PAY FOR PERCENTILE

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1. Introduction

In modern economies, most wealth is held in the form of human capital, and publicly funded schools play a key role in creating this wealth. Thus, reform proposals that seek to enhance the efficiency of schools are an omnipresent feature of debates concerning public policy and societal welfare. In recent decades, education policy makers have increasingly designed these reform efforts around measures of school output rather than school inputs. Although scholars and policy makers still debate the benefits of smaller classes, improved teacher preparation, or improved school facilities, few are willing to measure school quality using only measures of school inputs. During the 1990's many states adopted accountability systems that dictated sanctions and remediation for the staff of schools based on how their students performed on standardized assessments. In 2001, the No Child Left Behind Act (NCLB) mandated that all states adopt such systems or risk losing federal funds, and more recently, several states and large districts have introduced incentive pay systems that link the salaries of individual teachers to the performance of their students.

Table 1
Recent Performance Pay Schemes

Name	Location	Performance Indices Linked to Student Achievement
ProComp	Denver	Principals negotiate with teachers to set growth targets for each child. Teachers are judged based on how many targets they meet.
QComp	Minnesota	Schools submit plans that specify their own methods for measuring gains in student achievement.
TAP	14 States & DC	Value-added Model that attempts to measure deviations from expected achievement growth.
MAP	Florida	Districts are free to choose their own methods for measuring and weighting achievement gains and proficiency when forming indices of teacher performance.
STAR	Florida	Teachers receive points based on the changes in proficiency levels their students experience. A Value Table specifies the point allocations. Teachers receive more points when their students experience positive transitions that are rare.

Each of these systems attempts to evaluate multiple aspects of teacher performance. This table gives a brief description of the methods used to evaluate teacher performance with respect to student achievement. The MAP system has now replaced the STAR system in Florida. STAR created significant controversy when it was implemented in Hillsboro Co. Florida because the Value-Table method used in the first year of the program generated a strikingly high concentration of reward payments in economically advantaged schools.

Table 1 provides information about a number of performance pay systems introduced in recent years. Each of these systems contains a component that links teacher compensation to some measure of student achievement, and the optimal design of this component is our concern here. Although we propose a system that links measures of individual teacher performance to individual teacher compensation, our proposal will also work well as an incentive scheme for small teams of teachers if one assumes that peer monitoring is effective within teams.

When designing performance pay systems for teachers, policy makers must begin by constructing valid measures of teacher performance. A simple starting place is to assert that teacher A is judged to perform better than teacher B if teacher A contributes greater effective effort toward student learning than teacher B. However, policy makers have little objective data concerning teacher effort, and therefore must try to make inferences about the performance of teachers from the achievement of their students. Approaches that equate high student test scores with high teacher performance are not desirable because they ignore the fact that some teachers work with weaker students than others. For this reason, policy makers are now focused on ways to use measures of achievement growth as indices of teacher performance. Thus, each of the systems described in Table 1 creates a performance scale by employing procedures that translate the potential achievement changes that various students might experience into a metric of teacher performance.

The details of these procedures differ greatly but all involve choices about weights, scales, and statistical model specifications that are arbitrary in an important sense. None of the resulting performance scales are anchored to any metric of teacher effort or the value of student skill, e.g. the instruction time required by a representative teacher to create an increase from x to $x+k$ in a student's score or the changes in expected lifetime earnings for a student who experiences such a score increase. To illustrate what we mean by arbitrary, consider two teachers who both have only one student. Teacher A works with a student who began the year with a score of 130 on a particular developmental scale and ended the year with a score of 150. Teacher B works with a student who began at 270 and ended at 290. Did one teacher perform better than the other or should we conclude that they performed equally well? How would one gather the information required to determine whether or not an increase of 130 to 150 requires more, less, or the same expected teacher effort than an increase of 270 to 290? Is it possible that it makes no sense to ask which teacher performed better because the teaching techniques and skills required to help a student move from 130 to 150 are completely different than those required to help a student move from 270 to 290?

Because the systems described in Table 1 employ performance scales that are not anchored to cardinal measures of teacher effort or student benefit, it is difficult for policy makers to design pecuniary incentives around these performance indices. Unless the designers somehow manage to calibrate a system that makes the effort cost of increasing one's performance rating independent of the baseline achievement of one's students, teachers will face clear incentives to allocate their efforts to types of students that offer the highest return in terms of expected increases in their performance rating, and the resulting performance measures for teachers will reflect not only differences in teacher performance but also the assignment of students with different prior achievement levels to different teachers. In addition, when policy makers employ arbitrary performance scales, they have the freedom to manipulate these scales in ways that raise the performance ratings and reward pay of political allies. Policy makers who employ value-added or value table methods can always raise the performance ratings of those who teach in advantaged versus disadvantaged schools by rescaling assessments or assigning different performance weights to the various score improvements or proficiency level transitions that students may make.

In this paper, we propose an alternative performance pay scheme that is immune to these problems, and under suitable conditions elicits the efficient level of effort from all teachers. We call our system "pay for percentile," and it works as follows. For each student in a school system, first form a comparison set of students against which the student will be compared. Assumptions concerning the nature of instruction dictate exactly how to define this comparison set, but the general idea is to form a set that contains all other students in the system who begin the school year at the same level of baseline achievement in a comparable classroom setting. At the end of the year, give a cumulative assessment to all students. Then assign each student a percentile score based on his end of year rank among the students in his comparison set. For each teacher, sum these within-peer percentile scores over all the students she teaches, and denote this sum as a percentile performance index. Then, pay each teacher a common base salary plus a bonus that is proportional to her percentile performance index. Thus, teachers are rewarded the same for raising the within-peer percentile ranking of a student from p to $p + k$ regardless of the student and regardless of p.

Since our scheme only compares students with the same level of baseline achievement, we do not have to tackle the difficult task of translating gains at different points on a psychometric scale into a common metric of teacher performance. We are thus able to devise a system that is simpler and more transparent than those in Table 1. Further, because out system relies only on rank order comparisons, the performance indices and reward pay our system produces are invariant to any monotonic rescaling of student test scores. Thus, the results of our system cannot be manipulated through scaling choices, and no teacher is placed at a systematic advantage or disadvantage because of the types of students she teaches.

Some may find it surprising that the optimal performance pay system involves paying bonuses that are a linear function of percentile scores. However, it is important to realize that the percentile performance index we describe is simply the fraction of contests that a teacher wins when playing a large number of opponents simultaneously, where each contest involves comparing the final score of one of her student's to the final score of one the student's in her student's comparison set. Even though our scheme requires teachers to compete simultaneously in many contests, all contests share an important symmetry because each involves comparing the outcomes of two students who share the same baseline characteristics. Given this symmetry, it turns out to be optimal to pay each teacher a fixed prize for each of the tournaments she wins, and as the number of contests involving each student type grows large, this procedure becomes equivalent to paying a piece-rate linked to the sum of the within peer percentile scores received by all the students in a given teacher's class.

Lazear and Rosen (1981) demonstrate that tournaments can elicit efficient effort from workers even in settings where firms are only able to rank the performance of their workers. While their results apply to a setting where workers make one effort choice and compete in one contest, we construct a performance pay system for a setting where teachers make multiple effort choices and these choices simultaneously effect the outcomes of many contests. We find that, even in a setting with these effort spillovers among contests, a common prize for winning each tournament can induce efficient effort from all teachers.

Policy makers can choose this prize without any knowledge of the scale used to report the assessment results. Further, policy makers do not need to know the common function that describes teacher effort costs. In order to choose the efficient level of incentive pay, policy makers only need to know the social value of human capital and how marginal changes in effort affect the probability of winning the contests we describe.

We present our main results in a context where the private costs and social benefits of devoting effective instruction to a student are the same among students of different baseline achievement levels. After establishing our main results, we examine settings in which some students gain more than others from instruction. Here, the comparison sets we use in our scheme must condition not only on baseline achievement but also the distribution of baseline achievement among each student's peers.

Within our framework, it is natural to think of teachers as workers who perform complex jobs that require multiple tasks because each teacher must devote effort to general classroom instruction as well as one-on-one tutoring for each student. Our results offer new insight for the design of incentives in this setting. If employers can form an accurate ordinal ranking of worker performance on each task that defines their job, the percentiles from these rankings are a set of performance indices that may provide a basis for efficient incentive pay.[1]

However, our scheme does not address a different multi-tasking issue raised in Holmstrom and Milgrom (1991). In our model, the only way teachers can affect the assessment results of their students is by teaching. We assume that assessment results cannot be influenced by gaming, cheating, or other forms of manipulation. Our aim is to address the design of optimal performance pay systems in a setting where the ordinal information in assessments cannot be contaminated by actions that the designer cannot observe. We do not address the design of such assessments, but we readily stipulate that this design challenge is a daunting one.

2. Basic Model

Here, we describe our basic model and our key result concerning multiple, simultaneous tournaments. In the following section, we derive our pay for percentile system as a limiting result within this framework. Assume there are J classrooms, indexed by $j \in 1, ..., J$. Each classroom has one teacher, so j also indexes teachers. We will ultimately focus on the limit as $J \to \infty$, but for now we treat it as finite.

[1] Many forms of incentive pay in the private sector are based on performance evaluation systems that do not involve cardinal statistics but rather a set of rankings that describe how well a worker performs various tasks relative to other workers or a set of performance standards. Our results show how the ordinal information in these rankings may provide the basis for efficient incentives in complex jobs. Such rankings often involve subjective judgements by principals and may be relatively immune to the gaming behaviors that often contaminate objective performance statistics.

Each classroom has N students, indexed by $i \in 1, ..., N$. Let a_{ij} denote the initial human capital of the i-th student in the j-th class. Students within each class are ordered from least to most able, i.e.

$$a_{1j} \leq a_{2j} \leq \cdots \leq a_{Nj}$$

For now, we assume all classes are identical, i.e. $a_{ij} = a_i$ for all $j \in \{1, ..., J\}$ Later, we discuss how differences in classroom composition affect our results.

Teachers can undertake two types of efforts to help students acquire additional human capital. They can teach the class as a whole or tutor individual students. The tutoring instruction is student-specific, and any effort spent on teaching student i will not directly affect any student $i' \neq i$. Classroom teaching benefits all students in the class. Examples include tasks like lecturing or planning assignments.

Let e_{ij} denote the effort teacher j spends on individual instruction of student i, and t_j denote the effort she spends on classroom teaching. The human capital of a student at the end of the period, denoted a'_{ij}, depends on his initial ability a_{ij}, the efforts of his teacher e_{ij} and t_j, and a shock ε_{ij} that does not depend on teacher effort, e.g. random disruptions to the student's life at home. We assume the production of human capital is separable between the student's initial human capital and all other factors and takes the form

$$a'_{ij} = g(a_i) + t_j + \alpha e_{ij} + \varepsilon_{ij} \tag{1}$$

where $g(\cdot)$ is an increasing function and $\alpha > 0$ measures the relative productivity of classroom teaching versus individual instruction. The shock ε_{ij} is pairwise independent for any pair (i, j). Let $F(x) \equiv \Pr(\varepsilon_{ij} \leq x)$. We assume there is an associated density distribution $f(x) = \dfrac{dF(x)}{dx}$ that is unimodal, symmetric around 0, and differentiable.

Let X_j denote teacher j's expected income. Then her utility is assumed to be

$$U_j = X_j - C(e_{1j}, ..., e_{Nj}, t_j) \tag{2}$$

where $C(\cdot)$ denotes the teacher's cost of effort. We assume $C(\cdot)$ is increasing in all of its arguments and is strictly convex. We further assume it is symmetric with respect to individual effort, i.e. let $\mathbf{e_j}$ be any vector of tutoring efforts $(e_{1j}, ..., e_{Nj})$ for teacher j, and let $\mathbf{e'_j}$ be any permutation of $\mathbf{e_j}$, then

$$C(\mathbf{e_j}, t_j) = C(\mathbf{e'_j}, t_j)$$

We also impose the usual boundary conditions on marginal costs. The lower and upper limits of the marginal costs with respect to each dimension of effort are 0 and ∞ respectively. These conditions ensure the optimal plan will be interior. Although we do make it explicit, $C(\cdot)$ also depends on N. Optimal effort decisions will vary with class size, we can use our framework to derive the socially optimal class size. However, the tradeoffs between scales economies and congestion externalities at the center of such an analysis have been explored by others. Our present goal is to analyze the optimal provision of incentives given a fixed class size, N, and here, we suppress reference to N in the cost function .

2.1. **Social Optimum.** If we assume that each teacher has an outside option equal to U_0 The social planner's objective function is given by

$$\max_{e_{ij},t_j} E\sum_{j=1}^{J}[R[\sum_{i=1}^{N} g(a_i) + t_j + \alpha e_{ij} + \varepsilon_{ij}] - C(e_{1j}, ..., e_{Nj}, t_j) - U_0]$$

Here, R is the social value of a unit of a'. Given our production technology, one can also think of R as the gross social return per student when one unit of teacher time is effectively devoted to classroom instruction. Since $C(\cdot)$ is strictly convex, the first-order conditions are necessary and sufficient for an optimum. Since all teachers share the same cost of effort, the optimal allocation will dictate the same effort levels in all classrooms, i.e. $e_{ij} = e_i$ and $t_j = t$ for all j. Hence, the optimal effort levels $e_1, ..., e_N$ and t will solve the following system of equations:

$$\frac{\partial C(e_1, ..., e_N, t)}{\partial e_i} = R\alpha \qquad \text{for } i = 1, ..., N$$

$$\frac{\partial C(e_1, ..., e_N, t))}{\partial t} = RN$$

Given our assumptions, the cost and returns associated with devoting instruction time to a student are not a function of the student's baseline characteristics. Thus, the social optimum dictates not only homogeneous effort vectors among classrooms but also homogeneous tutoring effort among students in the same classroom.[2]

2.2. **Tournaments.** Here, we show that a simultaneous tournament mechanism can implement the socially optimal allocations of effort. Consider the following compensation scheme: Each teacher receives a base pay of X_0 per student. Further, each teacher knows that she is competing against one other teacher and that the results of this contest will determine her bonus. Any teacher j does not know whom her opponent will be when she makes her effort choice. She knows only that her opponent will be randomly chosen from the set of other teachers in the system and that her opponent will be facing the same compensation scheme that she faces. At the end of the year when teacher j is matched with some other teacher j', teacher j will receive a bonus $(X_1 - X_0)$ for each student i whose human capital is higher than the corresponding student in teacher j''s class, i.e. if $a'_{ij} \geq a'_{ij'}$. The total compensation for teacher j is then

[2]The proof of this claim is straightforward given that α is the same for all students. In section 4, we turn to a variation of the model in which students of different baseline achievement levels receive different benefits from receiving instruction, and here, the social optimum will involve different tutoring levels for students of different achievement levels.

$$NX_0 + (X_1 - X_0)\sum_{i=1}^{N}\mathbb{I}(a'_{ij} \geq a'_{ij'})$$

where $\mathbb{I}(E)$ is an indicator that equals 1 if event E is true and 0 otherwise. We note here that teacher j's compensation is not affected by the scale used to report assessment results. Because ordinal comparisons determine all payoffs, teacher behavior and teacher welfare are invariant to any re-scaling of the assessment results that preserves ordering.

We assume that each student in teacher j's class is compared with the comparable student in the class of teacher j'. However, this is not essential. More generally, the teacher against whom teacher j is compared can vary across students, e.g. we can let the identity of j' be a function of i, but this would be notationally more cumbersome.

For each $i \in 1, ..., N$, let us define a new variable $\nu_i = \varepsilon_{ij} - \varepsilon_{ij'}$ as the difference in the shock terms for students in the two classes whose initial human capital is a_i. Let $H(x) \equiv \Pr(\nu_i \leq x)$ denote the distribution of ν_i. We define $h(x) = dH(x)/dx$, and note that given our assumptions about $F(.)$, $H(.)$ is also unimodal, mean zero, and symmetric.

Since the initial ability of the students who are compared to each other is identical, the maximization problem for teacher j is

$$\max_{e_{ij}, t_j} NX_0 + (X_1 - X_0)\sum_{i=1}^{N} H(\alpha(e_{ij} - e_{ij'}) + t_j - t_{j'}) - C(e_{1j}, ..., e_{Nj}, t_j) - U_0$$

The first order conditions for each teacher are given by

$$\frac{\partial C(t_j, e_{1j}, ..., e_{Nj})}{\partial e_{ij}} = \alpha h(\alpha(e_{ij} - e_{ij'}) + t_j - t_{j'})(X_1 - X_0) \text{ for } i = 1, 2..N \tag{3}$$

$$\frac{\partial C(t_j, e_{1j}, ..., e_{Nj})}{\partial t_j} = \sum_{i=1}^{N} h(\alpha(e_{ij} - e_{ij'}) + t_j - t_{j'})(X_1 - X_0) \tag{4}$$

Consider setting the bonus $X_1 - X_0 = R/h(0)$ and suppose both teachers j and j' chose the same effort levels, i.e. $e_{ij} = e_{ij'}$ for all i. Then (**??**) and (**??**) become

$$\begin{aligned}\frac{\partial C(e_1, ..., e_N, t)}{\partial e_i} &= R\alpha \qquad \text{for } i = 1, ..., N \\ \frac{\partial C(e_1, ..., e_N, t)}{\partial t} &= RN\end{aligned}$$

Recall that these are the first order conditions for the planner's problem, and thus, the socially optimal effort levels $e_1 = ... = e_N = e^*$ and $t = t^*$ solve these first order conditions. Nonetheless, the fact that these levels satisfy teacher j's first order conditions is not enough to show that they are in fact optimal responses to the effort decisions of the other teacher. Since $H(\cdot)$ is neither strictly convex nor strictly concave everywhere, we have not shown that $e_{ij} = e^*$ and $t_j = t^*$ is a global best response to itself. Appendix A provides proofs for the following two propositions that summarize our main results for two teacher contests:

Proposition 1: *Let σ denote the variance of ε_{ij}. There exists $\overline{\sigma}$ such that $\forall \sigma > \overline{\sigma}$, both teachers choosing the socially optimal effort levels, $e_1 = ... = e_N = e^*$ and $t = t^*$, is a pure strategy Nash equilibrium of the two teacher contest.*

The variance restriction in Proposition 1 is needed to rule out cases where, given that the other teacher is choosing $e_1 = ... = e_N = e^*$ and $t = t^*$, teacher $j's$ expected gain from responding with $e_1 = ... = e_N = e^*$ and $t = t^*$ as opposed to some lower effort level does not cover the incremental cost. The total cost of optimal effort is a constant as is the change in total cost associated with moving from any other effort vector to $e_1 = ... = e_N = e^*$ and $t = t^*$. However, if σ is too small, the change in expected prize winnings associated with moving from a given lower effort level to the socially optimal levels may not cover the associated change in total cost.[3] However, if the element of chance in these contests is important enough, a unique pure strategy Nash equilibrium exists which involves both teachers choosing the socially optimal effort vectors, $e_1 = ... = e_N = e^*$ and $t = t^*$.

Proposition 2: *In the two teacher contest described here, whenever a pure strategy Nash equilibrium exists, it involves both teachers choosing the socially optimal effort levels, $e_1 = ... = e_N = e^*$ and $t = t^*$.*

Taken together, these propositions imply that our tournament scheme can elicit efficient effort from teachers who compete against each other in seeded competitions. Thus, the efficiency properties of two person contests involving a single dimension of effort carry over to two person contest involving an N+1 dimensional effort choice.

Finally, to ensure that teachers are willing to participate in this scheme, we need to make sure that

$$NX_0 + \frac{RN}{2h(0)} - C(e^*, ..., e^*, t^*) \geq U_0$$

Thus, let

$$X_0 = \frac{U_0 + C(e^*, ..., e^*, t^*)}{N} - \frac{R}{2h(0)}$$
$$X_1 = \frac{U_0 + C(e^*, ..., e^*, t^*)}{N} + \frac{R}{2h(0)}$$

We noted in our introduction that the rewards teachers earn under our scheme are independent of the scale used to report assessment results. We also want to stress that, because the bonus payments are tied to ranks, the planner does not need to know what psychometric scale is used to report the assessment results in order to pick the bonus scheme that generates first best effort. Further, the planner does not need to know the function $g(a_{ij})$, which determines the role of initial human capital in producing human

[3] Lazear and Rosen (1981) require a similar condition for existence in their single task, two person game

capital, because ranks are measured within sets of students who began the year with the same initial human capital. In order to choose the optimal prize, the planner needs to know the social values of e_{ij} and t_j, and he needs to know $h(0)$, the marginal return to effort for contestants who consider deviating from our candidate equilibrium. The planner also needs to know the outside option of teachers and the total private cost of (e^*, t^*) to choose a value of X_0 that extracts all the surplus from the teaching market.

3. Pay for Percentile

Although the procedure we describe above can achieve first best, in practice, performance contests against a single opponent create opportunities for corruption. In our setting, the opponent for teacher j is announced at the end of the year after students are tested. Thus, some teachers may respond to this system by offering bribes to school officials in exchange for being paired with a teacher whose students performed poorly. If one tried to avoid these bribes by announcing the pairs of contestants at the beginning of the year, then one would worry about collusion on low effort levels among contestants. Given these considerations, we now turn to performance contests that involve large numbers of teachers competing anonymously against one another. We then turn to the limiting case in which all teachers compete against all other teachers.

Suppose that each teacher now competes against $K > 1$ teachers who have also N students. To simplify notation, we assume again that the initial distribution of human capital is the same in all classes. Each teacher knows that K other teachers will be drawn randomly from the entire population of teachers to serve as her contestants, but teachers make their effort choices without knowing whom they are competing against. In this setting, teacher j's problem is

$$\max_{e_{ij}, t_j} NX_0 + \sum_{k=1}^{K} \sum_{i=1}^{N} H(\alpha(e_{ij} - e_{ik}) + t_j - t_k)(X_1 - X_0) - C(e_{1j}, ..., e_{Nj}, t_j) - U_0$$

The first order conditions are given by

$$\frac{\partial C(e_{1j}, ..., e_{Nj}, t_j)}{\partial e_{ij}} = \sum_{k=1}^{K} \alpha h(\alpha(e_{ij} - e_{ik}) + t_j - t_k)(X_1 - X_0) \text{ for } i = 1, ..., N \tag{5}$$

$$\frac{\partial C(e_{1j}, ..., e_{Nj}, t_j)}{\partial t_j} = \sum_{k=1}^{K} \sum_{i=1}^{N} h(\alpha(e_{ij} - e_{ik}) + t_j - t_k)(X_1 - X_0) \tag{6}$$

As before, suppose all teachers put in the same effort level, i.e. $e_{ij} = e_{ik}$ for all $k = 1, ..., K$. In this case, the right-hand side of (**??**) and (**??**) reduce to $\alpha K h(0)(X_1 - X_0)$ and $NKh(0)(X_1 - X_0)$, respectively. Thus, if we set $X_1 - X_0 = \frac{R}{Kh(0)}$ and assume that all teachers choose the socially optimal effort levels, the first order conditions for each teacher are satisfied. Further, Proposition 1 extends trivially to contests among $K > 2$ teachers.

Therefore, we know that given the prize, $\frac{R}{Kh(0)}$, a pure strategy Nash equilibrium in which all K teachers chose the socially optimal levels of effort exists.

Next, assume that $K = J$. This means that, among our population of teachers whose students share the same initial distribution of baseline achievement, each teacher competes against every other teacher. It is immediate that given a prize of $\frac{R}{Kh(0)}$ and a sufficient level of σ, there exists a pure strategy Nash equilibrium of this contest that involves all $K = J$ choosing socially efficient effort levels.

Now let $K = J \to \infty$ and let A'_i denote a terminal score chosen at random and uniformly from the set of all terminal scores $(a'_{i1}, ..., a'_{iJ})$. Since the distribution $(a'_{i1}, ...a_{i,j-1}, a_{i,j+1}, ..., a'_{iJ})$ converges to the distribution $(a'_{i1}, ...a_{i,j-1}, a_{ij,}a_{i,j+1}, ..., a'_{iJ})$ as $K \to \infty$, it follows that

$$\lim_{K\to\infty} \sum_{k=1}^{K} \frac{\mathbb{I}(a'_{ij} \geq a'_{ik})}{K} = \Pr(a'_{ij} \geq A'_i)$$

and the teacher's maximization problem reduces to

$$\max_{e_{ij},t_j} NX_0 + \frac{R}{h(0)} \sum_{i=1}^{N} \Pr(a'_{ij} \geq A'_i) - C(e_{1j}, ..., e_{Nj}, t_j) - U_0$$

Pay for percentile is a limiting case of our simultaneous contests scheme as the number of teachers grows large.

So far, we have assumed identical classes, but given our assumptions that teacher time produces the same increase in human capital for each student regardless of his initial human capital and that each teacher's cost of effort is symmetric over students, the composition of each class does not matter for either the first best effort levels or the effort levels that teachers choose under our proposed scheme.

Suppose each class $j \in 1, ..., J$ has N students, whose resepctive ability levels a_{ij} for $i \in 1, ..., N$ satisfy

$$a_{1j} \leq a_{2j} \leq \cdots \leq a_{Nj}$$

Rather than require that $a_{ij} = a_i$ for all j, we need only require that for each a_{ij}, there exists at least K students in other classrooms $(i\prime, j\prime)$ such that $a_{ij} = a_{i'j'}$. If K is large, our pay for percentile scheme can be applied to any classroom by calculating the final assessment percentile score of each student i within the set of all students who share $i's$ baseline achievement level.

This result holds because nothing that depends on a_{ij} appears in the first-order conditions for the planner's problem. The optimal allocation of effort $(\mathbf{e}^*, t^*)$ is the same for all teachers regardless of the distribution of baseline achievement in their classrooms. Further, given the symmetry in our cost function, the cost associated with teacher j choosing $(\mathbf{e}^*, t^*)$ as a best response to all teachers of all other students choosing $(\mathbf{e}^*, t^*)$ is not influenced by the distribution of baseline achievement in $j's$ class.

still working on uniqueness in this case

4. Heterogeneous Gains from Instruction

In this section, we turn to the case where the same amount of teacher instruction generates different expected gains in human capital depending on the baseline achievement levels of particular students. Here, the socially efficient vector of effort choices for teacher j is a function of the distribution of baseline achievement in her class. In this setting, we cannot base comparison sets on the baseline achievement of individual students alone. If two teachers both have one student who begins at a_i, the marginal cost of tutoring effort for such a student, given that the teachers are providing efficient effort to other students in their classes, depends on the distribution of baseline achievement among his peers.

Properly seeded tournaments pair contestants who face the same total and marginal costs of effort. Thus, for our scheme to work in this more general case, we require a stronger assumption concerning the availability of comparison sets. Above, we assume that for each student, there exists K other student with the same initial achievement level. We now require that for each classroom, there exists K other classes with the same composition of initial achievement level. For each j, there exists at least K j' such that $a_{ij} = a_{ij'}$ for all $i \in \{1, ..., N\}$. Given such comparison sets, our pay for percentile scheme can elicit efficient effort to all student and classrooms even in the presence of heterogeneous returns to instruction.

Consider the following generalization of our human capital production technology:

$$a'_{ij} = g(a_{ij}) + \gamma(a_{ij})t_j + \alpha(a_{ij})e_{ij} + \varepsilon_{ij} \tag{7}$$

We maintain our assumption that the cost of spending time teaching students does not depend on their identity, but now the productivity of teacher effort differs with initial baseline achievements.

In this setting, consider the planner's problem. We assume the planner takes the composition of classes as given, and must choose the effort decisions in each classroom given the students in that class. One could imagine a more general problem where the planner choose the composition of classrooms and the effort vector for each classroom. However, given the optimal composition of classrooms, the planner still needs to choose the optimal levels of effort in each class, and this second step is our focus because we are analyzing the provision of incentives for educators taking as given the sorting of students among schools and classrooms.

Within each class j, the planner chooses effort to solve

$$\max_{\mathbf{e}_j, t_j} R \sum_{i=1}^{N} [g(a_{ij}) + \gamma(a_{ij})t_j + \alpha(a_{ij})e_{ij} + \varepsilon_{ij}] - C(\mathbf{e}_j, t_j) - U_0$$

This problem is strictly concave, and so the first-order conditions are both necessary and sufficient for an optimum. These are given by

$$
\begin{aligned}
\frac{\partial C(\mathbf{e},t)}{\partial e_i} &= R\alpha(a_{ij}) \qquad \text{for } i = 1, ..., N \\
\frac{\partial C(\mathbf{e},t)}{\partial t} &= R\sum_{i=1}^{N}\gamma(a_{ij})
\end{aligned}
$$

For any composition of baseline achievement, there will be a unique effort vector $(\mathbf{e}, t)$ that solves this equation, but this vector will differ for classes with different compositions. In particular, two students with the same benchmark ability but different peers could receive different effort levels because differences in effort allocations to their peers create differences in the cost of effort applied to each of them.

We now show that a slight variation on our pay for percentile scheme can elicit socially optimal effort vectors from all teachers. The bonus scheme is the same as before, but now each student is compared to a set of students who not only have the same initial ability but also peers with the same distribution of baseline achievement. That is, each class k in teacher j's comparison set for a given student must satisfy $a_{ij} = a_{ik}$ for all $i \in \{1, ..., N\}$.

Teacher j's problem is given by

$$
\max_{e_{ij},t_j} NX_0 + \sum_{k=1}^{K}\sum_{i=1}^{N} H(\alpha(a_{ij})(e_{ij} - e_{ik}) + \gamma(a_{ij})(t_j - t_k))(X_1 - X_0) - C(t_j, e_{1j}, ..., e_{Nj})
$$

Once again, suppose we set $X_1 - X_0 = \frac{R}{Kh(0)}$. If all teachers provide the same effort levels, the first order conditions collapse to

$$
\begin{aligned}
\frac{\partial C(\mathbf{e},t)}{\partial e_i} &= R\alpha(a_{ij}) \qquad \text{for } i = 1, ..., N \\
\frac{\partial C(\mathbf{e},t)}{\partial t} &= R\sum_{i=1}^{N}\gamma(a_{ij})
\end{aligned}
$$

which are the same as the planner's first order condition. If we assume that other teachers are choosing the socially optimal levels of effort and invoke our usual restriction on σ, these first-order conditions are necessary and sufficient for an optimal response. Thus , a Nash equilibrium exists such that all teachers choose the first best effort levels.

Although our scheme requires that all teachers receive the same bonus rate regardless of classroom composition, teachers who incur greater costs of effort under the optimal scheme require a higher base pay to participate. Thus, our analyses show that differences in socially efficient effort levels among classrooms do not imply a need for differences in incentive pay among classrooms but rather compensating differences in base pay.

5. Conclusion

It is useful to divide the pay for performance systems described in Table 1 into two categories. Some attempt to use statistical models to place all teachers in an entire system on a common performance scale. These approaches necessarily lean heavily on modeling assumptions and the scales used to report assessment results. The goal of universal rankings requires the designers of these system to take particular stands concerning whether or not a teacher whose student experienced a score gain of 130 to 160 actually performed better or worse than a teacher whose student experienced a gain of 250 to 275. These stands are difficult to defend because the units of psychometric scales do not easily translate into pecuniary measures of student benefit or teacher effort.

Other approaches, like the Denver ProComp system, involve setting achievement targets for each child in each classroom. This approach has the advantage of using only the ordinal properties of the assessment results, but it invites collusion on low effort. One can imagine choosing target levels for each a_i that would elicit efficient effort from teachers, but how do policy makers gain the information needed to set these targets, and why should we expect that individual negotiations between teachers and principals will result in efficient achievement targets for each student.

Our approach relies only on the ordinal properties of assessments, and it uses seeded competition among teachers to implicitly set efficient achievement targets for each student. Because the assignment of reward pay to teachers is invariant to the scale used to report assessment results and the outcome of competition among a large number of teachers, there is little opportunity for collusion among administrators, principals, or teachers to contaminate the results. For each level of student achievement, our use of relative performance pay guarantees that there will be teachers who are winners because their student achieved a high percentile score within his peer group and there will be losers whose student achieved a low percentile score within the same peer group.

Our results suggest that efficient performance pay systems should be built around rank order competitions among teachers who teach in similar settings. Systems based on ordinal performance targets without real competition among teachers are subject to manipulation and invite outcomes which imply that all teachers provided above average performance, while competitive systems built around universal performance rankings rest on dubious assumptions concerning the meaning of pyschometric scales and the specification of statistical models. Designers of performance pay systems require enormous amounts of information in order to either set performance targets for each student or create a universal performance ranking for all teachers. We show here that policy makers need not attempt to solve either of these intractable handicapping problems. Properly seeded contests can elicit socially efficient effort levels and reveal the socially efficient achievement targets for students in various classrooms.

Appendix A

Our analysis of two teacher contests involving pairwise comparisons of outcomes for N students yields the following existence result:

Proposition 1: *Let σ denote the variance of ε_{ij}. There exists $\overline{\sigma}$ such that $\forall\sigma > \overline{\sigma}$, both teachers choosing the socially optimal effort levels, $e_1 = ... = e_N = e^*$ and $t = t^*$, is a pure strategy Nash equilibrium of the two teacher contest.*

Proof of Proposition 1: Define $\widetilde{v}_i = \widetilde{\varepsilon}_{ij} - \widetilde{\varepsilon}_{ij'}$, and let $\widetilde{H}(x) \equiv \Pr(\widetilde{v}_i \leq x) = H(x/\sigma)$. Similarly, define $\widetilde{h}(x) \equiv \dfrac{d\widetilde{H}(x)}{dx} = \dfrac{1}{\sigma}h(x/\sigma)$. Note that

$$\widetilde{h}(0) = \frac{1}{\sigma}h(0)$$

and

$$\begin{aligned}\widetilde{H}(\alpha e_i - \alpha e^* + t - t^*) &= \int_{-\infty}^{\alpha e_i - \alpha e^* + t - t^*} \widetilde{h}(x)dx \\ &= \int_{-\infty}^{\alpha e_i - \alpha e^* + t - t^*} \frac{1}{\sigma}h(x/\sigma)dx\end{aligned}$$

The teacher's objective function is given by

$$\max_{\mathbf{e}_j, t_j} NX_0 + (X_1 - X_0)\sum_{i=1}^{N}\widetilde{H}(\alpha e_{ij} - \alpha e^* + t_j - t^*) - C(\mathbf{e}_j, t_j) - U_0$$

If we set $X_1 - X_0 = R/\widetilde{h}(0)$, this objective function reduces to

$$\max_{\mathbf{e}_j, t_j} NX_0 + R\sum_{i=1}^{N}\left[\int_{-\infty}^{\alpha e_{ij} - \alpha e^* + t_j - t^*} \frac{h(x/\sigma)}{h(0)}dx\right] - C(\mathbf{e}_j, t_j) - U_0 \tag{8}$$

We first argue that the solution to this problem is bounded in a way that does not depend on σ. Since $h(\cdot)$ is unimodal with a peak at zero, it follows that $\dfrac{h(x/\sigma)}{h(0)} \leq 1$, and so

$$\begin{aligned}\int_{-\infty}^{\alpha e_{ij} - \alpha e^* + t_j - t^*} \frac{h(x/\sigma)}{h(0)}dx &= \int_{-\infty}^{-\alpha e^* - t^*} \frac{h(x/\sigma)}{h(0)}dx + \int_{-\alpha e^* - t^*}^{\alpha e_{ij} - \alpha e^* + t_j - t^*} \frac{h(x/\sigma)}{h(0)}dx \\ &\leq \int_{-\infty}^{-\alpha e^* - t^*} \frac{h(x/\sigma)}{h(0)}dx + \alpha e_{ij} + t_j\end{aligned}$$

The objective function in (**??**) is thus bounded above by

$$NX_0 + R\int_{-\infty}^{-\alpha e^* - t^*} \frac{h(x/\sigma)}{h(0)}dx + R\sum_{i=1}^{N}(\alpha e_{ij} + t_j) - C(\mathbf{e}_j, t_j) - U_0$$

Next, define the set $U = \left\{ \mathbf{u} \in \mathbb{R}_+^{N+1} : \sum_{i=1}^{N+1} u_i^2 = 1 \right\}$. Any vector $(\mathbf{e}_j, t_j)$ can be uniquely expressed as $\lambda\mathbf{u}$ for some $\lambda \geq 0$ and some $\mathbf{u} \in U$. Given our assumptions on $C(\cdot,\cdot)$, for any vector $\mathbf{u}$ it must be the case that $C(\lambda\mathbf{u})$ is increasing and convex in λ and satsifies the limit $\lim_{\lambda\to\infty} \frac{\partial C(\lambda\mathbf{u})}{\partial\lambda} = \infty$. Since $\lambda R \sum_{i=1}^{N} (\alpha e_{ij} + t_j)$ is linear in λ, for any $\mathbf{u} \in U$ there exists a finite cutoff $\lambda^*(\mathbf{u})$ such that $\lambda R \left[\sum_{i=1}^{N} \alpha u_i + u_{N+1}\right] - C(\lambda\mathbf{u}) < 0$ for all $\lambda > \lambda^*(\mathbf{u})$. Since U is compact, $\lambda^* = \sup\{\lambda^*(\mathbf{u}) : \mathbf{u} \in U\}$ is well defined and finite. Given that $\lambda R \left[\sum_{i=1}^{N} \alpha u_i + u_{N+1}\right] - C(\lambda\mathbf{u})$ for $\mathbf{u} = \mathbf{0}$, it follows that the solution to (**??**) lies in the bounded set $[0, \lambda^*]^{N+1}$.

Next, we argue that there exists a $\overline{\sigma}$ such that for $\sigma > \overline{\sigma}$, the Hessian matrix of second order partial derivatives for this objective function is negative definite over the bounded set $[0, \lambda^*]^{N+1}$. Define $\pi(t, e_1, ..., e_N) \equiv R \sum_{i=1}^{N} \left[\int_{-\infty}^{\alpha e_{ij} - \alpha e^* + t - t^*} \frac{h(x/\sigma)}{h(0)} dx\right]$. Then the Hessian matrix is the sum of two matrices, $\mathbf{\Pi} - \mathbf{C}$, where

$$\mathbf{C} \equiv \begin{bmatrix} \frac{\partial^2 C}{\partial e_1^2} & \frac{\partial^2 C}{\partial e_2 \partial e_1} & \cdots & \frac{\partial^2 C}{\partial t \partial e_1} \\ \frac{\partial^2 C}{\partial e_1 \partial e_2} & \frac{\partial^2 C}{\partial e_2^2} & \cdots & \frac{\partial^2 C}{\partial t \partial e_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 C}{\partial e_1 \partial t} & \frac{\partial^2 C}{\partial e_2 \partial t} & \cdots & \frac{\partial^2 C}{\partial t^2} \end{bmatrix}$$

and

$$\mathbf{\Pi} \equiv \begin{bmatrix} \frac{\partial^2 \pi}{\partial e_1^2} & \frac{\partial^2 \pi}{\partial e_2 \partial e_1} & \cdots & \frac{\partial^2 \pi}{\partial t \partial e_1} \\ \frac{\partial^2 \pi}{\partial e_1 \partial e_2} & \frac{\partial^2 \pi}{\partial e_2^2} & \cdots & \frac{\partial^2 \pi}{\partial t' \partial e_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 \pi}{\partial e_1 \partial t} & \frac{\partial^2 \pi}{\partial e_2 \partial t} & \cdots & \frac{\partial^2 \pi}{\partial t^2} \end{bmatrix}$$

Since the function $C(\cdot)$ is strictly convex, $-\mathbf{C}$ must be a negative definite matrix. Turning to $\mathbf{\Pi}$, we have

$$\mathbf{\Pi} = \frac{R}{\sigma h(0)} \times \begin{bmatrix} \alpha^2 h'(\frac{\alpha e_1 - \alpha e^* + t - t^*}{\sigma}) & 0 & \cdots & \alpha h'(\frac{\alpha e_1 - \alpha e^* + t - t^*}{\sigma}) \\ 0 & \alpha^2 h'(\frac{\alpha e_2 - \alpha e^* + t - t^*}{\sigma}) & \cdots & \alpha h'(\frac{\alpha e_2 - \alpha e^* + t - t^*}{\sigma}) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha h'(\frac{\alpha e_1 - \alpha e^* + t - t^*}{\sigma}) & \alpha h'(\frac{\alpha e_2 - \alpha e^* + t - t^*}{\sigma}) & \cdots & \sum_{i=1}^{N} h'(\frac{\alpha e_i - \alpha e^* + t - t^*}{\sigma}) \end{bmatrix}$$

But $\widetilde{h}'(x) = \dfrac{1}{\sigma^2}h'(x/\sigma)$. For a fixed x, all of the elements in $\mathbf{\Pi}$ converge to multiples of $h'(0)$ as $\sigma \to \infty$, which is just 0. Hence, within the bounded set $[0, \lambda^*]^{N+1}$, we have $\mathbf{\Pi} \to \mathbf{0}$ as $\sigma \to \infty$. Since $\mathbf{C}$ is positive definite and $\mathbf{\Pi} \to 0$, it follows that there exists a $\overline{\sigma}$ such that for all $\sigma > \overline{\sigma}$, the matrix $\mathbf{\Pi} - \mathbf{C}$ is negative definite for all values of $(\mathbf{e}_j, t_j) \in [0, \lambda^*]^{N+1}$. Hence, the objective function is strictly concave in the region that contains the global optimum, ensuring the first-order conditions are both necessary and sufficient to define a global maximum. ■

Proposition 2: *In the two teacher contest described here, if a pure strategy Nash equilibrium exists, it involves both teachers choosing the socially optimal effort levels,* $e_1 = ... = e_N = e^*$ *and* $t = t^*$.

Proof of Proposition 2:

We begin our proof by establishing the following Lemma:

Lemma: Suppose $C(\cdot)$ is a convex differentiable function which satisfies standard boundary conditions concerning the limits of the marginal costs of each dimension of effort as effort on each dimension goes to 0 or ∞. Then for any positive real numbers $a_1, ..., a_N$ and b, there is a unique solution to the system of equations

$$\begin{aligned}\frac{\partial C(e_1, ..., e_N, t)}{\partial e_i} &= a_i \qquad\qquad \text{for } i = 1, ..., N \\ \frac{\partial C(e_1, ..., e_N, t)}{\partial t} &= b\end{aligned}$$

Proof: Define a function $bt + \sum_{i=1}^{N} a_i e_i - C(e_1, .., , e_N, t)$. Since $C(\cdot)$ is strictly convex, this function is strictly concave, and as such has a unique maximum. The boundary conditions, together with the assumption that $a_1, ..., a_N$ and b are positive, ensure that this maximum must be at an interior point. Because the function is strictly concave, this interior maximum and the solution to the above equations is unique, as claimed. ■

Armed with this lemma, we can demonstrate that any pure strategy Nash equilibrium of the two teacher contest involves both teachers choosing the socially optimal effort levels. Note that, given any pure strategy Nash equilibrium, both teacher's choices will satisfy the first order conditions for a best response to the other teacher's actions. Further, since h(.) is symmetric, we know that given the effort choices of j and j',

$$h(\alpha(e_{ij} - e_{ij'}) + t_j - t_{j'}) = h(\alpha(e_{ij'} - e_{ij}) + t_{j'} - t_j)$$

In combination, these observations imply that any Nash equilibrium strategies, $(e_{1j}, ..e_{Nj}, t_j)$ and $(e_{1j'}, ..e_{Nj}, t_j')$, must satisfy:

$$h(0)\frac{\partial C(e_{1j}, ., e_{Nj}, t_j)}{\partial e_{ij}} = R\alpha h(\alpha(e_{ij} - e_{ij'}) + t_j - t_{j'}) = R\alpha h(\alpha(e_{ij'} - e_{ij}) + t_{j'} - t_j) = h(0)\frac{\partial C(e_{1j'}, ., e_{Nj'}, t_{j'})}{\partial e_{ij'}}$$

$$h(0)\frac{\partial C(e_{1j},.,e_{Nj},t_j)}{\partial t_j} = RNh(\alpha(e_{ij}-e_{ij'})+t_j-t_{j'}) = RNh(\alpha(e_{ij'}-e_{ij})+t_{j'}-t_j) = h(0)\frac{\partial C(e_{1j'},.,e_{Nj'},t_{j'})}{\partial t_{j'}}$$

Our lemma implies that these equations cannot be satisfied unless $e_{ij} = e_{ij'} = e^*$ for all $i = 1, ..., N$ and that $t_j = t_{j'} = t^*$. The only pure-strategy equlibrium possible in our two teacher contests is one where teachers invest the classroom instruction effort and common level of tutoring that are socially optimal. ■