Pass-Through Determines the Division of Surplus under Monopoly*

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October 2008

Abstract

We show that the division between producer and consumer surplus at the optimal price of a linear cost monopolist is determined by an average of the pass-through rate over prices above this optimum. The result also strengthens the link between the log-curvature of densities and their integrals.

Consider a monopolist with linear cost of production c choosing a linear price to charge consumers, whose continuous demand $D : \mathbb{R} \to \mathbb{R}$ depends only on this price. We assume that D is everywhere finite, weakly positive and is strictly decreasing and twice differentiable wherever it is strictly positive. Let $\overline{p} \equiv \sup\{p : D(p) > 0\}$ and assume $\overline{p} > -\infty$. The monopolist's profits are $\pi(p) = (p - c)D(p)$ and her first-order condition for maximizing these are

$$p - c = m(p) \equiv -\frac{D(p)}{D'(p)}$$

*Fabinger and Weyl are respectively grateful to the Society of Fellows for its support of his research and to Charles Fefferman for useful comments.

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We assume a standard, weak second-order condition (Weyl, 2008a) called *mark-up contraction*

$$m'(p) < 1 \quad \forall p < \overline{p}$$

ensuring that the first-order condition is sufficient for her maximization. We additionally assume that if $\overline{p} = \infty$ then $\lim_{p \to \infty} m'(p)$ exists and is less than 1. Two natural questions in the economics of monopoly are how the monopolist reacts to increases in her cost and how the benefits of trading are shared between the monopolist and consumers. The first issue, the *pass-through rate*, plays an important role in a wide variety of problems in industrial organization (Weyl, 2008a), from differentiated Cournot and Bertrand competition to the theory of two-sided markets. The pass-through rate is given by

$$\rho(p) \equiv \frac{1}{1 - m'(p)}$$

The second issue, the division of *surplus* between the monopolist and consumers, is important to, among other things, the design patent and monopoly right auction policy, firms' selection of products (Spence, 1976; Lancaster, 1975; Dixit and Stiglitz, 1977; Salop, 1979). Consumer surplus as a function of the price p charged is

$$V(p) \equiv \int_{p}^{\overline{p}} D(q)dq$$

Lemma 1. *V is well-defined and finite.*

Proof. See supporting information. □

$m' < 1$ (globally and in the limit) ensures the convergence of the integral. The ratio of surplus earned by consumers to that of the monopolist (profits) is

$$r(p) \equiv \frac{\overline{V}(p)}{m(p)}$$

where $\overline{V} \equiv \frac{V}{D}$. Weyl (2008a) shows that if $\rho(p)$ is the constant function ρ then $r(p) = \rho$ for all p and Weyl (2008b) shows that if $\rho < (>)1$ globally then under weak technical conditions $r < (>)1$. We substantially generalize this result to show that for arbitrary demand functions satisfying our conditions, the ratio of consumer to producer surplus under monopoly is given by an average of the pass-through rates over prices above the monopolist's optimum. This is useful both because it links two important parameters of monopoly pricing and because pass-through is much easier to measure than the division of surplus

Lemma 2. $\lim_{p \to \overline{p}_-} m(p)D(p) = 0$

Proof. See supporting information. □

Theorem 1. *For* $p < \overline{p}$

$$r(p) = \frac{\int_p^{\overline{p}} \mu(q)\rho(q)dq}{\int_p^{\overline{p}} \mu(q)dq} \tag{1}$$

where $\mu(q) \equiv \frac{D(p)}{\rho(q)}$.

Proof. The result states that

$$\frac{\int_p^{\overline{p}} D(q)dq}{m(p)D(p)} = \frac{\int_p^{\overline{p}} D(q)dq}{\int_p^{\overline{p}} \frac{D(q)}{\rho(q)}dq}$$

and so we must show that for $p < \overline{p}$

$$m(p)D(p) = \int_p^{\overline{p}} \frac{D(q)}{\rho(q)}dq$$

Substituting into the right hand side expression and invoking Lemma 2

$$\int_p^{\bar{p}} \frac{D(q)}{\rho(q)}dq = \int_p^{\overline{p}} D(q)\left(1 - m'(q)\right)dq = \int_p^{\bar{p}} \left(1 - \left(-\frac{D(q)}{D'(q)}\right)'\right) D(q)dq =$$

$$\int_p^{\bar{p}} \left(1 + \frac{D'^2(q) - D(q)D''(q)}{D'^2(q)}\right) D(q)dq = \int_p^{\bar{p}} \frac{2D(q)D'^2(q) - D^2(q)D''(q)}{D'^2(q)}dq =$$

$$\int_{p}^{\bar{p}} \left(\frac{D^2(q)}{D'(q)}\right)' dq = \int_{\bar{p}}^{p} (m(q)D(q))' \, dq = m(p)D(p) - \lim_{q \to \bar{p}_-} m(p)D(p) = m(p)D(p)$$

$\square$

This result has two direct corollaries.

Corollary 1. $\forall p < \bar{p},\ \rho(p) < (>) k \implies r(p) < (>) k,\ \forall p < \bar{p}$

Corollary 2. $\forall p < \bar{p},\ \rho'(p) < (>) 0 \implies \rho(p) > (<) r(p)$ *and* $r'(p) < (>) 0,\ \forall p < \bar{p}$.

Our result may also have some applications in probability theory and statistics. Log-curvature, particularly log-concavity, of distributions plays an important role in physics (Brascamp and Lieb, 1976), mathematics (Brent, 1992; Bobkov, 1999), economics (Bagnoli and Bergstrom, 2005), computer science (Stanley, 1989) and statistics (Fieze et al., 1994). In a classic result (Prékopa, 1971) showed that log-concave densities have log-concave cumulative distribution and survival functions. An (1998) generalized Prékopa's result to show that log-convex densities with supports unbounded from above (below) have log-convex survivor (cumulative distribution) functions. Under the special conditions we consider, our result goes much farther, establishing a *quantitative*, not merely qualitative, connection between the log-curvature of a distribution and the log-curvature of its integral. A generalization of our result might therefore be a tool for generating inequalities linking degrees of log-curvature of densities to those of their integrals. Because our primary interest is in monopoly pricing we do not pursue this generalization further. However we conclude by noting the direct implications of our result in a probabilistic context, showing how it confirms and strengthens Prékopa and An's results.

For any positive twice differentiable function $g : \mathbb{R} \to \mathbb{R}$ let its *log-curvature* be $\lambda_g(x) \equiv (log\,[g(x)])''$. Consider a continuous density f whose support has interior $(\underline{x}, \overline{x}) \subseteq \overline{\mathbb{R}}$. Suppose there is some $\tilde{x} \in (\underline{x}, \overline{x})$ such that for all $x \in (\tilde{x}, \overline{x})$ we have f being twice differentiable,

$f'(x) < 0$, $f'^2(x) > \lambda_f(x)f^2(x)$ and $\lim_{x\to\overline{x}_-} f'^2(x) > \lim_{x\to\overline{x}_-} \lambda_f(x)f^2(x)$ assuming these both exist. Let the survivor function of f be $S(x) \equiv \int_x^{\overline{x}} f(y)dy$ which is well defined by the fact that f is a density.

Corollary 3.

$$\frac{\lambda_S(x)S^2(x)}{f^2(x)} + 1 = \frac{\int_x^{\overline{x}} \frac{\eta(y)}{1-\frac{\lambda_f(y)f^2(y)}{f'^2(y)}}dy}{\int_x^{\overline{x}} \eta(y)dy}$$

for all $x \in (\tilde{x}, \overline{x})$*, where* $\eta(y) = \frac{f(y)}{f'^2(y)}\left(f'^2(y) - \lambda_f(y)f^2(y)\right)$*. In particular,* $\forall x \in (\tilde{x}, \overline{x}), \lambda_f(x) < (>)0 \implies \lambda_S(x) < (>)0, \forall x \in (\tilde{x}, \overline{x})$*.*

The second part of this corollary is essentially a special case of Prékopa (1971) and An (1998)'s result, except that it has no support restrictions and therefore generalizes An (1998)'s results to finite support distributions that monotonically decrease to zero as they reach the upper bound of their support. The first part makes their qualitative result quantitative. Our results are restricted, though, to the strictly monotonically decreasing region of densities (if they have one) and only apply to densities satisfying $f'^2(x) > \lambda_f(x)f^2(x)$ in that region. Given that survivor functions (more analogous to demands) are always strictly monotonically decreasing in their support, our results are more directly useful for linking their log-curvature to the log-curvature of their upper-tail integral. Our results can just as easily be applied to lower tail integrals, however, in regions where densities monotonically increase, or to linking the log-curvature of a CDF to the log-curvature of its lower-tail integral.

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A Proof of Lemmas 1 and 2

We first establish both results in the case that $\overline{p} \in \mathbb{R}$ and then in the case when $\overline{p} = \infty$.

$\overline{p} < \infty$

In this case Lemma 1 is trivial, as on the interval $[p, \overline{p}]$, D is bounded above by $D(p)$ and below by 0 and so its integral is finite. To prove Lemma 2 note that for any $p < \overline{p}$

$$\lim_{q \to \overline{p}_-} m(q) = m(p) + \lim_{q \to \overline{p}} \int_p^q m'(r) dr < m(p) + \overline{p} - p$$

The integral exists as m' is bounded above by 1 and the integral is bounded below by $-m(p)$ as for all $q < \overline{p}$, $D(q) > 0 > D'(q)$ and thus $m(q) > 0$. Thus

$$\lim_{q \to \overline{p}_-} m(q) D(q) \leq \lim_{q \to \overline{p}_-} \left(m[p] + \overline{p} - p \right) D(q) = 0$$

as $D(\overline{p}) = 0$.

$\overline{p} = \infty$

The assumption

$$\lim_{p \to \infty} m'(p) < 1$$

implies that there will be numbers p_0 and $a \in (0, 1)$ such that for all $p > p_0$,

$$m'(p) < a.$$

Using the definition of $m(p)$,

$$\left(-\frac{D(p)}{D'(p)}\right)' < a,$$

$$\frac{D(p)D''(p) - D'^2(p)}{D'^2(p)} < a,$$

$$\frac{D(p)D''(p) - (1+a)D'^2(p)}{D'^2(p)} < 0,$$

$$\frac{D(p)^{1+a}D''(p) - (1+a)D'^2(p)D(p)^a}{D^{2+2a}(p)} < 0,$$

$$\left(\frac{D'(p)}{D(p)^{1+a}}\right)' < 0.$$

By integration,

$$\frac{D'(p)}{D(p)^{1+a}} < \frac{D'(p_0)}{D(p_0)^{1+a}}.$$

For later purposes, rewrite this as

$$m(p)D(p)^a < m(p_0)D(p_0)^a.$$

At this point, however, the following form of the inequality will be more useful.

$$D'(p)D(p)^{-1-a} < D'(p_0)D(p_0)^{-1-a},$$

$$-\frac{1}{a}(D(p)^{-a})' < D'(p_0)D(p_0)^{-1-a}.$$

Integrating this inequality further,

$$D(p)^{-a} > D(p_0)^{-a} - aD'(p_0)D(p_0)^{-1-a}(p - p_0),$$

$$0 < D(p) < \frac{1}{[D(p_0)^{-a} - aD'(p_0)D(p_0)^{-1-a}(p-p_0)]^{\frac{1}{a}}}.$$

The fact that

$$\lim_{p\to\infty} \frac{1}{[D(p_0)^{-a} - aD'(p_0)D(p_0)^{-1-a}(p-p_0)]^{\frac{1}{a}}} = 0$$

implies that

$$\lim_{p\to\infty} D(p)$$

exists and is equal to zero. Moreover, since

$$\int_{p_0}^{\infty} \frac{1}{[D(p_0)^{-a} - aD'(p_0)D(p_0)^{-1-a}(p-p_0)]^{\frac{1}{a}}} dp$$

is finite, we see that

$$V(p) = \int_p^{\infty} D(q)dq$$

is well-defined, finite, and positive establishing Lemma 1.

Now return to inequality

$$m(p)D(p)^a < m(p_0)D(p_0)^a.$$

It implies that

$$0 < m(p)D(p) < m(p_0)D(p_0)^a D(p)^{1-a},$$

for all $p > p_0$. Because

$$\lim_{p\to\infty} m(p_0)D(p_0)^a D(p)^{1-a} = m(p_0)D(p_0)^a \lim_{p\to\infty} D(p)^{1-a} = 0,$$

we can conclude that

$$\lim_{p \to \infty} m(p)D(p)$$

exists and is equal to zero. This proves Lemma 2.