when the consumer-investor makes his current choices he recognizes that his future choices will not be the ones he would currently like to commit himself to make in the future. Thus, the consumer-investor chooses a consumption plan for the future that is, as Strotz asserted, "the best plan among those he will actually follow." Our approach can also be interpreted from the perspective of Peleg and Yaari [1973] and is also similar to that taken in the literature on durable goods monopoly and the Coase conjecture; see, for example, Coase [1972], Stokey [1979] and Bulow [1982]. In following Peleg and Yaari and the durable goods monopoly literature, we view the current consumer-investor as a leader in a leader follower game in which the followers are the same consumer-investor at future time periods. His current choices are a best response to the choices he knows he will want to make in the future. The result is a Nash equilibrium of a Stackleberg game in which each "player" is the consumer-investor at a particular consumption period.

We refer to our approach as one of "consistent planning" by a "forward looking" expected utility maximizing consumer-investor. Using this approach we follow Mehra-Prescott and Epstein-Zin and Weil and investigate the risk premium implied by equilibrium in the Lucas asset pricing model. We do find that, in contrast to the Epstein-Zin and Weil approach, the risk premium obtained from our approach is affected by the elasticity of substitution as well as the risk aversion measure. It thus does not suffer from the criticism of Kocherlakota and of Weil himself. In spite of this, it must be added that our approach does not appear to have great promise for explaining the Mehra-Prescott equity premium puzzle when consumption growth is assumed to be *i.i.d.,* and that is the case we investigate. This is true for two reasons. One is that our model yields a higher risk premium than the standard additively separable model only when the elasticity of substitution in our model is exceeded by that of the additively separable model. It will be recalled from the discussion above that in the additively separably model, the elasticity of substitution is the inverse of the risk aversion measure. The other problem is that the conditions required for existence of equilibrium in our approach put severe lower bound restrictions on the elasticity of substitution. These conditions combine to put an upper bound the risk aversion measure that limits the ability of our model to deliver a large risk premium. Even in the absence of these specific problems, Mehra and Prescott suggested that there is good reason to be pessimistic about the promise of models based on CES preferences to imply a large equity premium. In their paper, they asserted that "We doubt whether non-time-additivity separable preferences will resolve the puzzle, for that would require consumptions near in time to be poorer substitutes than conumptions at widely separated dates."

Labadie [1986] has also investigated the implications of the Kihlstrom-Mirman approach to risk aversion for the equity premium obtained in a Lucas asset pricing model equilibrium. Her work which was motivated by the Diamond-Stiglitz paper avoided the problem of dynamic inconsistency by using an overlapping generations model in which the consumer-investors lived for two periods.

### 1.1 Outline of the Paper

We introduce the Kihlstrom-Mirman approach to risk aversion with many commodities by considering the consumption-saving portfolio problem of a consumer-investor who lives for two periods. The Kihlstrom-Mirman approach assumes that the consumer-investor maximizes expected utility. Our discussion of the Kihlstrom-Mirman approach based on expected utility maximization is followed by a description of the Epstein-Zin, Weil, Kreps-Porteus analysis of the two period consumption-saving portfolio problem. We focus on the fact that their approach is not based on expected utility maximization and emphasize the contrast between the two approaches.

We then proceed to introduce our analysis of the dynamic model of forward looking von Neuman-Morganstern preferences by considering the three period case. In this section we focus on the dynamic inconsistency. But the three period case also provides a useful setting for the introduction of the consistent planning approach.

Next we consider the infinite horizon problem and contrast the equilibrium that results from the forward looking von Neuman-Morganstern preferences with that obtained from the EZW approach. Finally, we use the forward looking von Neuman-Morganstern preferences to obtain an equilibrium of Lucas asset pricing model and contrast this with the equilibrium of Lucas asset pricing model obtained using the EZW approach and using additively separable preferences. The equity premium that results from the equilibrium obtained with the forward looking von Neuman-Morganstern preferences exceeds that obtained from the additively separable model only when the elasticity of substitution falls below the level it is restricted to take on in the additively separable case.

## 2 The Two Period Case:

## 2.1 The Kihlstrom-Mirman Approach to Risk Aversion

In this section, we illustrate the use of the Kihlstrom-Mirman approach to risk aversion with many commodities by considering a two-period consumption-saving portfolio problem in which the consumer-investor chooses the fraction, $c$, of his initial wealth, $W$, to consume in the first consumption period. First period savings, which equal $(1-c)\,W$ can be invested in a safe and a risky asset. Thus, the consumer-investor must also choose the fraction, $\gamma$, of savings to invest in the risky asset. The safe asset return is $r_f$. The realized return on the risky asset is $r_m$. The random variable that denotes the risky asset return is $\tilde{r}_m$. It is useful to define the realized excess return on the risky asset as

$$x_m = r_m - r_f.$$

This is, of course, simply the realization of the random variable

$$\tilde{x}_m = \tilde{r}_m - r_f.$$

Second period consumption is then

$$W\,[1-c]\,[r_f + \gamma\,(\tilde{r}_m - r_f)]\,.$$

Following von Neumann-Morganstern, we assume that the consumer-investor solves this two-period consumption-saving portfolio problem by simply maximizing expected utility and choosing

$$(\hat{c},\hat{\gamma}) = \underset{(c,\gamma)}{\arg\max} E\left[U\left(cW, W\,[1-c]\,[r_f + \gamma\,(\tilde{r}_m - r_f)]\right)\right]$$

where

$$U\,(C_1, C_2)$$

is some strictly concave function of the two-period consumption stream

$$(C_1, C_2)\,.$$

This is the standard general approach to the two period problem. This was also the approach taken in Kihlstrom-Mirman [1974] and in a related paper, Diamond-Stiglitz [1974], in the same journal. In those papers, there was, however, no riskless asset and hence no portfolio choice problem.

In what follows we describe the Kihlstrom-Mirman approach to risk aversion with many commodities and indicate how it can be applied to study the effect of risk aversion on the $(\hat{c},\hat{\gamma})$ choice in the two period problem. The general Kihlstrom-Mirman approach to risk aversion with many goods restricts the risk aversion comparisons of utility functions to those which represent the same ordinal preferences. Thus, the risk aversion of two utility functions $U^1$ and $U^2$ of two-period consumption streams are compared only if they are related by a monotonically increasing transformation. The utility function $U^1$ is defined to be more risk averse than $U^2$ if

$$U^1\,(C_1, C_2) = h\left(U^2\,(C_1, C_2)\right)$$

and the transformation $h$ is strictly concave.

To define the absolute or relative risk aversion of a utility function of consumption streams

$$(C_1, C_2).$$

attention is restricted to those ordinal preferences for which there exists a "least concave representation," $U^0$, for which every concave representation of the preferences is a concave transformation of $U^0$. This definition of "least concave" was introduced in Debreu [1976]. When the preferences are homothetic, the least concave representation is homogeneous of degree one. Thus, when the approach is applied to CES ordinal preferences,

$$U^0(C_1, C_2) = (C_1^\rho + \beta C_2^\rho)^{\frac{1}{\rho}},$$

if $\rho \neq 0$. In the Cobb-Douglas case corresponding to $\rho = 0$,

$$U^0(C_1, C_2) = C_1^{\frac{1}{1+\beta}} C_2^{\frac{\beta}{1+\beta}}.$$

If $U(C_1, C_2)$ is a strictly concave representation of some ordinal preferences for which a least concave least concave representation, $U^0$, exists then

$$U(C_1, C_2) = h\left(U^0(C_1, C_2)\right)$$

where $h(\cdot)$ is monotone increasing and strictly concave. The absolute risk aversion of $U$ at $(C_1, C_2)$ is defined to be

$$-\frac{h''\left(U^0(C_1, C_2)\right)}{h'\left(U^0(C_1, C_2)\right)}.$$

The relative risk aversion of $U$ at $(C_1, C_2)$ is defined to be

$$-\frac{h''\left(U^0(C_1, C_2)\right) u^0(C_1, C_2)}{h'\left(U^0(C_1, C_2)\right)}.$$

This means that for CES ordinal preferences with $\rho \neq 0$ the relative risk aversion of the representation

$$U^\alpha(C_1, C_2) = \frac{(C_1^\rho + \beta C_2^\rho)^{\frac{1-\alpha}{\rho}}}{1-\alpha}$$

is $\alpha$ which we have assumed to be unequal to one. The case of $\alpha = 1$ is of course

$$U^\alpha(C_1, C_2) = \log\left((C_1^\rho + \beta C_2^\rho)^{\frac{1}{\rho}}\right).$$

In the Cobb-Douglas case corresponding to $\rho = 0$, the relative risk aversion of

$$U^\alpha(C_1, C_2) = C_1^{\frac{1-\alpha}{1+\beta}} C_2^{\frac{\beta(1-\alpha)}{1+\beta}}$$

is $\alpha$, where once again we have assumed that $\alpha \neq 1$. In the Cobb-Douglas case when $\alpha = 1$,

$$U^{\alpha}(C_1, C_2) = \log C_1 + \beta \log C_2.$$

In this approach, $U^0$ is the risk neutral representation of the preferences even when $U^0$ is not homogeneous of degree one. In the remainder of the discussion in this section we will, however, assume that $U^0$ is homogeneous of degree one.

## 2.2 Applying the Kihlstrom Mirman Approach to the Consumer-Investor's Problem

For the consumer-investor who faces the two-period consumption-saving portfolio problem and chooses

$$(\hat{c}, \hat{\gamma}) = \underset{(c,\gamma)}{\arg\max} E\left[U^0\left(cW, \left(W\left[1-c\right]\left[r_f + \gamma\tilde{x}_m\right]\right)\right)\right],$$

the value function $J_0(\cdot)$ defined by

$$J_0(W) = \max_{(c,\gamma)} E\left[U^0\left(cW, W\left[1-c\right]\left[r_f + \gamma\tilde{x}_m\right]\right)\right]$$

is

$$J_0(W) = \kappa_0 W.$$

where

$$\kappa_0 = \max_{(c,\gamma)} E\left[U^0\left(c, \left[1-c\right]\left[r_f + \gamma\tilde{x}_m\right]\right)\right].$$

The linearity of $J_0$ in $W$ implies risk-neutrality in the face of wealth risks. The risk neutrality of the value function is inherited from the risk-neutrality of $U^0$.

Also when $U^0$ is homogeneous of degree one, if the consumer-investor chooses

$$\begin{aligned}(\hat{c}, \hat{\gamma}) &= \underset{(c,\gamma)}{\arg\max} E\left[\left(\frac{1}{1-\alpha}\right)\left(U^0\left(cW, W\left[1-c\right]\left[r_f + \gamma\tilde{x}_m\right]\right)\right)^{1-\alpha}\right] \qquad (2)\\ &= \underset{(c,\gamma)}{\arg\max}\left(E\left[\left(U^0\left(c, \left[1-c\right]\left[r_f + \gamma\tilde{x}_m\right]\right)\right)^{1-\alpha}\right]\right)^{\left(\frac{1}{1-\alpha}\right)}\end{aligned}$$

then his value function $J_\alpha(\cdot)$ defined by

$$J_\alpha(W) = \max_{(c,\gamma)} E\left[\left(\frac{1}{1-\alpha}\right)\left(U^0\left(cW, W\left[1-c\right]\left[r_f + \gamma\tilde{x}_m\right]\right)\right)^{1-\alpha}\right]$$

is

$$J_\alpha(W) = \kappa_\alpha W^{1-\alpha}$$

where

$$\kappa_\alpha = \max_{(c,\gamma)} \frac{E\left[\left(U^0\left(c, \left[1-c\right]\left[r_f + \gamma\tilde{x}_m\right]\right)\right)^{1-\alpha}\right]}{1-\alpha}.$$

and $\alpha$ is the relative risk aversion of $J_\alpha$. In this case, the value function $J_\alpha(\cdot)$ inherits the risk aversion of

$$U^\alpha(C_1, C_2) = \left(\frac{1}{1-\alpha}\right)\left(U^0(C_1, C_2)\right)^{1-\alpha}.$$

Following Epstein and Zin we can define the "certainty equivalent utility" as

$$V(W) = \max_{(c,\gamma)}\left(E\left[\left(\frac{1}{1-\alpha}\right)\left(U^0(cW, W[1-c][r_f + \gamma\tilde{x}_m])\right)^{1-\alpha}\right]\right)^{\frac{1}{1-\alpha}}.$$

Clearly the certainty equivalent utility is linear in wealth since the linear homogeneity of $U^0(\cdot,\cdot)$ implies that

$$V(W) = vW$$

where

$$v = \left(E\left[\left(U^0(\hat{c}, [1-\hat{c}][r_f + \hat{\gamma}\tilde{x}_m])\right)^{1-\alpha}\right]\right)^{\frac{1}{1-\alpha}}$$

and where $(\hat{c}, \hat{\gamma})$ is given by (2).

In spite of the argument just outlined it is true that, in the two-period consumption-savings portfolio choice problem, the risks the consumer-investor faces are risks involving second period consumption and $U^0(C_1, C_2)$ is not linear in $C_2$. Thus, when choosing a portfolio, the consumer investor who maximizes the expected value of $U^0\left(C_1, \tilde{C}_2\right)$ is not risk-neutral. Specifically in the CES case with

$$\rho \neq 0,$$

$$U^0(C_1, C_2) = \left(C_1^\rho + \beta C_2^\rho\right)^{\frac{1}{\rho}}$$

and with

$$\rho = 0,$$

$$U^0(C_1, C_2) = C_1^{\frac{1}{1+\beta}} C_2^{\frac{\beta}{1+\beta}}$$

In both cases,

$$U^0_{C_2,C_2}(C_1, C_2) < 0$$

so that $U^0(C_1, C_2)$ is strictly concave in $C_2$.

In general, it is difficult to get comparative static results for $(\hat{c}, \hat{\gamma})$ as the risk aversion of the representation changes. It is true however that for $c$ fixed the expected utility maximizing $\gamma$ decreases if risk aversion or relative risk aversion increases. Also, if the elasticity of substitution is uniformly less (greater) than one, then for $\gamma$ fixed the expected utility maximizing $c$ decreases (increases) if risk aversion or relative risk aversion increases. The former result is a corollary to the well known results proven by Arrow [1971] and Pratt [1964]. This latter result is a consequence of a result proven in Kihlstrom-Mirman [1974].

### 2.2.1 The CES Case

We only consider the case of $\alpha \neq 1$. If, in this case, the elasticity of substitution, $\sigma$, is unequal to one

$$\rho \neq 0$$

and

$$\begin{aligned} V(W) &= \max_{(c,\gamma)} \left( E\left[ \left( [cW]^{\rho} + \beta \left[ W(1-c)(r_f + \gamma \tilde{x}_m) \right]^{\rho} \right)^{\frac{1-\alpha}{\rho}} \right] \right)^{\frac{1}{1-\alpha}} \quad (3) \\ &= vW \end{aligned}$$

where

$$v = \left( E\left[ \left( \hat{c}^{\rho} + \beta \left[ (1-\hat{c}) \tilde{r} \right]^{\rho} \right)^{\frac{1-\alpha}{\rho}} \right] \right)^{\frac{1}{1-\alpha}} \quad (4)$$

and

$$\tilde{r} = r_f + \hat{\gamma} \tilde{x}_m. \quad (5)$$

Note that $\hat{\gamma}$ typically does depends on $\rho$ as well as on the relative risk aversion measure $\alpha$.

Note also that, if there is no riskless asset,

$$\hat{c} = \arg\max_{c} \left( E\left[ \left( [cW]^{\rho} + \beta \left[ W(1-c) \tilde{r}_m \right]^{\rho} \right)^{\frac{1-\alpha}{\rho}} \right] \right)^{\frac{1}{1-\alpha}},$$

$$\begin{aligned} V(W) &= \max_{c} \left( E\left[ \left( [cW]^{\rho} + \beta \left[ W(1-c) \tilde{r}_m \right]^{\rho} \right)^{\frac{1-\alpha}{\rho}} \right] \right)^{\frac{1}{1-\alpha}} \\ &= vW \end{aligned}$$

and $v$ is given by (4) in which $\tilde{r} = \tilde{r}_m$.

When the elasticity of substitution, $\sigma$, equals one

$$\rho = 0,$$

$$\begin{aligned} V(W) &= \max_{(c,\gamma)} \left( E\left[ cW \right]^{\frac{1-\alpha}{1+\beta}} \left[ W(1-c)(r_f + \gamma \tilde{x}_m) \right]^{\frac{\beta(1-\alpha)}{1+\beta}} \right)^{\frac{1}{1-\alpha}} \quad (6) \\ &= vW \end{aligned}$$

where now

$$v = \left[ E\left( \tilde{r}^{\frac{\beta(1-\alpha)}{1+\beta}} \right) \right]^{\frac{1}{1-\alpha}} \hat{c}^{\frac{1}{1+\beta}} (1-\hat{c})^{\frac{\beta}{1+\beta}}, \quad (7)$$

$$\hat{c} = \arg\max_{c} c^{\frac{1}{1+\beta}} (1-c)^{\frac{\beta}{1+\beta}}, \quad (8)$$

$\tilde{r}$ is given by (5) and

$$\hat{\gamma} = \arg\max_{\gamma} \left[ E\left( (r_f + \gamma \tilde{x}_m)^{\frac{\beta(1-\alpha)}{1+\beta}} \right) \right].$$

If there is no riskless asset, $\hat{c}$ is again the solution to (8) and

$$\begin{aligned} V(W) &= \max_{(c,\gamma)} \left( E\left[cW\right]^{\frac{1-\alpha}{1+\beta}} \left[W(1-c)\tilde{r}_m\right]^{\frac{\beta(1-\alpha)}{1+\beta}} \right)^{\frac{1}{1-\alpha}} \\ &= vW \end{aligned}$$

where now $v$ is given by (7) with $\tilde{r} = \tilde{r}_m$.

## 2.3 A Two Period Problem: EZW, Kreps-Porteus

In this subsection, we illustrate the difference between our expected utility maximizing approach and that taken by EZW Kreps-Porteus. Following Epstein-Zin and Weil we assume that the preferences are CES and that relative risk aversion is constant and denoted by $\alpha$. As in the above discussion, we only consider the case $\alpha \neq 1$. In this case, when the elasticity of substitution, $\sigma$, is unequal to one and

$$\rho \neq 0,$$

the EZW Kreps-Porteus approach is to solve the problem

$$\max_{(c,\gamma)} \left[ \left( (cW)^{\rho} + \beta \left[ E\left( (W[1-c][r_f + \gamma \tilde{x}_m])^{1-\alpha} \right) \right]^{\frac{\rho}{1-\alpha}} \right)^{\frac{1}{\rho}} \right].$$

The solution is

$$(\hat{c}, \hat{\gamma}) = \arg\max_{(c,\gamma)} \left[ \left( c^{\rho} + \beta [1-c]^{\rho} \left[ E\left( [r_f + \gamma \tilde{x}_m]^{1-\alpha} \right) \right]^{\frac{\rho}{1-\alpha}} \right)^{\frac{1}{\rho}} \right].$$

Note that $\hat{\gamma}$ is simply the solution to

$$\begin{aligned} \hat{\gamma} &= \arg\max_{\gamma} \left[ E\left[r_f + \gamma \tilde{x}_m\right]^{1-\alpha} \right]^{\frac{1}{1-\alpha}} \\ &= \arg\max_{\gamma} \left[ \frac{E\left[r_f + \gamma \tilde{x}_m\right]^{1-\alpha}}{1-\alpha} \right], \end{aligned} \tag{9}$$

and does depend on $\alpha$ but not $\rho$. In this case,

$$\begin{aligned} \hat{c} &= \arg\max_{c \in [0,1]} \left[ \left( c^{\rho} + \beta [1-c]^{\rho} \left( E\tilde{r}^{1-\alpha} \right)^{\frac{\rho}{1-\alpha}} \right)^{\frac{1}{\rho}} \right] \\ &= \frac{\left( \beta \left( E\tilde{r}^{1-\alpha} \right)^{\frac{\rho}{1-\alpha}} \right)^{\frac{1}{\rho-1}}}{\left( 1 + \left( \beta \left( E\tilde{r}^{1-\alpha} \right)^{\frac{\rho}{1-\alpha}} \right)^{\frac{1}{\rho-1}} \right)} \end{aligned} \tag{10}$$

$\tilde{r}$ is given by (5).

In the EZW, Kreps-Porteus approach

$$V(W) = \max_{(c,\gamma)} \left[ \left( (cW)^{\rho} + \beta \left[ E\left( \left( W [1-c] [r_f + \gamma \tilde{x}_m] \right)^{1-\alpha} \right) \right]^{\frac{\rho}{1-\alpha}} \right)^{\frac{1}{\rho}} \right]$$

and

$$V(W) = vW \tag{11}$$

where

$$v = \left[ \left( \hat{c}^{\rho} + \beta [1-\hat{c}]^{\rho} \left[ E\left( \tilde{r}^{1-\alpha} \right) \right]^{\frac{\rho}{1-\alpha}} \right)^{\frac{1}{\rho}} \right]. \tag{12}$$

Equations (11) and (12) are the EZW, Kreps-Porteus analogs of (3) and (4.)

When there is no riskless asset, $\hat{c}$ and $v$ are still given by (10) and (12) respectively, but, in this case,

$$\tilde{r} = \tilde{r}_m.$$

When the elasticity of substitution, $\sigma$, equals one and

$$\rho = 0$$

the EZW Kreps-Porteus approach is used the consumer-investors problem is

$$\max_{(c,\gamma)} \left[ \log(cW) + \beta \log \left( \left[ E\left( \left( W [1-c] [r_f + \gamma \tilde{x}_m] \right)^{1-\alpha} \right) \right]^{\frac{1}{1-\alpha}} \right) \right].$$

The solution is

$$\begin{aligned} (\hat{c}, \hat{\gamma}) &= \arg\max_{(c,\gamma)} \left[ \log c + \beta \log \left[ [1-c] \, E\left( [r_f + \gamma \tilde{x}_m]^{1-\alpha} \right) \right]^{\frac{1}{1-\alpha}} \right] \\ &= \arg\max_{(c,\gamma)} \left[ c^{\frac{1}{1+\beta}} [1-c]^{\frac{\beta}{1+\beta}} \left[ E\left( \left( [r_f + \gamma \tilde{x}_m] \right)^{1-\alpha} \right) \right]^{\frac{\beta}{(1+\beta)(1-\alpha)}} \right]. \end{aligned}$$

$\hat{c}$ is again the solution to (8) and $\hat{\gamma}$ is again given by (9). In this case,

$$V(W) = \max_{(c,\gamma)} \left[ c^{\frac{1}{1+\beta}} [1-c]^{\frac{\beta}{1+\beta}} \left[ E\left( \left( [r_f + \gamma \tilde{x}_m] \right)^{1-\alpha} \right) \right]^{\frac{\beta}{(1+\beta)(1-\alpha)}} \right]$$

and $V(W)$ is given by (11) where now

$$v = \left[ E\left( \tilde{r}^{1-\alpha} \right) \right]^{\frac{\beta}{(1+\beta)(1-\alpha)}} \hat{c}^{\frac{1}{1+\beta}} [1-\hat{c}]^{\frac{\beta}{1+\beta}} \tag{13}$$

and $\tilde{r}$ is given by (5). Equation (13) is the EZW, Kreps-Porteus analog of (7).

When there is no riskless asset, $\hat{c}$ is again the solution to (8) and $v$ is given by (13) in which $\tilde{r} = \tilde{r}_m$.

When there is no risky asset, then EZW, Kreps-Porteus and Kihlstrom Mirman consumer-investors both choose

$$\begin{aligned} \hat{c} &= \arg\max_{c \in [0,1]} \left[ \left( (cW)^{\rho} + \beta [W(1-c) r_f]^{\rho} \right)^{\frac{1}{\rho}} \right] \\ &= \arg\max_{c \in [0,1]} \left[ \left( c^{\rho} + \beta [(1-c) r_f]^{\rho} \right)^{\frac{1}{\rho}} \right] \end{aligned}$$

when

$$\rho \neq 0.$$

# 3 The Three Period Case: The "Consistent Planning" Approach with "Forward Looking" von Neuman Morganstern Preferences

We now suppose that the consumer-investor lives for three periods and begins the first period with initial wealth, $W_1$. His consumption in period $t$ is $C_t$. In period 3,

$$C_3 = W_3.$$

In periods $t = 1$ and $t = 2$,

$$C_t = c_t W_t.$$

where $c_t$ is the fraction of period $t$ wealth, $W_t$, consumed. So in periods $t = 1$ and $t = 2$, the consumer-investor saves equal $(1 - c_t) W_t$, and can invest in a safe and a risky asset. Thus, in each of the first two periods, the consumer-investor must also choose the fraction, $\gamma_t$, of first period savings to invest in the risky asset. The safe asset return is $r_f$. The realized return on the risky asset is $r_t$. The random variable that denotes the risky asset return is $\tilde{r}_t$. We again define the realized excess return on the risky asset as

$$x_t = r_t - r_f.$$

This is again the realization of the random variable

$$\tilde{x}_t = \tilde{r}_t - r_f.$$

Period $t+1$ wealth is then

$$W_{t+1} = W_t [1 - c_t] [r_f + \gamma_t \tilde{x}_t]. \tag{14}$$

In discussing, forward looking preferences we are going to assume that, in each period, the preferences are CES and exhibit CRRA risk aversion in the Kihlstrom-Mirman sense. Furthermore, the elasticity of substitution and the level of relative risk aversion are the same each period. Thus, in the first period, the consumer-investor maximizes

$$EU\left(C_1, \tilde{C}_2, \tilde{C}_3\right) = \frac{1}{1-\alpha} E\left[\left(C_1^\rho + \beta \tilde{C}_2^\rho + \beta^2 \tilde{C}_3^\rho\right)^{\frac{1-\alpha}{\rho}}\right].$$

In the second period, the consumer-investor maximizes

$$EU\left(C_2, \tilde{C}_3\right) = \frac{1}{1-\alpha} E\left[\left(C_2^\rho + \beta \tilde{C}_3^\rho\right)^{\frac{1-\alpha}{\rho}}\right].$$

In this situation, the need for "consistent planning" arises when

$$1 - \rho \neq \alpha$$

and when the consumer cannot commit to the choice of $(c_2, \gamma_2)$ in period one. To see why this is so, let's consider the case in which commitment is possible. In that case, the consumer would, in the first period, choose

$$\begin{aligned} & \left(\hat{c}_1, \hat{\gamma}_1, \hat{c}_2(\cdot), \hat{\gamma}_2(\cdot)\right) \\ = & \underset{(c_1,\gamma_1,c_2(\cdot),\gamma_2(\cdot))}{\arg\max}\; W_1^\rho \left[\frac{1}{1-\alpha}\right] E\left[\left(c_1^\rho + \beta\left[1-c_1\right]^\rho \left[r_f + \gamma_1 \tilde{x}_1\right]^\rho \left[c_2^\rho + \beta\left[1-c_2\right]^\rho \left[r_f + \gamma_2 x_2\right]^\rho\right]\right)^{\frac{1-\alpha}{\rho}}\right] \end{aligned}$$

where

$$C_1 = c_1 W_1,$$

$$\begin{aligned} C_2 &= c_2 W_2 \\ &= c_2 W_1 \left[1-c_1\right]\left[r_f + \gamma_1 x_1\right] \end{aligned}$$

and

$$\begin{aligned} C_3 &= W_2\left[1-c_2\right]\left[r_f+\gamma_2 x_2\right] \\ &= W_1\left[1-c_1\right]\left[r_f+\gamma_1 x_1\right]\left[1-c_2\right]\left[r_f+\gamma_2 x_2\right]. \end{aligned}$$

The definition of $\left(\hat{c}_1, \hat{\gamma}_1, \hat{c}_2(\cdot), \hat{\gamma}_2(\cdot)\right)$ can be more compactly stated as

$$\begin{aligned} & \left(\hat{c}_1, \hat{\gamma}_1, \hat{c}_2(\cdot), \hat{\gamma}_2(\cdot)\right) \\ = & \underset{(c_1,\gamma_1,c_2(\cdot),\gamma_2(\cdot))}{\arg\max} \left[\frac{1}{1-\alpha}\right] E\left[\left(c_1^\rho + \beta\left[1-c_1\right]^\rho\left[r_f+\gamma_1\tilde{x}_1\right]^\rho v\left(c_2(\tilde{x}_1), \gamma_2(\tilde{x}_1), \tilde{x}_2\right)\right)^{\frac{1-\alpha}{\rho}}\right] \end{aligned}$$

where

$$v\left(c_2, \gamma_2, x_2\right) = c_2^\rho + \beta\left[1-c_2\right]^\rho\left[r_f+\gamma_2 x_2\right]^\rho.$$

Note that, for each $x_1$,

$$\begin{aligned} & \left(\hat{c}_2(x_1), \hat{\gamma}_2(x_1)\right) \\ = & \underset{(c_2,\gamma_2)}{\arg\max} \left[\frac{1}{1-\alpha}\right] E\left[\left(\hat{c}_1^\rho + \beta\left[1-\hat{c}_1\right]^\rho\left[r_f+\hat{\gamma}_1 x_1\right]^\rho\left(c_2^\rho+\beta\left[1-c_2\right]^\rho\left[r_f+\gamma_2\tilde{x}_2\right]^\rho\right)\right)^{\frac{1-\alpha}{\rho}}\right] \\ = & \underset{(c_2,\gamma_2)}{\arg\max} \left[\frac{1}{1-\alpha}\right] E\left[\left(A(x_1) + \left(c_2^\rho+\beta\left[1-c_2\right]^\rho\left[r_f+\gamma_2\tilde{x}_2\right]^\rho\right)\right)^{\frac{1-\alpha}{\rho}}\right] \end{aligned}$$

where

$$A(x_1) = \frac{\hat{c}_1^\rho}{\beta\left[1-\hat{c}_1\right]^\rho\left[r_f+\hat{\gamma}_1 x_1\right]^\rho}.$$

When

$$1-\rho \neq \alpha,$$

$$\left(\hat{c}_2(x_1), \hat{\gamma}_2(x_1)\right)$$

varies with $x_1$.

Now suppose that commitment is impossible. In that case, the forward looking von Neuman-Morganstern consumer-investor is able, in period 2, to choose

$$\begin{aligned}(c_2^*, \gamma_2^*) &= \arg\max_{(c_2,\gamma_2)} \left[\frac{1}{1-\alpha}\right] W_2^\rho E\left[\left(c_2^\rho + \beta\left[1-c_2\right]^\rho \left[r_f + \gamma_2 \tilde{x}_2\right]^\rho\right)^{\frac{1-\alpha}{\rho}}\right] \\ &= \arg\max_{(c_2,\gamma_2)} \left[\frac{1}{1-\alpha}\right] E\left[\left(c_2^\rho + \beta\left[1-c_2\right]^\rho \left[r_f + \gamma_2 \tilde{x}_2\right]^\rho\right)^{\frac{1-\alpha}{\rho}}\right].\end{aligned}$$

Since $(c_2^*, \gamma_2^*)$ is independent of $W_2$ it is also is independent of $x_1$,

$$(c_2^*, \gamma_2^*) \neq (\hat{c}_2(x_1), \hat{\gamma}_2(x_1))$$

unless

$$1 - \rho = \alpha,$$

in which case, it is easily seen that the usual dynamic programming argument implies that

$$(c_2^*, \gamma_2^*) = (\hat{c}_2(x_1), \hat{\gamma}_2(x_1)).$$

How should the forward looking von Neuman-Morganstern consumer-investor make his choice in the first period if he knows that he will choose $(c_2^*, \gamma_2^*)$ in the second period? As Strotz suggests we assume that he simply treats his "future misbehavior" as a constraint and makes a first period choice that is, in the game theoretic language of Peleg and Yaari, a best response to $(c_2^*, \gamma_2^*)$. The best response to $(c_2^*, \gamma_2^*)$ is

$$\begin{aligned}&(c_1^*, \gamma_1^*) \\ =\ & \arg\max_{(c_1,\gamma_1)} W_1^\rho \left[\frac{1}{1-\alpha}\right] E\left[\left(c_1^\rho + \beta\left[1-c_1\right]^\rho \left[r_f + \gamma_1 \tilde{x}_1\right]^\rho v\left(c_2^*, \gamma_2^*, \tilde{x}_2\right)\right)^{\frac{1-\alpha}{\rho}}\right] \\ =\ & \arg\max_{(c_1,\gamma_1)} \left[\frac{1}{1-\alpha}\right] E\left[\left(c_1^\rho + \beta\left[1-c_1\right]^\rho \left[r_f + \gamma_1 \tilde{x}_1\right]^\rho v\left(c_2^*, \gamma_2^*, \tilde{x}_2\right)\right)^{\frac{1-\alpha}{\rho}}\right].\end{aligned}$$

It is natural to ask how the consumer-investor's second period choice is affected by his ability to commit to a choice in the first period. In fact we can demonstrate the following proposition.

**Proposition 1** *Suppose that*

$$1 - \rho > (<)\, \alpha.$$

*In that case, for all $x_1$, the consumer-investor acts as if he were more (less) risk averse in the Kihlstrom-Mirman sense when he commits to*

$$(\hat{c}_2(x_1), \hat{\gamma}_2(x_1))$$

*in the first period than he does when he chooses*

$$(c_2^*, \gamma_2^*)$$

*in the second period. Also, in this case, increases in $x_1$ cause the consumer-investor to act as if he were less (more) risk averse in the Kihlstrom-Mirman sense when he commits to*

$$\left(\hat{c}_2\left(x_1\right), \hat{\gamma}_2\left(x_1\right)\right)$$

*in the first period.*

The proof of this proposition is in Appendix 1.

# 4 The Infinite Horizon Case:

We now suppose that the consumer-investor looks forward to an infinite lifetime. He begins the period $t$ with initial wealth, $W_t$. His consumption in period $t$ is $C_t$. The fraction of $W_t$ consumed in the first consumption period is $c_t$. In each period, savings, which equal $(1-c_t)\,W\;_t$, can be invested in a safe and a risky asset. Thus, in each period, the consumer-investor must also choose the fraction, $\gamma_t$, of period $t$ savings to invest in the risky asset. In each period, the safe asset return is $r_f$. The realized return on the risky asset is $r_t$. The random variable of which $r_t$ is the realization is $\tilde{r}_t$. We assume that the $\tilde{r}_t$'s are *i.i.d.*. We again define the realized excess return on the risky asset as

$$x_t = r_t - r_f.$$

This is again the realization of the random variable

$$\tilde{x}_t = \tilde{r}_t - r_f.$$

Period $t+1$ wealth is

$$W_{t+1} = W_t\left[1-c_t\right]\left[r_f + \gamma_t \tilde{x}_t\right].$$

and consumption in the period $t$ is

$$C_t = c_t W_t.$$

## 4.1 The "Consistent Planning" Approach with "Forward Looking" von Neumann Morganstern Preferences

In discussing, forward looking preferences in this infinite horizon setting, we are once again going to assume that, in each period, the preferences are CES and exhibit CRRA risk aversion in the Kihlstrom-Mirman sense. Furthermore, the elasticity of substitution and the level of relative risk aversion are the same each period.

Thus, in period $t$, the forward looking von Neuman-Morganstern consumer-investor maximizes

$$EU\left(C_t, \left\{\tilde{C}_{t+\tau}\right\}_{\tau=1}^{\infty}\right) = \frac{1}{1-\alpha} E\left[\left(C_t^{\rho} + \sum_{\tau=1}^{\infty} \beta^{\tau} \tilde{C}_{t+\tau}^{\rho}\right)^{\frac{1-\alpha}{\rho}}\right].$$

The risk neutral case is that in which

$$U\left(C_t, \{C_{t+\tau}\}_{\tau=1}^{\infty}\right) = \left(C_t^{\rho} + \sum_{\tau=1}^{\infty} \beta^{\tau} C_{t+\tau}^{\rho}\right)^{\frac{1}{\rho}}.$$

The Kihlstrom-Mirman relative risk aversion of

$$U\left(C_t, \{C_{t+\tau}\}_{\tau=1}^{\infty}\right) = \frac{1}{1-\alpha}\left(C_t^{\rho} + \sum_{\tau=1}^{\infty} \beta^{\tau} C_{t+\tau}^{\rho}\right)^{\frac{1-\alpha}{\rho}} \tag{15}$$

is $\alpha$. For all $\alpha$, the elasticity of substitution of (15) is

$$\sigma = \frac{1}{1-\rho}.$$

In this situation, the need for "consistent planning" again arises when

$$1-\rho \neq \alpha$$

and when the consumer cannot commit to the choice of $\left(c_{t+\tau}, \gamma_{t+\tau}\right)$ in period $\tau > 0$. Thus, when the consumer chooses $(c_t, \gamma_t)$ in period $t$ he anticipates his future choices $\left\{\left(c_{t+\tau}, \gamma_{t+\tau}\right)\right\}_{\tau=1}^{\infty}$. Because the consumer investor faces the same problem in every period, the equilibrium is one in which his choice in every period is the same. Specifically, for all $t$, $(c_t, \gamma_t) = (\hat{c}, \hat{\gamma})$ where $(c_t, \gamma_t) = (\hat{c}, \hat{\gamma})$ is a best response to the fact that for all, $\tau > 0$, $\left(c_{t+\tau}, \gamma_{t+\tau}\right) = (\hat{c}, \hat{\gamma})$. Thus,

$$\begin{aligned} & (\hat{c}, \hat{\gamma}) \qquad (16) \\ = \;& \arg\max_{(c,\gamma)} \left(E\left[\left((cW_t)^{\rho} + \beta W_t^{\rho}\left[1-c\right]^{\rho}\left[r_f + \gamma \tilde{x}_t\right]^{\rho}\left(v\left(\{\tilde{x}_{t+\tau}\}_{\tau=1}^{\infty}, (\hat{c}, \hat{\gamma})\right)\right)\right)^{\frac{1-\alpha}{\rho}}\right]\right)^{\frac{1}{1-\alpha}} \\ = \;& \arg\max_{(c,\gamma)} \left(E\left[\left(c^{\rho} + \beta\left[1-c\right]^{\rho}\left[r_f + \gamma \tilde{x}_t\right]^{\rho} v\left(\{\tilde{x}_{t+\tau}\}_{\tau=1}^{\infty}, (\hat{c}, \hat{\gamma})\right)\right)^{\frac{1-\alpha}{\rho}}\right]\right)^{\frac{1}{1-\alpha}} \end{aligned}$$

where, for each possible sample path of excess returns, $\{x_{t+\tau}\}_{\tau=1}^{\infty}$,

$$\begin{aligned} & v\left(\{x_{t+\tau}\}_{\tau=1}^{\infty}, (c,\gamma)\right) \qquad (17) \\ = \;& c^{\rho} + \beta\left[1-c\right]^{\rho}\left[r_f + \gamma x_{t+1}\right]^{\rho} v\left(\{x_{t+1+\tau}\}_{\tau=1}^{\infty}, (c,\gamma)\right). \end{aligned}$$

It is easy to check that when

$$v\left(\{x_{t+\tau}\}_{\tau=1}^{\infty}, (c,\gamma)\right) = c^{\rho}\left[1 + \sum_{\tau=1}^{\infty} \beta^{\tau}\left[1-c\right]^{\rho\tau} \prod_{s=1}^{\tau}\left[r_f + \gamma x_{t+s}\right]^{\rho}\right]$$

(17) holds.

Since we are going to apply this approach to derive the equilibrium in the Lucas asset pricing model, it is useful to describe the consistent planning equilibrium for the case in which there is no riskless asset. In this case, the consumer simply chooses $c_t$ in each period. Once again, the consumer-investor faces the same problem in every period, and the equilibrium is one in which his choice in every period is the same. Specifically, for all $t$, $c_t = \hat{c}$ where $c_t = \hat{c}$ is a best response to the fact that for all, $\tau > 0$, $c_{t+\tau} = \hat{c}$. Thus,

$$\begin{aligned}
\hat{c} &= \arg\max_{c} \left( E\left[ \left( (cW_t)^{\rho} + \beta W_t^{\rho} [1-c]^{\rho} \tilde{r}_t^{\rho} \left( v\left( \{\tilde{r}_{t+\tau}\}_{\tau=1}^{\infty}, \hat{c} \right) \right) \right)^{\frac{1-\alpha}{\rho}} \right] \right)^{\frac{1}{1-\alpha}} \\
&= \arg\max_{(c,\gamma)} \left( E\left[ \left( c^{\rho} + \beta [1-c]^{\rho} \tilde{r}_t^{\rho} \left( v\left( \{\tilde{r}_{t+\tau}\}_{\tau=1}^{\infty}, \hat{c} \right) \right) \right)^{\frac{1-\alpha}{\rho}} \right] \right)^{\frac{1}{1-\alpha}}
\end{aligned}$$

where, for each sequence, $\{r_{t+\tau}\}_{\tau=1}^{\infty}$,

$$v\left( \{r_{t+\tau}\}_{\tau=1}^{\infty}, c \right) = c^{\rho} + \beta [1-c]^{\rho} r_{t+1}^{\rho} v\left( \{r_{t+1+\tau}\}_{\tau=1}^{\infty}, c \right). \tag{18}$$

It is easy to check that when

$$v\left( \{r_{t+\tau}\}_{\tau=1}^{\infty}, c \right) = c^{\rho} \left[ 1 + \sum_{\tau=1}^{\infty} \beta^{\tau} [1-c]^{\rho\tau} \prod_{s=1}^{\tau} r_{t+s}^{\rho} \right] \tag{19}$$

(18) holds.

Andrew Postlewaite has pointed out that when we follow Peleg and Yaari and interpret the equilibrium just described as an equilibrium of the Stackleberg game between the current consumer and the future consumer-investors, this equilibrium is not unique. The problem is that, in infinite games of this type, subgame perfect equilibria can exist in which the current consumer-investor anticipates punishment by the future consumer-investors if he does not deviate in particular ways from the equilibrium just described. Nevertheless, the equilibrium just described does seem to be the natural one to consider. In fact, Andreu Mas Colell has pointed out that this non-uniqueness problem arises even in the additively separable case. In that case, it has never been an obstacle to focusing on the outcome just described. We describe that case in the next subsection.

### 4.1.1 The Additively Separable Case

As we have noted, the additively separable case is a special case of the model just described. This case arises in our approach when

$$1 - \alpha = \rho$$

which, of course, implies that

$$\sigma = \frac{1}{\alpha}.$$

In that case, the fact the $\{\tilde{x}_{t+\tau}\}_{\tau=0}^{\infty}$ are *i.i.d.* implies that

$$\begin{aligned}
& E\left[ \left( (cW_t)^{\rho} + \beta W_t^{\rho} [1-c]^{\rho} [r_f + \gamma \tilde{x}_t]^{\rho} \left( v\left( \{\tilde{x}_{t+\tau}\}_{\tau=1}^{\infty}, (\hat{c}, \hat{\gamma}) \right) \right) \right)^{\frac{1-\alpha}{\rho}} \right] \\
= & \left[ \left( (cW_t)^{1-\alpha} + \beta W_t^{1-\alpha} [1-c]^{1-\alpha} E[r_f + \gamma \tilde{x}_t]^{1-\alpha} E\left( v\left( \{\tilde{x}_{t+\tau}\}_{\tau=1}^{\infty}, (\hat{c}, \hat{\gamma}) \right) \right) \right) \right].
\end{aligned}$$

Thus, (16) reduces to

$$
\begin{aligned}
& (\hat{c}, \hat{\gamma}) && (20) \\
= \; & \arg\max_{(c,\gamma)} \left[ (cW_t)^{1-\alpha} + \beta v^{1-\alpha} W_t^{1-\alpha} [1-c]^{1-\alpha} E\left[r_f + \gamma \tilde{x}_t\right]^{1-\alpha} \right]^{\frac{1}{1-\alpha}} \\
= \; & \arg\max_{(c,\gamma)} \left[ c^{1-\alpha} + \beta v^{1-\alpha} [1-c]^{1-\alpha} E\left[r_f + \gamma \tilde{x}_t\right]^{1-\alpha} \right]^{\frac{1}{1-\alpha}} \\
= \; & \arg\max_{(c,\gamma)} \left[ \frac{1}{1-\alpha} \right] \left[ c^{1-\alpha} + \beta v^{1-\alpha} [1-c]^{1-\alpha} E\left[r_f + \gamma \tilde{x}_t\right]^{1-\alpha} \right]
\end{aligned}
$$

where

$$
\begin{aligned}
v^{1-\alpha} &= E\left(v\left(\{\tilde{x}_{t+\tau}\}_{\tau=1}^{\infty}, (\hat{c}, \hat{\gamma})\right)\right) && (21) \\
&= \hat{c}^{1-\alpha} \left[ 1 + \sum_{\tau=1}^{\infty} \beta^{\tau} [1-\hat{c}]^{(1-\alpha)\tau} \left(E\tilde{r}^{1-\alpha}\right)^{\tau} \right] \\
&= \hat{c}^{1-\alpha} \left[ \frac{1}{1 - \beta [1-\hat{c}]^{(1-\alpha)} E\tilde{r}^{1-\alpha}} \right],
\end{aligned}
$$

$$\hat{\gamma} = \arg\max_{\gamma} \left[ \frac{1}{1-\alpha} \right] E\tilde{r}^{1-\alpha}.$$

and

$$\tilde{r} = r_f + \hat{\gamma} \tilde{x}_t$$

The solution is

$$\hat{c} = \frac{1}{1 + \left(\beta v^{1-\alpha} E\tilde{r}^{1-\alpha}\right)^{\frac{1}{\alpha}}}. \tag{22}$$

Substituting (22) in (21) and solving for $v$ the result is

$$v^{1-\alpha} = \left[ 1 - \left(\beta E\tilde{r}^{1-\alpha}\right)^{\frac{1}{\alpha}} \right]^{-\alpha}. \tag{23}$$

Substituting this in (22) yields

$$\hat{c} = \left[ 1 - \left(\beta E\tilde{r}^{1-\alpha}\right)^{\frac{1}{\alpha}} \right]. \tag{24}$$

Note that $\hat{c}$ is positive only if

$$\frac{1}{\beta} > E\tilde{r}^{1-\alpha}.$$

When $v$ and $\hat{c}$ satisfy (23) and (24)

$$v^{1-\alpha} = \left[ \hat{c}^{1-\alpha} + \beta v^{1-\alpha} [1-\hat{c}]^{1-\alpha} E\tilde{r}^{1-\alpha} \right]$$

so that

$$v = \max_{(c,\gamma)} \left[ c^{1-\alpha} + \beta v^{1-\alpha} [1-c]^{1-\alpha} E\left[r_f + \gamma \tilde{x}_t\right]^{1-\alpha} \right]^{\frac{1}{1-\alpha}}. \tag{25}$$

## 4.2 Epstein-Zin and Weil

In the current setting, the EZW approach yields an equilibrium in which the optimal choice for $(c_t, \gamma_t)$ in each period, $t$, is

$$(\hat{c}, \hat{\gamma}) = \underset{(c,\gamma)}{\arg\max} \left[ \left( (cW)^{\rho} + \beta \left[ E \left( \left[ V \left( W \left[ 1 - c \right] \left[ r + \gamma \tilde{x}_t \right] \right) \right]^{1-\alpha} \right) \right]^{\frac{\rho}{1-\alpha}} \right)^{\frac{1}{\rho}} \right]$$

where $V(\cdot)$ solves the functional equation

$$V(W) = \max_{(c,\gamma)} \left[ \left( (cW)^{\rho} + \beta \left[ E \left( \left[ V \left( W \left[ 1 - c \right] \left[ r_f + \gamma \tilde{x}_t \right] \right) \right]^{1-\alpha} \right) \right]^{\frac{\rho}{1-\alpha}} \right)^{\frac{1}{\rho}} \right]. \tag{26}$$

The solution to the functional equation is

$$V(W) = vW. \tag{27}$$

Using (27), (26) becomes

$$vW = \max_{(c,\gamma)} \left[ \left( (cW)^{\rho} + \beta \left[ E \left( \left( vW \left[ 1 - c \right] \left[ r_f + \gamma \tilde{x}_t \right] \right)^{1-\alpha} \right) \right]^{\frac{\rho}{1-\alpha}} \right)^{\frac{1}{\rho}} \right].$$

So solving the functional equation reduces to finding $v$ that solves the equation

$$v = \max_{(c,\gamma)} \left[ \left( c^{\rho} + \beta v^{\rho} \left[ 1 - c \right]^{\rho} \left( E \left[ r_f + \gamma \tilde{x}_t \right]^{1-\alpha} \right)^{\frac{\rho}{1-\alpha}} \right)^{\frac{1}{\rho}} \right]. \tag{28}$$

A well known advantage of the EZW is its tractability. In this example,

$$(\hat{c}, \hat{\gamma}) = \underset{(c,\gamma)}{\arg\max} \left[ \left( c^{\rho} + \beta v^{\rho} \left[ 1 - c \right]^{\rho} \left( E \left[ r_f + \gamma \tilde{x}_t \right]^{1-\alpha} \right)^{\frac{\rho}{1-\alpha}} \right)^{\frac{1}{\rho}} \right]. \tag{29}$$

and the solution can be found by first noting that

$$\begin{aligned} \hat{\gamma} &= \underset{\gamma}{\arg\max} \left[ E \left[ r_f + \gamma \tilde{x}_t \right]^{1-\alpha} \right]^{\frac{1}{1-\alpha}} \\ &= \underset{\gamma}{\arg\max} \left[ \frac{E \left[ r_f + \gamma \tilde{x}_t \right]^{1-\alpha}}{1-\alpha} \right], \end{aligned} \tag{30}$$

and

$$\begin{aligned} \hat{c} &= \underset{c}{\arg\max} \left[ \left( c^{\rho} + \beta v^{\rho} \left[ 1 - c \right]^{\rho} \left( E \tilde{r}^{1-\alpha} \right)^{\frac{\rho}{1-\alpha}} \right)^{\frac{1}{\rho}} \right] \\ &= \frac{1}{1 + \left( \beta v^{\rho} \left( E \tilde{r}^{1-\alpha} \right)^{\frac{\rho}{1-\alpha}} \right)^{\frac{1}{1-\rho}}}, \end{aligned} \tag{31}$$

where

$$\tilde{r} = r_f + \hat{\gamma}\tilde{x}_t. \tag{32}$$

It is clear from (30) that $\hat{\gamma}$ depends only on the relative risk aversion measure $\alpha$ and is independent of $\rho$. As mentioned in the introduction, this is a feature of the EZW optimal portfolio has been noted by a number of authors.

Using the expression (31) that determines $\hat{c}$ we can solve (28) for $v$ to get

$$v = \left[1 - \left(\beta\left(E\tilde{r}^{1-\alpha}\right)^{\frac{\rho}{1-\alpha}}\right)^{\frac{1}{1-\rho}}\right]^{\frac{\rho-1}{\rho}} \tag{33}$$

which is positive if and only if

$$\frac{1}{\beta} > \left(E\tilde{r}^{1-\alpha}\right)^{\frac{\rho}{1-\alpha}}.$$

Thus, $V(W)$ becomes

$$V(W) = \left[1 - \left(\beta\left(E\tilde{r}^{1-\alpha}\right)^{\frac{\rho}{1-\alpha}}\right)^{\frac{1}{1-\rho}}\right]^{\frac{\rho-1}{\rho}} W.$$

Also the expression for $\hat{c}$ simplifies to

$$\hat{c} = \left[1 - \left(\beta\left(E\tilde{r}^{1-\alpha}\right)^{\frac{\rho}{1-\alpha}}\right)^{\frac{1}{1-\rho}}\right]. \tag{34}$$

In this approach, $\alpha$ is interpreted as the measure of relative risk aversion. The elasticity of substitution is of course

$$\sigma = \frac{1}{1-\rho}.$$

When the EZW approach is applied to derive the equilibrium in the Lucas asset pricing model, there is no riskless asset. For this case, in each period, the consumer simply chooses $c_t = \hat{c}$ where $\hat{c}$ satisfies (34) and

$$\tilde{r} = \tilde{r}_t. \tag{35}$$

In this case, (28) is replaced by

$$v = \max_c \left[\left(c^\rho + \beta v^\rho \left[1-c\right]^\rho \left(E\tilde{r}_t^{1-\alpha}\right)^{\frac{\rho}{1-\alpha}}\right)^{\frac{1}{\rho}}\right].$$

In this case, $v$ is again given by (33) where now $\tilde{r}$ is as in (35).

### 4.2.1 The Riskless Rate When the Riskless Asset is in Zero Net Supply

Before leaving the discussion of the EZW approach, it is useful to interpret the EZW equilibrium just described for the case of no riskless asset as the equilibrium of a constant returns to scale production economy in which $\tilde{r}$ in the marginal product of capital. When a risk-free asset is introduced but is in zero net supply, the return on the risk-free asset, $r_f$,will have to be such that

$$1 = \hat{\gamma} = \arg\max_{\gamma} \left[ E\left[r_f + \gamma \tilde{x}_t\right]^{1-\alpha} \right]^{\frac{1}{1-\alpha}}.$$

The first-order condition satisfied at $1 = \hat{\gamma}$ implies that

$$r_f = E^* \tilde{r}_t \tag{36}$$

where

$$E^* \tilde{r}_t = \frac{E\tilde{r}_t^{1-\alpha}}{E\tilde{r}_t^{-\alpha}} \tag{37}$$

is the risk neutral expected value of $\tilde{r}_t$. Note that, as Weil asserted, the equity risk premium

$$\frac{E\tilde{r}_t}{r_f} = \frac{E\tilde{r}_t}{E^*\tilde{r}_t}$$

is independent of $\rho$. This is the reason that Weil asserted that "If the dividend growth process is i.i.d., the risk-premium, when appropriately defined, is independent of the intertemporal elasticity of substitution, and thus is the same whether or not the time-additive, expected utility restriction is imposed."

### 4.2.2 The Additively Separable Case

This case arises when

$$1 - \alpha = \rho$$

which, of course, implies that

$$\sigma = \frac{1}{\alpha}.$$

In this case, (33) and (34) reduce to (23) and (24) respectively. Also, (28) and (29) reduce to (25) and (20) respectively.

## 5 The Infinite Period Lucas Asset Pricing Equilibrium

Assume that the Lucas tree dividends are $\tilde{s}_t$ and that

$$\tilde{s}_{t+1} = \tilde{g}_{t+1} s_t.$$

We assume that the dividend growth rates $\tilde{g}_t$ are *i.i.d.* and that

$$\Pr\left(\tilde{g}_t > 0\right) = 1.$$

Let $\tilde{g}$ be a random variable with the same distribution as $\tilde{g}_t$ for all $t$. Let $P\left(s_t\right)$ be the period $t$ price of the tree. Then the return on savings invested at time $t$ is

$$r\left(s_{t+1}, s_t\right) = \frac{s_{t+1} + P\left(s_{t+1}\right)}{P\left(s_t\right)} \tag{38}$$

Now, in equilibrium, wealth in period $t$ is

$$W_t\left(s_t\right) = s_t + P\left(s_t\right),$$

consumption in period $t$ is $s_t$ and savings in period $t$ is $P\left(s_t\right)$.

In what follows, the assumption that the growth rates $\tilde{g}_t$ are *i.i.d.* will make it possible to describe an equilibrium in which $c_t$ is the same in every period $t$. When, for all $t$, $c_t = \hat{c}$, then

$$\hat{c} = \frac{s_t}{s_t + P\left(s_t\right)}$$

and

$$1 - \hat{c} = \frac{P\left(s_t\right)}{s_t + P\left(s_t\right)}.$$

This implies that the price dividend ratio is

$$\frac{P\left(s_t\right)}{s_t} = \frac{1-\hat{c}}{\hat{c}}$$

and that

$$P\left(s_t\right) = \frac{1-\hat{c}}{\hat{c}} s_t. \tag{39}$$

Substituting (39) in (38) we get that

$$\begin{aligned} r_{t+1} &= r\left(s_{t+1}, s_t\right) \\ &= \frac{1}{1-\hat{c}} \frac{s_{t+1}}{s_t} \\ &= \frac{1}{1-\hat{c}} g_{t+1}. \end{aligned} \tag{40}$$

## 5.1 The "Consistent Planning" Approach with "Forward Looking" von Neumann Morganstern Preferences

In this case, for all $t$, $c_t = \hat{c}$ where

$$\hat{c} = \underset{(c,\gamma)}{\arg\max} \left( E\left[ \left(c^\rho + \beta\left[1-c\right]^\rho \tilde{r}_t^\rho \left(v\left(\left\{\tilde{r}_{t+\tau}\right\}_{\tau=1}^\infty, \hat{c}\right)\right)\right)^{\frac{1-\alpha}{\rho}} \right] \right)^{\frac{1}{1-\alpha}}$$

where, for each sequence, $\{r_{t+\tau}\}_{\tau=1}^{\infty}$,

$$v\left(\{r_{t+\tau}\}_{\tau=1}^{\infty}, c\right) = c^{\rho}\left[1 + \sum_{\tau=1}^{\infty}\beta^{\tau}\left[1-c\right]^{\rho\tau}\prod_{s=1}^{\tau} r_{t+s}^{\rho}\right]. \tag{41}$$

Using (40), (41) reduces to

$$\begin{aligned} & v\left(\{r_{t+\tau}\}_{\tau=1}^{\infty}, c\right) \\ = \; & v\left(\left\{\frac{1}{1-\hat{c}} g_{t+\tau}\right\}_{\tau=1}^{\infty}, c\right) \\ = \; & c^{\rho}\theta\left(\{g_{t+\tau}\}_{\tau=1}^{\infty}\right) \end{aligned} \tag{42}$$

where

$$\theta\left(\{g_{t+\tau}\}_{\tau=1}^{\infty}\right) = \left[1 + \sum_{\tau=1}^{\infty}\beta^{\tau}\prod_{s=1}^{\tau} g_{t+s}^{\rho}\right]. \tag{43}$$

In what follows we will assume that for all possible sample paths of growth rates $\{g_{t+\tau}\}_{\tau=1}^{\infty}$

$$\theta\left(\{g_{t+\tau}\}_{\tau=1}^{\infty}\right)$$

is finite. As it turns out this is quite a restrictive assumption. We devote a separate brief subsection below to the discussion of this issue.

The first order condition for $\hat{c}$ is

$$E\left[\left(\hat{c}^{\rho} + \beta\left[1-\hat{c}\right]^{\rho}\tilde{r}_t^{\rho}\left(v\left(\{\tilde{r}_{t+\tau}\}_{\tau=1}^{\infty}, \hat{c}\right)\right)\right)^{\frac{1-\alpha}{\rho}-1}\left(\hat{c}^{\rho-1} - \beta\left[1-\hat{c}\right]^{\rho-1}\tilde{r}_t^{\rho} v\left(\{\tilde{r}_{t+\tau}\}_{\tau=1}^{\infty}, \hat{c}\right)\right)\right] = 0. \tag{44}$$

Substituting (42) in (44) yields

$$E\left[\left(1 + \beta\tilde{g}_t^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}-1}\left(1 - \beta\left[\frac{\hat{c}}{1-\hat{c}}\right]\tilde{g}_t^{\rho}\tilde{\theta}\right)\right] = 0$$

where

$$\tilde{\theta} = \theta\left(\{\tilde{g}_{t+\tau}\}_{\tau=1}^{\infty}\right).$$

Solving this we get

$$\begin{aligned} \frac{P(s_t)}{s_t} &= \left[\frac{1-\hat{c}}{\hat{c}}\right] \\ &= \frac{\beta E\left[\left(1+\beta\tilde{g}_t^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}-1}\left(\tilde{g}_t^{\rho}\tilde{\theta}\right)\right]}{E\left(1+\beta\tilde{g}_t^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}-1}}. \end{aligned} \tag{45}$$

which implies that

$$\hat{c} = \frac{E\left(1+\beta\tilde{g}_t^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}-1}}{E\left(1+\beta\tilde{g}_t^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}}}$$

and

$$1-\hat{c}=\frac{\beta E\left[\left(1+\beta\tilde{g}_t^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}-1}\left(\tilde{g}_t^{\rho}\tilde{\theta}\right)\right]}{E\left(1+\beta\tilde{g}_t^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}}}.$$

To begin to interpret the expression (45) suppose that a risk-free asset is introduced but that it is in zero net supply. Then $r_f$, the return on the risk-free asset will have to be such that

$$1=\hat{\gamma}=\arg\max_{\gamma}\left(E\left[\left(\hat{c}^{\rho}+\beta\left([1-\hat{c}]\left[r_f+\gamma\left(\tilde{r}_t-r_f\right)\right]\right)^{\rho}\left(v\left(\{\tilde{r}_{t+\tau}\}_{\tau=1}^{\infty},\hat{c}\right)\right)\right)^{\frac{1-\alpha}{\rho}}\right]\right)^{\frac{1}{1-\alpha}}.$$

The first-order condition satisfied at $1=\hat{\gamma}$ implies that we must have

$$r_f=\frac{1}{m}. \tag{46}$$

where

$$\begin{aligned} m &= \frac{\beta\hat{c}E\left[\left(1+\beta\tilde{g}_t^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}-1}\tilde{g}_t^{\rho-1}\tilde{\theta}\right]}{E\left(1+\beta\tilde{g}_t^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}-1}} \\ &= \frac{\beta E\left[\left(1+\beta\tilde{g}_t^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}-1}\tilde{g}_t^{\rho-1}\tilde{\theta}\right]}{E\left(1+\beta\tilde{g}_t^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}}} \end{aligned} \tag{47}$$

Since, in equilibrium the riskfree rate $r_f$ must equal the marginal rate of substitution between a sure dollar of additional of additional growth in period $t$ and a sure dollar of consumption in period $t-1$, $\frac{1}{m}$ must measure this MRS.

Letting

$$E^{*}\tilde{g}_t=\frac{E\left[\left(1+\beta\tilde{g}_t^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}-1}\tilde{g}_t^{\rho}\tilde{\theta}\right]}{E\left[\left(1+\beta\tilde{g}_t^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}-1}\tilde{g}_t^{\rho-1}\tilde{\theta}\right]} \tag{48}$$

we can interpret $E^{*}\tilde{g}_t$ as a "risk-neutral" expected growth rate of consumption and

$$E^{*}\tilde{r}_t=\frac{E^{*}\tilde{g}_t}{(1-\hat{c})}$$

as the "risk-neutral" expected return to the Lucas tree. We can also easily show that

$$E^{*}\tilde{g}_t<E\tilde{g}_t. \tag{49}$$

and we can rewrite (45) as

$$\begin{aligned} \frac{P\left(s_t\right)}{s_t} &= \left[\frac{1-\hat{c}}{\hat{c}}\right] \\ &= \frac{m}{\hat{c}} E^* \tilde{g}_t \\ &= \frac{E^* \tilde{g}_t}{r_f \hat{c}} \end{aligned} \tag{50}$$

which implies that

$$1-\hat{c} = \frac{E^* \tilde{g}_t}{r_f}. \tag{51}$$

Equations (49), (50) and (51) also imply that

$$r_f = E^* \tilde{r}_t = \frac{E^* \tilde{g}_t}{(1-\hat{c})} < \frac{E \tilde{g}_t}{(1-\hat{c})} = r_f \frac{E \tilde{g}_t}{E^* \tilde{g}_t} = E \tilde{r}_t. \tag{52}$$

The risk premium on the risky asset represented by the Lucas tree is

$$\frac{E \tilde{r}}{r_f} = \frac{E \tilde{g}}{E^* \tilde{g}}. \tag{53}$$

This is our main result. It implies that the equity risk premium is determined by

$$E^* \tilde{g} = \frac{E\left[\left(1+\beta \tilde{g}^\rho \tilde{\theta}\right)^{\frac{1-\alpha}{\rho}-1} \tilde{g}^\rho \tilde{\theta}\right]}{E\left[\left(1+\beta \tilde{g}^\rho \tilde{\theta}\right)^{\frac{1-\alpha}{\rho}-1} \tilde{g}^{\rho-1} \tilde{\theta}\right]} \tag{54}$$

which, in general, depends on both the elasticity of substitution

$$\sigma = \frac{1}{1-\rho}$$

and the risk aversion $\alpha$.

### 5.1.1 The Additively Separable Case

Note that when we are in the additively separable case and

$$\alpha = 1-\rho$$

(54) reduces to

$$E^* \tilde{g} = \frac{E \tilde{g}^{1-\alpha}}{E \tilde{g}^{-\alpha}}. \tag{55}$$

and

$$m = \beta E \tilde{g}^{-\alpha}.$$

so that

$$r_f = \frac{1}{\beta E \tilde{g}_t^{-\alpha}}.$$

#### 5.1.2 When is $\theta$ Finite? An Example:

As noted above, our approach is based on the assumption that, for all possible sample paths of growth rates $\{g_{t+\tau}\}_{\tau=1}^{\infty}$,

$$\theta\left(\{g_{t+\tau}\}_{\tau=1}^{\infty}\right)=\left[1+\sum_{\tau=1}^{\infty}\beta^{\tau}\prod_{s=1}^{\tau}g_{t+s}^{\rho}\right]$$

is finite. We also noted that this turns out to be quite a restrictive assumption. To see why this is so let's consider the example in Mehra and Prescott [1985], in which $g_t\in\{.982,1.054\}$. In their formulation the *i.i.d.* case arises when these two growth rates are equally likely. Assume also that $\beta=.97$. On the sample path for which, for all $\tau$, $g_{t+\tau}=.982$, we have

$$\theta\left(\{g_{t+\tau}\}_{\tau=1}^{\infty}\right)=1+\Sigma_{\tau=1}^{\infty}\left(.97\cdot .982^{\rho}\right)^{\tau},$$

and this sum is finite only when $.97\cdot .982^{\rho}<1$. Thus, for this sample path, $\theta\left(\{g_{t+\tau}\}_{\tau=1}^{\infty}\right)$ will be finite only if $\rho>-1.63$ and $\sigma=\frac{1}{1-\rho}>.38$. In the Mehra-Prescott example, this is the lower bound restriction on the elasticity of substitution required for existence of equilibrium in our approach.

#### 5.1.3 The Equity Risk Premium in the Non-Additively Separable Case

Since the equity premium in (53) rises as $E^{*}\tilde{g}$ falls, it is natural to ask when

$$\frac{E\left[\left(1+\beta\tilde{g}^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}-1}\tilde{g}^{\rho}\tilde{\theta}\right]}{E\left[\left(1+\beta\tilde{g}^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}-1}\tilde{g}^{\rho-1}\tilde{\theta}\right]}<\frac{E\tilde{g}^{1-\alpha}}{E\tilde{g}^{-\alpha}}. \tag{56}$$

The left hand side of (56) is $E^{*}\tilde{g}$ in our model when

$$\alpha\neq 1-\rho$$

and the right hand side of (56) is $E^{*}\tilde{g}$ in the additively separable case that arises as a special case of our model when

$$\alpha=1-\rho.$$

Because the equity premium in (53) is inversely related to $E^{*}\tilde{g}$, (56) asserts that the equity premium implied by our model exceeds the equity premium obtained from the additively separable case. Kothcerlakota has pointed out that the following proposition is true.

**Proposition 2** *(Kocherlakota) Inequality (56) holds when and only when*

$$\alpha < 1 - \rho. \tag{57}$$

The proof is given in Appendix 2.

It is perhaps interesting to note that (57) is also the condition under which the consumer-investor acts as if he were more risk averse in the Kihlstrom-Mirman sense when he commit to future consumption-portfolio plans for than he does when he makes his consumption-portfolio choice in the future. In the Epstein-Zin approach (57) implies that late resolution of uncertainty is preferred to early resolution of uncertainty. It must be emphasized, however, that unless

$$\alpha = 1 - \rho.$$

the Epstein-Zin risk-aversion measure has a different interpretation than the risk-aversion measure of our approach.

This proposition implies that, as noted in the introduction, our model yields a higher risk premium than the standard additively separable model only when the elasticity of substitution, $\sigma = \frac{1}{1-\rho}$ in our model is exceeded by the elasticity of substitution $\frac{1}{\alpha}$ of the additively separable model. When we combine the inequality (57) with the inequality

$$\rho > -1.63$$

required for existence of equilibrium in the example of the previous subsection we have

$$\alpha < 2.63. \tag{58}$$

This is the upper bound on the risk aversion measure that limits the ability of our model to deliver a large risk premium. It is only when (58) holds that equilibrium in our model exists (for the example) and our model yields a higher risk premium than the additively separable model.

#### 5.1.4 The Effect of Risk Aversion on Savings in the Non-Additively Separable Case

In the analysis of the two period consumption savings model without a riskless asset Kihlstrom-Mirman [1974] savings were shown to increase (decrease) with risk aversion if the elasticity of substitution is less (greater) than one. Essentially the same result was obtained by Diamond-Stiglitz [1974]. The proposition of this section obtains an analogous result for the equilibrium savings level in the Lucas asset pricing model just described. We demonstrate that $1 - \hat{c}$ is a decreasing (increasing) function of the risk aversion parameter $\alpha$ when the elasticity of substitution is greater (less) than one. Because of (40) the equilibrium return will rise for each possible level of growth if saving decreases. Thus, an increase in the risk aversion parameter will raise the equilibrium return for each possible level of growth if the elasticity of substitution is greater (less) than one.

Let $\hat{c}(\alpha)$ be defined as the solution to

$$E\left[\left(1+\beta\tilde{g}^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}-1}\left(1-\beta\left[\frac{\hat{c}(\alpha)}{1-\hat{c}(\alpha)}\right]\tilde{g}^{\rho}\tilde{\theta}\right)\right]=0.$$

**Proposition 3** $1-\hat{c}(\alpha)$ *is a decreasing (increasing) function of* $\alpha$ *if* $\sigma(\rho)=\frac{1}{1-\rho}>(<)\,1$.

The proof is given in Appendix 3.

**Corollary 4** *For each* $g$*, the equilibrium return*

$$r=\frac{g}{1-\hat{c}(\alpha)}$$

*is an increasing (decreasing) function of* $\alpha$ *if* $\sigma(\rho)=\frac{1}{1-\rho}>(<)\,1$*. This, of course implies that the equilibrium expected return*

$$E\tilde{r}=\frac{E\tilde{g}}{1-\hat{c}(\alpha)}$$

*is an increasing (decreasing) function of* $\alpha$ *if* $\sigma(\rho)=\frac{1}{1-\rho}>(<)\,1$.

### 5.1.5 The Effect of Risk Aversion on E*g and the Equity Premium?

We would expect an increase in the risk aversion measure $\alpha$ to result in a decrease in the risk-neutral expected growth rate

$$E^{*}\tilde{g}$$

and an increase in the equity premium

$$\frac{E\tilde{g}}{E^{*}\tilde{g}}.$$

As the next proposition demonstrates this turns out to be true under some conditions on the parameter values.

Let

$$E_i^{*}\tilde{g}=\frac{E\left[\left(1+\beta\tilde{g}^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha_i}{\rho}-1}\left(\tilde{g}^{\rho}\tilde{\theta}\right)\right]}{E\left[\left(1+\beta\tilde{g}^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha_i}{\rho}-1}\left(\tilde{g}^{\rho-1}\tilde{\theta}\right)\right]}$$

**Proposition 5** *Suppose that* $\rho<0$ *and*

$$1-\alpha_2<\rho$$

*or that* $\rho>0$ *and*

$$1-\alpha_2>\rho$$

*Then*

$$\alpha_2 < \alpha_1$$

*implies*

$$E_2^*\tilde{g} > E_1^*\tilde{g}. \tag{59}$$

The proof is in Appendix 4.

## 5.2 Epstein-Zin and Weil

In this case we have already noted that when a risk-free asset is introduced but is in zero net supply, the return on the risk-free asset, $r_f$,will have to will be given by (36). Substituting (40) in (37) yields

$$E^*\tilde{r}_t = \frac{1}{(1-\hat{c})}E^*\tilde{g}_t. \tag{60}$$

where

$$E^*\tilde{g}_t = \frac{E\tilde{g}_t^{1-\alpha}}{E\tilde{g}_t^{-\alpha}} \tag{61}$$

which can once again be interpreted as a risk-neutral expected growth rate. Substituting (60) in (36) yields

$$r_f = \frac{1}{(1-\hat{c})}E^*\tilde{g}_t. \tag{62}$$

We again have

$$E^*\tilde{g}_t < E\tilde{g}_t.$$

Equation (40) also implies

$$E\tilde{r}_t = \frac{E\tilde{g}_t}{(1-\hat{c})}. \tag{63}$$

Combining (62) and (63) we obtain the expression

$$\frac{E\tilde{r}_t}{r_f} = \frac{E\tilde{g}_t}{E^*\tilde{g}_t} \tag{64}$$

for the equity risk premium. Note that the expression (61) for $E^*\tilde{g}_t$ is the same as in the additively separable case. This expression also implies that the EZW equity risk premium in (64) is independent of $\rho$.

(34) implies that

$$1-\hat{c} = \left(\beta\left(E\tilde{r}_t^{1-\alpha}\right)^{\frac{\rho}{1-\alpha}}\right)^{\frac{1}{1-\rho}} \tag{65}$$

Substituting (40) in (65) yields

$$(1-\hat{c}) = \beta\left(E\tilde{g}_t^{1-\alpha}\right)^{\frac{\rho}{1-\alpha}}.$$

When this expression is substituted in (62) the result is

$$r_f = \frac{\left[\left(E\tilde{g}_t^{1-\alpha}\right)^{\frac{1}{1-\alpha}}\right]^{1-\alpha-\rho}}{\beta E\tilde{g}_t^{-\alpha}}.$$

# 6 Summary

In this paper we have proposed a dynamic consumption-savings-portfolio choice model in which the consumer-investor maximizes the expected utility of a non-additively separable utility function of current and future consumption. The non-additive separability coupled with the fact that the consumer-investor's expected utility is independent of past consumption implies that his choices are not dynamically consistent. If he could commit himself to future consumption plans he would make plans that he would not choose to carry out in the future if he were then free of past commitments. In the face of this dynamic inconsistency we follow the "consistent planning" approach of Strotz [1956] and assume that, when making his current choice, the consumer-investor will "take account of future disobedience;" he chooses a consumption-portfolio plan for the future that is, as Strotz asserted, "the best plan among those he will actually follow." Our approach can also be interpreted from the perspective taken by Peleg and Yaari [1973] if we view the current consumer-investor as a leader in a leader follower game in which the followers are the same consumer-investor at future time periods. Current choices are best responses to the choices he expects to make in the future.

Our model uses the measure of risk aversion suggested by Kihlstrom and Mirman [1974] and [1981]. Preferences for consumption streams are CES and the elasticity of substitution can be chosen independently of the risk aversion measure. When the elasticity of substitution is the inverse of the risk aversion measure our model reduces to the additively separable model.

We refer to our approach as one of "consistent planning" by a "forward looking" expected utility maximizing consumer-investor. We introduce the approach and discuss the dynamic inconsistency in a three period setting. We then describe the infinite horizon version of the model for the case in which risky asset returns are *i.i.d.* Since the standard additively separable model is a special case of ours we can easily compare that special case to our general model by simply noting the form our model takes when the elasticity of substitution equals the inverse of the risk aversion measure. Since our approach is an alternative to the widely used non-expected utility, recursive approach of Epstein-Zin and Weil we also compare our model to theirs.

Finally we apply our approach to investigate the equilibrium risk premium obtained in the Lucas asset pricing model. We find that, in contrast to the Epstein-Zin and Weil approach, the risk premium obtained from our approach is affected by the elasticity of substitution as well as the risk aversion measure. Indeed, our model yields a higher risk premium than the additively separable model only when the elasticity of substitution in our model is exceeded by that of the additively separable model. This parameter restriction together with the parameter restrictions required for the existence of equilibrium in our model impose limits the size of the equity premium implied by our model when consumption growth is assumed to be *i.i.d.* We have yet to investigate the case of non- *i.i.d.* consumption growth.

# 7 References


Arrow, K. J., *Essays in the Theory of Risk Bearing*, Markham, Chicago, (1971).

Abel, A.,"Asset Prices under Habit Formation and Catching Up with the Joneses," *American Economic Review*, 80 (1990), 38-42.

________, "Risk Premia and Term Premia in General Equilibrium," *Journal of Monetary Economics*, 43 (1999), 3-33

Bhamra, H. and R. Uppal, "The Role of Risk Aversion and Intertemporal Substitution in Dynamic Consumption-Portfolio Choice with Recursive Utility," *Journal of Economic Dynamics and Control, 30 (2006) 967-91.*

Breeden, D., "An Intertemporal Asset Pricing Model with Stochastic Consumption and Investment Opportunities," *Journal of Financial Economics,* 7 (1979), 265-96.

Bulow, J., "Durable Goods Monopolists," *Journal of Political Economy*, 90 (1982), 314-32.

Campbell, J. and J. Cochrane, "By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior" *Journal of Political Economy* 107 (1999), 205-51.

Coase, R., "Durability and Monopoly," *Journal of Law and Economics*, 15 (1972), 143-49.

Constantinides, G., "Habit Formation: A Resolution of the Equity Premium Puzzle," 98 *Journal of Political Economy* (1990), 519-43.

Debreu, G., "Least Concave Utility Functions," *Journal of Mathematical Economics* 3 (1976), 121-9.

Diamond, P. and J. Stiglitz, "Increasing Risk and Risk Aversion," *Journal of Economic Theory* 8 (1974), 337-60.

Epstein, L., "Risk Aversion and Asset Prices," *Journal of Monetary Economics,* 22 (1988), 179-92.

Epstein, L., and S. Zin, "Substitution, Risk Aversion and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework," *Econometrica*,57 (1989), 937-69.

___________________, "'First-order' Risk Aversion and the Equity Premium Puzzle," *Journal of Monetary Economics*, 26 (1990) 387-407.

___________________, "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: An Empirical Analysis," *Journal of Political Economy* 99, (1991), 263-286.

Grossman, S., and R. Shiller, "The Determinants of the Variability of Stock Market Prices," *American Economic Review,Papers and Proceedings, 71, (1981)*

*222-7.*

Grossman, S., A. Melino, and R. Shiller, "Estimating the Continuous-Time Consumption-Based Asset-Pricing Model," *Journal of Business and Economic Statistics, 5, (1987) 315-27.*

Hakansson, N., "Optimal Investment and Consumption Strategies Under Risk for a Class of Utility Functions," *Econometrica*, 38 (1970), 587-607.

Hansen, L. P., and K. Singleton, "Stochastic Consumption, Risk Aversion and the Temporal Behavior of Asset Returns," *Journal of Political Economy* 91 (1983), 249-65.

Harris, C., and D. Laibson "Dynamic choices of Hyperbolic Consumers," *Econometrica*, 69 (2001), 935-57.

Kihlstrom, R. and L. Mirman, "Risk Aversion with Many Commodities," *Journal of Economic Theory* 8 (1974), 361-88.

_______________________, "Constant, Increasing and Decreasing Risk Aversion with Many Commodities" *Review of Economics Studies,* 48 (1981)271-80.

Kocherlakota, N., "Disentangling the Coefficient of Relative Risk Aversion from the Intertemporal Elasticity of Substitution," *Journal of Finance*, 45 (1990) 175-90.

Kreps, D. M. and E. L. Porteus, "Temporal Resolution of Uncertainty and Dynamic Choice Theory," *Econometrica*, 46 (1978), 185-200.

___________________________, "Temporal von Neumann Morgenstern and Induced Preferences," *Journal of Economic Theory*, 20 (1979a), 81-109.

___________________________, "Dynamic Choice Theory and Dynamic Programming," *Econometrica*, 47 (1979b), 91-100.

Labadie, "Comparative Dynamics and Risk Premia in an Overlapping Generations Model," *The Review of Economic Studies*, 53 (1986), 139-52.

Laibson, D. "Golden Eggs and Hyperbolic Discounting," *Quarterly Journal of Economics*, 112 (1997) 443-77.

Levhari, D. and T. N. Srinavasan, "Optimal Savings Under Uncertainty," *The Review of Economic Studies*, 36 (1969), 153-63.

Lucas, R., "Asset Prices in an Exchange Economy," *Econometrica*, 46 (1978), 1429-45.

Lucas, R., and N. Stokey, "Optimal Growth with Many Consumers," *Journal of Economic Theory* 32 (1984), 139-71.

Luttmer, E., and T. Mariotti, "Subjective Discounting in an Exchange Economy," *Journal of Political Economy 111 (2003) 959-89.*

_______________________, "Efficiency and Equilibrium when Preferences are Time Inconsistent," *Journal of Economic Theory 132 (2007) 493-506.*

Merton, R., "Lifetime Portfolio Selection Under Uncertainty: The Continuous Time Case," *The Review of Economics and Statistics,* 51 (1969), 247-57.

__________, "Optimal Consumption and Portfolio Rules in a Continuous-Time Model," *Journal of Economic Theory,* 3 (1971), 373-413.

__________, "An Intertemporal Capital Asset Pricing Model," *Econometrica*, 41 (1973), 867-87..

Mehra, R., and E. Prescott, "The Equity Premium: A Puzzle," *Journal of Monetary Economics,* 15 (1985), 145-61.

Peleg, B., and M. Yaari, "On the Existence of a Consistent Course of Action When Tastes are Changing," *Review of Economic Studies*, 40 (1973) 391-401.

Phelps, E. S. and R. Pollak, "On Second Best National Saving and Game-Equilibrium Growth," *Review of Economic Studies*, 35 (1968) 185-99.

Pollak, R., "Consistent Planning," *Review of Economic Studies*, 35 (1968) 201-8.

Pratt, J., "Risk Aversion in the Large and in the Small," *Econometrica*, 32 (1964) 122-36.

Samuelson, P. "Lifetime Portfolio Selection by Dynamic Stochastic Programming," *The Review of Economics and Statistics,* 51 (1969), 239-46.

Stokey, N., "Intertemporal Price Discrimination," *Quarterly Journal of Economics*, 93 (1979), 355-71.

Strotz, R., "Myopia and Inconsistency in Dynamic Utility Maximization," *Review of Economic Studies*, 23 (1956) 165-80.

Van den Heuvel, S., "Notes on Temporal Risk Aversion," Working Paper, Department of Finance, The Wharton School, University of Pennsylvania (2007).

Swensson, L.E.O., "Portfolio Choice with Non-Expected Utility in Continuous Time," *Economic Letters*, 30 (1989) 313-7.

Weil, P., "The Equity Premium Puzzle and the Risk Free Rate Puzzle," *Journal of Monetary Economics,* 24 (1989), 401-21.

______, "Nonexpected Utility in Macroeconomics," *Quarterly Journal of Economics,* 105 (1990), 29-42.

# 8 Appendix 1:

**Proof.** Proposition 1: Let

$$h(U) = \left[\frac{1}{1-\alpha}\right](A^{\rho} + U^{\rho})^{\frac{1-\alpha}{\rho}}$$

Then

$$\begin{aligned} R &= -\frac{h''(U)\,U}{h'(U)} \\ &= \frac{[\alpha U^{\rho} + (1-\rho)\,A^{\rho}]}{(A^{\rho} + U^{\rho})} \end{aligned}$$

and

$$\frac{\partial R}{\partial A} = \frac{[1-\rho-\alpha]\,\rho U^{\rho} A^{\rho-1}}{\left(A^{\rho}+U^{\rho}\right)^{2}} > (<)\, 0$$

when

$$1-\rho > (<)\, \alpha.$$

Note that when $A = 0$, maximizing

$$Eh\left(\tilde{U}\right) = E\left[\frac{1}{1-\alpha}\right] \tilde{U}^{1-\alpha}.$$

which is what the consumer-investor maximizes when commitment is impossible and he chooses

$$\left(c_2^*, \gamma_2^*\right).$$

When he can commit

$$A = A\left(x_1\right) = \frac{\hat{c}_1}{\beta^{\frac{1}{\rho}}\left[1-\hat{c}_1\right]\left[r_f + \hat{\gamma}_1 x_1\right]} > 0.$$

So the consumer-investor acts as if he were more (less) risk averse in the Kihlstrom-Mirman sense when he chooses

$$\left(\hat{c}_2\left(x_1\right), \hat{\gamma}_2\left(x_1\right)\right)$$

in the first period than he does when he chooses

$$\left(c_2^*, \gamma_2^*\right)$$

in the second period.

Also since

$$A'\left(x_1\right) < 0,$$

$$\frac{\partial R}{\partial x_1} < (>)\, 0$$

when

$$1-\rho > (<)\, \alpha.$$

This implies that, when

$$1-\rho > (<)\, \alpha,$$

increases in $x_1$ cause the consumer-investor to act as if he were less (more) risk averse in the Kihlstrom-Mirman sense when he commits to

$$\left(\hat{c}_2\left(x_1\right), \hat{\gamma}_2\left(x_1\right)\right)$$

in the first period. ■

# 9 Appendix 2:

**Proof.** Proposition 2: This follows simply from the fact that

$$\left(1+\beta g^{\rho}\theta\right)^{\frac{1-\alpha}{\rho}-1} g^{\rho-1}\theta = \left(g^{-\rho}+\beta\theta\right)^{\frac{1-\alpha-\rho}{\rho}} g^{-\alpha}\theta$$

which implies that

$$\frac{E\left[\left(1+\beta\tilde{g}^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}-1}\tilde{g}^{\rho}\tilde{\theta}\right]}{E\left[\left(1+\beta\tilde{g}^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha}{\rho}-1}\tilde{g}^{\rho-1}\tilde{\theta}\right]} = \frac{E\left[\left(\tilde{g}^{-\rho}+\beta\tilde{\theta}\right)^{\frac{1-\alpha-\rho}{\rho}}\tilde{g}^{1-\alpha}\tilde{\theta}\right]}{E\left[\left(\tilde{g}^{-\rho}+\beta\tilde{\theta}\right)^{\frac{1-\alpha-\rho}{\rho}}\tilde{g}^{-\alpha}\tilde{\theta}\right]}. \tag{66}$$

When (57) holds

$$g < (>)\,\frac{E\tilde{g}^{1-\alpha}}{E\tilde{g}^{-\alpha}}$$

implies

$$\left(g^{-\rho}+\beta\theta\right)^{\frac{1-\alpha-\rho}{\rho}} > (<) \left(\left[\frac{E\tilde{g}^{1-\alpha}}{E\tilde{g}^{-\alpha}}\right]^{-\rho}+\beta\theta\right)^{\frac{1-\alpha-\rho}{\rho}}.$$

So, for each $\theta$,

$$\left(g^{-\rho}+\beta\theta\right)^{\frac{1-\alpha-\rho}{\rho}}\left(g-\frac{E\tilde{g}^{1-\alpha}}{E\tilde{g}^{-\alpha}}\right)\theta < \left(\left[\frac{E\tilde{g}^{1-\alpha}}{E\tilde{g}^{-\alpha}}\right]^{-\rho}+\beta\theta\right)^{\frac{1-\alpha-\rho}{\rho}}\left(g-\frac{E\tilde{g}^{1-\alpha}}{E\tilde{g}^{-\alpha}}\right)\theta$$

and

$$\begin{aligned} E\left[\left(\tilde{g}^{-\rho}+\beta\theta\right)^{\frac{1-\alpha-\rho}{\rho}}\left(\frac{\tilde{g}^{1-\alpha}}{E\tilde{g}^{-\alpha}}-\frac{\tilde{g}^{-\alpha}}{E\tilde{g}^{-\alpha}}\left[\frac{E\tilde{g}^{1-\alpha}}{E\tilde{g}^{-\alpha}}\right]\right)\theta\right] &< E\left(\left[\frac{E\tilde{g}^{1-\alpha}}{E\tilde{g}^{-\alpha}}\right]^{-\rho}+\beta\theta\right)^{\frac{1-\alpha-\rho}{\rho}}\left(\frac{E\tilde{g}^{1-\alpha}}{E\tilde{g}^{-\alpha}}-\frac{E\tilde{g}^{1-\alpha}}{E\tilde{g}^{-\alpha}}\right)\theta \\ &= 0. \end{aligned}$$

Thus, for each $\theta$,

$$\frac{E\left[\left(\tilde{g}^{-\rho}+\beta\theta\right)^{\frac{1-\alpha-\rho}{\rho}}\tilde{g}^{1-\alpha}\theta\right]}{E\left[\left(g^{-\rho}+\beta\theta\right)^{\frac{1-\alpha-\rho}{\rho}}\tilde{g}^{-\alpha}\theta\right]} < \frac{E\tilde{g}^{1-\alpha}}{E\tilde{g}^{-\alpha}}. \tag{67}$$

Taking expectations over $\theta$ and using (66), (67) implies (56). Note that when (57) is reversed a parallel argument implies that the inequality (56) is reversed.
■

# 10 Appendix 3:

**Proof.** Proposition 3: Assume that

$$\alpha_2 < \alpha_1.$$

Use the fact that, for each

$$\theta,$$

$$\begin{aligned}
& (1+\beta g^{\rho}\theta)^{\frac{1-\alpha_1}{\rho}-1} \\
= \quad & (1+\beta g^{\rho}\theta)^{\frac{\alpha_2-\alpha_1}{\rho}} (1+\beta g^{\rho}\theta)^{\frac{1-\alpha_2}{\rho}-1}
\end{aligned}$$

If $\rho > (<)\, 0$, then

$$(1+\beta g^{\rho}\theta)^{\frac{\alpha_2-\alpha_1}{\rho}} < (>) \left(1+\left[\frac{1-\hat{c}(\alpha_2)}{\hat{c}(\alpha_2)}\right]\right)^{\frac{\alpha_2-\alpha_1}{\rho}}$$

when

$$1 < \beta\left[\frac{\hat{c}(\alpha_2)}{1-\hat{c}(\alpha_2)}\right] g^{\rho}\theta$$

and

$$(1+\beta g^{\rho}\theta)^{\frac{\alpha_2-\alpha_1}{\rho}} > (<) \left(1+\left[\frac{1-\hat{c}(\alpha_2)}{\hat{c}(\alpha_2)}\right]\right)^{\frac{\alpha_2-\alpha_1}{\rho}}$$

when

$$1 > \beta\left[\frac{\hat{c}(\alpha_2)}{1-\hat{c}(\alpha_2)}\right] g^{\rho}\theta.$$

So, for all $g$ and $\theta$

$$\begin{aligned}
& (1+\beta g^{\rho}\theta)^{\frac{\alpha_2-\alpha_1}{\rho}} (1+\beta g^{\rho}\theta)^{\frac{1-\alpha_2}{\rho}-1}\left[1-\beta\left[\frac{\hat{c}(\alpha_2)}{1-\hat{c}(\alpha_2)}\right] g^{\rho}\theta\right] \\
> \quad & (<) \left(1+\left[\frac{1-\hat{c}(\alpha_2)}{\hat{c}(\alpha_2)}\right]\right)^{\frac{\alpha_2-\alpha_1}{\rho}} (1+\beta g^{\rho}\theta)^{\frac{1-\alpha_2}{\rho}-1}\left[1-\beta\left[\frac{\hat{c}(\alpha_2)}{1-\hat{c}(\alpha_2)}\right] g^{\rho}\theta\right]
\end{aligned}$$

if $\rho > (<)\, 0$. Thus,

$$\begin{aligned}
& E\left(1+\beta\tilde{g}^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha_1}{\rho}-1}\left[1-\beta\left[\frac{\hat{c}(\alpha_2)}{1-\hat{c}(\alpha_2)}\right]\tilde{g}^{\rho}\tilde{\theta}\right] \\
= \quad & E\left(1+\beta\tilde{g}^{\rho}\tilde{\theta}\right)^{\frac{\alpha_2-\alpha_1}{\rho}}\left(1+\beta\tilde{g}^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha_2}{\rho}-1}\left[1-\beta\left[\frac{\hat{c}(\alpha_2)}{1-\hat{c}(\alpha_2)}\right]\tilde{g}^{\rho}\tilde{\theta}\right] \\
> \quad & (<) \left(1+\left[\frac{1-\hat{c}(\alpha_2)}{\hat{c}(\alpha_2)}\right]\right)^{\frac{\alpha_2-\alpha_1}{\rho}} E\left(1+\beta\tilde{g}^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha_2}{\rho}-1}\left[1-\beta\left[\frac{\hat{c}(\alpha_2)}{1-\hat{c}(\alpha_2)}\right]\tilde{g}^{\rho}\tilde{\theta}\right] \\
= \quad & 0 \\
= \quad & E\left(1+\beta\tilde{g}^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha_1}{\rho}-1}\left[1-\beta\left[\frac{\hat{c}(\alpha_1)}{1-\hat{c}(\alpha_1)}\right]\tilde{g}^{\rho}\tilde{\theta}\right].
\end{aligned}$$

This implies that

$$\left[\frac{\hat{c}(\alpha_1)}{1-\hat{c}(\alpha_1)}\right] > \left[\frac{\hat{c}(\alpha_2)}{1-\hat{c}(\alpha_2)}\right]$$

if $\rho > (<)\, 0$. ■

## 11 Appendix 4:

**Proof.** Proposition 4: Note that

$$(1+\beta g^{\rho}\theta)^{\frac{1-\alpha_1}{\rho}-1} = (1+\beta g^{\rho}\theta)^{\frac{\alpha_2-\alpha_1}{\rho}} (1+\beta g^{\rho}\theta)^{\frac{1-\alpha_2}{\rho}-1}.$$

When

$$\alpha_2 < \alpha_1$$
$$(1+\beta g^{\rho}\theta)^{\frac{\alpha_2-\alpha_1}{\rho}}$$

is a decreasing function of $g$. Thus,

$$g > (<)\, E_2^* \tilde{g}$$

implies

$$(1+\beta g^{\rho}\theta)^{\frac{\alpha_2-\alpha_1}{\rho}} < (>) \left(1+\beta \left(E_2^*\tilde{g}\right)^{\rho}\theta\right)^{\frac{\alpha_2-\alpha_1}{\rho}}.$$

We then have, for all $g$ and $\theta$,

$$\begin{aligned} & (1+\beta g^{\rho}\theta)^{\frac{1-\alpha_1}{\rho}-1}\theta g^{\rho-1}\left(g-E_2^*\tilde{g}\right) \\ = \; & (1+\beta g^{\rho}\theta)^{\frac{\alpha_2-\alpha_1}{\rho}}(1+\beta g^{\rho}\theta)^{\frac{1-\alpha_2}{\rho}-1}\theta g^{\rho-1}\left(g-E_2^*\tilde{g}\right) \\ < \; & \left(1+\beta\left(E_2^*\tilde{g}\right)^{\rho}\theta\right)^{\frac{\alpha_2-\alpha_1}{\rho}}(1+\beta g^{\rho}\theta)^{\frac{1-\alpha_2}{\rho}-1}\theta g^{\rho-1}\left(g-E_2^*\tilde{g}\right). \end{aligned}$$

Taking expectations over $g$,

$$E\left[(1+\beta\tilde{g}^{\rho}\theta)^{\frac{1-\alpha_1}{\rho}-1}\theta\tilde{g}^{\rho-1}\left(\tilde{g}-E_2^*\tilde{g}\right)\right] < \left(1+\beta\left(E_2^*\tilde{g}\right)^{\rho}\theta\right)^{\frac{\alpha_2-\alpha_1}{\rho}} E\left[(1+\beta\tilde{g}^{\rho}\theta)^{\frac{1-\alpha_2}{\rho}-1}\theta\tilde{g}^{\rho-1}\left(\tilde{g}-E_2^*\tilde{g}\right)\right]. \tag{68}$$

and taking expectations over $\tilde{\theta}$ we have

$$E\left[\left(1+\beta\tilde{g}^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha_1}{\rho}-1}\tilde{\theta}\tilde{g}^{\rho-1}\left(\tilde{g}-E_2^*\tilde{g}\right)\right] < E\left(\left(1+\beta\left(E_2^*\tilde{g}\right)^{\rho}\tilde{\theta}\right)^{\frac{\alpha_2-\alpha_1}{\rho}}\left[\left(1+\beta\tilde{g}^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha_2}{\rho}-1}\tilde{\theta}\tilde{g}^{\rho-1}\left(\tilde{g}-E_2^*\tilde{g}\right)\right]\right). \tag{69}$$

We will have (**??**) if we can demonstrate that

$$E\left(\left(1+\beta\left(E_2^*\tilde{g}\right)^{\rho}\tilde{\theta}\right)^{\frac{\alpha_2-\alpha_1}{\rho}}\left[\left(1+\beta\tilde{g}^{\rho}\tilde{\theta}\right)^{\frac{1-\alpha_2}{\rho}-1}\tilde{\theta}\tilde{g}^{\rho-1}\left(\tilde{g}-E_2^*\tilde{g}\right)\right]\right) < 0. \tag{70}$$

Now let $\theta_0$ be defined by

$$E\left[(1+\beta\tilde{g}^{\rho}\theta_0)^{\frac{1-\alpha_2}{\rho}-1}\theta_0\tilde{g}^{\rho-1}\left(\tilde{g}-E_2^*\tilde{g}\right)\right] = 0.$$

Then

$$\frac{E\left[(1+\beta\tilde{g}^{\rho}\theta_0)^{\frac{1-\alpha_2}{\rho}-1}\tilde{g}^{\rho}\right]}{E\left[(1+\beta\tilde{g}^{\rho}\theta_0)^{\frac{1-\alpha_2}{\rho}-1}\tilde{g}^{\rho-1}\right]} = E_2^*\tilde{g}$$

We will show below that

$$1-\alpha_2 > (<)\, \rho$$

implies that

$$\frac{E\left[(1+\beta\tilde{g}^{\rho}\theta)^{\frac{1-\alpha_2}{\rho}-1}\tilde{g}^{\rho}\right]}{E\left[(1+\beta\tilde{g}^{\rho}\theta)^{\frac{1-\alpha_2}{\rho}-1}\tilde{g}^{\rho-1}\right]}$$

is an increasing (decreasing) function of $\theta$. Thus, when

$$1-\alpha_2 > \rho$$

$$\frac{E\left[(1+\beta\tilde{g}^{\rho}\theta)^{\frac{1-\alpha_2}{\rho}-1}\tilde{g}^{\rho}\right]}{E\left[(1+\beta\tilde{g}^{\rho}\theta)^{\frac{1-\alpha_2}{\rho}-1}\tilde{g}^{\rho-1}\right]} > (<)\, E_2^*\tilde{g} = \frac{E\left[(1+\beta\tilde{g}^{\rho}\theta_0)^{\frac{1-\alpha_2}{\rho}-1}\tilde{g}^{\rho}\right]}{E\left[(1+\beta\tilde{g}^{\rho}\theta_0)^{\frac{1-\alpha_2}{\rho}-1}\tilde{g}^{\rho-1}\right]}.$$

and

$$E\left[(1+\beta\tilde{g}^{\rho}\theta)^{\frac{1-\alpha_2}{\rho}-1}\theta\tilde{g}^{\rho-1}\left(\tilde{g}-E_2^*\tilde{g}\right)\right] > (<)\, 0$$

if

$$\theta > (<)\, \theta_0.$$

When

$$1-\alpha_2 < \rho$$

we have

$$\frac{E\left[(1+\beta\tilde{g}^{\rho}\theta)^{\frac{1-\alpha_2}{\rho}-1}\tilde{g}^{\rho}\right]}{E\left[(1+\beta\tilde{g}^{\rho}\theta)^{\frac{1-\alpha_2}{\rho}-1}\tilde{g}^{\rho-1}\right]} < (>)\, E_2^*\tilde{g} = \frac{E\left[(1+\beta\tilde{g}^{\rho}\theta_0)^{\frac{1-\alpha_2}{\rho}-1}\tilde{g}^{\rho}\right]}{E\left[(1+\beta\tilde{g}^{\rho}\theta_0)^{\frac{1-\alpha_2}{\rho}-1}\tilde{g}^{\rho-1}\right]}.$$

and

$$E\left[(1+\beta\tilde{g}^{\rho}\theta)^{\frac{1-\alpha_2}{\rho}-1}\theta\tilde{g}^{\rho-1}\left(\tilde{g}-E_2^*\tilde{g}\right)\right] < (>)\, 0$$

if

$$\theta > (<)\, \theta_0.$$

If we also assume that

$$\rho < (>)\, 0$$

then the fact that

$$\alpha_2 < \alpha_1$$

implies that

$$\left(1+\beta\left(E_2^*\tilde{g}\right)^{\rho}\theta\right)^{\frac{\alpha_2-\alpha_1}{\rho}}$$

is an increasing (decreasing) function of $\theta$. Thus, when

$$\rho < 0$$

and

$$1-\alpha_2 < \rho$$

or when

$$\rho > 0$$

and

$$1 - \alpha_2 > \rho$$

the inequality

$$\begin{aligned} & \left(1 + \beta \left(E_2^* \tilde{g}\right)^\rho \theta\right)^{\frac{\alpha_2 - \alpha_1}{\rho}} E\left[\left(1 + \beta \tilde{g}^\rho \theta\right)^{\frac{1-\alpha_2}{\rho} - 1} \theta \tilde{g}^{\rho - 1} \left(\tilde{g} - E_2^* \tilde{g}\right)\right] \\ < \quad & \left(1 + \beta \left(E_2^* \tilde{g}\right)^\rho \theta_0\right)^{\frac{\alpha_2 - \alpha_1}{\rho}} E\left[\left(1 + \beta \tilde{g}^\rho \theta\right)^{\frac{1-\alpha_2}{\rho} - 1} \theta \tilde{g}^{\rho - 1} \left(\tilde{g} - E_2^* \tilde{g}\right)\right] \end{aligned}$$

holds for all $\theta$.Taking expectations over $\theta$

$$\begin{aligned} & E\left(\left(1 + \beta \left(E_2^* \tilde{g}\right)^\rho \tilde{\theta}\right)^{\frac{\alpha_2 - \alpha_1}{\rho}} \left[\left(1 + \beta \tilde{g}^\rho \tilde{\theta}\right)^{\frac{1-\alpha_2}{\rho} - 1} \tilde{\theta} \tilde{g}^{\rho - 1} \left(\tilde{g} - E_2^* \tilde{g}\right)\right]\right) \quad (71) \\ < \quad & \left(1 + \beta \left(E_2^* \tilde{g}\right)^\rho \theta_0\right)^{\frac{\alpha_2 - \alpha_1}{\rho}} E\left[\left(1 + \beta \tilde{g}^\rho \tilde{\theta}\right)^{\frac{1-\alpha_2}{\rho} - 1} \tilde{\theta} \tilde{g}^{\rho - 1} \left(\tilde{g} - E_2^* \tilde{g}\right)\right]. \end{aligned}$$

The right hand side of the inequality in (71) is zero by definition of $E_2^* \tilde{g}$. So (70) holds if

$$\rho < 0$$

and

$$1 - \alpha_2 < \rho$$

or

$$\rho > 0$$

and

$$1 - \alpha_2 > \rho.$$

The proof is complete if we can show that

$$\frac{E\left[\left(1 + \beta \tilde{g}^\rho \theta\right)^{\frac{1-\alpha_2}{\rho} - 1} \tilde{g}^\rho\right]}{E\left[\left(1 + \beta \tilde{g}^\rho \theta\right)^{\frac{1-\alpha_2}{\rho} - 1} \tilde{g}^{\rho - 1}\right]}$$

is increasing (decreasing) in $\theta$ when

$$1 - \alpha_2 > (<) \, \rho.$$

Using a generalization the argument employed above to obtain (68) we can show that

$$\frac{E u_g \left(g; \theta\right) \tilde{g}}{E u_g \left(g; \theta\right)}$$

is increasing in $\theta$ if the absolute risk aversion measure

$$-\frac{u_{gg} \left(g; \theta\right)}{u_g \left(g; \theta\right)}$$

is decreasing in $\theta$.Note that when

$$u(g;\theta) = (1+\beta g^{\rho}\theta)^{\frac{1-\alpha_2}{\rho}},$$

the risk aversion measure

$$-\frac{u_{gg}(g;\theta)}{u_g(g;\theta)} = \frac{g(1-\alpha_2-\rho)}{(1+\beta g^{\rho}\theta)} + \alpha_2 g$$

is decreasing (increasing) in $\theta$ when

$$1-\alpha_2 > (<)\, \rho$$

Thus,

$$\frac{Eu'(g;\theta)\,\tilde{g}}{Eu'(g;\theta)} = \frac{E\left[(1+\beta\tilde{g}^{\rho}\theta)^{\frac{1-\alpha_2}{\rho}-1}\,\tilde{g}^{\rho}\right]}{E\left[(1+\beta\tilde{g}^{\rho}\theta)^{\frac{1-\alpha_2}{\rho}-1}\,\tilde{g}^{\rho-1}\right]}$$

is increasing (decreasing) in $\theta$ when

$$1-\alpha_2 > (<)\, \rho.$$

■