Robust Mechanism Design of Exchange

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Abstract

We provide a robust (prior-free) strategic foundation for the Walrasian Equilibrium: mechanisms for exchange economies with asymmetric information and interdependent values that are ex-post individually rational, incentive compatible, generate budget surplus and are ex-post nearly efficient, when there are many agents. The level of inefficiency is proportional to the sum of $\frac{1}{N}$ and the degree of value interdependence. Conversely, we show that mechanisms generating smaller efficiency losses must violate some of the constraints. The mechanisms can be implemented using a novel discriminatory conditional double auction.

Keywords: exchange economy, asymptotic efficiency, interdependent values, prior-free mechanisms, robustness

1 Introduction

The analysis of exchange economies and their efficient allocations is the paradigmatic problem of economics. This prominence is well deserved, as the value of trade on centralized stock, bond or commodity exchanges dwarves the world’s GNP. In a symmetric information environment, one way to achieve efficiency is to post market clearing (Walrasian) prices and let agents buy and sell freely. The problem becomes much harder if agents have private information, especially when values are interdependent, as in the case of asset trade. Indeed, in this setting, no feasible prior-free mechanism that guarantees a near-efficient allocation is known.

In a private information environment, restriction to prior-free mechanisms is a means to capture the robustness of posted prices, in the following sense. Bayesian mechanism design presumes knowledge of a joint distribution of agents’ payoff types, beliefs, beliefs
about beliefs etc. and builds a different mechanism for each. The prior-free approach is more demanding in that it asks for one mechanism that works for all distributions, and satisfies the incentive constraints and is near-efficient ex-post, for any realization of payoff types (see Wilson (1985b), Chung and Ely (2002), Bergemann and Morris (2005)). Crucially, the mechanism does not depend on the details of the information structure, and thus, is robust to its misspecification.

The goal of this paper is to provide such mechanisms. Specifically, we propose a class of mechanisms for exchange economies with asymmetric information and interdependent values, with no symmetry restrictions, that are ex-post individually rational, incentive compatible, generate budget surplus and result in an ex-post nearly efficient allocation, when there are many agents. We also show that the rate at which they approximate the Walrasian Equilibrium allocation is tight and cannot be improved upon by any other mechanisms. Finally, we provide an implementation using a novel discriminatory conditional double auction.

We motivate our results using a stylized example (see Rostek and Weretka (2012) and Example 2). Suppose that an economy has a large number of English and Americans and one stock traded. Each agent gets a noisy signal of his private per-unit value of the stock, and values are correlated among compatriots. How should one design a prior-free trade mechanism that achieves a near-efficient allocation? One may try a double auction: let each agent submit a demand schedule specifying how much he wants to buy or sell at any price, and have the designer choose the price that clears the market. While this game mirrors the real-life trading protocol on a centralized exchange, it will not lead to a near-efficient allocation, no matter how large the economy. The problem is that a single price cannot convey all the relevant public information needed to make the right choice: a high price might be a signal of high demand by the foreigners, and thus, a perfect selling opportunity; or a signal of high demand by the compatriots and, therefore, agent’s own value. An alternative mechanism would elicit private information via appropriate scoring rules, much in the spirit of Cremer and McLean (1985) and (1988), and then use it to set the Walrasian price. This approach, however, works only if the designer knows the distribution of signals. Finally, one could also try to use prior-free Vickrey-Clark-Groves (VCG) mechanisms, but in exchange environments they run a budget deficit, and so these mechanisms are infeasible. What else can be done?

1 Alternatively, one group of agents trades for fundamental, and one for liquidity reasons.
2 Scoring rules may extract the agent’s beliefs about others. However, exact knowledge of the joint distribution is needed to map such beliefs (“marginals”) to the agent’s payoff type, see Neeman (2004).
More precisely, in this paper, we look at exchange economies with \( N \) agents that have quasilinear utilities over a single divisible good and money. Each agent has a one-dimensional payoff type, and his utility depends on the whole profile of types (interdependent values). In addition to uniform bounds on the derivatives and a global single crossing property, we also assume that the average effect of any single agent’s type on marginal utilities of others converges to zero as the economy grows (small pairwise interdependence). It is a near analog of the informational smallness assumption from the literature on Bayesian strategic foundations of Walrasian Equilibrium in our prior-free setting (see Gul and Postlewaite (1992), McLean and Postlewaite (2002) and (2015)). For example, in the American/English example, pairwise interdependence is of order \( \frac{1}{N} \).

A direct trading mechanism in this environment consists of an allocation and a transfer rule. The main result of the paper is a construction of \( \sigma - \) Walrasian Equilibrium mechanisms that satisfy all the constraints (individual rationality, incentive compatibility, budget surplus, market clearing) and are almost efficient ex-post. For an economy of size \( N \), the maximal distance to the efficient allocation, across all agents and type profiles, is proportional to the sum of \( \frac{1}{N} \) and the bound on pairwise interdependence (Proposition 1). The mechanisms also guarantee vanishing deadweight loss, as the economy grows, when pairwise interdependence is, at most, of order \( \sqrt{\frac{1}{N}} \). Conversely, we show that a faster rate of convergence to efficiency is not possible without violating the constraints (Proposition 2 i)). In other words, our positive result is tight.

\( \sigma - \) Walrasian Equilibrium mechanisms are based on allocations that introduce a wedge between the market price and each agent’s marginal utility, which is linear in the quantity traded with slope \( \sigma \). It can be thought of as a mechanism design version of demand reduction, familiar from the double auction literature (Ausubel et al. (2014)). In the paper, we show that appropriate linear wedges guarantee the fastest rate of convergence to efficiency, given that the accompanying incentivizing payments result in budget surplus. Moreover, linear wedges are strictly optimal, in the following sense. In order to match the optimal rate of convergence, a wedge must increase at least linearly in quantity (Proposition 2 ii)). For example, incentive compatible mechanisms based on a fixed bid-ask spread (constant per-unit wedge) or a fixed entry fee (no wedge for sufficiently large trade) perform strictly worse.

\( \sigma - \) Walrasian Equilibrium mechanisms are direct mechanisms, in which the designer knows how to pick an allocation and transfers given reported signals. In the paper, we provide an implementation using a discriminatory conditional double auction, which, in the spirit of Dasgupta and Maskin (2000), requires no such knowledge. In the game,
each player submits his inverse demand, which specifies marginal prices at which he is willing to clear the market, conditional on not just a total quantity, but on a vector of quantities that the other agents trade. The mechanism computes the market clearing quantity vector and price, as well as, for every trader, the residual demand curve that he is facing. Transfers are discriminatory: each trader pays the integral under his residual demand curve, together with a tax quadratic in quantity. We show that there is an ex-post equilibrium of this game that implements the $\sigma-$Walrasian Equilibrium allocation (Proposition 3). In the equilibrium, each agent submits his marginal utility, adjusted by the linear wedge, conditional on the efficient allocation profile.

The auction format bears three differences from a standard double auction. First, conditioning on the whole allocated vector is a finer instrument that allows an agent to infer more information from the trading behavior on the market and to adjust his position accordingly. In the American/English example, agents submit prices conditional on the total quantity to be cleared and the average trade by their compatriots, with price increasing in each variable. Second, prices are discriminatory and not uniform. Intuitively, discriminatory pricing guarantees incentive compatibility of truthful reporting of agent’s own marginal value for the good, with information rents expressed directly in terms of lower inframarginal prices. Third, transfers include a quadratic tax, which guarantees a budget surplus and, thus, the feasibility of the auction.

Equilibrium inverse demands in a discriminatory conditional double auction may condition on the entire high-dimensional allocated vector. However, this complexity only mirrors the fact that our framework allows for arbitrarily asymmetric interdependent utilities. In the case when, realistically, utilities depend on a low-dimensional statistic of types, as in the American/English example, equilibrium inverse demands condition on a low-dimensional statistic of the allocated vector (Lemma 2). Indeed, equilibrium inverse demands are considerably easier to characterize than in a standard double auction (see Section 4). Conditioning on all relevant information makes the complicated task of filtering the endogenous informative content of prices redundant. Discriminatory pricing obviates any form of strategizing and demand reduction, much like a second-price auction obviates shading from a first-price auction, in the classical auction theory. While harder to quantify than efficiency gain, we consider this simplicity a crucial virtue of our mechanisms.

**Relationship to the Literature.** Strategic foundations of Walrasian Equilibrium have been researched extensively in the Bayesian setting. One strand of literature focuses on the question of when double auctions guarantee asymptotic efficiency. Important
contributions for private values include Wilson (1985a), Palfrey and Srivastava (1986), Postlewaite and Schmeidler (1986), Klemperer and Meyer (1989), Gresik and Satterthwaite (1989), Satterthwaite and Williams (1989), Rustichini et al. (1994), Fudenberg et al. (2007) and Cripps and Swinkels (2006); and Hellwig (1980), Reny and Perry (2006) and Vives (2011) for interdependent values. Going beyond double auctions, Gul and Postlewaite (1992) and McLean and Postlewaite (2002) construct asymptotically efficient mechanisms drawing from the insights of Cremer and McLean (1985) and (1988). In our paper, we require stronger ex-post incentive compatibility and individual rationality constraints. Moreover, double auctions guarantee asymptotic efficiency only with symmetric agents (and so not in the American/English example; see Rostek and Weretka (2012)) and under additional assumptions on the distribution of signals, such as conditional independence.

A different approach has been to analyze the limits of manipulability of the efficient mechanisms when there are many agents. Roberts et al. (1976), Jackson (1992) and Jackson and Manelli (1997) looked at manipulability in the symmetric information setting. In the asymmetric information case, McLean and Postlewaite (2015) show that under information smallness, the benefits of ex-post deviations are small. We work with asymmetric information economies and require full ex-post incentive compatibility, which requires going beyond the efficient mechanisms.

In the standard auction context, under common knowledge of benefits of trade, VCG mechanisms generate budget surplus. In the private-value case the mechanism is the celebrated Vickrey (second-price) auction. Dasgupta and Maskin (2000), Perry and Reny (2002) and (2005) and Ausubel (2004) show how to implement VCG mechanisms with interdependent values. Whereas VCG mechanisms are not feasible in our exchange setting, conditional demands (on the allocation vector) in our implementation bear resemblance to the conditional demands (on the bid vector) in Dasgupta and Maskin (2000) and Perry and Reny (2002). In a context of dynamic trade with private values Du and Zhu (2013) and Sannikov and Skrzypacz (2016) have also recently used conditional demands.

Few studies have offered mechanisms for exchange environments that are both ex-post incentive compatible and asymptotically almost efficient. In a seminal paper, McAfee (1992) provided a simple mechanism that achieves this with private values and unit demands and supplies, while Baliga and Vohra (2003), Loertscher and Mezzetti (2014) and Loertscher and Marx (2015) deal with multiunit trade. Kojima and Yamashita (2016) allow for interdependent values when types of the agents are independently distributed and agents are symmetric. In this paper, we have multiunit trade and asymmetric interde-
ependent values. Moreover our framework is entirely prior-free: $\sigma$-Walrasian mechanisms are almost efficient not just in expectation, but also ex-post, and, in particular, without independence assumptions.

2 Setup

Payoffs. An exchange economy consists of $N$ agents who can trade a single good (or asset) for money. Agent $i$’s payoff type is $s_i \in S_i$, where $S_i$ is a possibly infinite subset of the real line. Each agent $i$ has a quasilinear utility,

$$U_i(q_i, t_i, s) = u_i(q_i, s) - t_i,$$

where $q_i \in \mathbb{R}$ is the quantity of the good $i$ gets and $t_i$ is what he pays.\(^3\) We emphasize that utilities may depend on the whole vector of types (interdependent values) and that we are not making any symmetry assumptions: functions $u_i$ are indexed by the agents. We will write $s = (s_1, ..., s_N) \in S_1 \times ... \times S_N = S$, $q = (q_1, ..., q_N)$, $t = (t_1, ..., t_N)$,

$$mu_i(q_i, s) = \frac{\partial}{\partial q_i} u_i(q_i, s), \quad mu_{i,j}(q_i, s) = \frac{\partial^2}{\partial q_i \partial s_j} u_i(q_i, s) \quad \text{and} \quad mu_{i,q}(q_i, s) = \frac{\partial^2}{\partial q_i \partial q_j} u_i(q_i, s).$$

Throughout the paper we make the following assumptions. Functions $u_i$ are twice continuously differentiable, strictly concave in $q_i$ and moreover, to prevent efficient infinite trades, $\bigcap_{i=1}^N mu_i(\mathbb{R}, s) \neq \emptyset$, for every $s$. We let utilities increase in types, and strictly so in the case of own type, $mu_{i,j}(q_i, s) \geq 0$, $mu_{i,i}(q_i, s) > 0$, for every $i, j, q_i$ and $s$, and assume (global) single-crossing:

$$SC) \quad mu_{i,j}(q_i, s) \leq mu_{j,j}(q_j, s), \, \forall i, j, q_i, q_j, s$$

For the main results in the paper we must further restrict utilities as follows. First, we impose uniform bounds on the derivatives, for all $N$:

$$A1) \quad mu_i(0, s) \in [-m, m], \, \forall i, s$$
$$A2) \quad mu_{i,j}(q_i, s) \in [\overline{m}_i, \overline{m}_j] > 0, \, \forall i, q_i, s$$
$$A3) \quad mu_{i,q}(q_i, s) \in [-\overline{m}_q, -\overline{m}_q] < 0, \, \forall i, q_i, s$$

The uniform bounds are a fairly straightforward prerequisite for providing bounds on the “worst case” efficiency losses, as we intend here. Second, we make the following assumption of small pairwise interdependence:

\(^3\)Normalizing the initial endowment to zero is purely for notational convenience.
A4) \( \frac{1}{N-1} \sum_{i \neq j} m_{ij}(q_i, s) \leq \phi_N \), with \( \lim_{N \to \infty} \phi_N = 0 \), \( \forall j, q_j, s \)

The assumption says that as the number of agents in the economy grows, the average impact of each single agent on marginal utilities of other agents vanishes. Such “smallness” assumption is necessary for asymptotic efficiency under any mechanism: With few agents monopolizing publicly relevant information, their information rents would lead to inefficient allocations, no matter how large the economy. Working with economies of fixed size \( N \), we will be using the following counterpart of A4,

\[ \frac{1}{N-1} \sum_{i \neq j} m_{ij}(q_i, s) \leq \phi_N \], with \( 2\phi_N < \frac{N-2}{N-1} \rho \), \( \forall j, q_j, s \)

Example 1 (Fundamental Value Model, Vives (2011), Rostek and Weretka (2012)) Utilities are

\[ u_i(q_i, s) = (\alpha s_i + \beta \bar{s})q_i - \frac{\mu}{2} \frac{q_i^2}{q_i^2} \]

for some constants \( \alpha, \beta, \mu > 0 \), where \( \bar{s} \) is the arithmetic average of the types. Each agent is uncertain about the value of the first infinitesimal unit of the good, or intercept of the linear marginal utility function. The value is a weighted average of common and idiosyncratic shocks, \( \bar{s} \) and \( s_i \). For example, it may reflect both the expected cash flow of the asset, common to all the agents that are partially informed about it, and private hedging or liquidity needs. We have \( m_{ij}(q_i, s) = \frac{\beta}{N} \) for any \( j \neq i \), and so the small pairwise interdependence assumption A4 is satisfied; for fixed \( N \), A4N holds as long as \( \alpha > \frac{\beta}{N-2} \).

Example 2 (Group Model, Rostek and Weretka (2012)) Suppose agents are divided into two groups of the same size, “American” and “English”, and the utilities are

\[ u_i(q_i, t_i, s) = (\alpha s_i + \beta \bar{s}^G)q_i - \frac{\mu}{2} \frac{q_i^2}{q_i^2}, \quad \forall i \in G, G = A, E \]

where \( \bar{s}^A \) and \( \bar{s}^E \) are the average types in each group, \( \alpha, \beta, \mu > 0 \). Compared to Example 1, now beside idiosyncratic shocks there are two shocks, \( \bar{s}^A, \bar{s}^E \), common to the respective group and irrelevant to the other. Agents might be trading either for fundamental or liquidity reasons, and either private values or liquidity needs are correlated within a group.

Example 3 (Fundamental Value Model with Heterogenous Traders) There are two groups of agents, “small” and “big”, and utilities are

\[ u_i(q_i, s) = (\alpha s_i + \beta \bar{s})q_i - \frac{\mu_G}{2} \frac{q_i^2}{q_i^2}, \quad \forall i \in G, G = S, B \]
where $\mu_S = \mu, \mu_B = \mu, \Pi > \mu$. As in the Fundamental Value Model all the agents are partially and symmetrically informed about the fundamental value of the risky asset, but they differ now in their capacity to hold large position, with big agents relatively risk neutral (or bearing large cost of staying away from their ideal position).

A natural source of interdependence in preferences is informational, when agent’s type corresponds to a signal informative of all agents’ values of the asset. We have used this interpretation in the examples above, and will continue to do it throughout the paper. This informational interpretation can be further justified via Bayesian models. Indeed, suppose each agent $i$ has a private state space $\Theta_i$ and a continuously differentiable utility function $v_i(q_i; \theta_i)$ as well as a prior $\delta_i \in \Delta(\Theta_i \times S_1 \times ... \times S_N)$. If one defines

$$u_i(q_i, s) = \mathbb{E}_{\delta_i}[v_i(q_i, \theta_i)|s]$$

then the correlations of signal vectors $s$ and value $\theta_i$ translate to utilities $u_i$ with interdependent values.

In particular, utility functions in the examples above are expected utilities in the model in which each agent $i$ has linear-quadratic utilities,

$$v_i(q_i, \theta_i) = \theta_i q_i - \frac{\mu}{2} q_i^2; \quad (1)$$

and believes that $(\theta_i, s_1, ..., s_n)$ has a joint normal distribution, where the linear parameters $w_j, j \leq N$, are such that

$$\mathbb{E}_{\delta_i}[\theta_i|s_1, ..., s_N] = \sum_j w_j s_j, \quad (2)$$

from the Projection Theorem. Put otherwise, a fixed linear-quadratic utility function with parameters $w_j, j \leq N$ may be justified, for example, by any normal prior beliefs for which (2) holds. We stress here that many beliefs may give rise to the same utilities $u_i$; we discuss this Bayesian justification for the interdependent preferences in the context of robustness in Section 7 (see also Example 6).

Our notion of small pairwise interdependence is closely related to the informational smallness used in the context of such Bayesian models (see Gul and Postlewaite (1992), McLean and Postlewaite (2002) and (2015)). If $\Theta_i$ and $\frac{\partial v_i}{\partial \theta_i}$ are bounded for all $i, j$ then $u_i$ satisfy $A4$ precisely when$^4$.$^5$

$^4$We identify a distribution $\delta_i$ with its cdf.

$^5$The main difference is that McLean and Postlewaite (2002) require the bound in $A4^B$ to hold not
$A_4^{B)} \quad \frac{1}{N-1} \sum_{i \neq j} \left| \frac{\partial \delta_i(\theta_j|s)}{\partial s_j} \right| \leq \phi_N^B,$ with $\lim_{N \to \infty} \phi_N^B = 0 \ \forall j, \theta_j, s$

In other words, given a Bayesian justification, bound on pairwise interdependence $A_4$ is equivalent to any single agent $j$ having a small average impact on beliefs of others about the payoff relevant state.

**Mechanisms.** The efficient benchmark in our exchange economy setting is of course the *Walrasian Equilibrium (WE) allocation profile* $\{q^0(s)\}_{s \in S}$. For any vector of types $s$ the allocation $q^0(s)$ is defined jointly with the *Walrasian price* $p^0(s)$ by

$$mu_i(q^0_i(s), s) = p^0(s), \ \forall i$$

$$\sum_i q^0_i(s) = 0.$$ 

There is a unique WE allocation for any type profile $s$ (see Section 3), and, from strict concavity of the utility functions, any other allocation results in a positive deadweight loss. The goal of this paper is to consider asymmetric information setting and find mechanisms that robustly implement almost WE allocations, at each type vector, if $N$ is large.

Given a profile of utility functions $(u_1, ..., u_N)$ a *(direct) mechanism* in our setting is $\{(q(s), t(s))\}_{s \in S}$, where $\{q(s)\}_{s \in S}$ is the *allocation profile* and $\{t(s)\}_{s \in S}$ the *transfers profile*, $q(s), t(s) \in \mathbb{R}^N$. Below we list the constraints and the objective that we want mechanisms to satisfy. A mechanism $\{(q(s), t(s))\}_{s \in S}$ satisfies

- **Market Clearing**, if
  $$MC) \quad \sum_i q_i(s) = 0, \ \forall s$$

- **Budget Surplus**, if
  $$BS) \quad \sum_i t_i(s) \geq 0, \ \forall s$$

- **Individual Rationality**, if
  $$IR) \quad U_i(q_i(s), t_i(s), s) \geq U_i(0, 0, s), \ \forall i, \forall s$$

for every $s_{-j}$ but only for $s_{-j}$ with probability $1 - \phi_N^S$. The strengthening in our paper is dictated by a stronger, ex-post version of incentive compatibility.
• Incentive Compatibility, if

\[ IC \quad U_i(q_i(s), t_i(s), s) \geq U_i(q_i(s_i', s_{-i}), t_i(s_i', s_{-i}), s), \quad \forall i, s, s_i' \]

• \( \varepsilon \)-Efficiency, if

\[ \varepsilon - Eff \quad |q_i(s) - q_i^0(s)| \leq \varepsilon, \quad \forall i, s \]

For a function \( f : \mathbb{N}_+ \rightarrow \mathbb{R}_+ \) we say that a family of mechanisms \( \{(q(s), t(s))\}_{s \in S} \) is \textit{robustly asymptotically} \( O(f) \)-\textit{efficient} if there is \( C > 0 \) such that for \( N \) large enough and any \( (u_1, \ldots, u_N) \) the corresponding mechanism satisfies all the constraints \( MC, BS, IR, IC \) and is \( Cf_N \)-Efficient. Mechanisms are \textit{robustly asymptotically} \( o(f) \)-\textit{efficient} if the preceding is true for any \( C > 0 \).

Market clearing is a standard feasibility constraint. Budget surplus is similar, as we want mechanisms to run without an outside source of money. Individual rationality requires participation in a mechanism to be voluntary, and incentive compatibility means that truthful reporting of own type is optimal.

Crucially, all the constraints and the objective are required to be satisfied ex-post, for any vector of types. While this is rather natural in the case of the first two feasibility constraints, those are strong requirements in the case of IR, IC and \( \varepsilon \)-Efficiency, relative to their Bayesian counterparts. They imply that the resulting mechanisms are relatively simple, or \textit{detail-free}: they may not depend on the details of the information structure (see Example 6). Moreover, the mechanisms satisfy Bayesian IR and IC for any type space, describing agents’ beliefs, beliefs about beliefs etc. (see e.g. Chung and Ely (2002), Bergemann and Morris (2005)), as well as guarantee expected almost efficiency for any distribution of types. Thus, ex-post specification guarantees \textit{robustness} of the mechanisms, since they are immune to the misspecification of agents’ beliefs or a distribution of types.

Ex-post constraints also guarantee \textit{no regret}, and so will work in a weak contractual environment, when each agent can walk away from the transaction at any stage, even after observing the full allocation vector and possibly inferring types of other agents. This shares many of the features of a standard spot market. Relatedly, ex-post constraints nullify any benefits of (inefficient) spying on other agents to infer their types (Bergemann and Välimäki (2002)).

We note that, similarly as in most of the literature (see e.g., McAfee (1992), Kojima and Yamashita (2016)) our notion of efficiency does not penalize for budget surpluses.
One interpretation is that extra money need not be flushed down the drain, but can be used elsewhere. On the other hand, spelling out $\varepsilon-$Efficiency directly in terms of differences in allocations and not utilities is without loss of generality. Moreover, $\varepsilon$-Efficient mechanisms generate deadweight losses of order $\varepsilon^2 N$.

**Lemma 1** Suppose $A1$ and $A3$ hold. If a mechanism $\{(q(s), t(s))\}_{s \in S}$ is $\varepsilon$-Efficient and satisfies $MC$ then for every $s$

$$
\left| u_i(q_i(s), s) - u_i(q^0_i(s), s) \right| \leq \varepsilon (m + \varepsilon \overline{m}_q),
$$

$$
\left| \sum_{i=1}^N u_i(q_i(s), s) - \sum_{i=1}^N u_i(q^0_i(s), s) \right| \leq \varepsilon^2 \frac{N \overline{m}_q}{2}.
$$

Since the utility for each infinitesimal unit traded in the efficient allocation is bounded in absolute terms by $m$, from $A1$, it is bounded in any $\varepsilon$-Efficient allocation by $m + \varepsilon \overline{m}_q$, from $A3$. This implies the first part. The second part follows from $A3$ and

$$
\left| \sum_{i=1}^N u_i(q_i(s), s) - \sum_{i=1}^N u_i(q^0_i(s), s) \right| = \left| \int_{q^0_i(s)}^{q_i(s)} (mu_i(x, s) - p^0(s)) \, dx \right|.
$$

### 3 $\sigma$-Walrasian Equilibrium Mechanisms

Our results rely on the following family of mechanisms. We define the allocations and transfers separately.

**Definition 1** ($\sigma$-Walrasian Equilibrium Allocation) Fix a profile of utility functions $(u_1, ..., u_N), \sigma \geq 0$ and a vector of types $s$. A $\sigma$-Walrasian Equilibrium ($\sigma$-WE) allocation $q^\sigma(s)$ and a $\sigma$-Walrasian Equilibrium price $p^\sigma(s)$ are defined via

$$
mu(q^\sigma_i(s), s) = p^\sigma(s) + \sigma \times q^\sigma_i(s), \quad \forall i
$$

$$
\sum_i q^\sigma_i(s) = 0.
$$

Had the information been symmetric, $\sigma$-WE allocation could be implemented by posting a $\sigma$-WE price $p^\sigma(s)$ and imposing a tax $\frac{\sigma}{2} q^2_i$ quadratic in the quantity traded. The price $p^\sigma(s)$ is set to clear the market. Of course $0$-WE allocations agree with the efficient WE allocations. Strictly positive $\sigma$, however, results in insufficient trade and compromises efficiency of the allocation: a marginal utility of a buyer ($q^\sigma_i(s) > 0$) is strictly greater than a marginal utility of a seller ($q^\sigma_i(s) < 0$). This is related to demand
reduction in the double auction mechanisms (see e.g. Ausubel et al. (2014); see Example 4).

Given continuous differentiability and strict concavity of the utility functions, for any \( \sigma \geq 0, p \) and \( i \leq N \) there is at most one \( q_i(p, s) \) that satisfies

\[
mu_i(q_i(p, s), s) = p + \sigma \times q_i(p, s),
\]

and the functions \( q_i(\cdot, s) \) are continuous and strictly decreasing in its domains. This means that there is at most one \( p \) for which \( \sum_{i=1}^{N} q_i(p, s) = 0 \). That such a \( p \) exists follows from \( \bigcap_{i=1}^{N} mu_i(\mathbb{R}, s) \neq \emptyset \). All this establishes existence and uniqueness of \((p^\sigma(s), q^\sigma(s))\), for any \( \sigma \geq 0 \) and \( s \).

**Definition 2 (\( \sigma \)-Walrasian Equilibrium Transfers)** Fix a profile of utility functions \((u_1, \ldots, u_N)\), \( \sigma \geq 0 \) and a vector of types \( s \). \( \sigma \)-Walrasian Equilibrium transfers \( t^\sigma(s) \) are defined via

\[
t^\sigma_i(s) = \int_0^{q^\sigma_i(s)} p^\sigma(s_i(x), s_{-i}) \, dx + \frac{\sigma}{2} q^\sigma_i(s)^2,
\]

where \( \{q^\sigma(s)\}_{s \in S} \) and \( \{p^\sigma(s)\}_{s \in S} \) are the \( \sigma \)-Walrasian Equilibrium allocations and prices, and for any agent \( i \) and quantity \( x \) between \( 0 \) and \( q^\sigma_i(s) \) \( s_i(x) \) is the signal such that\(^6\)

\[
x = q^\sigma_i(s_i(x), s_{-i}),
\]

\[
s_i(x) = \begin{cases} 
\inf S_i & \text{if } q^\sigma_i(s_i(x), s_{-i}) > x \text{ for all } s'_i, \\
\sup S_i & \text{if } q^\sigma_i(s_i(x), s_{-i}) < x \text{ for all } s'_i.
\end{cases}
\]

Transfers by any agent \( i \) consist of a discriminatory part and a tax. Tax is quadratic in own quantity traded, or, alternatively, is a linear per-unit tax. It is a weakly positive contribution by everyone. Discriminatory part is the integral over per-unit prices for each inframarginal unit traded, with the price for unit \( x \) equal to the \( \sigma \)-WE price under a counterfactual report \( s_i(x) \) that results in \( i \) trading exactly \( x \). Thus, it is simply the integral under the residual demand curve \( i \) is facing. It follows that in the case of private values, at any type profile the discriminatory transfers by agent \( i \) are the integral under the aggregate linear tax adjusted marginal utilities of agents other than \( i \) (see Figure 1). With interdependent values the transfers are more discriminatory: say, a lower report by

\(^6\)The second and the third clause in the definition are relevant only in the case when \( S_i \) is finite. It follows form strict monotonicity of \( q^\sigma_i(s) \) in \( s_i \), established in the proof of Proposition 1, that \( s_i(x) \) is a function.
a buyer, which results in a lower quantity he trades, also decreases marginal utilities of
other agents, and so further depresses the prices for inframarginal units (see Figure 2).

For any $\sigma \geq 0$ a $\sigma$–Walrasian Equilibrium mechanism $\{(q^\sigma(s), t^\sigma(s))\}_{s \in S}$ consists of
the $\sigma$-Walrasian Equilibrium allocations and transfers.

![Figure 1: $\sigma$–WE allocation and discriminatory transfers in a two person economy with private values, for $\sigma > 0$. Dotted lines are the marginal utility curves, and solid lines adjust for the linear per unit tax. Shaded areas are the discriminatory payments by the buyer (left panel) and proceeds for the seller (right panel).](image)

The following is the main result of the paper. Recall that $\phi_N$ is the bound on pairwise
interdependence in $A_4$ and $A_4^N$.

**Proposition 1** Suppose assumptions $A_1$ – $A_4$ hold. For appropriate $\sigma_N$, $N \geq 1$, the
family of $\sigma_N$–WE mechanisms is robustly asymptotically $O(1/N + \phi_N)$–Efficient.

More precisely, fix $N$ and $\sigma \geq 0$.

i) $\sigma$–WE mechanisms satisfy MC, IR and IC. For a fixed $\sigma$–WE allocation, any
agent $i$ and $s_{-i}$ the IR and (locally) IC transfers $t_i(\cdot, s_{-i})$ are unique up to a constant.

ii) Suppose assumptions $A_2, A_3$ and $A_4^N$ hold. $\sigma$–WE mechanisms satisfy BS as
long as

$$\sigma \geq \sigma_N := \frac{\bar{m}_q}{N-1} \left( \frac{m_o}{N - 1} + \frac{\bar{m}_q}{\bar{m}_q} \phi_N \right). \quad (6)$$

iii) Suppose assumptions $A_1$ – $A_3$ hold. $\sigma$–WE mechanisms are $\gamma(\sigma)$–Efficient, where

$$\gamma(\sigma) = \frac{\sigma m}{\bar{m}_q(\sigma + \bar{m}_q)} \left( 2 + \frac{(\bar{m}_q - m_q)}{\bar{m}_q} \right). \quad (7)$$
Figure 2: $\sigma - WE$ allocation and discriminatory transfers in a two person economy with interdependent values, for $\sigma > 0$. Thin lines are the linear tax adjusted marginal utility curves for the realized type profile (solid), and lower types for the buyer on the left panel and higher types for the seller on the right panel (dotted). Thick solid lines are the residual demand curves and shaded areas are the discriminatory payments by the buyer (left panel) and proceeds for the seller (right panel).

The proposition says that for any number of agents $N$ and any vector of utility functions, the appropriate $\sigma - WE$ mechanism satisfies all the ex-post constraints. Moreover, it provides an explicit uniform (or “worst case”) bound, across all agents and states, on how much the implemented allocation differs from the optimal one. If assumption $A4$ of small pairwise interdependence is met, inefficiency vanishes as $N$ grows at rate $\frac{1}{N} + \phi_N$. In particular, as long as the interdependence parameter $\phi_N$ is of order smaller than $\sqrt{1/N}$, as in Examples 1-3, Lemma 1 implies that when $A1 - A4$ hold then $\sigma_N - WE$ mechanisms guarantee vanishing deadweight loss, uniformly across all states:

$$\left| \sum_{i=1}^{N} u_i(q_i^{\sigma_N}(s), s) - \sum_{i=1}^{N} u_i(q_i^{0}(s), s) \right| \leq \gamma(\sigma_N)^2 \frac{N\bar{m}q}{2} = O\left(\frac{1}{N} + \phi_N + N\phi_N^2\right),$$

for $\sigma_N$ and $\gamma(\sigma_N)$ as in Proposition 1.

Intuition behind the result is as follows. Part i) follows a standard logic behind incentivizing transfers that dates back to Vickrey (1961). In our case, fix a type vector $s$ and a buyer $i$, $q_i^{\sigma}(s) > 0$. In order to achieve incentive compatibility the price that $i$ pays for every $x'th$ infinitesimal inframarginal unit of the good, $x < q_i^{\sigma}(s)$, must equal his value for this unit had he reported the type that makes him pivotal. Agent $i$ is pivotal for $x'th$ unit of the good precisely when he reports the type $s_i(x)$ such that $q_i^{\sigma}(s_i(x), s_{-i}) = x$. 

\[14\]
and so his value for it equals \( \mu_i(x, (s_i(x), s_{-i})) \), which, given the definition of a \( \sigma - WE \) allocation in (3), equals \( p^\sigma(s_i(x), s_{-i}) + \sigma x \). Integrating over such per-unit payments gives rise to transfer functions in (4). Monotonicity of the allocation follows from the single crossing condition.

Uniform bounds \( A_1 - A_3 \) on utility functions guarantee that quantity traded by any agent in any \( \sigma - WE \) is uniformly bounded. From definition of the \( \sigma - WE \) allocation, this implies that agents’ marginal utilities are distorted proportionally to \( \sigma \) and, given \( A_3 \), so are the quantities traded, as in part iii).

Establishing budget surplus (part ii)) is the centerpiece of the result. Proof shows that, for sufficiently large slope \( \sigma \), each buyer’s average inframarginal price is above, and each seller’s average inframarginal price is below the \( \sigma \)-Walrasian price \( p^\sigma(s) \). First, effect of \( \iota \)'s type on the \( \sigma \)-Walrasian price equals a weighted average of its effect on marginal utilities of the agents, which is of order \( \frac{1}{N} + \phi_N \). Given strict monotonicity, \( \frac{\partial q_i}{\partial s_i} \gg 0 \), the price for \( \iota \)'s inframarginal units also changes at the rate \( \frac{1}{N} + \phi_N \). Setting quadratic taxes, or a per-unit tax of order \( \left( \frac{1}{N} + \phi_N \right) \times x \) on every inframarginal unit \( x \) has each agent pay in taxes at least what he gains via discriminatory pricing. For example, in the case of a buyer a high tax on the “last” units subsidizes low prices for the “early” units.

4 Examples

In the following we derive \( \sigma - WE \) mechanisms for the examples from Section 2. For each example we also look at information structures with normal beliefs that can be used to justify the utilities (see Section 2) and compare the \( \sigma - WE \) mechanisms to the Bayesian equilibria in standard double auctions.

Example 4. Consider the Fundamental Value Model from Example 1. For any \( \sigma \) the \( \sigma - WE \) price and allocation are

\[
p^\sigma(s) = (\alpha + \beta)\overline{s},
\]

\[
q^\sigma_i(s) = \frac{\alpha}{\mu + \sigma}(s_i - \overline{s}).
\]

Using Proposition 1 the slope that guarantees BS equals

\[
\sigma_N = \frac{\mu(\alpha + \beta)}{(N - 2)\alpha - \beta}.
\]
which is positive, given $A^4N$ (see Example 1). For $\sigma = \sigma_N$ the mechanism boils down to

$$q^{\sigma_N}_i(s) = \frac{(N-2)\alpha - \beta}{\mu(N-1)} (s_i - \overline{s}), \quad t^{\sigma_N}_i(s) = p^{\sigma_N}(s) \times q^{\sigma_N}_i(s), \quad (10)$$

and so satisfies budget balance. This follows from the fact that in this example $\frac{\partial p^{\sigma_N}(s|x, s_{-i})}{\partial x} = \sigma_N$, as can be verified from (8).

Consider now a Bayesian framework as in Vives (2011) and Rostek and Weretka (2012). Each agent $i$ has a linear-quadratic utility function $v_i(\theta_i, q_i)$ as in (1) and agents believe that their values $(\theta_1, \ldots, \theta_N)$ are jointly normally distributed with mean zero, variances $1$ and covariances $\rho \geq 0$. Each agent $i$ observes a signal $s_i = \theta_i + \varepsilon_i$, with noise $\varepsilon_i \sim N(0, \zeta)$, $\zeta > 0$, independent of all other variables. One may think of private value $\theta_i = \theta + \theta_{i}^{id}$ as consisting of a common shock $\theta \sim N(0, \rho)$ and an idiosyncratic shock $\theta_{i}^{id} \sim N(0, 1 - \rho)$.

Applying Proposition 2 from Rostek and Weretka (2012) to this information structure, we verify in Appendix that

$$\mathbb{E}[v_i(\theta_i, q_i)|s] = \left( \frac{1 - \rho}{1 - \rho + \zeta} s_i + \frac{N\rho\zeta}{(1 - \rho + \zeta)(1 + (N-1)\rho)} \overline{s} \right) q_i - \frac{\mu}{2} q_i^2, \quad (11)$$

and the linear Bayesian equilibrium in double auction results in the identical allocation and transfers as the $\sigma_N$–WE mechanism given by (10).

The main takeaway is that in the Fundamental Value Model the most efficient $\sigma_N$–WE mechanism, with the lowest slope $\sigma_N$ still consistent with BS, agrees with and so can be implemented by a double auction. Since a demand schedule allows an agent to condition his demand on own signal and the equilibrium price, given that the price is privately revealing (Rostek and Weretka (2012)), agents are effectively choosing ex-post optimal price-quantity pairs. Moreover, in this example the equilibrium demand reduction and the positive slope $\sigma_N$ of $\sigma_N$–WE mechanisms have the same effect on allocations and transfers. In particular, it follows that this Bayesian equilibrium remains an equilibrium for any information structure justifying payoffs as in Example 1.

**Example 5** Consider the Group Model from Example 2. For any $\sigma$ the $\sigma$–WE alloca-
tion and $\sigma - WE$ price satisfy
\[ p^\sigma(s) = (\alpha + \beta)s, \]
\[ q^\sigma_i(s) = \frac{\alpha s_i + \beta s^G - (\alpha + \beta)s}{\mu + \sigma}, \quad \forall i \in G, G = A, E \]

The slope that guarantees $BS$ equals (Proposition 1)
\[ \sigma_N = \frac{\mu(\alpha + \beta)}{(N-2)\alpha}, \]

and just as in Example 4 it is easy to verify that the corresponding transfers are $t^\sigma_N(s) = p^\sigma_N(s) \times q^\sigma_N(s)$, and so satisfy budget balance.

Suppose now agents have utility functions $v_i(\theta_i, q_i)$ as in (1) and believe that their values $(\theta_1, ..., \theta_N)$ are jointly normally distributed with mean zero, variances 1 and covariances $\rho$, $\rho \geq 0$, for the agents only in the same group. Thus, private value $\theta_i = \theta^G + \theta^id_i$ consists of a shock $\theta^G_i \sim N(0, \rho)$, $i \in G$, common to own group, and an idiosyncratic shock $\theta^id_i \sim N(0,1 - \rho)$. As before, each agent $i$ observes a signal $s_i = \theta_i + \varepsilon_i$, $\varepsilon_i \sim N(0, \zeta)$. Just as in Example (4) it follows that the expected utilities conditional on a profile of signals take the form of utilities in the Fundamental Value Model, for the subgroup of compatriots: for any $i \in G, G = A, E$,
\[ \mathbb{E}[v_i(\theta_i, q_i)|s] = \left( \frac{1 - \rho}{1 - \rho + \zeta} s_i + \frac{(N/2) \rho \zeta}{(1 - \rho + \zeta)(1 + (N/2 - 1)\rho)s^G} \right) q_i - \frac{\mu q_i^2}{2}. \]

Applying Proposition 2 from Rostek and Weretka (2012) to this information structure, we verify in Appendix that in the linear Bayesian equilibrium in double auction the equilibrium price and allocation are
\[ p^{l-N}(s) = \frac{c^N_s}{1 - c^N_p s}, \quad q^{l-N}_i(s) = \frac{c^N_s}{\mu} \left( \frac{\frac{N-2}{N-1} - c^N_p}{1 - c^N_p} \right) (s_i - \overline{s}), \]

for constants $c^N_s, c^N_p$ that depend on $\zeta, \rho, N$ such that
\[ \lim_{N \to \infty} c^N_s = \frac{2 - \rho}{2 - \rho + 2\zeta}, \quad \lim_{N \to \infty} c^N_p = \frac{2\zeta}{2 - \rho + 2\zeta}. \]

Allocations and transfers in the two mechanisms now differ. Figure 3 shows ratios of expected utilities in each mechanism, for different correlations of values $\rho$ within each
group, noise variances $\zeta$ and economy sizes $N$. In particular, when economy grows expected utilities converge to efficiency only in the case of $\sigma_N - WE$:

$$
\lim_{N \to \infty} \mathbb{E}[v_i(\theta_i, q_i^{\sigma_N}(s))|s] = \lim_{N \to \infty} \mathbb{E} \left[ q_i^{\sigma_N}(s) \left( \mathbb{E}[\theta_i|s] - \frac{\mu}{2} q_i^{\sigma_N}(s) \right) \right] 
$$

$$
= \mathbb{E} \left[ q_i^{0}(s) \left( \mathbb{E}[\theta_i|s] - \frac{\mu}{2} q_i^{0}(s) \right) \right] = \frac{\mu}{2} \mathbb{E} \left[ q_i^{0}(s)^2 \right] = \frac{1}{\mu} \left( \frac{\rho}{4} + \frac{(1 - \rho)^2}{2(1 - \rho + \zeta)} \right),
$$

$$
\lim_{N \to \infty} \mathbb{E} \left[ v_i(\theta_i, q_i^{\sigma_N}(s))|s \right] = \frac{\mu}{2} \mathbb{E} \left[ q_i^{\sigma_N}(s)^2 \right] = \frac{1}{\mu} \left( \frac{(2 - \rho)^2}{4(2 - \rho + 2\zeta)} \right) < \frac{1}{\mu} \left( \frac{\rho}{4} + \frac{(1 - \rho)^2}{2(1 - \rho + \zeta)} \right),
$$

for every $i \in G; G = A; E$, where replacing the order of limits under the expectation may be justified by the Dominated Convergence Theorem, and the fourth equality is justified by $\mathbb{E}[q_i^{0}(s)(\mathbb{E}[\theta_i|s] - \mu q_i^{0}(s))] = 0$.

When economy is large, in a $\sigma - WE$ mechanism agents can trade across groups to exploit the difference between the realized group shocks, $\theta_A$ and $\theta_E$, based on the arbitrarily precise estimates $\bar{s}_A$ and $\bar{s}_E$. This is captured by the first term in the limiting expected utility in (14). The second term represents the benefit of trading away the idiosyncratic part of agent $i$’s value, $\theta_i^{id}$, based on the idiosyncratic part of his signal, $\theta_i^{id} + \varepsilon_i$. In the Bayesian equilibrium in double auction price shifts each agent’s estimate of his value in step. Consequently, trade of each agent $i$ is based solely on the information about $\theta_i$ provided by his private signal $s_i$. In particular, large gains from intergroup trade are not realized. For example, large noise $\zeta$ of private signals may render trade in a double auction nearly useless, without wiping out the gains from trade in a $\sigma - WE$.

**Example 6** In the Fundamental Value Model with Heterogenous Traders from Example 3, for any $\sigma$ the $\sigma - WE$ allocations and prices satisfy

$$
q_i^{\sigma}(s) = \frac{\alpha s_i + \beta \bar{s} - p^{\sigma}(s)}{\mu_G + \sigma}, \quad \forall i \in G; G = S; B
$$

$$
p^{\sigma}(s) = \alpha \left( \gamma_S \bar{s}^S + \gamma_B \bar{s}^B \right) + \beta \bar{s},
$$

where $\bar{s}^S$ and $\bar{s}^B$ are average signals in each group and $\gamma_G = \frac{\sigma + \mu - \mu_G}{2\sigma + \mu_B + \mu_S}$, $G = S, B$. Compared to the Fundamental Value Model, now big agents trade more aggressively on their information and thus have a larger price impact. Given the allocations and prices we compute the transfers as in the previous examples (see Appendix).

While the same Bayesian model as in Example 4 can be used to justify the utility function, there are many more. Specifically, suppose the agents have the utilities $v_i(\theta_i, q_i)$
Figure 3: Ratios of $\sigma_N - WE$ mechanism to double auction expected utilities in a Group Model, for different correlations within group $\rho$, signal noise variances $\zeta$ and economy sizes $N$.

as in (1), and believe that, for $i \in G, G = S, B$,

$$\theta_i = \theta + \theta_G + \theta_i^{id},$$

$$s_i = \theta_i + \varepsilon_G + \varepsilon_i,$$

where all random variables $\theta, \theta_G, \theta_i^{id}, \varepsilon_G, \varepsilon_i^{id}$, for $G = S, B$ and $i \leq N$, are independently normally distributed with variances $\sigma^2_\theta, \sigma^2_{\theta_G}, \sigma^2_{\theta_i^{id}}, \sigma^2_{\varepsilon_G}, \sigma^2_{\varepsilon_i^{id}}$. In other words, there is a common shock $\theta$ to everyone's value, there are two separate group value shocks $\theta_S, \theta_B$ and idiosyncratic value shocks $\theta_i^{id}$. Likewise, noise consists of a group noise component $\varepsilon_G, G = S, B$, and idiosyncratic noise $\varepsilon_i$. Intuitively, agents in the same group care about
similar aspects of the asset and also observe signals from similar sources, giving rise to positively correlated noises.\footnote{Other Bayesian models are available. We assumed that agents that share a group value shock and have positively correlated noises also share the degree of risk aversion. The two splits could disagree, or there could be more groups with correlated values and signals. Likewise, variances of either group value shocks or idiosyncratic value components could differ (see Section 7).}

With no group value and noise shocks, we recover the model from Example 4 as a special case. However, for fixed parameters $\alpha$ and $\beta$ in utility function there is a continuum of Bayesian models from this class, with variances solving the system of linear equations (see Appendix):

\[
\begin{align*}
\sigma_{\theta_{id}}^2 &= \frac{\alpha \sigma_{\varepsilon_{id}}^2}{1 - \alpha}, \\
\sigma_{\theta_{G}}^2 &= \frac{\alpha \sigma_{\varepsilon_{G}}^2}{1 - \alpha}, \\
\sigma_{\theta}^2 &= \frac{\beta (2\sigma_{\varepsilon_{id}}^2 + \sigma_{\varepsilon_{G}}^2)}{2N(1 - \alpha)(1 - \alpha - \beta)}.
\end{align*}
\]  

(15)

For example, increasing common group noise $\sigma_{\varepsilon_{G}}^2$, which decreases the attractiveness of signals from own group, is compensated by the increase in correlation $\sigma_{\theta_{G}}^2$ of values within a group.

This continuum of models gives rise to a continuum of different linear Bayesian equilibria in double auctions. Moreover, for a fixed vector of parameters that solve (15), generically an analytical characterization of the equilibrium is unavailable (see Appendix). However, we may argue indirectly as follows. The same linear equilibria give rise to the same linear market clearing price $p^l(s) = \gamma_S \bar{s}^S + \gamma_B \bar{s}^B$, for appropriate $\gamma_S, \gamma_B$, and the same price impact $\frac{\partial p^l}{\partial q_i} = d_i \geq 0$, for any agent $i$. Agent $i$’s demand $q_i^l(s_i, p)$ given signal $s_i$ and price $p^l$ satisfies

\[
q^l_i(s_i, p) = \frac{E\left[\theta_i | s_i, p^l\right] - p^l}{\mu_G + d_i},
\]

(16)

for $i \in G, G = S, B$. As we show in Appendix, however, the expectation function $E\left[\theta_i | s_i, p^l\right]$ can agree for at most countably many models.

Despite the differences in risk aversion among the agents, in the model with uncorrelated noises ($\sigma_{\varepsilon_G}^2 = \sigma_{\theta_G}^2 = 0$) expected utilities converge to the efficient ones (Manzano and Vives (2016), see also Hellwig (1980)). This, however, is the only model featuring asymptotic efficiency: It follows from equation (16) that with vanishing price impacts, as long as $\mu_B \neq \mu_S$, the market clearing price $p^l(s) = \gamma_S \bar{s}^S + \gamma_B \bar{s}^B$ may not assign equal weights to the two group averages, $\gamma_S \neq \gamma_B$. As long as noises are positively correlated within a group ($\sigma_{\varepsilon_G}^2, \sigma_{\theta_G}^2 > 0$), neither $\bar{s}^S$ nor $\bar{s}^B$ and thus also not the price converges in probability to the average of all signals $\bar{s}$. It follows that $E\left[\theta_i | s_i, p^l\right] \neq E\left[\theta_i | s_i, \bar{s}\right]$ and so,
from (16), allocations do not converge to the efficient ones.

The example shows how a continuum of Bayesian models underlying a fixed utility profile may i) feature a continuum of Bayesian equilibria in double auctions that are sensitive to the details of the underlying information structure, and which ii) (generically) lack analytical characterization, and iii) (generically) do not yield asymptotic expected efficiency. All this contrasts with uniqueness, simplicity and asymptotic efficiency of the $\sigma_N - WE$ mechanism.

5 Other Mechanisms

One way to interpret a $\sigma - WE$ mechanism is this. Agents are charged with taxes that are quadratic in the quantity they trade. Then, assuming that the taxes are internalized by the agents and so their utilities properly adjusted, $\sigma - WE$ mechanisms are the (generalized) VCG mechanisms that implement the efficient allocation. More precisely, fix a tax $\tau : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, which is twice continuously differentiable, increasing and convex. A mechanism $\{(q(s), t(s))\}_{s \in S}$ is a VCG mechanism with taxes $\tau$ if

$$
q_i(s) = \tilde{q}_i(s),
$$

$$
t_i(s) = \tilde{t}_i(s) + \tau(|\tilde{q}_i(s)|),
$$

where $\{(\tilde{q}(s), \tilde{t}(s))\}_{s \in S}$ is an IR, IC, MC mechanism implementing the efficient allocation for utility functions $\tilde{u}_i$, $i \leq N$,

$$
\tilde{u}_i(q_i, s) = u_i(q_i, s) - \tau(|q_i|).
$$

VCG mechanism with taxes satisfy IR, IC and MC for every choice of tax function. Taxes, then, allow to strike a balance between efficiency and budget surplus: Trivial taxes result in the efficient VCG mechanism and budget deficits. Larger taxes distort efficiency of the allocation, relative to the underlying utilities, but mitigate budget deficit. Of course, given budget surplus as a constraint, some kinds of taxes are better than others in giving up efficiency. Before characterizing the optimal efficiency loss and mechanisms that may achieve it, below we look at two examples.

Example 7 (Linear tax: Bid-ask spread) Fix $\sigma \geq 0$ and consider a linear tax $\tau(q) = \sigma |q|$. For any type profile $s$ the allocation $q(s)$ of a VCG mechanism with such taxes is
determined jointly with bid-ask prices \( p_b(s), p_a(s) \) such that \( p_a(s) = p_b(s) + 2\sigma \), markets clear and

\[
mu_i(q_i(s), s) = p_a(s), \ \forall i \text{ such that } q_i(s) > 0
\]

\[
mu_i(q_i(s), s) = p_b(s), \ \forall i \text{ such that } q_i(s) < 0
\]

Transfers \( t(s) \) that guarantee incentive compatibility and individual rationality are

\[
t_i(s) = T_i(s_{-i}) + \left\{ \begin{array}{ll}
\int_0^{q_i(s)} p_a(s_i(x), s_{-i}) \, dx & \text{if } q_i(s) > 0,
\\
\int_0^{q_i(s)} p_b(s_i(x), s_{-i}) \, dx & \text{if } q_i(s) < 0,
\end{array} \right.
\]

and \( t_i(s) = T_i(s_{-i}) \) if \( q_i(s) = 0 \), where the \( s_i \) function is defined as in (5) and \( T_i(s_{-i}) \leq 0 \).

In other words, the allocation is as if market determined the market clearing bid-ask price pair with a fixed spread of \( 2\sigma \), each agent got all the relevant information and was free to trade at those prices. The discriminatory Vickrey transfers are then chosen to guarantee incentive compatibility and individual rationality.

**Example 8 (Entry Fee tax)** Fix \( \sigma \geq 0 \) and consider the entry fee tax \( \tau(q) = \sigma \times 1_{q \neq 0} \). For any type profile \( s \) the allocation \( q(s) \) of the VCG mechanism with such taxes is determined jointly with a price \( p(s) \) such that markets clear and

\[
mu_i(q_i(s), s) = p(s), \ \forall i \text{ such that } q_i(s) \neq 0
\]

\[
q_i(s) \neq 0 \iff \int_0^{q_i(s)} \mu_i(x, s) dx - p(s)q_i(s) \geq \sigma.
\]

The transfers \( t(s) \) that guarantee incentive compatibility are

\[
t_i(s) = T_i(s_{-i}) + \sigma + \int_0^{q_i(s)} p(s_i(x), s_{-i}) \, dx, \ \text{if } q_i(s) \neq 0
\]

and \( t_i(s) = T_i(s_{-i}) \) if \( q_i(s) = 0 \), where the \( s_i \) function is defined in (5) and \( T_i(s_{-i}) \leq 0 \).

Intuitively, now the allocation is as if market determined the market clearing price, each agent got all the relevant information and then could decide whether to pay a fixed entry fee \( \sigma \) in order to trade at this price.

The following result is the main result of this section. Given assumptions A1 and A3

\[
q_{\text{max}} = \frac{2m}{m_i}
\]

is the maximal quantity that can be traded by any agent in the efficient \( WE \).
Proposition 2 Suppose assumptions A1 – A4 hold.

i) No robustly asymptotically $o\left(\frac{1}{N} + \phi_N\right)$-Efficient mechanisms exist.

ii) If VCG mechanisms with taxes $\tau_N$ are robustly asymptotically $O\left(\frac{1}{N} + \phi_N\right)$-Efficient, then there is $C > 0$ such that for every $q > 0$

$$\tau'_N(q) - \tau'_N(0) \geq C \left(\frac{1}{N} + \phi_N\right) \times q, \quad \text{for all } q \in [0, q_{\text{max}}], N \geq N(q). \quad (19)$$

The first part is a converse to Proposition 1. It establishes that the rate at which $\sigma - WE$ allocations converge to efficiency is tight. The second part considers a special class of VCG mechanisms with taxes. It shows that if some mechanisms converge to efficiency at the optimal rate, then taxes must be at least quadratic in quantity, as in $\sigma - WE$ mechanisms. In this sense, quadratic taxes create strictly minimal distortions. In particular, VCG mechanisms with entry fees or bid-ask spreads perform strictly worse.

The minimal rate of efficiency loss $\frac{1}{N} + \phi_N$ is the marginal effect an agent may have on the Walrasian price in an economy of size $N$. In a near-efficient and incentive compatible allocation, this is the rate at which any buyer’s inframarginal prices decrease, or seller’s inframarginal prices increase. If a mechanism is too efficient and so any buyer’s and seller’s marginal prices are too close, it will run a budget deficit.

To gain intuition for the second part of the proposition, let us focus on the difference between quadratic taxes on the one hand and entry fees and bid-ask spreads on the other. When many agents have average signals and are not willing to pay a fee, or their marginal utilities fall within a spread, the few trading ones have large price impact. For example, a small fixed fee that a buyer pays results in a steep decrease in prices for his last inframarginal units, before other agents enter the market. Thus, the fee is dwarfed by the decrease in prices for all but the last inframarginal units, and so it may not patch up the budget. Similarly, a per-unit fee of $\sigma$ decreases the price for almost all inframarginal units by $\sigma$, which is compounded with the decrease in prices at signals when most agents trade.

6 Implementation

$\sigma - WE$ mechanisms are direct mechanisms: agents report their types to the mechanism designer, who enforces reallocation the of the good and the transfers. While robustness strips him of the responsibility to know the information structure, he must know the utility functions. In this section we present a way to implement $\sigma - WE$ mechanisms in
a way that does not require any such knowledge.

We will make the following two assumptions (see Section 7):

\[
\begin{align*}
\text{INF} & \quad S_i = \mathbb{R}, \quad \forall i \\
\text{INJ} & \quad \inf_{q, s} \det \left[ m_{i,j}(q_i, s) \right]_{i,j \leq N} > 0.
\end{align*}
\]

For the linear-quadratic utility functions the assumption \text{INJ} needs to be checked only at a fixed pair \( q = \emptyset, s = \emptyset \), and is easily verified in each of the examples from Section 3.

Fix slope \( \sigma \geq 0 \) and consider the following auction format.

**Definition 3** (\( \sigma \)-Discriminatory Conditional Double Auction, \( \sigma \text{DCDA} \))

**Actions.** Each agent \( i \) chooses a continuous conditional inverse demand correspondence

\[ d_i : \mathbb{R}^{N-1} \to \mathbb{R}, \]

where \( d_i(q_{-i}) \) is the set of marginal prices at which \( i \) is willing to clear the market and purchase \( \sum_{j \neq i} q_j \), when each other agent \( j \neq i \) is allocated \( q_j \).

**Allocation.** Mechanism finds the price-allocation pair \( (p, q) \) that clears the market:

\[ p \in d_i(q_{-i}), \quad \forall i \quad (20) \]

**Transfers.** A continuous function \( p_i : \mathbb{R} \to \mathbb{R} \) is a residual demand curve for \( i \), \( i \leq N \), if for every \( q_i \) there is \( q_{-i} \) such that the price-allocation pair \( (p_i(q_i), q_i, q_{-i}) \) clears the market. Transfers by \( i \) are defined as

\[ t_i = \int_0^{q_i} p_i(x) \, dx + \frac{\sigma}{2} q_i^2, \]

for the residual demand curve \( p_i \).

If for the submitted conditional inverse demands there is no unique residual demand curve for every \( i \leq N \), and so no unique market clearing \( (p, q) \), then the allocation and transfers are zero.

The first step in running a DCDA - determining the price-allocation vector - requires solving the system of the market clearing equation and \( N \) inclusions (20) in \( N + 1 \) unknowns \( (p, q) \). Aside from the added difficulty in allowing for set-valued demands, discussed below, this generalizes finding the market clearing price in a double auction. It boils down to it when inverse demands condition only on the total quantity to be cleared.
Again, much like in a double auction, reports by all agents other than \( i \) determine the residual demand curve that \( i \) is facing. It is the menu of (marginal) price-quantity pairs \((p,q)\) that \( i \) is choosing from (see Chung and Ely (2002)), which is pinned down by market clearing and \( N-1 \) demands submitted by other agents. The first term in the transfers are then the discriminatory payments: an integral over all the inframarginal units under the residual demand curve that \( i \) is facing. The second term is a tax quadratic in the quantity traded, paid by every agent.

Given little restrictions on bids they need not determine unique allocation or transfers, in which case the economy reverts to autarky. This is a standard proviso from double auctions, when multiple market clearing prices exist. Alternatives, such a picking an allocation or residual demand curves according to some pre-specified order would do just as well. Of course, it must be checked that with equilibrium bids multiplicity does not arise.

Ex-post equilibrium is the game theoretic counterpart of ex-post incentive compatibility. It means that each agent’s strategy is optimal at every type profile of other agents, and so irrespectively of the underlying information structure (see Bergemann and Morris (2005)). For any \( \sigma \geq 0 \) a \( \sigma - WE \) conditional inverse demand strategy in a \( \sigma \)DCDA for agent \( i \leq N \) is defined as

\[
d_i(s_i)(q_{-i}) = \left\{ mu_i(\sum_{j \neq i} q_j, s_i, s_{-i}) + \sigma \sum_{j \neq i} q_j \Big| q_{-i}(s_i, s_{-i}) = q_{-i} \right\}.
\]

\( \sigma - WE \) conditional inverse demands are reports of own linear tax adjusted marginal utilities (see Section 5), evaluated at a market clearing quantity, but under reparametrized types. The report conditions not on the profiles of original types, but on the efficient allocations.

**Proposition 3** Fix \( \sigma \geq 0 \) and consider a game induced by the \( \sigma \)DCDA. The profile of \( \sigma - WE \) conditional inverse demand strategies constitutes an ex-post equilibrium that results in transfers and allocation as in the \( \sigma - WE \) mechanism.

The idea behind the result is as follows. Assumption \( INJ \) is sufficient for the function from signal to marginal utility profiles, evaluated at any allocation, to be globally invertible. Crucially, given the definition of \( \sigma - WE \), this restriction on utilities is sufficient for the function from signal profiles to \( \sigma - WE \) allocation-price pairs to be invertible, for any \( \sigma \geq 0 \). But this, intuitively, establishes that the informational content of either is the same. More precisely, when each agent \( i \) reports the \( \sigma - WE \) conditional inverse demand
\(d_i(s_i)\), it is equivalent to reporting the set of \(\sigma - WE\) allocation-price pairs consistent with own signal \(s_i\). Injectivity implies that the intersection of the reports across all the agents is the unique \(\sigma - WE\) price-allocation \((p^\sigma(s), q^\sigma(s))\). Uniqueness of residual demand curves follows analogously, and ex-post optimality follows from IC of the \(\sigma - WE\) mechanism.

The following Example solves for the \(\sigma - WE\) conditional inverse demands in case of linear utilities.

**Example 9** In case of linear quadratic utilities, as in the previous examples, \(\sigma - WE\) allocations satisfy

\[
q^\sigma_i(s) = \frac{\sum_j w_{ij} s_j - p^\sigma(s)}{\mu_i + \sigma},
\]

where \(w_{ij} = m u_{i,j}(0, 0)\). This implies

\[
p^\sigma(s) \cdot \sum_j \tilde{w}_{ij} = \left( s_i - \sum_j \tilde{w}_{ij}(\mu_j + \sigma)q^\sigma_j(s) \right)
= \left( s_i - \sum_{j\neq i} \left( \tilde{w}_{ij}(\mu_j + \sigma) - \tilde{w}_{ii}(\mu_i + \sigma) \right)q^\sigma_j(s) \right),
\]

where \(\left[\tilde{w}_{ij}\right]_{i,j \leq N}\) is the inverse of the matrix \(\left[w_{ij}\right]_{i,j \leq N}\).

In the Fundamental Value Model, matrix inversion yields \(\sigma - WE\) conditional inverse demand functions

\[
d_i(s_i)(q_{-i}) = (\alpha + \beta) \left[ s_i + \frac{\mu + \sigma}{\alpha} \sum_{j\neq i} q_j(s) \right],
\]

which depend only on the total quantity an agent must clear.

In contrast, in the Group Model from Example 2 \(\sigma - WE\) conditional inverse demand functions are

\[
d_i(s_i)(q_{-i}) = (\alpha + \beta) s_i + \frac{\mu + \sigma}{\alpha} \left[ \left( \alpha + \beta \frac{N - 2}{N} \right) \sum_{j\neq i} q_j + \beta \overline{q}^{own} \right],
\]

Marginal price at which an agent is willing to clear the market is increasing in own signal, which increases his value of the good, and increasing in the quantity \(\sum_{j\neq i} q_j\) that he must clear, due to decreasing marginal utility of absorbing more units.\(^8\)

\(^8\)For example, if \(s_i\) goes up by \(\Delta\) but the allocated \(\overline{q}^{own}\) and \(\overline{q}\) do not change, this means that \(\overline{q}^{own}\) and \(\overline{q}\) must have gone up by \(\Delta\) as well. In this case the marginal valuation of the good by \(i\) goes up by \((\alpha + \beta)\Delta\).
inverse demand function, however, each agent conditions his price also on the average trade by the agents in his group. For a fixed own type and total quantity to be cleared, higher demand by own group indicates higher own group shock and so value of the good to the agent, thus increasing the price.

Similarly, in the Fundamental Value Model with Heterogenous Traders $\sigma - WE$ conditional inverse demand by a big agent $i$ is (and similarly for small agents):

$$d_i(s_i)(q_{-i}) = (\alpha + \beta) \left[ s_i + \frac{\mu + \sigma}{\alpha} \sum_{j \neq i} q_j(s) \right] + \frac{\beta (\mu_S - \mu_B)}{2\alpha} q^S.$$

While small agents have equally precise information as big agents, they trade less aggressively. Thus, keeping the total demand by all other agents fixed, high demand by small agents reveals high value of the good and so pushes up the price demanded to clear the market.

$\sigma - WE$ conditional inverse demands are own tax adjusted marginal utilities conditional on types of others, represented by efficient allocation, evaluated at a single market clearing quantity. Unlike demand reduction and filtering of the informational content of prices in the case of double auctions, tax slope $\sigma$ and value interdependencies are exogenous and not determined by endogenous equilibrium conditions (see Section 5). The extent to which $\sigma - WE$ conditional inverse demands are complicated thus reflects the relatively unrestricted nature of the primitives in our framework, with no symmetry assumptions. On the other hand, as Example 9 shows, when utilities depend only on low-dimensional statistic of the types of others, the inverse demands also condition on low-dimensional statistics of other trades. This is not a coincidence:

**Lemma 2** Fix $n \geq 0$ and suppose that for every $i$ there exist functions $w_i : S_{-i} \to \mathbb{R}^n$ and $\tilde{u}_i : \mathbb{R}^{n+2} \to \mathbb{R}$ such that

$$u_i(\cdot, s_i, s_{-i}) = \tilde{u}_i(\cdot, s_i, w_i(s_{-i})).$$

Then there is a function $\tilde{w}_i : \mathbb{R}^N \to \mathbb{R}^n$ such that $\sigma - WE$ conditional inverse demand satisfies

$$d_i(s_i)(q_{-i}) = \left\{ x | x = m\tilde{w}_i(-\sum_{j \neq i} q_j, s_i, \tilde{w}_i(q_{-i}, x)) + \sigma \sum_{j \neq i} q_j \right\}, \quad \forall i$$

(22)
If \( d_i(s_i) \) is a function then there is \( \tilde{w}_i : \mathbb{R}^{N-1} \rightarrow \mathbb{R}^n \) such that

\[
d_i(s_i)(q_{-i}) = m\tilde{u}_i(-\sum_{j \neq i} q_j, s_i, \tilde{w}_i(q_{-i})) + \sigma \sum_{j \neq i} q_j, \quad \forall i
\]

An agent may report several tax adjusted marginal utilities consistent with a given efficient allocation, and so a \( \sigma - \text{WE} \) conditional demand may be a correspondence. Intuitively, with a lot of interdependence, for a fixed agent’s type, high shocks of others can move the marginal utilities of all the agents in step, not affecting the allocation. This a familiar phenomenon from the rational expectation equilibrium or the double auction literature, where too much interdependence may imply demand upward sloping in price, resulting in the nonexistence of equilibrium (e.g., Kyle (1989)). The following lemma shows a sufficient condition (necessary for linear utilities) for this not to happen.

Consider the following assumption

\[
\text{FUN) } \quad \text{The sum of every row in } \left[ mu_{i,j}(q_i, s) \right]_{i,j \leq N}^{-1} \text{ is nonzero, } \forall q, s.
\]

**Lemma 3** Fix \( \sigma \geq 0 \). If FUN is satisfied then \( \sigma - \text{WE} \) conditional inverse demands are functions. In case of linear utilities the converse is also true.

Below we present two sufficient conditions for FUN. The first one requires, roughly, that there is not too much interdependence in preferences. Surprisingly, the second sufficient condition is *equicommonality* (Rostek and Weretka (2012)), satisfied when the average level of interdependence across the agents is constant, no matter its level. It is satisfied in all the examples considered in this paper. In the Bayesian framework with normal beliefs and equal quadratic coefficients the condition is sufficient for the existence of a symmetric linear equilibrium.

**Lemma 4** Either of the following conditions is sufficient for FUN. For every \( i, j, q_i, q_j, s \)

i) \( \mu_{i,i}(q_i, s) \geq 2N \cdot \mu_{j,i}(q_j, s) \),

ii) \( \sum_k \mu_{i,k}(q_i, s) = \sum_k \mu_{j,k}(q_j, s) \).

### 7 Discussion

**Weaker Assumptions.** Assumptions on the utilities can be weakened to fit specific applications: i) Assumptions A2 and A3 can be qualified to hold only when \( \mu_i(q_i, s) \in \)
\[ \min_{i \leq N} \mu_i(0, s), \max_{i \leq N} \mu_i(0, s) \]. In particular, this allows environments with non-negative marginal utilities at any allocation. ii) With large economies, when \( \sigma - WE \) allocations are nearly efficient, global single crossing SC can be weakened to hold only locally, at allocations \( q_i \) and \( q_j \) such that \( \mu_i(q_i, s) \) and \( \mu_j(q_j, s) \) are close. iii) Similarly, with large economies interdependence \( \frac{1}{N-1} \sum_i \mu_{i,j}(q_i, s) \) in \( A4 \) and \( A4^N \) needs to be bounded only at \( q_{-j}'s \) such that all \( \mu_i(q_i, s), i \neq j \), are close.

**Multidimensional Signals.** Without additional restrictions, our results cannot be extended to environments with multidimensional signals. Jehiel et al. (2006) have shown that with multidimensional signals and interdependence, typically no nontrivial allocation can be ex-post incentive compatible. There are, nevertheless, several cases when such analysis is not doomed. For example, results in Jehiel et al. (2008) suggest that our results may be extended to the case of multidimensional signals as long as marginal utilities are linear functions of signals (see also Bikhchandani (2006), Eso and Maskin (2002)). In case of private value environments, the impossibility has no bite, and results in Bikhchandani et al. (2006) suggest the right notion of monotonicity. Detailed analysis of each extension is beyond the scope of this paper.

**Tightness.** Efficiency losses in Proposition 1 are tight in that there is no other class of mechanisms that converges to efficiency at a faster rate, as economy grows (Proposition 2). More specifically, formula (38) in the Appendix implies that for large \( N \) no mechanism can attain efficiency losses \( \frac{2m_q}{m_q} \left( 2 + \frac{(m_q - m_\phi)}{m_q} \right) \) times smaller than \( \sigma - WE \).

While a tight rate of convergence is a standard optimality criterion in the asymptotic worst case analysis, we may strengthen the result as follows. Let’s assume that the marginal utilities decrease linearly in quantity,

\[ A3' \] \( \mu_{i,q}(q_i, s) = m_q < 0, \forall i, q_i, s \)

and consider the following alternative measure of ex-post efficiency loss:

\[ \varepsilon - qEff \] \( \left| q_i(s) - q_i^0(s) \right| \leq \varepsilon |q_i^0(s)|, \forall s \)

**Proposition 4** Suppose assumptions \( A1, A2, A3' \) and \( A4 \) hold. \( \frac{m_\phi}{m_\phi} (\frac{m_\phi}{N-1} + \phi_N) - WE \) mechanisms are robustly asymptotically \( (\frac{1}{N-1} + \frac{\phi_N}{m_\phi}) - qEfficient. Moreover, for any \( \gamma < 1 \), no robustly asymptotically \( \gamma (\frac{1}{N-1} + \frac{\phi_N}{m_\phi}) - qEfficient mechanisms exist.

**Signals as Types.** One source of interdependence in preferences is informational, when agent’s type corresponds to a signal informative of each agent’s value. In a Bayesian
model, interdependencies in expected utilities, conditional on a profile of signals, arise naturally when either signals or values are correlated (see Section 2). It seems like this informational motivation of the interdependence does not square with the prior-free approach of this paper. We note that despite this objection one may still view $\sigma -$ Walrasian mechanisms as particularly simple Bayesian mechanisms that attain asymptotic efficiency.

However, the problem is only apparent. It is true that ex-post utilities and a fixed distribution of correlated signals and values pin down expected utilities with interdependencies, but the reverse is not true. Since many distributions may give rise to one expected utility function, knowing only the the latter assumes strictly less than the Bayesian framework. The following example makes the point stark. Suppose agent $i$ has a linear ex-post utility function $v_i(q_i, \theta_i)$ as in (1) and fix a linear expected utility function

$$u_i(q_i, s) = q_i \sum w_{ij} s_j - \frac{\mu}{2} q_i^2,$$

for some $\mu, w_{ij}, j \leq N$. Prior-free analysis requires knowledge of $u_i$ only, while a Bayesian game complements $u_i$ with a fixed distribution $\delta_S \in \Delta(S)$. We may assume that $\delta_S$ is normal.

**Lemma 5** For any normal distribution $\delta_S \in \Delta(S)$ there is a normal distribution $\delta_{\Theta_i \times S} \in \Delta(\Theta_i \times S)$ such that

$$u_i(q_i, s) = \mathbb{E}_{\delta_{\Theta_i \times S}}[v_i(q_i, \theta_i)|s].$$

The proof is the statement of the Projection Theorem, read as a system of $N$ linear equations in $N$ variables, $\text{Cov}(s, \theta_i)$. In other words, expected utilities put no restrictions whatsoever on the distribution of signals $\delta_S$.

**Equilibrium Uniqueness.** Ex-post implementation implies that in every information structure the game induced by a $\sigma -$ WE mechanism has a truth-telling equilibrium. In the paper we are not requiring that this equilibrium is unique. The following, however, is a direct consequence of Theorem 1 in Bergemann and Morris (2009). In case of linear utilities, with $mu_{i,i}(q_i, s)$ normalized to 1, $i \leq N$, truth-telling is the unique equilibrium of a $\sigma -$ WE mechanism in every information structure precisely when the matrix

$$
\begin{bmatrix}
0 & |w_{1,2}| & \ldots & |w_{1,N}| \\
|w_{1,2}| & 0 & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
|w_{1,N}| & \ldots & \ldots & 0
\end{bmatrix}
$$

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with $w_{ij} = mu_{i,j}(q_i, s)$, has the largest eigenvalue $\lambda < 1$. Moreover, the condition is also necessary for the equilibrium uniqueness under any mechanism (Bergemann and Morris (2009), Section 3).\footnote{Since we are not insisting on responsive allocation functions and strict ex-post incentive compatibility, we may not use Theorem 2 in Bergemann and Morris (2009). However, the direct analysis in Section 3 of their paper applied to our linear setting implies that if the above condition is violated, then there is an agent $i$ and his beliefs over the types of his opponents such that all $i$'s types are indistinguishable.} In other words, whether ex-post near-efficient implementation can be strengthened to guarantee equilibrium uniqueness in every information structure is not a mechanism design problem, and the answer depends solely on the properties of an exchange economy.

*Conditions for implementation.* Assumptions \textsc{Inf} and \textsc{Inj} in Section 6 are sufficient for the function from the type to the marginal utility profiles (at any allocation) to be globally invertible.\footnote{There are alternative sufficient conditions for the global invertibility (see (38) in Dasgupta and Maskin (2000), chapter 6 in Krantz and Parks (2012)).} Given the definition of $\sigma - WE$ mechanisms, this guarantees that the mapping from type profiles to endogenous $\sigma - WE$ allocation and price profiles is globally invertible as well. The conditions help achieve a sensible implementation, but are not necessary. If they are violated, a profile of reported $\sigma - WE$ conditional inverse demands may result in multiple profiles of market clearing price-allocation pairs and transfers. In such event, instead of shutting down trade a mechanism may allow another stage with “all-out” communication: Have each agent report his type, and then agents pick the allocation and transfer profile, with the unanimous choice implemented. While such communication requires a daunting amount of coordination (see Dasgupta and Maskin (2000)), there seems to be little alternative if there is no simpler sufficient statistic for the information carried by the types.

\section*{Appendix}

\section{Proof of Proposition 1}

\textbf{i) Necessity.} Fix a $\sigma - WE$ allocation profile $\{q^\sigma(s)\}_{s \in S}$. \textsc{IC} implies that there are functions $\tilde{t}_i$, $i \leq N$, such that

$$t^\sigma_i(s) = \tilde{t}_i(q^\sigma_i(s), s_{-i}).$$

Fix a type vector $s$ and an agent $i$ and consider the allocation $q^\sigma_i(\cdot, s_{-i})$ as a function
of $i$'s signal. Local IC implies that

$$mu_i(q_i^*(s_i, s_{-i}), (s_i, s_{-i})) \frac{\partial q_i^*(s_i, s_{-i})}{\partial s_i} = \frac{\partial \tilde{t}_i(q_i^*(s_i, s_{-i}), s_{-i})}{\partial q_i} \frac{\partial q_i^*(s_i, s_{-i})}{\partial s_i},$$

$$\frac{\partial \tilde{t}_i(q_i^*(s_i, s_{-i}), s_{-i})}{\partial q_i} = mu_i(q_i^*(s_i, s_{-i}), (s_i, s_{-i})),$$

and so for any $s_i' < s_i$

$$t_i^*(s) = \tilde{t}_i(q_i^*(s), s_{-i}) = t_i^*(s_i', s_{-i}) + \int_{q_i(s_i', s_{-i})}^{q_i(s)} \frac{\partial \tilde{t}_i(x, s_{-i})}{\partial q_i} dx$$

$$= t_i^*(s_i', s_{-i}) + \int_{q_i(s_i', s_{-i})}^{q_i(s)} mu_i(x, (s_i(x), s_{-i})) dx$$

$$= t_i^*(s_i', s_{-i}) + \int_{q_i(s_i', s_{-i})}^{q_i(s)} [p^*(s_i(x), s_{-i}) + \sigma x] dx.$$

This establishes uniqueness of the transfer function up to a constant.

**Sufficiency.** Let us establish monotonicity of the allocation, $\frac{\partial \mu_i(s)}{\partial s_i} \geq 0$, for all $i$ and $s$. From the first order condition we have for any $j$ and $i$

$$mu_{j,i}(q_i^*(s), s) \frac{\partial q_j^*(s)}{\partial s_i} + mu_{j,i}(q_j^*(s), s) = \frac{\partial p^*(s)}{\partial s_i} + \sigma \frac{\partial q_j^*(s)}{\partial s_i}. \quad (24)$$

With $i = j$ this establishes that

$$\frac{\partial q_i^*(s)}{\partial s_i} = \frac{mu_{i,i}(q_i^*(s), s) - \frac{\partial p^*(s)}{\partial s_i}}{\sigma - mu_{i,i}(q_i^*(s), s)}. \quad (25)$$

On the other hand, since market clears for all $s_i'$, it follows that

$$0 = \sum_j \frac{\partial q_j^*(s)}{\partial s_i} = \sum_j \frac{mu_{j,i}(q_j^*(s), s) - \frac{\partial p^*(s)}{\partial s_i}}{\sigma - mu_{j,i}(q_j^*(s), s)},$$

and so

$$\frac{\partial p^*(s)}{\partial s_i} = \frac{\sum_j mu_{j,i}(q_j^*(s), s) \times (\sigma - mu_{j,i}(q_j^*(s), s))^{-1}}{\sum_j (\sigma - mu_{j,i}(q_j^*(s), s))^{-1}}. \quad (26)$$

Formulas (25) and (26) together with $SC$ establish monotonicity. Given local IC and $mu_{i,i}(q_i, s) > 0$ this implies global IC. By construction, IR is satisfied in each of the three (exclusive) cases: for the signal $s_i$ under which $q_i^*(s_i, s_{-i}) = 0$, signal $\inf S_i$ when
\( q_i^\sigma(S_i, s_{-i}) > 0 \) and \( \sup S_i \) when \( q_i^\sigma(S_i, s_{-i}) < 0 \). Global IC implies then global IR in each case.

ii) Fix \( s, i \) and \( q_i \) and let us write

\[
y_j = \left( \sigma - \frac{\partial u_j(q^\sigma_j(s_i(q_i), s_{-i}), s_i(q_i), s_{-i})}{\partial q_j} \right)^{-1}, \quad \tilde{y}_j = \frac{y_j}{\sum_{k=1}^{N} y_k},
\tag{27}
\]

with \( s_i(x) \) as in (5). Using formulas (25) and (26) we may bound the sensitivity of \( \sigma \)-Walrasian price with respect to \( i \)'s allocation:

\[
\frac{dp^\sigma(s_i(q_i), s_{-i})}{dq_i} = \frac{\frac{\partial p^\sigma(s_i(q_i), s_{-i})}{\partial q_i}}{\partial q_i} = \frac{\sum \mu_{j,i}(q^\sigma_j(s), s)\tilde{y}_j}{(\sum \mu_{j,i}(q^\sigma_j(s), s)\tilde{y}_j) y_i}
\]

\[
= \frac{\mu_{i,i}(q^\sigma_i(s), s)\sum_{j \neq i} y_j - \sum_{j \neq i} \mu_{j,i}(q^\sigma_j(s), s)y_j}{(\sum_{j \neq i} \mu_{j,i}(q^\sigma_j(s), s)\tilde{y}_j) y_i}
\]

\[
\leq \frac{m_o}{m_o - \frac{m_q}{m_q} \phi_N} \frac{N-1}{\sigma + m_q} \phi_N
\]

Given this bound the transfers satisfy

\[
t^\sigma_i(s) = \int_0^{q^\sigma_i(s)} \left[ p^\sigma(s_i(x), s_{-i}) + \sigma x \right] dx
\]

\[
\geq \int_0^{q^\sigma_i(s)} \left[ p^\sigma(s) - \frac{\sigma + m_q}{m_o} \left( \frac{m_o}{N-1} + \frac{m_q}{m_q} \phi_N \right) (q^\sigma_i(s) - x) + \sigma x \right] dx
\]

\[
= p^\sigma(s) q_i(s) + \frac{q^\sigma_i(s)^2}{2} \left[ \sigma - \frac{\sigma + m_q}{m_o} \left( \frac{m_o}{N-1} + \frac{m_q}{m_q} \phi_N \right) \right] \geq p^\sigma(s) q^\sigma_i(s),
\]

where the last inequality follows from (6). This means that the average per-unit price any buyer pays is above \( p^\sigma(s) \) and any seller’s average per-unit price is below \( p^\sigma(s) \), which implies BS.
iii) Fix a $\sigma-$Walrasian Equilibrium mechanism $\{(q^\sigma(s), t^\sigma(s))\}_{s \in S}$ and $s$. First, for any $i$ let $mu_{q,i} \in [m_q, m_q]$ be such that $mu_i(q^\sigma_i(s), s) = mu_i(0, s) - mu_{q,i} * q^\sigma_i(s)$. Using (26) we have

$$|q^\sigma_i(s)| = \left| \frac{mu_i(0, s) - p^\sigma(s)}{\sigma + mu_{q,i}} \right| = \left| \frac{mu_i(0, s) - \sum_j \frac{mu_j(0, s)}{\sigma + mu_{q,j}} / \sum_j \frac{1}{\sigma + mu_{q,j}}}{\sigma + mu_{q,j}} \right| \leq \frac{2m}{\sigma + m_q}. \quad (28)$$

Second, in order to bound the difference $|p^\sigma(s) - p^0(s)|$, let $q(s,p)$ be the allocation that solves the $\sigma-$Walrasian Equilibrium equations (3) but with $p$ in place of $p^\sigma(s)$. We have

$$\sum_{i \leq N} q_i(s, p^0(s)) = \sum_{i \leq N} q^0_i(s) \frac{mu^\#_{q,i}}{\sigma + mu^\#_{q,i}},$$

where $q^0_i(s)$ is the Walrasian allocation and for any $j$ $mu^\#_{q,j} \in [m_q, m_q]$ is such that $mu_j(q_j(s, p^0(s)), s) = mu_j(q^0_j(s), s) - mu^\#_{q,j} * (q_j(s, p^0(s)) - q^0_j(s))$. Thus, using bound in (28) with $\sigma = 0,$

$$\sum_{i \leq N} q_i(s, p^0(s)) \leq \sum_{i \leq N} q^0_i(s) \frac{-\sigma}{\sigma + mu^\#_{q,i}} \leq \sigma \times \left( \frac{1}{\sigma + m_q} - \frac{1}{\sigma + m_q} \right) \times \frac{1}{2} \sum_{i \leq N} |q^0_i(s)| \leq \frac{\sigma(m_q - m_q)}{(\sigma + m_q)(\sigma + m_q)} \frac{Nm}{m_q}.$$

Since on the other hand

$$\frac{\partial}{\partial p} \sum_{i \leq N} q_i(s, p) \geq -\frac{N}{\sigma + m_q},$$

we have

$$|p^\sigma(s) - p^0(s)| \leq \frac{\sigma(m_q - m_q)}{(\sigma + m_q)(\sigma + m_q)} \frac{Nm}{m_q}.$$

Using bounds (28)-(29) we have

$$|q^\sigma_i(s) - q^0_i(s)| \leq \frac{1}{m_q} |mu_i(q^\sigma_i(s), s) - mu_i(q^0_i(s), s)| \leq \frac{1}{m_q} \left( |q^\sigma_i(s)| + |p^\sigma(s) - p^0(s)| \right) \leq \frac{\sigma m}{m_q(\sigma + m_q)} \left( 2 + \frac{(m_q - m_q)}{m_q} \right). \quad (30)$$
B  Proofs for Section 4

Example 4. Proposition 2 (and the preceding discussion) in Rostek and Weretka (2012) implies that (using original notation)

\[ \alpha = c_s = \frac{1 - \rho}{1 - \rho + \zeta}, \]
\[ \beta = \frac{c_p}{\alpha + \beta} = \frac{(2 - N^{-2}) \rho}{1 - N^{-2} + \rho(1 - \rho + \zeta)} = \frac{N \rho \zeta}{(1 - \rho + \zeta)(1 + (N - 1) \rho)}, \]

and so

\[ \beta = \frac{N \rho \zeta}{(1 - \rho + \zeta)(1 + (N - 1) \rho)} = \frac{N \rho \zeta}{(1 - \rho + \zeta)(1 + \zeta + (N - 1) \rho)}. \]

The submitted inverse demands are

\[ \overline{p}(s_i) \left( \sum_{j \neq i} q_j \right) = c_s \frac{c_p}{1 - c_p} s_i + \frac{\mu}{N - 2} c_p \sum_{j \neq i} q_j = (\alpha + \beta) \left[ s_i + \frac{\mu(N - 1)}{\alpha(N - 2) - \beta} \sum_{j \neq i} q_j \right], \]

whereas the market clearing price \( p_l(s) \) and allocation \( q_l^i(s) \), for any \( s \), satisfy

\[ p_l(s) = \frac{c_s}{1 - c_p} s = (\alpha + \beta) s, \]
\[ q_l^i(s) = \frac{c_s}{\mu} \left( \frac{N - 2}{N - 1} - c_p \right) (s_i - \overline{s}) = \frac{(N - 2) \alpha - \beta}{\mu(N - 1)} (s_i - \overline{s}), \]

which means that the transfers and the allocation is the same as in the \( \sigma_N - \text{WE} \) mechanism (see (10)).

Example 5. Proposition 2 in Rostek and Weretka (2012) implies that for any \( s \) the equilibrium price \( p_l^i(s) \) equals \( \frac{c_s}{1 - c_p} \overline{s} \), and the linear equilibrium has the form (13) for constants (using the original notation):

\[ c_s = \frac{1 - \frac{N - 2}{2(N - 1)} \rho}{1 - \frac{N - 2}{2(N - 1)} + \zeta}, \]
\[ c_p = \frac{\frac{N - 1}{N - 1} - \frac{N - 2}{2(N - 1)} \rho}{1 - \frac{N - 2}{2(N - 1)} + \zeta} = \frac{N(N - 2)}{2(N - 1)} \rho \frac{\zeta}{1 + \frac{N - 2}{2(N - 1)} \rho - \frac{N - 2}{2(N - 1)} \rho + \zeta}. \]

Example 6. 1. With slope \( \sigma_N \) from Proposition 1 the transfers for \( i \in G, G = S, B, \)
become:

\[ t_i(s) = \int_0^{q_i(s)} \left[ p^{s_N}(s_i(x),s_{-i}) + \sigma_N x \right] dx = p^s(s_i)q_i(s) \]

+ \frac{q_i(s)^2}{2} \left[ \sigma_N - \frac{\sigma_N + \mu_G}{\alpha(N - 2\gamma_G)} \left( \beta + 2\alpha\gamma_G \right) \right] \]

= \frac{p^s(s_i)q_i(s)}{2} \left[ \sigma_N - \frac{\sigma_N + \mu_G}{\alpha(N - 2\frac{\sigma_N + \mu_G}{2\alpha + \mu_B + \mu_S})} \left( \beta + 2\alpha\frac{\sigma_N + \mu_G}{2\alpha + \mu_B + \mu_S} \right) \right].

2. Given a Bayesian model, the parameters \( \sigma_G^2, \sigma_{\theta_G}^2, \sigma_{\theta_{id}}^2, \sigma_{\varepsilon_G}^2, \sigma_{\varepsilon_{id}}^2 \) that give rise to utilities as in Example (3) are, from the Projection Theorem,

\[
\begin{pmatrix}
\alpha \\
\beta/2 \\
\beta/2 \\
0 \\
0 \\
0
\end{pmatrix} = \begin{pmatrix}
1 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix} - Var(\varepsilon_S + \varepsilon_i, \overline{s}^S, \overline{s}^B)_{3 \times 3}^{-1} \cdot Cov(\varepsilon_S + \varepsilon_i, (\varepsilon_S + \varepsilon_i, \overline{s}^S, \overline{s}^B)^T)_{3 \times 1},
\]

\[
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix} = Var(\varepsilon_S + \varepsilon_i, \overline{s}^S, \overline{s}^B)_{3 \times 3} \cdot \begin{pmatrix}
1 - \alpha \\
-\beta/2 \\
-\beta/2
\end{pmatrix} - Cov(\varepsilon_S + \varepsilon_i, (\varepsilon_S + \varepsilon_i, \overline{s}^S, \overline{s}^B)^T)_{3 \times 1}
\]

\[
= \begin{pmatrix}
A & B & \sigma_G^2 \\
B & B & \sigma_G^2 \\
\sigma_G^2 & \sigma_G^2 & B
\end{pmatrix} \cdot \begin{pmatrix}
1 - \alpha \\
-\beta/2 \\
-\beta/2
\end{pmatrix} - \begin{pmatrix}
\sigma_G^2 + \sigma_{\varepsilon_{id}}^2 \\
\sigma_G^2 + \frac{\sigma_{\varepsilon_{id}}^2}{2N} \\
0
\end{pmatrix},
\]

for \( i \in S \) and similarly for \( i \in B \), where

\[
A = \sigma_G^2 + \sigma_{\theta_G}^2 + \sigma_{\theta_{id}}^2 + \sigma_{\varepsilon_G}^2 + \sigma_{\varepsilon_{id}}^2,
\]

\[
B = \sigma_G^2 + \sigma_{\theta_G}^2 + \sigma_{\varepsilon_G}^2 + \frac{\sigma_{\theta_{id}}^2 + \sigma_{\varepsilon_{id}}^2}{N/2}.
\]

It is readily verified that the solution to the above system of linear equations is given by (15).

3. For a fixed set of parameters \( \sigma_G^2, \sigma_{\theta_G}^2, \sigma_{\theta_{id}}^2, \sigma_{\varepsilon_G}^2, \sigma_{\varepsilon_{id}}^2 \) consider a linear Bayesian equilibrium in double auction of the form

\[
q_i^l(s_i)(p) = a_Gs_i - b_Gp, \quad \forall i \in G, G = S, B,
\]
and so the equilibrium price \( p' \) is given by

\[
p'(s) = \frac{a_S}{b_S + b_B} s^s + \frac{a_B}{b_S + b_B} s^B.
\]

From the first order condition, demand schedules are given by equation (16), for

\[
d_i = \frac{1}{N-2 b_G + \frac{N}{2} b_{-G}}. \quad \forall i \in G, G = S, B
\]

From the Projection Theorem, we compute \( E[\theta_i | s_i, p] \), for \( i \in G \), as

\[
E[\theta_i | s_i, p'] = \tilde{a}_G s_i - \tilde{b}_G p',
\]

where

\[
\begin{bmatrix}
\tilde{a}_G \\
\tilde{b}_G
\end{bmatrix} = Var(s_i, p')^{-1} \cdot Cov(\theta_i, (s_i, p')^T) \cdot 2 \times 1,
\]

\[
Var((s_i, p'))_{2 \times 2} =
\begin{bmatrix}
\sigma^2 + \sigma^2_{\theta_G} + \sigma^2_{\theta_{id}} + \sigma^2_{\sigma_{id}} + \sigma^2_{\sigma_{id}} \\
\sigma^2 + \frac{\sigma^2}{(b_G + b_{-G})} \left( \sigma^2_{\theta_G} + \sigma^2_{\sigma_{id}} + \frac{2 \sigma^2_{\sigma_{id}}}{N} \right)
\end{bmatrix}
\]

\[
Cov(\theta_i, (s_i, p')^T)_{2 \times 1} =
\begin{bmatrix}
\sigma^2 + \sigma^2_{\theta_G} + \sigma^2_{\theta_{id}} + \frac{2 \sigma^2}{N}
\end{bmatrix}.
\]

Substituting (15) into these formulas, we get

\[
\tilde{a}_G = \frac{2(\alpha-1)\alpha (a_G^2(n-2)\sigma_{\sigma_{id}} + a_{-G}^2n(\sigma_{\sigma_{id}} + \sigma_{\sigma_{G}}) + 2(\sigma_{\sigma_{id}}^2 + \sigma_{\sigma_{G}}^2))}{2(\alpha-1)(a_G^2(n-2)\sigma_{\sigma_{id}} + a_{-G}^2n(\sigma_{\sigma_{id}} + \sigma_{\sigma_{G}}) + \beta(n-2)\sigma_{\sigma_{id}}(a_G - a_{-G})^2 + \beta(a_G - a_{-G})(\alpha a_G(n-2)\sigma_{\sigma_{id}} + a_{-G}(-\alpha n(\sigma_{\sigma_{id}} + \sigma_{\sigma_{G}}) + n\sigma_{\sigma_{G}} + 2\sigma_{\sigma_{id}}))}
\]

\[
\tilde{b}_G = \frac{(\alpha-1)\beta(b_G + b_{-G})(a_G^2(n-2)\sigma_{\sigma_{id}} + a_{-G}^2n(\sigma_{\sigma_{id}} + \sigma_{\sigma_{G}}) + \beta(n-2)\sigma_{\sigma_{id}}(a_G - a_{-G})^2)}{2(\alpha-1)(a_G^2(n-2)\sigma_{\sigma_{id}} + a_{-G}^2n(\sigma_{\sigma_{id}} + \sigma_{\sigma_{G}}) + \beta(n-2)\sigma_{\sigma_{id}}(a_G - a_{-G})^2)}
\]

Equilibrium is thus characterized by the solutions \( a_S, a_B, b_S, b_B \) of the system of equations

\[
a_G = \frac{\tilde{a}_G(a_G, a_{-G})}{\mu_G + d_G(b_G, b_{-G})}, \quad b_G = \frac{\tilde{b}_G(a_G, a_{-G}, b_G + b_{-G}) + 1}{\mu_G + d_G(b_G, b_{-G})}. \quad G = S, B
\]

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Note that \( a_G \neq a_{-G} \) as long as \( \mu_G \neq \mu_{-G} \)

\[
\begin{align*}
\frac{a_G}{a_{-G}} = 1 & \Rightarrow \frac{b_G}{b_{-G}} = 1 \Rightarrow \mu_G = \mu_{-G}.
\end{align*}
\]

In the asymmetric case (34) is a system of four equations of third order and so generically does not admit an analytical solution (Abel-Ruffini Theorem).

4. Finally, to see that the solution to the system (34) depends locally on the parameters \( \sigma_{\varepsilon id} \) and \( \sigma_{\varepsilon G} \), even when \( \alpha, \beta \) are fixed, observe that:

\[
\begin{align*}
\frac{\partial \delta_G}{\partial a_{id}} &= (\alpha - 1)\beta a_{-G}(n - 2)n\sigma_{\varepsilon id}(a_G - a_{-G}) (2(\alpha - 1) (a_G^2 + a_{-G}^2) + \beta(a_G - a_{-G})^2, \\
\frac{\partial \delta_{id}}{\partial \sigma_{\varepsilon id}^2} &= \frac{(\alpha - 1)\beta a_{-G}(n - 2)n\sigma_{\varepsilon id}(a_G - a_{-G}) (2(\alpha - 1) (a_G^2 + a_{-G}^2) + \beta(a_G - a_{-G})^2)}{(2(\alpha - 1) (a_G^2(n - 2)\sigma_{\varepsilon id} + a_{-G}^2 \sigma_{\varepsilon G}) + \beta(n - 2)\sigma_{\varepsilon id}(a_G - a_{-G})^2)^2} \neq 0,
\end{align*}
\]

as long as \( n > 2, \alpha \neq 1, \beta \neq 0, \sigma_{\varepsilon id} > 0, \sigma_{\varepsilon G} > 0 \) and \( a_G \neq a_{-G} \neq 0 \), which is true when \( \mu_G \neq \mu_{-G} \).

\[\text{C Proof of Proposition 2}\]

\[\text{Part 1.} \text{ Fix } N \text{ and consider the following utility functions:}\]

\[
\begin{align*}
u_1(q_1, t_1, s) &= (m_0 s_1 + \phi_N s_2) q_1 - \frac{m}{2} q_1^2, \quad \forall q_1, s \\
u_2(q_2, t_2, s) &= (m_0 s_2 + \phi_N s_1) q_2 - \frac{m}{2} q_2^2, \quad \forall q_2, s \\
u_i(q_i, t_i, s) &= (m_0 s_i + \phi_N (s_1 + s_2)) q_i - \frac{m}{2} q_i^2, \quad \forall i > 2, q_i, s
\end{align*}
\]

with \( S_i \) defined so that \( A1 \) holds with equality,

\[
\begin{align*}
S_1 &= S_2 = \left[-\frac{m}{m_0 + \phi_N}, \frac{m}{m_0 + \phi_N}\right], \\
S_i &= \left[-\frac{m}{m_0 + 2\phi_N}, \frac{m}{m_0 + 2\phi_N}\right], \quad \forall i > 2
\end{align*}
\]

In this environment derivative of the \( WE \) price with respect to \( WE \) quantity allocated
to player 1 satisfies

\[
\frac{dp^0(q_1^0(s_1), s_{-1})}{dq_1} = \frac{\frac{d}{ds_1}(s_1^0(q_1), s_{-1})}{\frac{d}{ds_1}(s_1^0(q_1), s_{-1})} = \frac{1}{N} \sum_j m u_{j,1}(q_1, s) - \frac{1}{N} \sum_j m u_{j,1}(q_1, s) = \frac{1}{m_q} \left( m_o + \phi(N) - \phi(N-1) \right)
\]

(36)

where the function \( s_0^1(x) \) is defined so that \( q_1^0(s_1^0(x), s_{-1}) = x \), for every \( x \).

Consider the signal profile \( s^* \),

\[
\begin{align*}
    s^*_1 &= \frac{m}{m_o + \phi(N)}, \\
    s^*_2 &= -\frac{m}{m_o + \phi(N)}, \\
    s^*_i &= 0, \quad \forall i > 2.
\end{align*}
\]

Fix \( \delta > 0 \) small and a mechanism that satisfies \( IR, IC, MC \) and is \( \delta \)-Efficient. From \( \delta \)-Efficiency it follows that there is a signal \( s_1(0) \) such that \( q_1(s_1(0), s_{-1}) = 0 \), and moreover \( q_1(s^*) \geq q_1^0(s^*) - \delta > 0 \). For the function \( s_1(x) \) defined as in (5) transfers by player 1 satisfy\(^{11}\)

\[
\begin{align*}
    t_1(s^*) &\leq \int_{0}^{q_1(s^*)} m u_1(x, s_1(x), s^*_{-1}) dx \\
    &\leq \int_{0}^{q_1(s^*)} m u_1(x, s_1^0(x + \delta), s^*_{-1}) dx \\
    &= \int_{0}^{q_1(s^*)} \left[ m u_1(x + \delta, s_1^0(x + \delta), s^*_{-1}) + \delta m_q \right] dx \\
    &= \int_{0}^{q_1(s^*)} \left[ p^0(s_1^0(x + \delta), s^*_{-1}) + \delta m_q \right] dx,
\end{align*}
\]

where the first inequality follows from local \( IC \) and \( IR \) at \( s_1(0) \), and the second one

\(^{11}\)Note that the allocation \( q_1(\cdot, s^*_{-1}) \) must be weakly increasing, from \( IC \). If \( q_1(\cdot, s^*_{-1}) = q^*_1 \) is constant over a range of signals \([s, \bar{s}]\), then \( s_1(\bar{s}_1) \) can be defined as any selection from \([s, \bar{s}]\).
follows from $\delta$-Efficiency. Consequently,

$$t_1(s^*) \leq \int_0^{q_1(s^*)} \left[ p^0(s_0^0(x + \delta), s_{-1}^*) + \delta m_q \right] \, dx$$

$$= \int_0^{q_1(s^*)} \left[ p^0(s_1^* + s_1^0(x + \delta) - s_1^*, s_{-1}^*) + \delta m_q \right] \, dx$$

$$= \int_0^{q_1(s^*)} \left[ p^0(s^*) + p^0_{q_1} \times (x + \delta - q_1^0(s^*)) + \delta m_q \right] \, dx$$

$$\leq q_1(s^*) \times \left[ p^0(s^*) + p^0_{q_1} \times \left( \frac{q_1^0(s^*) + \delta}{2} + \delta - q_1^0(s^*) \right) + \delta m_q \right].$$

Since $p^0(s^*) = 0$ and the first order condition determining $q_1^0(s^*)$ is

$$\frac{m}{m_o + \phi_N} (m_o - \phi_N) - q_1^0(s^*) \times m_q = 0,$$

it follows that as long as

$$\delta < \frac{p^0_{q_1}}{m_q + 1.5p^0_{q_1}} \times \frac{q_1^0(s^*)}{2} = \frac{p^0_{q_1}}{m_q + 1.5p^0_{q_1}} \times \frac{m}{2m_q} \times \frac{m_o - \phi_N}{m_o + \phi_N} =: \delta_N. \quad (37)$$

then

$$t_1(s^*) < q_1(s^*) \times p^0(s^*) = 0.$$

Analogous argument shows that as long as (37) holds then also $t_2(s^*) < q_2(s^*) \times p^0(s^*) = 0$. Since $p^0(s^*) = 0$ and $q_i^0(s^*) = 0$ for $i > 2$, the IR implies that $\sum_{i>2} t_i(s^*) \leq 0$. Altogether this establishes that there is no $\delta$-Efficient mechanism satisfying all the constraints for $\delta < \delta_N$ as in (37). The bound on $\delta$ satisfies (all the approximations are of order $o \left( \frac{1}{N} + \phi_N \right)$)

$$\delta_N = \frac{p^0_{q_1}}{m_q + 1.5p^0_{q_1}} \frac{m}{2m_q} \frac{m_o - \phi_N}{m_o + \phi_N} \approx \frac{p^0_{q_1}}{m_q} \frac{m}{2m_q} \approx \frac{m}{2m_q} \frac{m_o}{N - 1 + \phi_N}, \quad (38)$$

which establishes the first part of the proof.

**Part 2.** Fix a sequence of tax functions $\{\tau_N\}_{N \in \mathbb{N}}$ so that the respective VCG mechanism with transfers $\tau_N$ are $D \left( \frac{1}{N} + \phi_N \right)$-Efficient, for some $D > 0$. $D \left( \frac{1}{N} + \phi_N \right)$-Efficiency and convexity implies that taxes have derivatives uniformly bounded by $D \left( \frac{1}{N} + \phi_N \right)$. Therefore, from the Arzela-Ascoli Theorem, condition (19) is equivalent to pointwise convergence, for every $q \in (0, q_{\text{max}}]$.

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Suppose that the condition (19) is violated, and so there is \( q \in (0, q_{\text{max}}] \) and a sequence \( \{\varepsilon_N\}_{N \in \mathbb{N}} \) of positive numbers converging to zero such that

\[
\tau_N'(q) - \tau_N'(0) \leq \varepsilon_N \left( \frac{1}{N} + \phi_N \right) \times q. \tag{39}
\]

We will establish that for sufficiently large \( N \) the mechanisms violate \( BS \).

Consider utility functions as in (35), large \( N \) (to be determined later) and a profile of signals \( s^* = (s_1^*, s_2^*, 0, \ldots, 0) \) such that \( s_2^* = -s_1^* \) and

\[
u_1(q, s^*) = s_1^* m_q + \phi_N s_2^* - m_q q = \tau_N'(q),
\]

so that \( q^* = (q, -q, 0, \ldots, 0) \). Let \( a_N = \lim_{q \to 0} \tau_N(q) - \tau_N(0) \geq 0 \) be the entry fee of the tax and, with slight abuse of notation, let \( \tau_N'(0) = \lim_{q \to 0} \tau_N'(q) \) be the initial slope of the tax scheme. As in Proposition 1 the transfers satisfy

\[
t_1(s^*) \leq \int_0^q p^T(s_1(x), s_{-1}^*) dx + \tau_N(q),
\]

\[
t_2(s^*) \leq \int_0^{-q} p^T(s_2(x), s_{-2}^*) dx + \tau_N(q),
\]

\[
t_i(s^*) \leq 0, \quad \forall i > 2,
\]

where \( p^T(s) \) are defined, analogously to \( \sigma\)-WE prices, as the \( WE \) prices for the economy with distorted utilities \( \tilde{u}_i \), and \( s_1(x) \) and \( s_2(x) \) are defined analogously as in (5):

\[
u_i(q_i^T(s), s) = p^T(s) + \tau_N(q_i^T(s)), \quad \forall i, s
\]

\[
x = q_i^T(s_i(x), s_{-i}^*). \quad i = 1, 2, \forall x \in [-q, q]
\]

Below we establish that \( t_1(s^*) < 0 \); as the proof of \( t_2(s^*) > 0 \) is analogous, this will establish the result.

Let \( s_1' \) and \( s_1'' \) be the signals for player 1, \( s_1' \leq s_1'' \leq s_1^* \), such that

\[
\max_{q_3} \{ u_3(q_3, s_1', s_{-i}^*) - q_3 \times p^T(s_1', s_{-i}^*) - a_N \} = 0, \tag{40}
\]

\[
\max_{q_3} \{ u_3(q_3, s_1'', s_{-i}^*) - \int_0^{q_3} (p^T(s_1'', s_{-i}^*) + \tau_N'(x)) dx - a_N \} = 0.
\]

In other words, if the tax scheme consisted of only the entry fee \( a_N \) then agents \( i > 2 \) would start trading (buying from agent 2) if 1’s signal drops below \( s_1' \). Similarly, agents
\[ i > 2 \text{ start trading given the original tax scheme } \tau_N \text{ when } 1 \text{'s signal drops below } s''_1. \]

Let us compute prices \( p^N(s_1(x), s^*-1) \) for the inframarginal units, \( x, x \in [0, q] \). For \( s_1 \geq s''_1 \) only players 1 and 2 trade nonzero quantities, and we have

\[
m_{u_1}(q^N_1(s_1, s^*-1), s_1, s^*_1) = s_1 m_o + \phi_N s^*_2 - m_q q^N_1(s_1, s^*_1) = p^N(s_1, s^*_1) + \tau'_N(q^N_1(s_1, s^*_1)),
\]

\[
m_{u_2}(q^N_2(s_1, s^*_1), s_1, s^*_1) = s^*_2 m_o + \phi_N s_1 + m_q q^N_2(s_1, s^*_1) = p^N(s_1, s^*_1) + \tau'_N(q^N_2(s_1, s^*_1)),
\]

and so

\[
\frac{\partial q^N_1(s_1, s^*_1)}{\partial s_1} = \frac{m_o - \frac{\partial p^N(s_1, s^*_1)}{\partial s_1}}{m_q + \tau''_N(q^N_1(s_1, s^*_1))},
\]

\[
\frac{\partial q^N_2(s_1, s^*_1)}{\partial s_1} = \frac{\phi_N}{m_q + \tau''_N(q^N_2(s_1, s^*_1))}.
\]

Using \( MC \), we thus have

\[
\frac{\partial p^N(s_1, s^*_1)}{\partial s_1} = \frac{m_o + \phi_N}{2(m_q + \tau''_N(x))} = \frac{m_o + \phi_N}{2},
\]

and so, for any \( x \) such that \( s_1(x) \geq s''_1 \)

\[
\frac{\partial p^N(s_1(x), s^*_1)}{\partial x} = -\frac{\partial p^N(s_1, s^*_1)}{\partial s_1} \frac{\partial q^N_1(s_1, s^*_1)}{\partial s_1} = \frac{m_o + \phi_N}{2} = \frac{(m_q + \tau''_N(x)) m_o + \phi_N}{m_q - \phi_N}.
\]  \( \text{(41)} \)

Similarly, for \( s_1 < s''_1 \) and \( x \geq 0 \) such that \( s_1(x) < s''_1 \), when all agents trade nonzero quantities, we have

\[
\frac{\partial p^N(s_1, s^*_1)}{\partial s_1} = w_1(s_1) m_o + (1 - w_1(s_1)) \phi_N,
\]  \( \text{(42)} \)

\[
\frac{\partial q^N_1(s_1, s^*_1)}{\partial s_1} = \frac{m_q - \frac{\partial p^N(s_1, s^*_1)}{\partial s_1}}{m_q + \tau''_N(q^N_1(s_1, s^*_1))},
\]

\[
\frac{\partial p^N(s_1(x), s^*_1)}{\partial x} = -\frac{1}{w_1(s_1(x))} \frac{m_q + \tau''_N(x)}{(m_q - \phi_N)} (w_1(s_1(x)) m_o + (1 - w_1(s_1(x))) \phi_N),
\]

where

\[
w_1(s_1) = \frac{1}{m_q + \tau''_N(q^N_1(s_1, s^*_1)) + \frac{1}{m_q + \tau''_N(q^N_2(s_1, s^*_1)) + \frac{N-2}{m_q + \tau''_N(q^N_3(s_1, s^*_1))}}.\]
Since \( \tau'_N(0) \geq 0 \), \( \tau'_N \) is increasing and bounded by \( D \left( \frac{1}{N} + \phi_N \right) \), it follows that for \( x \in [0, q] \) \( \tau''_N(x) \) is bounded above by 1 on an interval of measure \( q - D \left( \frac{1}{N} + \phi_N \right) \). It follows from the definition of \( w_1(s_1) \) and (42) that there are \( c, c' > 0 \) such that \( w_1(s_1(x)) \) is bounded from below by \( \frac{c}{N} \), and so

\[
\frac{\partial p^{\tau N}(s_1(x), s^*_i)}{\partial x} \geq c \times \left( \frac{1}{N} + \phi_N \right), \tag{43}
\]

for a subset of \( x \in [0, q] \) of measure \( q - D \left( \frac{1}{N} + \phi_N \right) \), for every \( N \).

Moreover, it follows from \( D \left( \frac{1}{N} + \phi_N \right) \)-Efficiency of the allocation for agents \( i > 2 \) that

\[
\tau'_N(0), p^{\tau N}(s''_1, s^*_i), p^{\tau N}(s'_1, s^*_i) = O \left( \frac{1}{N} + \phi_N \right),
\]

and so, from (41),

\[
q^{\tau N}_1(s'_1, s^*_i), q^{\tau N}_1(s''_1, s^*_i) = q - O \left( \frac{1}{N} + \phi_N \right).
\]

Finally, since \( \tau'_N \geq 0 \) and \( u_3(q_3, s''_1, s^*_i) \leq u_3(q_3, s'_1, s^*_i) \) for every \( q_3 \), equations in (40) imply that

\[
p^{\tau N}(s'_1, s^*_i) - p^{\tau N}(s''_1, s^*_i) \geq \tau'_N(0).
\]

The integral of the inframarginal prices agent 1 pays is

\[
\int_0^q p^{\tau N}(s_1(x), s^*_i) dx = I + II + III,
\]

\[
I = q^{\tau N}_1(s'_1, s^*_i) \times p^{\tau N}(s'_1, s^*_i) + \int_{q^{\tau N}_1(s'_1, s^*_i)}^q p^{\tau N}(s_1(x), s^*_i) dx,
\]

\[
II = q^{\tau N}_1(s''_1, s^*_i) \times \left( p^{\tau N}(s''_1, s^*_i) - p^{\tau N}(s'_1, s^*_i) \right) + \int_{q^{\tau N}_1(s''_1, s^*_i)}^{q^{\tau N}_1(s'_1, s^*_i)} p^{\tau N}(s_1(x), s^*_i) - p^{\tau N}(s'_1, s^*_i) dx,
\]

\[
III = \int_0^{q^{\tau N}_1(s'_1, s^*_i)} \left( p^{\tau N}(s_1(x), s^*_i) - p^{\tau N}(s''_1, s^*_i) \right) dx.
\]

Below we bound from above each of the three terms \( I, II \) and \( III \).

First, since

\[
a_N = u_3(q'_3, s'_1, s^*_i) - q'_3 \times p^{\tau N}(s'_1, s^*_i) = \frac{p^{\tau N}(s'_1, s^*_i)^2}{2m_i},
\]

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where $q'_3$ is the maximizer, we have, for $N$ sufficiently high

$$ I < q_1^N(s'_1, s^*_1) \times p^N(s'_1, s^*_1) = \left(q - O\left(\frac{1}{N} + \phi_N\right)\right) \times p^N(s'_1, s^*_1) < a_N. $$

Second,

$$ II = q_1^N(s'_1, s^*_1) \times (p^N(s''_1, s^*_1) - p^N(s'_1, s^*_1)) $$

$$ + \int_{q_1^N(s'_1, s^*_1)} p^N(s_1(x), s^*_1) - p^N(s'_1, s^*_1))dx $$

$$ = q_1^N(s'_1, s^*_1) \times (p^N(s''_1, s^*_1) - p^N(s'_1, s^*_1)) + O\left(\frac{1}{N} + \phi_N\right)^2 $$

$$ < q_1^N(s'_1, s^*_1) \times (-\tau'_N(0)) + O\left(\frac{1}{N} + \phi_N\right)^2 = -q \times \tau'_N(0) + O\left(\frac{1}{N} + \phi_N\right)^2. $$

Finally, the bound (43) yields, for $c > 0$ independent of $N$,

$$ III = \int_{q_1^N(s''_1, s^*_1)} (p^N(s_1(x), s^*_1) - p^N(s''_1, s^*_1))dx $$

$$ < -c \times \left(q_1^N(s''_1, s^*_1) - O\left(\frac{1}{N} + \phi_N\right)\right) \times \left(\frac{1}{N} + \phi_N\right) $$

$$ = -c \times q \times \left(\frac{1}{N} + \phi_N\right) + O\left(\frac{1}{N} + \phi_N\right)^2. $$

The three bounds imply

$$ t_1(s^*) = I + II + III + \tau_N(q) = I + a_N + II + \tau'_N(0) \times q + III + \int_0^q (\tau'_N(x) - \tau'_N(0)) dx $$

$$ < -c \times q \times \left(\frac{1}{N} + \phi_N\right) + \int_0^q (\tau'_N(x) - \tau'_N(0)) dx + O\left(\frac{1}{N} + \phi_N\right)^2 $$

$$ < -c \times q \times \left(\frac{1}{N} + \phi_N\right) + \varepsilon_N \left(\frac{1}{N} + \phi_N\right) \times q^2 + O\left(\frac{1}{N} + \phi_N\right)^2 $$

$$ = q \times \left(\frac{1}{N} + \phi_N\right) \times (\varepsilon_Nq - c) + O\left(\frac{1}{N} + \phi_N\right)^2, $$

where the second inequality follows from assumption (39). It follows that for $N$ sufficiently high the transfers $t_1(s^*)$ are negative, which establishes the proof.
D Proofs for Section 6

Proof of Proposition 3. Fix \((u_1, ..., u_N)\) and \(\sigma \geq 0\). We first establish that the mapping \(\varphi : \mathbb{R}^N \to \mathbb{R} \times \mathbb{B}\), where \(\mathbb{B} \subset \mathbb{R}^N\) has coefficients add up to 0,

\[
\varphi(s) = (p^\sigma(s), q^\sigma(s)),
\]

is a diffeomorphism. For any \(q \in \mathbb{R}^N\) the function \(\zeta_q : \mathbb{R}^N \to \mathbb{R}^N\),

\[
\zeta_q(s) = (mu_1(q_1, s), ..., mu_N(q_N, s)),
\]

is a diffeomorphism. This follows from local invertibility assumption \(INJ\) together with the assumptions \(A2\) and \(A4\) (Hadamard, see e.g., Theorem 6.2.4 in Krantz and Parks (2012)). Moreover, the mapping \(\xi : \mathbb{R}^N \to \mathbb{R} \times \mathbb{B}\), \(\xi(mu_1, ..., mu_N) = (p, q)\) such that

\[
mu_i = p + \sigma q_i, \quad \forall i
\]

is a diffeomorphism as well. This is because \(\xi\) satisfies

\[
\xi(mu_1, ..., mu_N) = \left[ \frac{1}{N} \cdot e^T \right] \cdot (mu_1, ..., mu_N)^T,
\]

for \(N \times N\) identity matrix \(I\) and \(N \times 1\) unit vector \(e\), and so is linear with rank \(N\).

It follows that if \(\varphi(s) = \varphi(s') = (p^\sigma(s), q^\sigma(s))\), then

\[
s = \zeta_{p^\sigma(s)}^{-1} \circ \xi^{-1}(p^\sigma(s), q^\sigma(s)) = s'..
\]

On the other hand, for any \((p, q) \in \mathbb{R} \times \mathbb{B}\) we have \((p, q) = \varphi(s)\) for

\[
s = \zeta_q^{-1} \circ \xi^{-1}(p, q).
\]

We may verify now that that at every \(s\) the profile of \(\sigma - WE\) conditional inverse demands yields \((p^\sigma(s), q^\sigma(s))\) as the unique market clearing price-allocation pair. An alternative representation of a conditional inverse demand correspondence \(d_i\) is via the set \(D(d_i)\) of market clearing price-allocation pairs consistent with it (see Perry and Reny (2002)),

\[
D_i(d_i) = \bigcup_{q_{-i}} \left(d_i(q_{-i}), -\sum_{j \neq i} q_j, q_{-i}\right).
\]
Market clearing price-allocation pairs are then represented by the intersection \( \bigcap_i D_i(d_i) \).
In the case of the \( \sigma - \text{WE} \) conditional inverse demand of \( i \) at \( s_i \) we have
\[
D_i(d_i(s_i)) = \bigcup_{\varphi} \bigcup_{s_{-i}} (p^{\varphi}(s_i, s_{-i}), q^{\varphi}(s_i, s_{-i})) = \bigcup_{s_{-i}} (p^{\varphi}(s_i, s_{-i}), q^{\varphi}(s_i, s_{-i})).
\]
The injectivity of \( \varphi \) implies then that
\[
\bigcap_i D_i(d_i(s_i)) = (p^{\varphi}(s), q^{\varphi}(s)),
\]
and so indeed \( (p^{\varphi}(s), q^{\varphi}(s)) \) is the unique market clearing price-allocation pair.

Fix \( s \) and a player \( i \). If all the players \( j \neq i \) play the equilibrium strategies, the graph \( G \) of any residual demand curve for a player \( i \) satisfies
\[
G \subseteq \text{Proj}_{P \times Q} \bigcap_{j \neq i} D_j(d_j(s_j)) = \text{Proj}_{P \times Q} \bigcup_{s'_i} (p^{s'_i}(s_i, s_{-i}), q^{s'_i}(s_i, s_{-i})).
\]
In other words the unique residual demand curve \( p_i \) that \( i \) is facing is given by
\[
p_i(q_i) = p^{s'_i}(s_i, s_{-i}), \text{ for } s'_i \text{ such that } q_i = q^{s'_i}(s_i, s_{-i}).
\]

Given the definition of transfers, at \( s \) agent \( i \) is choosing between either 1) a zero quantity and transfer pair (when he submits an incompatible conditional demand function) or 2) pairs \( (q^{s'_i}(s_i, s_{-i}), t^{s'_i}(s'_i, s_{-i})) \), for some \( s'_i \), when he submits the \( \sigma - \text{WE} \) conditional inverse demand at \( s'_i \). The result thus follows from IC and IR of the \( \sigma - \text{WE} \) mechanism.

**Proof of Lemma 2** Fix \( s_i \). Given assumption \( \text{INJ} \), the function \( f_i : S_{-i} \to \mathbb{R}^N \), \( f_i(s_{-i}) = (p^{s_i}(s_i, s_{-i}), q^{s_i}(s_i, s_{-i}),) \) is injective. We have
\[
d_i(s_i)(q_{-i}) = \{ x | x = p^{s_i}(s_i, s_{-i}) \text{ and } q^{s_i}(s_i, s_{-i}) = q_{-i} \}
= \{ x | x = m\hat{u}_i(-\sum_{j \neq i} q_j, s_i, s_{-i}) - \sigma \left| \sum_{j \neq i} \right. q_j \text{ and } q^{s_i}(s_i, s_{-i}) = q_{-i} \}
= \{ x | x = m\hat{w}_i(-\sum_{j \neq i} q_j, s_i, w_i(s_{-i})) - \sigma \left| \sum_{j \neq i} q_j \right. \text{ and } q^{s_i}(s_i, s_{-i}) = q_{-i} \}
= \{ x | x = m\hat{w}_i(-\sum_{j \neq i} q_j, s_i, w_i(f^{-1}_i(q^{s_i}_i, x))) - \sigma \left| \sum_{j \neq i} q_j \right. \text{ and } q^{s_i}_i = q_{-i} \},
\]
and so the result follows for \( \hat{w}_i = w_i \circ f^{-1}_i \).

If \( \sigma - \text{WE} \) inverse demands are functions then there are no \( s_{-i} \) and \( s'_{-i} \) with \( q^{s_i}(s_i, s_{-i}) = q^{s'_i}(s_i, s_{-i}) \).
Equivalently, since the system of 

equations is injective. Following the steps as above establishes the proof, with 

\[ \tilde{w}_i = w_i \circ g_i^{-1}. \]

**Proof of Lemma 3.** Fix \( \sigma \geq 0 \) and let 

\[
J(s) = \left[ mu_{i,j}(q_\sigma^i(s), s) \right]_{i,j \leq N}.
\]

be the \( N \times N \) Jacobian matrix of the function \( \zeta(s) = (mu_1(q_1, s), ..., mu_N(q_N, s)) \) for 

\( q = q_\sigma(s) \). Totally differentiating equations (3) that define the \( \sigma - WE \) allocation we get the system of \( N \) equations 

\[
J(s)ds = \left( \sigma I - \left[ mu_{i,q}(q_\sigma^i(s), s) \right]_{i \leq N} \cdot I \right) dq + e \cdot dp
\]

or equivalently, since \( J(s) \) is invertible by \( INJ \),

\[
ds = J(s)^{-1} \left( \sigma I - \left[ mu_{i,q}(q_\sigma^i(s), s) \right]_{i \leq N} \cdot I \right) dq + J(s)^{-1}e \cdot dp \quad (44)
\]

where \( I \) is the \( N \times N \) identity matrix and \( e \) is the unit vector.

Consider two profiles of signals \( (s_k, s_{-k}) \) and \( (s'_k, s'_{-k}) \) with \( s_{-k} \neq s'_{-k} \) such that 

\( q_\sigma^i(s_k, s_{-k}) = q_\sigma^i(s'_k, s'_{-k}) \) but \( p_\sigma^i(s_k, s_{-k}) \neq p_\sigma^i(s'_k, s'_{-k}) \); we need to establish that \( s_k \neq s'_k \). Let \( \gamma(t), t \in [0, 1] \) be the straight line in \( \mathbb{R}^{N+1} \) such that \( \gamma(0) = (q_\sigma^i(s_k, s_{-k}), p_\sigma^i(s_k, s_{-k})) \) and \( \gamma(1) = (q_\sigma^i(s'_k, s'_{-k}), p_\sigma^i(s'_k, s'_{-k})) \). Since \( \varphi(s) = (p_\sigma^i(s), q_\sigma^i(s)) \) is a diffeomorphism onto \( \mathbb{R} \times \mathcal{B} \), where \( \mathcal{B} \subset \mathbb{R}^N \) has coefficients add up to 0 (see proof of Proposition 3), \( \varphi^{-1}(\gamma(t)) \) is also a path in \( \mathbb{R}^N \) from \( (s_k, s_{-k}) \) to \( (s'_k, s'_{-k}) \). However, it follows from (44) that along this path

\[
ds = J(s)^{-1}e \cdot dp,
\]

and so in particular, from \( FUN \) and the continuity of \( J(s)^{-1}e \) in \( s, ds_k \neq 0 \) and does not change the sign along the whole path. It follows that \( s_k = \varphi^{-1}(\gamma(0))_k \neq \varphi^{-1}(\gamma(1))_k = s'_k \).

Suppose now that utilities are linear and \( (J(s)^{-1}e)_k = 0 \) for some agent \( k \). It follows that for a vector \( v = J(s)^{-1}e \) we have

\[
mu(0, v) = mu(0, 0) + J(0) \cdot v = mu(0, 0) + e,
\]
and so for any \( \sigma \geq 0 \), from the definition of \( \sigma - WE \) mechanism,

\[
q^\sigma(v) = q^\sigma(0),
\]
\[
p^\sigma(v) = p^\sigma(0) + 1,
\]

implying that the \( \sigma - WE \) conditional inverse demand of agent \( k \) at \( s_k = 0 \) is a correspondence.

**Proof of Lemma 4** i) Let matrices \( A \) and \( a \) be the diagonal and off-diagonal parts of matrix \( \left[ mu_{i,j}(0,0) \right]_{i,j \leq N} \), and let \( b = \left[ mu_{i,j}(0,0) \right]_{i,j \leq N}^{-1} - A^{-1} \). We have

\[
(A^{-1} + b)(A + a) = I
\]
\[
A^{-1}a + bA + ba = 0
\]
\[
aA^{-1} + Ab(1 + aA^{-1}) = 0
\]
\[
Ab = -aA^{-1}(1 + aA^{-1})^{-1}
\]

Since the infinity norm is both sub-multiplicative and sub-additive:

\[
||Ab||_\infty \leq \frac{||aA^{-1}||_\infty}{1 - ||aA^{-1}||_\infty}.
\]

Finally, by assumption, each element of the \( aA^{-1} \) matrix is less then \( 1/2n \), therefore \( ||aA^{-1}||_\infty < 1/2 \). By the inequality above \( ||Ab||_\infty < 1 \) and therefore matrix \( W^{-1} = A^{-1}(I + Ab) \) has positive row sums, since \( A^{-1} \) has only positive diagonal elements.

ii) Equicommonality means that \( \left[ mu_{i,j}(0,0) \right]_{i,j \leq N} \cdot e = [\alpha]_{N \times 1} \), for unit vector \( e \) and some \( \alpha \neq 0 \). Left-multiplying by \( \left[ mu_{i,j}(0,0) \right]_{i,j \leq N}^{-1} \) yields the result.

**E Proof of Proposition 4**

One direction follows immediately from (30) in the proof of Proposition 1 as well as the fact that \( A3' \) implies \( p^0(s) = p^\sigma(s) \), for every type profile \( s \) and \( \sigma \geq 0 \).

To establish tightness, fix \( N \) and consider the same utility functions and signal profile \( s \) as in the proof of Proposition 2. Fix \( \delta > 0 \) small and a mechanism that satisfies \( IR, IC, MC \) and is \( \delta \)-Efficient. For the function \( s_1(x) \) defined as in (5) and \( s_1^0(x) \) defined
the same way but for the efficient allocation \( q^0(s) \) the transfers by player 1 satisfy

\[
t_1(s) \leq \int_0^{q_1(s)} mu_1(x, s_1(x), s_{-1}) dx \leq \int_0^{q_1(s)} mu_1(x, s_1^0(x(1 + \delta) + \delta), s_{-1}) dx
\]

\[
= \int_0^{q_1(s)} \left[ mu_1(x(1 + \delta), s_1^0(x + \delta), s_{-1}) + x\delta m_q \right] dx
\]

\[
= \int_0^{q_1(s)} \left[ p^0(s_1(x + \delta), s_{-1}) + x\delta m_q \right] dx
\]

\[
\leq \int_0^{q_1(s)} \left[ p^0(s_1 + s_1^0(x(1 + \delta)) - s_1, s_{-1}) + x\delta m_q \right] dx
\]

\[
= \int_0^{q_1(s)} \left[ p^0(s) + p^0_{q_1} \times (x(1 + \delta) - q_1^0(s)) + x\delta m_q \right]
\]

\[
= q_1(s) \times \left[ p^0(s) + p^0_{q_1} \times \left( \frac{q_1^0(s)(1 + \delta)^2}{2} - q_1^0(s) \right) + \frac{q_1^0(s)(1 + \delta)\delta m_q}{2} \right]
\]

\[
= q_1(s)p^0(s) + q_1(s)\frac{q_1^0(s)}{2} \left[ (1 + \delta)\delta m_q - p^0_{q_1} (1 - 2\delta - \delta^2) \right],
\]

where \( p^0_{q_1} = \frac{m_q}{m_{N-1} - \phi_N} \left( \frac{m_q}{N-1} + \frac{\phi_N}{m_{N-1}} \right) \) is defined as in (36).

This implies that if \( \delta < \left( \frac{1}{N-1} + \frac{\phi_N}{m_{N-1}} \right) \) then for \( N \) sufficiently large we have \( t_1(s) < 0 \), and, by an analogous argument, \( t_2(s) < 0 \). \( p^0(s) = 0 \) and \( q_i^0(s) = 0 \) for \( i > 2 \) together with \( IR \) imply that \( \sum_{i>2} t_i(s) \leq 0 \), and so establish that the mechanism violates BS.

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