REGULATING COLLATERAL-REQUIREMENTS
WHEN MARKETS ARE INCOMPLETE∗

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Abstract

In this paper we examine the effects of default and collateral on risk sharing. We assume that there is a large set of assets which all promise a risk less payoff but which distinguish themselves by their collateral requirements. In equilibrium agents default, the assets have different payoffs, and there are as many linearly independent assets available for trade as there are states of the world. We derive necessary and sufficient conditions for equilibria to be Pareto-efficient in the presence of uncertainty. We explore some examples for which the collateral equilibrium allocation is identical to the Arrow-Debreu allocation, either when agents have a high preference for the durable good, or when the endowment distribution of the durable good is relatively homogeneous. We examine a series of examples to understand which collateral-levels prevail in equilibrium and under which conditions there is scope for regulating margin-requirements, that is, restricting the sets of tradable assets through government intervention. In these examples equilibrium is always sub-optimal but regulation never leads to a Pareto-improvement. While the competitive equilibria are constrained efficient, there do exist regulations which make large groups of agents in the economy better off. These regulations typically restrict all trades to

∗We are honored to have the opportunity to contribute to this volume dedicated to the memory of David Cass. He has always inspired us in our scientific work, and particularly in the broad subject of this paper, general equilibrium with incomplete markets and finite states of nature, to which David dedicated much of his research in his last years. We thank the seminar participants at IMPA, EPGE/FGV, the Latin American and Caribbean Economic Association 2008 at Rio de Janeiro, the Latin American meeting of the Econometric Society 2009 at Buenos Aires, the 5th Annual Caress-Cowles 2009 at Yale, the University of Chicago, Federal Reserve Bank of Minneapolis, Columbia University and Princeton University, and in particular Yves Balasko, John Geanakoplos, Patrick J. Kehoe, Michael Magill and Martine Quinzii for helpful comments and suggestions. We thank two anonymous referees for helpful comments that led to substantial improvements in this work.

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take place in the low-collateral loans and benefit the poor and the rich agents in the economy through their effects on the equilibrium interest rate and the equilibrium prices of the durable goods.

**Keywords:** General equilibrium, Incomplete markets, Collateral, Default, Risk sharing, Government intervention, Pareto efficiency, Welfare, Regulation, Identical homothetic utility

**JEL classification:** D52, D53, D58, G18

1. Introduction

The vast majority of debt, especially if it extends over a long period of time, is guaranteed by tangible assets called collateral. For example, residential homes serve as collateral for short-term and long-term loans to households, equipment and plants are often used as collateral for corporate bonds, and investors can borrow money to establish a position in stocks, using these as collateral. Dubey, Geanakoplos and Shubik [6] and Geanakoplos and Zame [9] incorporate default and collateral into the standard two-period general equilibrium model with incomplete markets. In a two period model, default can be prevented by collateral or by utility-penalties that can be thought of as a reduced form representation of reputation effects present in more realistic dynamic models. While it is true that in developed economies it is also possible to take substantial positions in unsecured debt and that default on collateralized obligations has consequences beyond the loss of the collateral, to simplify our analysis we follow Geanakoplos and Zame [9] and abstract from these considerations by assuming that collateral constitutes the only enforcement mechanism.

In our model, there are two periods, with uncertainty over the states of the world in the second period. There are two commodities, one perishable and the second durable. The durable good serves as collateral. An asset in this model is characterized by its state-contingent promises in the second period and by its collateral- (or margin-) requirement which specifies how much of the durable good needs to be used (i.e. held by the borrower) to back a short position in this asset. The actual payoff of the asset will be the minimum of its promise and the value of the associated collateral. The margin (or collateral) requirement that dictates how much collateral one has to hold in order to borrow one dollar therefore also determines the payoff of an asset via the possibility of default. We assume that the only assets available for trade are promising one unit of the perishable good across all states. While it appears somewhat arbitrary to limit the set of available securities to assets that promise a safe payout, this assumption can be motivated by the observation that few individuals hold a short position in assets other than debt.

While in principle agents could write contracts with any collateral requirement, we characterize a minimal set of assets that suffices to ‘span’ all other possible assets so that in equilibrium it is without loss of generality to focus on this set of contracts. In the standard model without collateral it is a trivial observation that $S$ assets suffice to span the payoff of any possible security. In our model with collateral, this is not obvious since one not only has to replicate
the payoffs of any available security but also its collateral requirement. It turns out that the minimal set still consists of $S$ assets but it is crucial that the margin requirements to these assets are chosen to ensure that for each asset there is one state where the spot-price of the underlying collateral is exactly equal to the promise of the asset.

If everybody owns ‘enough’ of the collateralizable durable good, all assets in this minimal set are traded. Nevertheless, we show that even then equilibrium allocations are often not Pareto-optimal. In fact, it turns out to be difficult to find sufficient conditions for the competitive equilibrium to be Pareto-optimal. We derive necessary conditions which are fairly restrictive and illustrate that the fact that $S$ linearly independent assets are available for trade does not ensure Pareto-optimality even when there is a lot of collateral in the economy.

On the other hand, with the scarce collateralizable durable good, only very few assets in this set are traded and markets are endogenously incomplete. As Geanakoplos and Zame [9] and Geanakoplos [7] point out, ‘scarce collateral’ rations the volume of trade since there will always be a gap between utility of buying and disutility of selling an asset. The rationing does not reduce volume of trade proportionally but chokes off all trade in many contracts. We show, through numerical examples, that with several states and several agents there are generally still more than one contract being actively traded in equilibrium. However, usually not all contracts are traded and many agents trade only in one of the available assets.

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It is easy to see that with scarce collateral the resulting equilibrium allocation is not Pareto optimal. This simply follows from the fact that not every agent trades every security, but would also be true in a model with certainty, if collateral requirements are binding. This in itself is not very surprising. Given the simple ‘contracting’ technology that does not allow for agents to deliver on promises without collateral, it is clear that with too little collateralizable goods in the economy, there cannot be complex trade in securities and full risk sharing is impossible.

A more interesting question is whether the allocation is constrained sub-optimal in the sense that a government intervention can lead to a Pareto-improvement. There are several interventions one can think of (see e.g. Geanakoplos and Polemarchakis [8] or Cass and Citanna [4]). The most interesting one in this model is clearly to consider the effects of a regulation of margin-requirements. In other words, whether it can be Pareto-improving to force agents to trade only a subset of the available assets. Geanakoplos and Zame [9] are the first to address this question. They show that without price effects, competitive equilibrium allocations are constrained efficient and a regulation can never be Pareto-improving. We prove that in particular this implies that identical homothetic utility is a sufficient condition for constraint optimality. However, even under this condition it is possible that groups of agents benefit from a regulation. Moreover, without identical homothetic utility, the result does not apply.

It is a quantitative question who in the economy gains and who loses through a regulation of collateral-requirements. We provide a series of examples, some of them illustrative and some calibrated in order to address this quantitative question.

Our examples illustrate that regulation of margin requirements generally does not lead to
Pareto-improvements. However, often a large group of agents would favor a regulation since it is welfare improving for them.

In our model, we can interpret the asset with the lowest collateral-requirement as a ‘subprime loan’. What is typically understood as subprime loans are mortgage loans for borrowers who do not qualify for prime loans, due to weak credit history, low incomes or missing collateral. To compensate for these credit risks subprime loans carry higher interest rates compared to prime loans. In our collateral model, assets with low margin-requirements do carry higher interest rates, and are bought by agents who lack collateralizable durable goods in the present. It is clear that understanding the effect of subprime loans on the economy is of fundamental importance for today’s highly sophisticated financial markets. In this paper we ask whether the existence of subprime loans can lead to a Pareto-improvement and who in the economy gains and who loses if subprime loans are regulated.

In our examples, it is never optimal to regulate the market for subprime loans. In some cases, the rich and the poor agents gain if only subprime loans can be traded and markets for prime loans are shut down. However, the middle-class loses from such a regulation and it is therefore not Pareto-improving.

The examples also elucidate that only subprime loans are optimal (in the sense that no other assets are traded) when the borrower owns almost no collateralizable goods. In one example, where all borrowers own substantial amounts of the durable good agents endogenously avoid subprime loans, even when there is little certainty about future income.

In our model, default does not have any negative externalities and all agents are completely rational and agree on the probabilities of default. Our results can therefore be interpreted as benchmarks in which subprime loans play an important role for the economy.

Related literature

In this paper, we study a specialization of the general equilibrium model with incomplete markets and collateral (GEIC), introduced in Geanakoplos and Zame [9]: We do consider a particular asset structure, but all collateral levels are assumed to be available. While we will be primarily interested in the effects of uncertainty, we should note that, in the absence of uncertainty, Geanakoplos and Zame [9] derive necessary and sufficient conditions for the Pareto-optimality of competitive equilibria. The theoretical contributions of our paper that go beyond Geanakoplos and Zame [9] are as follows. We characterize a minimal set of assets that can be traded in equilibrium. This allows us to state necessary and sufficient conditions for the equilibrium to be Pareto-optimal in the presence of uncertainty. These conditions clarify that ‘plentiful collateral’ is only a necessary condition for optimality and that in our model equilibrium might be inefficient no matter how much collateral there is available to back trade in bonds. We also clarify the notion of constrained sub-optimality in this model. We characterize an important class of economies for which equilibrium is constrained optimal and argue that outside of this class equilibrium will not be generically sub-optimal.
The main contribution of our paper consists of the quantitative results about who gains and who loses through regulation. We illustrate a surprising, yet potentially important channel through which some agents might favor a regulation of financial markets.

Our paper is also closely related to Kilenthong [10] and Kilenthong and Townsend [11]. However these papers consider a model with capital and they do not assume that all assets promise a safe payoff. Kilenthong and Townsend [11] show that in their model equilibrium is generally constrained sub-optimal but they consider a slightly different notion of constrained sub-optimality than we do in this paper. Therefore the results cannot really be compared.

In our model, collateral levels are exogenously given, but since all possible collateral levels are in principal available for trade, one can think of the market picking out the collateral levels. Araújo, Orrillo and Páscoa [2] and Araújo, Fajardo and Páscoa [1] develop a model where collateral levels are determined endogenously and set by the lender. It is subject to further research to compare the welfare consequences and assets traded in our model to their analysis.

Clearly our focus on a two period model has important implications for our welfare analysis. In a dynamic model (see e.g. Araújo, Páscoa and Torres [3] or Kubler and Schmedders [12]), where agents can accumulate the durable good over time, the distribution of collateral is endogenous. It is an important open question to evaluate the welfare consequences in such a model.

Outline of the paper

The paper is organized as follows. In Section 2 we present the basic equilibrium model with collateral (GEIC). In Section 3, we present our theoretical results. In Section 4 we provide examples that illustrate our main points.

2. The model

We consider a pure exchange economy over two time periods $t = 0, 1$ with uncertainty over the state of nature in period 1 denoted by the subscript $s \in S = \{1, \ldots, S\}$. For convenience, the first period will sometimes be called state 0 so that in total there are $S^* = S + 1$ states.

The economy consists of a finite number of $H$ agents denoted by the superscript $h \in H = \{1, \ldots, H\}$ and $L = 2$ goods or commodities, denoted by the subscript $l \in L = \{1, 2\}$. Throughout the analysis we assume that good 1 is perishable and good 2 is durable, i.e. there is a storage technology for commodity 2. Consuming one unit of the durable good in the first period yields 1 units of the good in each state $s = 1, \ldots, S$. Each agent has an initial endowment of the two goods in each state, $e^h \in \mathbb{R}^{2S^*}_+$. The preference ordering of agent $h$ is represented by a utility function $u^h : \mathbb{R}^{2S^*}_+ \to \mathbb{R}$, defined over consumption $x^h = (x^h_0, x^h_1, \ldots, x^h_S) \in \mathbb{R}^{2S^*}_+$. The characteristics of agent $h$ are summarized by a utility function and endowment vector $(u^h, e^h)$ satisfying standard assumptions.
In each state \( s = 0, \ldots, S \) there are complete spot markets - the spot prices of the commodities across states are denoted by \( p(s) \in \mathbb{R}_{++}^2, s = 0, 1, \ldots, S \). Throughout, we normalize the price of the non-durable commodity to be equal to one in all states, i.e. \( p(s) = (1, p_2(s)) \). There are \( J \) real assets denoted by the subscript \( j \in \mathcal{J} = \{1, \ldots, J\} \), we assume each asset promises payments of one unit of commodity 1 in each state \( s = 1, \ldots, S \), i.e. each asset is a bond with default-risk.

We associate with each asset \( j \in \mathcal{J} \) a collateral requirement \( C_j \geq 0 \). We assume that the collateral always has to be held by the borrower, in order to simplify our analysis. Agents have to hold \( C_j \) units of good 2 in order to sell one unit of asset \( j \). We assume that collateral being held to secure some asset \( j \) cannot be used for any other asset and long position of other assets cannot be used to secure a short position. This will turn out to be a strong assumption with important welfare implications.

Agents default on their promises whenever the market value of the durable good they hold as collateral is lower than the face value of their promise. Given equilibrium prices \( p \), the actual payoff of asset \( j \) in states \( s \) is therefore \( \min(1, p_2(s)C_j) \). Let \( q \in \mathbb{R}_+^J \) denote the prices of assets in period zero. It is useful for our analysis to define the collateral requirement in terms of units. It is often economically more intuitive to consider the margin requirement on asset \( j \) which we define as \( \mu_j = \frac{C_j p_2(0) - q_j}{q_j} \). Throughout the paper, we report collateral-requirements, not margin-requirements.

Finally, let \( \theta^h = (\theta^h_1, \ldots, \theta^h_J) \in \mathbb{R}_+^J \) denote the number of units of each of the \( J \) assets bought by agent \( h \), and \( \varphi^h = (\varphi^h_1, \ldots, \varphi^h_J) \in \mathbb{R}_+^J \) the short-positions in the assets.

The economy with collateral, \( E_{GEIC} \), is characterized by the agents’ utility functions \( u = (u^h)_{h \in \mathcal{H}} \), the agents’ endowment process \( e = (e^h)_{h \in \mathcal{H}} \), the asset structure \( (C_j)_{j \in \mathcal{J}} \).

Given \( p \in \mathbb{R}_+^{2S} \), and \( q \in \mathbb{R}_+^J \) the agent \( h \) chooses consumption and portfolios \((x^h, \theta^h, \varphi^h)\), to maximize utility subject to the budget constraints.

\[
\max_{x \geq 0, \theta \geq 0, \varphi \geq 0} u^h(x^h) \\
\text{s.t.} \quad p(0) \cdot (x^h(0) - e^h(0)) + q \cdot (\theta - \varphi) \leq 0; \\
p(s) \cdot (x^h(s) - e^h(s)) - p_2(s)x_2^h(0) - \sum_{j \in \mathcal{J}} (\theta_j - \varphi_j) \min\{1, p_2(s)C_j\} \leq 0; \quad \forall s \in \mathcal{S} \\
x_2^h(0) - \sum_{j \in \mathcal{J}} \varphi_j C_j \geq 0. \tag{1}
\]

In state \( s \), an asset \( j \) pays \( \min\{1, p_2(s)C_j\} \), an agent has endowments and receives \( x_2(0) \) units from his ‘investment’ in the first period. We refer to the last inequality constraint in the agent’s problem as the ‘collateral-constraint’.

A competitive equilibrium is defined as usual by agents’ optimality and market clearing.

**Definition 1.** A GEIC equilibrium for the economy \( E_{GEIC} \) is a vector \([x, \theta^\mathcal{H}, \varphi^\mathcal{H}]; (p, q)] \) with \((x^h, \theta^h, \varphi^h)_{h \in \mathcal{H}} \) such that:
(i) \((\overline{x}^h, \overline{\theta}^h, \overline{\varphi}^h)\) solves problem (1).
(ii) \(\sum_{h=1}^{H} (x^h(0) - e^h(0)) = 0\)
(iii) \(\sum_{h=1}^{H} (x^h_1(s) - e^h_1(s)) = 0\) and \(\sum_{h=1}^{H} (x^h_2(s) - e^h_2(s) - \overline{x}^h_2(0)) = 0\), \(s = 1, ..., S\)
(iv) \(\sum_{h=1}^{H} (\theta^h - \varphi^h) = 0\)


As a benchmark for our welfare-analysis, we also consider the Arrow-Debreu (AD) equilibrium in the examples below. An AD equilibrium consists of an allocation \((x^h)_h\) and prices \(\rho \in \mathbb{R}^{2S_*++}\) such that each agent maximizes his utility subject to the budget constraint \(\rho \cdot (x^h - e^h) \leq 0\) and such that markets clear. We impose a normalization condition, by requiring prices to lie on the unit simplex, \(\sum_{s \in S_*} \sum_{l \in L} \rho_{sl} = 1\).

3. Theoretical results

Throughout the analysis we want to think of the set of assets \(J\) as a finite but very large set. It is easy to see that ‘generically’ (e.g. for an open and full-measure set of individual endowments) for each GEIC equilibrium we have \(p_2(s) \neq p_2(s')\) (with \(p_1(s)\) normalized to 1 for all \(s\)) for each \(s, s' \in S\). For the remainder of this paper, we will only consider this generic case.

Since we assume that each asset \(j\) promises one unit of good 1 in each state \(s = 1, ..., S\) assets distinguish themselves only by their collateral requirement \(C_j\) and not by their promises. Of course, given the assumptions on default, this will also imply that the assets have different payoffs. We write \((1, C_j)_{j \in J}\) to characterize all assets.

The first insight is that if the set \(J\) is very large many assets are collinear and not all assets are being traded in equilibrium, or differently put, there exists an equivalent equilibrium with trade in only a few assets. In fact, it is clear that if for two assets \(j\) and \(j'\), \(C_j < C_{j'} \leq \min_s p_2(s)\), assets \(j\) and \(j'\) are collinear and it is without loss of generality to only consider equilibria with \(\varphi_j = 0\).

More interestingly, if there are \(S\) states and there is a set of asset \(J^{CC}\) (CC standing for complete set of collateral requirements) that consists of \(S\) assets \(j \in J^{CC} \subset J\) such that for each state \(s = 1, ..., S\) there is a \(j \in J^{CC}\) with \(C_j p_2(s) = 1\), then it is without loss of generality to assume that only these \(S\) assets are traded. In other words, the set \(J^{CC}\) denotes a minimal set of assets such that a portfolio of these assets can replicate any other possible asset using the same or less collateral. If \(J^{CC} \subset J\), we say there is a complete set of collateral requirements and we can restrict attention to the assets in \(J^{CC}\).

The following proposition formalizes this issue.

Proposition 1. Given an economy \(((u^h)_h, (e^h)_h, (1, C_j)_{j \in J})\) and a GEIC equilibrium \([\overline{x}, \overline{\theta}, \overline{\varphi}; (\overline{p}, \overline{q})]\), suppose that for each \(s\) there is a \(j \in J\) with \(C_j p_2(s) = 1\), then \(\overline{x}\) and \(\overline{p}, \overline{q}\) are GEIC equilibrium consumptions and prices for any economy \(((u^h)_h, (e^h)_h, (1, C_j)_{j \in J})\) if \(J \subset \tilde{J}\).
Proof. Let \( \delta \in \mathbb{R}^{HS^+} \) denote the vector of multipliers associated with the \( S+1 \) spot budget constraints and let \( \kappa \) denote the multiplier associated with the collateral constraints. Agent \( h \) chooses positive \( \theta_j^h \) if
\[
-\delta_0^h q_j + \sum_{s \in S} \delta_s^h \min\{1, p_2(s)C_j\} = 0;
\] (2)

Agent \( h \) chooses positive \( \phi_j^h \) if
\[
\delta_0^h q_j - \sum_{s \in S} \delta_s^h \min\{1, p_2(s)C_j\} - \kappa C_j = 0;
\] (3)

Assume now that \( i, j \in J, C_i < C_j \) and that there is no \( k \in J \) with \( C_i < C_k < C_j \). If an individual holds an asset \( j \in \tilde{J} \) with \( C_i < C_j < C_j \), there obviously exists a \( \mu > 0 \) such that \( \mu C_i + (1-\mu)C_j = C_j \). The individual can then simply hold \( \mu \) units of asset \( i \) and \( (1-\mu) \) units of asset \( j \), obtaining the same payoff in the second period and holding the exact same collateral. By Equation (2), the first order condition of the lender, the individual is indifferent between holding asset \( i \) at price \( q_i \) and asset \( j \) at price \( q_j \) or holding asset \( j \) at price \( \mu q_i + (1-\mu)q_j \). The cost to the borrower of holding asset \( j \) is the same as holding the portfolio of \( i \) and \( j \) that gives the same payoff. Finally the collateral the borrower has to hold is the same and he is indifferent as well. □

We denote a GEIC equilibrium in which the set of assets available for trade contains \( J^{CC} \) by GEICC, a GEIC equilibrium with ‘complete collateral’. Note that one cannot determine ex ante which collateral requirements will be used, without simultaneously solving for the actual equilibrium.

We will also consider the case where the set of assets available for trade is regulated exogenously and might not contain all potential assets we would like to trade. We refer to a GEIC equilibrium with an exogenously fixed set of collateral requirements as a GEIRC equilibrium – GEIC with regulated collateral. For a given economy, while there might be a unique GEICC equilibrium, there are obviously always infinitely many GEIRC equilibrating on different available assets. The GEICC equilibrium can be viewed as that specific GEIRC equilibrium for which adding assets (differing only by the collateral requirement) does not change the equilibrium allocation.

3.1. Necessary and sufficient condition for markets to be complete

We first consider the situation where there is an ‘abundance’ of collateralizable goods in the economy and each individual owns enough of them to back all the promises he wants to make. Since there are as many assets as states of the world, markets should be complete and the equilibrium allocation should be identical to the Arrow-Debreu equilibrium allocation. While this sounds intuitive, it is surprisingly difficult to formalize this idea and it turns out that the necessary conditions for markets to be complete are very restrictive.
By definition, the collateral constraint does not allow agents to transfer endowments from the second period to the first and it is clear that a necessary condition for the Arrow-Debreu equilibrium to be a GEICC equilibrium is that for all agents \( h \) and all states \( s \),
\[
\rho(s) \cdot (x^h(s) - e^h(s)) \geq 0. \tag{4}
\]
It is impossible for an agent to transfer any part of his endowments in any shock \( s \) to another shock – since utility is increasing, he will then never consume less than the value of the endowments. However, clearly, if he has sufficient durable goods, he might consume more.

Condition (4) is a formal way to state that collateral ‘is plentiful’. We will see in the examples below that it is quite difficult to make assumptions on fundamentals that guarantee that the condition holds. This condition is also discussed in [9] for the case of certainty where it is a necessary and sufficient condition for the GEICC equilibrium to be Pareto-efficient. Unfortunately, the condition is not sufficient in our framework with uncertainty. We investigate this issue in more detail and give economically meaningful necessary and sufficient conditions.

Given an Arrow-Debreu equilibrium \((\rho, (x^h)_{h \in \mathcal{H}})\), assume without loss of generality that:
\[
\rho_2(1) > \rho_1(1) > \rho_2(2) > \rho_1(2) > \ldots > \rho_2(S) > \rho_1(S).
\]
We also assume that \( C_1 > C_2 > \ldots > C_S \). Then there is a GEICC equilibrium with the Arrow-Debreu allocation \((x^h)_{h \in \mathcal{H}}\) if and only if portfolios \( \phi_j = -\min(0, \eta^h_j) \) and \( \theta_j = \max(0, \eta^h_j) \) with \( \eta^h \in \mathbb{R}^S \) defined by
\[
\eta^h = \left( \begin{array}{cccc}
1 & 1 & \cdots & 1 \\
1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & \rho_2(S)/\rho_1(S) & \cdots & \rho_2(S)/\rho_1(S) \\
\end{array} \right) \left( \begin{array}{c}
x_1(1) - e^h_1(1) + \frac{\rho_2(1)}{\rho_1(1)} (x_2(1) - e^h_2(1) - x_2(0)) \\
\vdots \\
x_1(S) - e^h_1(S) + \frac{\rho_2(S)}{\rho_1(S)} (x_2(S) - e^h_2(S) - x_2(0)) \\
\end{array} \right)
\]
satisfy the collateral constraints of the agent, given Arrow-Debreu prices and consumptions of the durable good. This condition itself is not very illuminating, but it highlights two important problems.

First, if prices vary little across states, large positions and therefore large amounts of collateral are necessary to achieve the Arrow-Debreu allocation. Since the durable good that can be used as collateral is also a consumption good in the first period and since there are no rental markets, it is unlikely that in the Arrow-Debreu equilibrium all agents choose to consume the ‘right’ amount of the durable good in the first period (i.e. the amount needed to satisfy the collateral constraints). This problem can easily be avoided by introducing rental markets or (less satisfactory from an economic point but much simpler) by assuming that no agent derives utility from consumption of the durable good in the first period. If we introduce a third good in the first period that is perishable, this framework can replicate a model with a rental market since one can view the second good as being split into the asset that pays in the second period.
and the third good that provides the rental-services. However, for simplicity we assume in this section that there are only two goods and that first period utility does not depend on consumption of the durable good. Introducing a third good and therewith modeling rental markets will not change the results derived. With this, we only need to consider $S - 1$ bonds, the asset with the lowest collateral requirement now does not need to be traded since it is identical to the durable good.

Secondly, consumption of the durable good at 0 appears both in the collateral-constraint and in the definition of $\eta$ – it is far from obvious that a solution exists no matter how valuable the durable good is in the second period.

The following proposition states a simple necessary condition on an Arrow-Debreu equilibrium for there to exist a GEICC equilibrium with identical allocation.

**Proposition 2.** Given an Arrow-Debreu equilibrium $(\rho, (x^h)_{h \in H})$, a necessary condition for there to exist a GEICC equilibrium with the same allocation is that for all agents $h$ and all pairs of states $s$ and $s'$ with $\frac{\rho_2(s)}{\rho_1(s)} > \frac{\rho_2(s')}{\rho_1(s')}$ we have

$$x^h_1(s) - e^h_1(s) + \frac{\rho_2(s)}{\rho_1(s)}(x^h_2(s) - e^h_2(s)) \geq x^h_1(s') - e^h_1(s') + \frac{\rho_2(s')}{\rho_1(s')}(x^h_2(s') - e^h_2(s')) \geq 0 \quad (5)$$

**Proof.** Define $\tilde{\rho}_h = \frac{\rho_2(s)}{\rho_1(s)}$ and suppose that

$$x^h_1(s) - e^h_1(s) + \tilde{\rho}_h(x^h_2(s) - e^h_2(s)) < x^h_1(s') - e^h_1(s') + \tilde{\rho}_h'(x^h_2(s') - e^h_2(s')).$$

Then in any GEICC equilibrium, the portfolio of agent $h$, $\eta^h \in \mathbb{R}^{S-1}$ together with the durable good holding $x^h_2(0)$ must satisfy

$$\sum_{j=1}^{S-1} \eta^h_j \min(1, C_j \tilde{\rho}_h) + \tilde{\rho}_h x^h_2(0) < \sum_{j=1}^{S-1} \eta^h_j \min(1, C_j \tilde{\rho}_h') + \tilde{\rho}_h' x^h_2(0).$$

Setting $\varphi^h_j = - \min(0, \eta^h_j)$ and $\theta^h_j = \max(0, \eta^h_j)$ this can be written as

$$\left( - \sum_j \varphi^h_j \min(1, \tilde{\rho}_h C_j) + \tilde{\rho}_h x^h_2(0) \right) + \sum_j \theta^h_j \min(1, \tilde{\rho}_h C_j) < $n

$$\left( - \sum_j \varphi^h_j \min(1, \tilde{\rho}_h' C_j) + \tilde{\rho}_h' x^h_2(0) \right) + \sum_j \theta^h_j \min(1, \tilde{\rho}_h' C_j) \quad (6)$$

However, if $\frac{\rho_2(s)}{\rho_1(s)} > \frac{\rho_2(s')}{\rho_1(s')}$, clearly

$$\sum_j \theta^h_j \min(1, \tilde{\rho}_h C_j) \geq \sum_j \theta^h_j \min(1, \tilde{\rho}_h' C_j).$$
Moreover, we have

\[- \sum_{j} \varphi^h_j \min(1, \tilde{\rho}_s C_j) = - \sum_{j; \tilde{\rho}_s C_j < 1} \varphi^h_j \tilde{\rho}_s C_j - \sum_{j; \tilde{\rho}_s C_j \geq 1} \varphi^h_j \]

\[\geq - \sum_{j; \tilde{\rho}_s C_j < 1} \varphi^h_j \tilde{\rho}_s C_j - \sum_{j; \tilde{\rho}_s C_j \geq 1} \varphi^h_j \]

and

\[- \sum_{j} \varphi^h_j \min(1, \tilde{\rho}'_s C_j) = - \sum_{j; \tilde{\rho}'_s C_j < 1} \varphi^h_j \tilde{\rho}'_s C_j - \sum_{j; \tilde{\rho}'_s C_j \geq 1} \varphi^h_j \]

Since by collateral constraint \(- \sum_j \varphi^h_j C_j + x^h_2(0) \geq 0\), we therefore obtain \(- \sum_j \varphi^h_j \min(1, \tilde{\rho}_s C_j) + \tilde{\rho}_s x^h_2(0) \geq - \sum_j \varphi^h_j \min(1, \tilde{\rho}_s' C_j) + \tilde{\rho}_s' x^h_2(0)\) and hence (6) cannot possibly hold. □

It turns out that when there are only two states (5) is also a sufficient condition for the GEICC equilibrium to be Pareto-efficient and markets to be complete. To see this, define for each agent \( h \), \( v^h \) by

\[ v^h = x^h_1(s) - e^h_1(s) + \tilde{\rho}_s (x^h_2(s) - e^h_2(s)), \quad s = 1, \ldots, S \]

and consider the system of budget constraints for all states.

\[ \eta_1 + \tilde{\rho}_1 x^h_2(0) = v^h_1 \]
\[ \eta_1 + \tilde{\rho}_2 x^h_2(0) = v^h_2 \]

Since the durable good now does not enter utility in the first period, we can drop the second bond in this expression. Direct computation yields that \( x^h_2(0) = \frac{v^h_2 - v^h_1}{\tilde{\rho}_1 - \tilde{\rho}_2} \geq 0 \), by Condition (5). This condition also implies that \( v^h_2 \geq 0 \) and hence the second budget constraint ensures that collateral constraints hold.

Unfortunately for more than 2 states the analysis is much more complicated and Condition (5) is no longer sufficient for complete markets. It is easy to see that \( x_2(0) \) always gets pinned down by \( v^h_1, v^h_2 \) as well as \( \tilde{\rho}_1 \) and \( \tilde{\rho}_2 \) – if the resulting \( x_2(0) \) is small, it cannot be guaranteed that the collateral constraint holds for all other assets.

Even the necessary condition appears quite restrictive. The crucial problem is that the long position of some bonds cannot be used as collateral for short positions in others. Short positions have to be backed by the durable good and not all payoffs can be replicated – in fact they are tied to the prices of the durable good across states.

While clearly being restrictive, the question of how likely the condition is to hold in calibrated examples arises. We do not have a complete answer to this, but the following three examples illustrate that if Condition (4) holds, Condition (5) is also likely to hold if agents have identical von Neumann Morgenstern utility. In the examples, we do assume that the durable good enters first period utility – even under this strong assumption, complete markets can be achieved in our model.
3.1.1. Examples with coinciding Arrow-Debreu and Collateral equilibria

High preferences for the durable good

As a first example, we consider the simplest two period model with two states, \( S^* = \{0, 1, 2\} \), and with two agents, \( H = \{1, 2\} \). The two individuals \( h = 1, 2 \) have identical homothetic utility of the form:

\[
u^h(x) = \alpha \log(x_1(0)) + (1-\alpha) \log(x_2(0)) + \frac{1}{2} \sum_{s=1}^{2} \alpha \log(x_1(s)) + (1-\alpha) \log(x_2(s))\]

We suppose that endowments are as follows

\[
e^1 = (e^1_1(0), e^1_2(0); e^1_1(1), e^1_2(1); e^1_1(2), e^1_2(2)) = (4, 2; 4, 0; 4, 0);
\]

\[
e^2 = (e^2_1(0), e^2_2(0); e^2_1(1), e^2_2(1); e^2_1(2), e^2_2(2)) = (2, 2; 6, 0; 2, 0).
\]

Clearly the parameter \( \alpha \) will determine the Arrow-Debreu price of the durable good and whether complete markets are feasible in this example. Since we only have two states, Condition (5) is necessary and sufficient. We first consider \( \alpha = 0.8 \), in this case \( \frac{\rho_2(1)}{\rho_1(1)} = 0.625 > 0.375 = \frac{\rho_2(2)}{\rho_1(2)} \) and the Arrow-Debreu allocations are:

\[
x^1 = (x^1_1(0), x^1_2(0); x^1_1(1), x^1_2(1); x^1_1(2), x^1_2(2)) = (3.5, 2.3; 5.8, 2.3; 3.5, 2.3);
\]

\[
x^2 = (x^2_1(0), x^2_2(0); x^2_1(1), x^2_2(1); x^2_1(2), x^2_2(2)) = (2.5, 1.7; 4.2, 1.7; 2.5, 1.7).
\]

Direct computation shows that \( v^1_1 = 3.25 > 0.35 = v^1_2 \) but that \( v^2_1 = -0.75 < 1.15 = v^2_2 \) and that therefore Condition (5) fails to hold for agent 2. The GEICC equilibrium is not Pareto-efficient.

On the other hand, for \( \alpha = 1/3 \) we obtain \( \frac{\rho_2(1)}{\rho_1(1)} = 5 > 3 = \frac{\rho_2(2)}{\rho_1(2)} \) and \( v^1_1 = 12 > 5.6 = v^1_2 \) and \( v^2_1 = 8 > 6.4 = v^2_2 \) – Condition (5) holds for both agents and the GEICC equilibrium is efficient.

As the preference for the durable good increases, its price increases and the GEICC equilibrium becomes efficient. In this example, it appears as if the fact that the collateral has sufficiently high value is the crucial stumbling block.

For the second example we assume that there are four states in period 1 \( S^* = \{0, 1, \ldots, 4\} \) and we suppose that endowments are as follows

\[
e^1 = (4, 2; 4, 0; 4, 0; 4, 0; 4, 0);
\]

\[
e^2 = (2, 2; 6, 0; 4, 0; 3, 0; 2, 0).
\]

In this example, Condition (5) is necessary but no longer sufficient for optimality. When the value of collateral is sufficiently high, e.g. for \( \alpha = 1/3 \) the condition is satisfied and we obtain

\[
\frac{\rho_2(1)}{\rho_1(1)} = 5 > 4 = \frac{\rho_2(2)}{\rho_1(2)} \quad \frac{\rho_2(3)}{\rho_1(3)} = 3.5 > 3 = \frac{\rho_2(4)}{\rho_1(4)} \quad \text{and} \quad v^1_1 = 12 > 8.8 = v^1_2 > v^1_3 = 7.2 > 5.6 = v^1_4,
\]

\[
\text{and} \quad v^2_1 = 8 > 7.2 = v^2_2 > v^2_3 = 6.8 > 6.4 = v^2_4.
\]

Although with 4 states this condition is
only a necessary condition, it turns out that GEICC equilibrium allocation is identical to the Arrow-Debreu allocation which is given by

\[ x^1 = (3.2, 2.1; 5.3, 2.1; 4.3, 2.1; 3.7, 2.1; 3.2, 2.1); \]
\[ x^2 = (2.8, 1.9; 4.7, 1.9; 3.7, 1.9; 3.3, 1.9; 2.8, 1.9). \]

**Even distribution of the durable good among agents**

Clearly, even if the value of the durable good is high, it is important that each agent owns a sufficient amount. This is illustrated by our third example. We fix \( \alpha = 1/3 \) but vary endowments of the durable good in the first period as follows:

\[ e^1 = (4, (4 - \omega); 4, 0; 4, 0); \]
\[ e^2 = (2, \omega; 6, 0; 2, 0). \]

We first consider \( \omega = 0 \), in this case \( \rho^2_1(1) = 5 > 3 = \rho^2_1(2) \), but \( v^1_1 = 22 > 11.6 = v^1_2 \) and \( v^2_1 = -2 < 0.4 = v^2_2 \) so and the Condition (5) for agent 2 fails. On the other hand, for \( \omega = 1.5 \) we obtain \( \rho^2_1(2) = 5 > 3 = \rho^2_1(1) \) and \( v^1_1 = 14.5 > 7.1 = v^1_2 \) and \( v^2_1 = 5.5 > 4.9 = v^2_2 \) that Condition (5) holds for both agents.

If the value of the durable good is sufficiently high (i.e. collateral is plentiful) and if each agent in the economy owns enough collateral, the GEICC allocation turns out to be Pareto-efficient in a model with 2 states.

In the three examples, the GEICC equilibrium allocation is Pareto-optimal whenever the weak necessary Condition (4) holds. This suggests that with homothetic preferences and von Neumann Morgenstern utility, Condition (5) might not be as restrictive as it first seems.

### 3.2. Constrained Pareto optimality

It is clear that even with an abundance of collateral, GEICC equilibria might not be Pareto-efficient. The more interesting case results if the amount of durable goods is small or some agents own no durable goods at all. In this case not all \( S \) contracts in \( J^{CC} \) will be traded in equilibrium. Instead, the examples below show that only very few assets are traded. Just like in the General Equilibrium model with incomplete markets (GEI), allocations will not be Pareto-efficient.

In the situation of scarce collateralizable goods, the collateral constraints will be binding and \( \kappa > 0 \) in Equation (3). Whether a particular contract will now be traded depends on the multipliers \( \delta^h \) across agents. Generally they will not be collinear and some assets are more attractive to both agents than others. These will be the only asset traded in equilibrium.

In particular, it is easy to see that if an agent is poor in the first period and owns no durable good, he wants to finance his first period consumption in the durable good by selling an asset which promises to hand the durable good over to the lender in the second period. As
Geanakoplos and Zame [9] observe, this is essentially a rental contract. The agent buys the durable good and borrows as much money as possible on an asset that defaults for sure, i.e. hands the durable good to the lender in all states of the world tomorrow.

If the agent is very poor and his marginal utility for the durable good is large, this will be the only asset he will sell. He will not trade in any asset with a large collateral requirement since he must forgo consumption in the first period to come up with the down-payment. More interestingly, we give an example below, where two agents trade in a unique asset that does not default in all states. We will argue that this asset has the best risk-sharing characteristics for the two agents.

Following Geanakoplos and Zame [9] it is a natural question whether it is efficient that precisely the assets in $J$ are traded or if it is possible to make everybody in the economy better off by restricting trade to take place in other (possible fewer) assets. As Geanakoplos and Zame [9] put it, ‘Given that the markets choose the asset structure, we are compelled to ask whether the market chooses the asset structure efficiently’.

Neither Geanakoplos and Zame [9] nor we can provide a complete answer theoretical to the problem, but we can prove when all agents have identical homothetic utility (an assumption often made in applied work), the answer is simple. The market chooses the asset structure efficiently. Geanakoplos and Zame [9] show that the standard proof of the constrained first welfare theorem applies to this model if one assumes that prices do not change through the intervention. The following result follows from this, since identical homothetic utility is sufficient for spot-prices in the second period to be independent of the wealth distribution.

**Proposition 3.** If all agents have identical homothetic utility, given a GEICC equilibrium with actively traded assets $J^{CC}$, there is no other set of assets $J'$ such that in the resulting GEIRC equilibrium all agents are better off.

**Proof.** Identical homothetic utility implies that at all states $s = 1, ..., S$ spot prices are independent of the assets traded since this only affects the distribution of wealth across agents. But then, the standard argument shows that there cannot be a GEIRC equilibrium that is Pareto-better than the GEICC equilibrium. □

Note that spot prices in period 0 are affected by which assets are traded, as are asset prices. Therefore, allocations will not be Pareto-ranked even if agents have identical homothetic utility.

Below, we give examples where agents do not have identical utility but the GEICC equilibrium allocation is still constrained efficient. The examples suggest that there is certainly no generic sense in which GEICC allocations are always constrained inefficient when preferences are heterogeneous.

4. Risk sharing with scarce collateralizable goods

In this section, we describe three numerical examples that illustrate how scarce collateralizable good leads to a situation where only very few assets are traded and welfare losses due to
imperfect risk-sharing are large. In these examples, allocations are always far from the Arrow-Debreu allocation, yet it is impossible to Pareto-improve by regulating collateral. We use the algorithm described in Schommer [13] to approximate equilibrium numerically.

Throughout the rest of the paper, we report welfare numbers in terms of wealth equivalence compared to the Arrow-Debreu allocation. That is, for log-utility and the case of no discounting (these are the preferences considered throughout the paper), if $u^{hGEIC}$ denotes an agent’s utility in the GEIC equilibrium and $u^{hAD}$ denotes his utility in the Arrow-Debreu equilibrium, we compute Welfare Rate $WR_h = \exp\left(\frac{u^{hGEIC} - u^{hAD}}{2}\right)$. If we multiply consumption in the Arrow-Debreu equilibrium by $WR$ in all states, we obtain an allocation that gives the agent the same utility as in the GEIC equilibrium. That is a number of say 0.95 means that an agent would be willing to decrease his consumption in the Arrow-Debreu allocation by 5 percent in each state to avoid the incomplete markets consumption.

4.1. Example 1: Scarce durable goods markets ‘appear’ incomplete

We consider an example with four states in period 1 $S^* = \{0, 1, \ldots, 4\}$ and two agents, $\mathcal{H} = \{1, 2\}$, each with identical utility,

$$u^h(x) = \log(x_1(0)) + \log(x_2(0)) + \frac{1}{4} \sum_{s=1}^{4} (\log(x_1(s)) + \log(x_2(s)))$$

We consider a variety of profiles of endowments, differing by the distribution of the durable (collateralizable) good in the first period.

$$e^1(0) = (4, \omega), e^1(1) = e^1(2) = (1, 0), e^1(3) = e^1(4) = (2, 0);$$
$$e^2(0) = (1, (1 - \omega)), e^2(1) = e^2(3) = (1, 0), e^2(2) = e^2(4) = (2, 0.2).$$

In the first period agent 1 is rich (the natural lender in the example) and agent 2 is poor. In the second period both agents face identically distributed shocks to endowments of good 1 that are independent across agents. In addition, agent 2 has random endowments in the durable good. The parameter $\omega$ determines how the collateralizable durable good is distributed between the two agents. We consider $\omega \geq 1/2$.

Since we assume identical homothetic utility, spot-prices do not depend on $\omega$. The set $\mathcal{J}^{CC}$ consists of the four assets with collateral requirements $C_1 = 0.5$, $C_2 = 0.4$, $C_3 = 0.333$ and $C_4 = 0.3$. The assets’ payment in the states defined by $\min\{1, p_2(s)C_j\}$ follows in Table I.

Obviously, if agents would not face a collateral constraint in period 0, markets would be complete and the Arrow-Debreu allocation (which is unique since we assume Cobb-Douglas utility) would be achieved. However, since agents do face collateral constraints, the Arrow-Debreu allocation is not achieved for any value of $\omega$. 

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4.1.1. Assets traded

We first examine how portfolios depend on the distribution of the durable good and how with an unequal distribution, the fact that collateralizable goods is scarce implies that only one of the four assets is traded and the collateral requirement is uniquely determined endogenously.

The following Table II denotes the portfolio-holding $\theta - \varphi$ of agent 1 for different values of $\omega$.

If the borrower owns almost nothing of the durable good in period zero, the only asset traded in equilibrium is the one with the lowest possible margin requirement. The borrower buys the durable good and borrows as much as possible for it at the lowest possible margin – resulting in a situation where he will default in all states in the second period. Any asset with a higher margin requirement is not optimal since the additional price the lender would be willing to pay to get a payoff above one in states 1, 2 or 3 is not sufficiently high for the borrower to forgo extra consumption in period zero. The little collateralizable goods he owns he needs to use to finance the margin on the loan he takes out just to buy more of the durable good.

For $\omega = 0.85$, two assets are traded, the asset that defaults in all states, but also asset 2, that pays back in full in states 2, 3 and 4 (see Table I). This is traded to improve risk sharing in the second period. In state 2, the borrower is rich (has endowments of 2) while the lender is poor (has endowments of 1). So the lender values this asset relatively more than the borrower,
making its price high enough so that the borrower is willing to take on the extra collateral (compared to assets 3 and 4). Around $\omega = 0.81$ there is a robust region of endowments in good 2, for which in fact asset 2 is the only asset traded. For these distributions of endowments, remarkably there is a unique collateral-requirement that is not equal to the lowest collateral available. Since the two agents want to share the risk in the second period, if agent 2 has a sufficient amount of durable goods he sells only asset 2 to finance first period consumption. This asset fetches the relatively best price among the 4 available assets.

For $\omega = 0.8$, the borrower, agent 2, starts buying assets in the first period (i.e. starts holding long-short positions). He holds a long-position in asset 3 and borrows money in asset 2. Asset 2 pays in full in states 2, 3 and 4 but defaults in state 1. In state 3, the borrower is poor (endowments of 1) while the lender is rich (endowments of 2), so buying asset 3 is a way for the borrower to insure himself against the second period risk. As the Table II shows this is true for all $\omega$ between 0.75 and 0.8.

Finally, for $\omega = 0.5$ both agents have sufficient endowments in the collateralizable goods to establish large short positions in some assets, however, the collateral requirement is still binding for both agents. Asset 4 (which obviously would be traded in a situation with collateral and default) is still not traded in equilibrium. Instead, agent 1 has relatively large short positions in assets 1 and 3. Agent 2 only uses asset 3 to borrow and finance first period consumption. He takes long-positions in assets 1 and 3 to insure against the second period risk. Even with both agents owning substantial amounts of the durable good in the first period, trade only takes place in 3 of the four available assets.

This obviously raises the question of how large the welfare losses due to this trade in a restricted set of assets are and if welfare can be improved by ‘forcing’ agents to trade in other assets through a regulation of the margin requirements.

4.1.2. Welfare

We first consider the case of a complete set of collateral requirements and ask how large the welfare losses are that are implied by the fact that agents cannot commit to pay their promises, i.e. we compare the GEICC equilibrium welfares to the welfares agents would obtain in an Arrow-Debreu equilibrium. The following Table III shows this welfare loss that due to default and collateral for different values of $\omega$.

As one would expect, the welfare losses due to default and collateral are large when the borrower has little collateralizable goods. In particular for $\omega > 0.9$, the possible welfare gains from better enforcement of intertemporal contracts are very large. Note that for each value of $\omega$, we compare the GEICC welfare to the Arrow-Debreu equilibrium welfare for that given economy. So for $\omega = 0.95$, agent 2 would gain more than 6 percent if he could commit to pay back all promises and trade in all assets without holding any collateral. This would allow the agent to buy more of the durable good in the first period and to ensure against his endowment risk in the second period. Since with log-utility risk aversion is relatively low, the endowment risk is actually not the main source of the welfare losses. They are mostly due to the fact
Table III: Welfare rate for distribution of durable good: agent 1 and 2

<table>
<thead>
<tr>
<th>ω</th>
<th>Lender (agent 1)</th>
<th>Borrower (agent 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.9930</td>
<td>0.9959</td>
</tr>
<tr>
<td>0.75</td>
<td>0.9933</td>
<td>0.9861</td>
</tr>
<tr>
<td>0.8</td>
<td>0.9933</td>
<td>0.9823</td>
</tr>
<tr>
<td>0.85</td>
<td>0.9916</td>
<td>0.9789</td>
</tr>
<tr>
<td>0.9</td>
<td>0.9898</td>
<td>0.9683</td>
</tr>
<tr>
<td>0.95</td>
<td>0.9887</td>
<td>0.9334</td>
</tr>
</tbody>
</table>

that the agent cannot afford to consume very much of the durable good. In fact, agent 2’s GEICC equilibrium consumption in period zero is $x_2^1(0) = 0.76$ and $x_2^2(0) = 0.15$ while in the Arrow-Debreu equilibrium it is $x_1^1(0) = 1.10$ and $x_2^2(0) = 0.22$. Of course, his second period consumption is higher in all states, but this shows how collateral skews consumption away from the efficient Arrow-Debreu allocation.

Even for the case of an equal distribution of collateralizable durable goods, the welfare losses are still substantial. As we argued in the previous section, not all assets are traded and collateralizable goods are still scarce. Moreover, it is surprising that the fact that the natural borrower (agent 2) has more collateralizable goods available to finance large short positions, does not necessarily bring the other agent closer to the complete markets welfare. Between $\omega = 0.8$ and $\omega = 0.5$ the welfare losses remain more or less constant for agent 1, while there are still substantial improvements for agent 2.

The fact that even for $\omega = 0.5$, the welfare losses due to default and collateral are still substantial (certainly significantly positive) highlights that it is not clear ex ante what ‘plentiful collateralizable goods’ means. In this example, agents spend 50 percent of their income on the consumption of the durable good. That seems to be a large fraction but is not sufficient to create so much collateralizable durable goods that markets are complete.

4.1.3. Regulating the margin requirement

What happens if, instead of allowing the agents to trade in assets with arbitrary margin requirements, we consider a situation where there is a fixed set of assets with ‘optimally’ set margin requirements?

As explained above, under the assumption made in this example that all agents have identical homothetic utility, equilibrium allocations must be constrained efficient, and it is impossible to make both agents better off by exogenously selecting margin requirements. This obviously does not imply, however, that all possible margin-requirements are Pareto-ranked.

To illustrate these points, we consider two cases. First we assume $\omega = 0.95$. In this case, it seems likely that GEIC equilibrium allocations are in fact Pareto-ranked, with the GEICC allocation yielding the highest utility for both agents: Forcing both agents to trade in an asset
that does not default in all states will reduce trade in the single asset traded and most likely make most agents worse off. This intuition turns out to be correct. In Figure 1, we show a few different points in utility (measured in terms of welfare rate) space for a sample 895 values of collateral between 0 and 1.

Figure 1: Welfare Rate for regulated collateral ($\omega = 0.95$)

The figure clearly shows that all equilibria are Pareto-ranked with the largest utility arising from the GEICC allocation.

We also consider the case $\omega = 0.85$. In this case, the GEIC equilibria are not Pareto-ranked as Figure 2 shows for a sample of 350 values for collateral-requirements between 0 and 1.

While it is true that there is no GEIC equilibrium that is Pareto-better than the complete collateral equilibrium, there are GEIC equilibria that make agent 1 better and there are other GEIC equilibria that make agent 2 better off. In particular, somewhat counter-intuitively, the lender, agent 1, would be better off if trade only takes place in the asset that defaults in all states. The point in the graph that gives him the highest utility corresponds to a situation where only the full default asset is available for trade. In this case, all agents of type 2 borrow heavily, since the collateral requirement is not an issue. The equilibrium interest rate is so high that agent 1 is compensated by the high interest rate for the fact that the only available asset
Figure 2: Welfare Rate for regulated collateral ($\omega = 0.85$)

has bad risk-sharing properties.

On the other hand, the point that gives agent 2 the highest utility corresponds to the case where only one asset with collateral requirement 0.4 is traded. This asset only defaults in one state. If it is the only asset traded in equilibrium, its interest rate is so favorable that agent 2 is well off. Agent 1 naturally is hurt by the low interest rate.

*Subprime loans*

As explained above, if the borrower owns almost no durable goods in the first period (case when $(1 - \omega) = 0.05$), the GEIC equilibrium allocations are Pareto-ranked and the only asset traded is the one with lowest collateral requirement. Figure 1 illustrates this. If we want to interpret these securities as ‘subprime loans’ we see that in situations where the lender has very little collateral these loans play an important role. On the other hand, when the borrower has more of the durable good (e.g. $(1 - \omega) = 0.15$), the GEIC equilibria are not Pareto-ranked. Restricting trade only to take place in the subprime loan turns out to be good for the lender, agent 1, while not allowing subprime loans benefits the borrower.
4.2. Example 2: Heterogeneous utility

We now consider an example with 3 agents. It seems clear that enough heterogeneity among agents should lead to trade in several assets even when collateralizable goods are scarce and unequally distributed among agents. As in Example 1, we first discuss portfolios, then report welfare rate due to collateral and finally show how the GEICC equilibrium welfare compares to welfare achieved in regulated economies.

To keep the example relatively simple, we assume that there are $S = 3$ states in the second period. The three agents’ endowments are given by:

$e_1^0 = (4, \omega), \quad e_1^1 = (1, 0), \quad e_1^2 = (4, 0), \quad e_1^3 = (2, 0);
\quad e_2^0 = (1, \gamma), \quad e_2^1 = (1, 0), \quad e_2^2 = (2, 0), \quad e_2^3 = (4, 0);
\quad e_3^0 = (2, 1 - \omega - \gamma), \quad e_3^1 = (2, 0.2), \quad e_3^2 = (2, 0), \quad e_3^3 = (2, 0.2).$

We assume that agents have heterogeneous utility. Under this assumption, Proposition 3 from above does not hold and there could be GEIRC allocations that are Pareto-better than the GEICC allocation. However, in this example this turns out not to be the case, suggesting that one can not generically improve on the GEICC allocation even when preferences are heterogeneous.

Utility functions are:

$u_h(x) = \alpha_h \log(x_1(0)) + (1 - \alpha_h) \log(x_2(0)) + \frac{1}{3} \sum_{s=1}^{3} (\alpha_h \log(x_1(s)) + (1 - \alpha_h) \log(x_2(s)),$

with $\alpha_1 = 0.7, \quad \alpha_2 = 0.77, \quad \alpha_3 = 0.625$.

With these preferences, the collateral requirements of the traded assets (i.e. of the assets in $\mathcal{J}^{CC}$) obviously vary with $\omega$ and $\gamma$ since spot prices will vary. However, since preferences are quite similar, the variation is relatively small. In most cases, the collateral requirements for the three assets traded are around $C_1 = 0.64, C_2 = 0.29$ and $C_3 = 0.36$. In all the cases we consider, asset 1 pays back in full in all states, asset 2 pays one unit in state 2, but defaults in states 1 and 3 and asset 3 defaults only in state 1, but pays one unit in both states 2 and 3.

4.2.1. Portfolios

As before, we first examine which assets are actively traded in the GEICC equilibrium, depending on the distribution of collateralizable goods which we parametrize by $\omega, \gamma$. We report portfolios of agent 1 and of agent 2 in Table IV.

Although now, there are almost always more than just one asset being traded in equilibrium, the same logic as in Example 1 above can be applied to understand which assets are traded. In terms of risk sharing, agent 1 would ideally want to hold an asset that pays a lot in state 1, little in state 2 and substantial in state 3. Among the available assets, asset 1 comes the closest to this pattern in the sense that it is the only asset that pays in full in state 1. Asset 2 seems
Table IV: Portfolio agent 1 and 2

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>$\gamma$</th>
<th>Portfolio agent 1</th>
<th>Portfolio agent 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.9</td>
<td>0.05</td>
<td>(0.00, 0.34, 0.76)</td>
<td>(0.00, 0.34, 0.76)</td>
</tr>
<tr>
<td>0.8</td>
<td>0.1</td>
<td>(0.06, 0.42, 0.74)</td>
<td>(0.06, 0.42, 0.74)</td>
</tr>
<tr>
<td>0.8</td>
<td>0.2</td>
<td>(0.00, 0.00, 1.15)</td>
<td>(0.00, 0.00, 1.15)</td>
</tr>
<tr>
<td>0.6</td>
<td>0.3</td>
<td>(0.18, -0.15, 0.98)</td>
<td>(0.18, -0.15, 0.98)</td>
</tr>
<tr>
<td>0.6</td>
<td>0.2</td>
<td>(0.64, 0.04, 0.24)</td>
<td>(0.64, 0.04, 0.24)</td>
</tr>
<tr>
<td>0.4</td>
<td>0.3</td>
<td>(0.49, -0.32, 0.52)</td>
<td>(0.49, -0.32, 0.52)</td>
</tr>
</tbody>
</table>

the worst since it only pays in full in state 2. Agent 2, on the other hand, would like to borrow since he is poor today and pay back more in state 3 than in state 2. He is also poor in state 1, so asset 1 is not a good asset to borrow in. Agent 3 is relatively rich in states 1 and 3, but wants to pay less in state 2. So asset 1 seems to be a good asset to be traded between agent 1 and 3, while asset 3 seems to be a good asset for trade between agent 1 and 2. But asset 3 is also attractive for agent 3 since he pays in full in state 3, where he is relatively rich. Asset 2 is the worst for agent 3.

Unfortunately, in the first case ($\omega = 0.9$, $\gamma = 0.05$) agent 2 is so poor that he only borrows to finance consumption in the durable good, therefore he trades in asset 2 which defaults in all states and, as we pointed out, can be interpreted as a rental contract. In this case, agent 3, although poor, still trades in asset 3 which gives him a relatively better price although it requires him to hold more collateralizable good.

In the second case ($\omega = 0.8$, $\gamma = 0.1$), both agents become more wealthy, but still agent 2 is stuck with asset 2 (he is still too poor to trade in any other asset). Instead agent 3 starts borrowing both in asset 1 and in asset 3, which are both good assets for him to share risk with agent 1.

In the third case ($\omega = 0.8$, $\gamma = 0.2$), agent 3 has no durable goods. Agent 2 is not yet rich enough to trade in anything but asset 3. There is a unique asset being traded in equilibrium by all three agents.

Let us now consider $\omega = 0.6$ – there are a lot of collateralizable goods both for agents 2 and 3. In the first case ($\gamma = 0.3$), agent 2 owns a lot of collateralizable goods, now instead of using asset 2 to borrow, he actually goes long in asset 2 and borrows only in asset 3. Agent 3 borrows in assets 1 and 3. Agent 1 goes short in asset 2, to share risk with agent 2. For $\gamma = 0.2$ agent 2 does not have enough collateralizable good anymore to borrow enough so that he can take long-positions in some assets. He exclusively borrows in asset 3, which with some collateralizable good is the best way to at the same time borrow and share risk with agent 1. For $\gamma = 0.1$, agent 2 is quite poor again and has to do some of the borrowing in the full default
Table V: Welfare rate for the distribution of durable good: agent 1, 2 and 3

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>$\gamma$</th>
<th>Agent 1</th>
<th>Agent 2</th>
<th>Agent 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.9</td>
<td>0.05</td>
<td>0.9756</td>
<td>0.9470</td>
<td>0.9773</td>
</tr>
<tr>
<td>0.8</td>
<td>0.1</td>
<td>0.9775</td>
<td>0.9718</td>
<td>0.9845</td>
</tr>
<tr>
<td>0.8</td>
<td>0.2</td>
<td>0.9783</td>
<td>0.9973</td>
<td>0.9612</td>
</tr>
<tr>
<td>0.6</td>
<td>0.3</td>
<td>0.9815</td>
<td>0.9926</td>
<td>0.9810</td>
</tr>
<tr>
<td>0.6</td>
<td>0.2</td>
<td>0.9831</td>
<td>0.9856</td>
<td>0.9906</td>
</tr>
<tr>
<td>0.6</td>
<td>0.1</td>
<td>0.9804</td>
<td>0.9662</td>
<td>0.9966</td>
</tr>
<tr>
<td>0.4</td>
<td>0.3</td>
<td>0.9836</td>
<td>0.9886</td>
<td>0.9928</td>
</tr>
</tbody>
</table>

Finally, for the case $\omega = 0.4, \gamma = 0.3$, we have a situation where the three available assets go a long way to share second period risk for the three agents. We now ask, how much of their Arrow-Debreu welfare the agents can achieve for the different distributions of collateralizable durable goods.

4.2.2. Welfare

As in Example 1, we now want to examine how scarce collateralizable goods lead to welfare loss. In this example, we in addition want to point out how one agent’s welfare-losses can depend on the distribution of collateralizable durable goods between the other agents. The following Table V shows this welfare rate due to default and collateral for different values of $\omega$ and $\gamma$.

For all distributions of collateralizable goods, welfare losses are substantial for all three agents. In the first case, despite the fact that agent 1 is the lender and his collateral constraint is not binding, his welfare losses due to collateral are substantial. The fact that with scarce collateralizable durable goods agents 2 and 3 are not willing to trade in asset 1 leads to little risk sharing in the second period. The table also shows that even for $\omega = 0.4, \gamma = 0.3$, i.e. in a situation where every agent owns substantial collateralizable good, welfare losses compared to incomplete markets are fairly large, in particular for agents 1 and 2.

Agent 2 is actually the one who seems to be hurt relatively least by default and collateral requirements. This is surprising since he is relatively poor in the first period, having the greatest need to borrow. However, the payoffs of the available assets are best for him - in states where he is poor he has to pay back relatively less than in states where he is rich.

4.2.3. Regulating the collateral requirements

As before, we ask how regulating the collateral requirements and not allowing agents to trade in arbitrary assets can influence welfare. Note that since preferences are heterogeneous,
Proposition 3 no longer applies and it could be possible that the GEICC allocation is constrained optimal. To investigate this question, we search for GEIRC equilibria that could be Pareto better.

We examine the case $\omega = 0.8, \gamma = 0.1$. The Figure 3 shows a three dimensional scatter plot of different utility (measured in terms of welfare rate) levels corresponding to different GEIRC allocations for a sample 813 values of collateral between 0 and 1.

![Figure 3: Welfare Rate for regulated collateral ($\omega = 0.8$ and $\gamma = 0.1$)](image)

While it is still impossible to Pareto-improve on the GEICC allocation, both agents 2 and 3 can obtain relative gains through a regulation. It is clear that the allocations are not Pareto-ranked, but the Figure 3 also shows that the GEICC allocation cannot be Pareto-dominated. It is a bit difficult to see in a 3D scatter plot, but it turns out that both agent 2 and 3 prefer a regulated equilibrium. Figure 4 illustrates this.

While the GEICC allocation is not Pareto-dominated, the figure shows that agents 2 and 3 can do much better. At the GEICC equilibrium the collateral requirements are $C_1 = 0.644, C_2 = \ldots$
0.288 and $C_3 = 0.362$. The point denoted by GEIRC1 in the figure corresponds to the equilibrium where trade is restricted to take place only in two assets and collateral levels are set exogenously to $C_2 = 0.286$ and $C_3 = 0.45$. Since trade takes place in two assets, there is still significant risk sharing possible, however, the equilibrium interest rate decreases making it easier for the borrowers to finance first period consumption.

The first asset has been eliminated and the collateral requirement on the third asset is higher than its endogenous level. While agents 2 and 3 benefit from this, the next figure shows that agent 1 loses.

Figure 5 shows that agent 1 is best off in a GEIRC where all agents are only allowed to trade in a asset that fully defaults. The point GEIRC2 in the figure corresponds to this point.

The situation here is similar to the previous example. The lender is better off in a situation where all trade takes place in the full default asset (i.e. essentially only rental contracts are traded). While trade in this asset does not allow for any risk sharing, if all borrowers are forced to borrow only in this asset, the equilibrium interest rate rises so dramatically that in fact the lender is compensated for the lack of risk sharing by a high interest rate.

Finally, restricting all agents to trade in an asset that never defaults makes all agents worse off. The point GEIRC3 shows the equilibrium that arises if there is only one asset available for trade and its collateral requirement ensures (exactly) full delivery in all three states.

4.3. Example 3: Regulation might benefits groups of agents

As a last and somewhat more elaborate example, we assume that there are 4 types of agents whose endowments we calibrated to (roughly) match the income and wealth distribution in US
Figure 5: Welfare Rate for regulated collateral ($\omega = 0.8$ and $\gamma = 0.1$) - Agents 1 and 2 (top) and 1 and 3 (below)
data. We want to investigate the role of subprime loans for risk sharing as well as who in the
economy gains and who loses through regulation.

We assume that there are 4 states and as before preferences are Cobb-Douglas with:

\[ u^h(x) = \alpha^h \log(x_1(0)) + (1 - \alpha^h) \log(x_2(0)) + \sum_{s=1}^{4} \pi_s(\alpha^h \log(x_1(s)) + (1 - \alpha^h) \log(x_2(s))) \]

We interpret endowments in the non-durable as income while endowments in the durable
good are interpreted as wealth. The estimates on wealth and income distribution in the US
for 1995 and 2004 follow (Di [5], Table 1). We choose \( \alpha^h \) to roughly match a relative price of
durable to non-durable good of 1/2 and take endowments to be:

\[
\begin{align*}
 e^1(0) &= (0.61, 0.84), & e^1(1) &= e^1(3) = (0.63, 0), & e^1(2) &= e^1(4) = (0.21, 0); \\
 e^2(0) &= (0.22, 0.12), & e^2(1) &= e^2(3) = (0.21, 0), & e^2(2) &= e^2(4) = (0.63, 0); \\
 e^3(0) &= (0.12, 0.04), & e^3(1) &= e^3(2) = (0.11, 0), & e^3(3) &= e^3(4) = (0.05, 0); \\
 e^4(0) &= (0.05, 0.00), & e^4(1) &= e^4(2) = (0.05, 0), & e^4(3) &= e^4(4) = (0.11, 0).
\end{align*}
\]

Probabilities are given by: \( \pi = (0.6, 0.18, 0.18, 0.04) \).

While this example does a good job of matching income and wealth inequality at \( t = 0 \) it is
very simplistic in the stochastic dynamics of income over time. To keep things simple we merely
assume that agents 1 and 2 and agents 3 and 4 can interchange. A more realistic calibration
where there is a large number of ex ante identical agents in each income class is computationally
not feasible.

To fix ideas, suppose first that:

\[ \alpha^h = (\alpha^1, \alpha^2, \alpha^3, \alpha^4) = (0.5, 0.4, 0.3, 0.6). \]

This results in prices that are slightly off in the first period but match the data well for the
second period (first period prices are \( p(0) = (0.30, 0.70) \) but second period prices all lie between
0.45 and 0.55). In the GEICCC equilibrium only the subprime loan that defaults in all states and
the safe asset that defaults in no state are traded. The rich agent 1 lends in the subprime asset
(0.37 units) and borrows (0.16 units) in the safe asset. Agents 2-4 borrow exclusively subprime,
while agents 2 and 3 (the middle-class) actually saves some money in the safe bond.

From this situation, one can ask which possible regulation of collateral requirements are
improving. As Figures 6-8\(^1\) show, it turns out that agents 2 and 3 cannot be made better off
through any regulation, while agents 1 and 4 gain simultaneously if only trade in the asset is
allowed that defaults in all states. The point denoted by GEIRC1 in Figures 6-8 corresponds

\(^1\)In these Figures, we show different points in welfare space for a sample of 3,798 values of collateral between
0 and 1.5.
to this regulated equilibrium. Figure 8 shows that agents 1 and 4 are best off in a GEIRC. The rich agent 1 benefits from lending more units in the subprime asset (0.43 units), due to a higher interest rate compared to the GEICC equilibrium, while agents 2 and 3 are now allowed to save some money in safe bonds. Agent 4 benefits from the increased borrowing in the subprime asset.

The point denoted by GEIRC2 corresponds to the equilibrium with default occurring only for asset 4 in the states 1 and 3 (i.e. in this economy the subprime loans are not available). Agents 1 and 4 lose, since they can’t lend and borrow, respectively, in the subprime asset (see Figure 8). Agents 2 and 3 are better off in GEIRC2, through lending in the safe asset.

Finally, restricting all agents to trade in an asset that never defaults makes all agents worse off (point GEIRC3) as in Example 2 above.

![Figure 6: Welfare Rate for regulated collateral (\(\alpha_h = (0.5,0.4,0.3,0.6)\)) - Agents 1 and 2](image)

### 4.3.1. Robustness analysis

In order to verify if the previous specification are robust we consider eight more specifications for preferences.

If we consider the opposite case, when the durable good is very expensive in the first period (i.e. if the rich agents propensity to consume the durable good is high) then regulation of the collateral requirement does not make anyone better off. For example we considered:

\[
\alpha_h = (0.3,0.4,0.5,0.8), \quad \alpha_h = (0.3,0.4,0.5,0.6) \quad \text{or} \quad \alpha_h = (0.2,0.4,0.6,0.7).
\]

In the GEICC equilibria two assets are traded, one that defaults in all states and other that ensures full delivery in four states. The rich agent 1 lends in the safe asset and borrows
in the subprime asset. Agents 2 and 3 lend in the subprime asset, while agents 2 and 3 (the middle-class) actually save some money in the safe bond.

In [15] preferences over housing and other goods consumption are represented by the Cobb-Douglas function. They estimate the housing preference \((1 - \alpha^h)\) as 0.2 for the US in 2001 based on the average proportion of household housing expenditure according to the Bureau of Labor Statistics (BLS) of the US Department of Labor. Here we estimate the durable good preference \((1 - \alpha^h)\) for each type of agent, based on the proportions of housing, furniture and vehicle purchases in the Consumer Expenditure Survey of BLS for 2004\(^2\). In this case the preference is:

\[
\alpha^h = (0.74, 0.73, 0.72, 0.71).
\]

As in the above example, both agents 1 and 4 can be made better off when trade is restricted to be in the subprime asset. The following values for \(\alpha\) also give this result:

\[
\alpha^h = (0.7, 0.6, 0.5, 0.4), \quad \alpha^h = (0.4, 0.3, 0.5, 0.7), \quad \alpha^h = (0.5, 0.4, 0.3, 0.2) \quad \text{or} \quad \alpha^h = (0.5, 0.3, 0.4, 0.6).
\]

indicating that what seems to be the most robust case is a case as in the above example.

Note that in all cases the subprime asset is traded actively and it always hurts all agents if trade in this asset is restricted. As mentioned in the introduction, this is not meant to say that markets in subprime loans are always Pareto-improving as our model is clearly very specialized,

\(^2\text{In Table 46 of U.S. Department of Labor (2004) consumers are split into nine groups of uneven size. In order to reach four classes, we regroup and perform an weighted average.}\)
default is rationally anticipated by all market participants and there are no externalities of default. However, this result should still provide a benchmark since it shows that in this economy (where there are no rental markets) subprime loans provide the only possibility for the poor to purchase the durable good.

5. Conclusion

In this paper we consider a model with default and collateral and demonstrate how scarcity of collateralizable durable goods can lead to large welfare losses when other mechanisms to enforce intertemporal contracts are absent.

We provide new necessary and sufficient conditions for the GEICC equilibrium to be Pareto-efficient and we discuss examples for which the Arrow-Debreu and GEICC equilibria coincide.

In our examples, equilibrium is constrained efficient in the sense that a regulation of collateral requirements never leads to a Pareto-improvement for all agents. However, we show that equilibria corresponding to different regulated collateral requirements are often not Pareto-ranked, and some agents can be benefited with regulation.

We show, through examples, that when the borrower owns almost nothing of the durable good a low-collateral loan is optimal for both the lender and the borrower. The lender always benefits from low-collateral loans, his risk being completely absorbed by the higher interest rate carried by these loans.
References


