

Nonparametric Surrogate Binary Classification^{*}

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Abstract

We develop a novel surrogate-loss approach to cost-sensitive empirical binary classification, a framework that encompasses maximum score estimation, prediction of utility-maximizing actions, and policy learning. These problems pose well-known challenges for identification, computation, and statistical inference. To address them, we introduce a smooth and strictly convex surrogate loss that uniquely selects a representative function from the generally nonunique set of optimal binary classifiers. We show that this representative function induces the same classification boundary, and hence the same policy value, as any optimal classifier, thereby providing a point-identified functional representation of the original classification problem. The empirical representative function can be readily computed using standard optimization routines and admits Gaussian inference at the standard nonparametric rate. More importantly, it serves as an intermediate device for estimating the original optimal policy value, for which we establish root- n consistency and asymptotic normality. Our approach therefore resolves key identification issues while substantially simplifying computation and inference relative to working directly with the original classification problem. We apply the method to the National JTPA Study to conduct inference on the optimal treatment assignment policy and its associated welfare.

Keywords: cost-sensitive binary classification, Gaussian approximation and limit, nonparametric inference, statistical decision, strictly convex surrogate loss.

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1 Introduction

Cost-sensitive binary classification problems encompass important classes of econometric models: e.g., the maximum score estimator (Manski, 1975, 1985); utility maximization with a binary action (Elliott and Lieli, 2013); and a recent strand of research on treatment choice problems (Manski, 2000, 2004; Stoye, 2009, 2012; Hirano and Porter, 2009; Bhattacharya and Dupas, 2012; Tetenov, 2012; Kitagawa and Tetenov, 2018; Athey and Wager, 2021; Mbakop and Tabord-Meehan, 2021; Manski, 2021; Ponomarev and Semenova, 2024; Viviano, 2025).

Despite its wide range of potential applications, binary classification is well known to pose challenges along several dimensions, including identification of the optimal classification function, statistical inference, and empirical implementation. When attention is restricted to the class of linear classification functions, as in maximum score estimation, Manski (1975, 1985) show that the optimal classification function is identified up to a scale normalization under the conditional median independence condition. However, computation and inference remain difficult because of the discontinuous 0–1 (or indicator) loss function. Kim and Pollard (1990) show that the estimated policy parameters converge at the cube-root- n rate to a nonstandard, non-Gaussian limiting distribution. Moreover, the naive bootstrap is invalid in this setting (see Abrevaya and Huang, 2005; Léger and MacGibbon, 2006). Consequently, statistical inference for cost-sensitive binary classification remains challenging in practice, even when one restricts attention to a parametrized policy class.¹

These difficulties become even more pronounced when researchers allow for more flexible classification functions. While parametric models can typically be point-identified after an appropriate normalization, there is no obvious normalization that generally guarantees point identification when the classification function belongs to a nonparametric class. The nonuniqueness is intrinsic to the classification criterion: the 0–1 loss determines the optimal classification region, but generally does not determine a unique real-valued function defining the regime. Any transformation that preserves the sign of an optimal classifier generates the same classification decision. The resulting partial-identification feature poses substantial challenges for developing valid statistical inference procedures. Finally, because of the 0–1 nature of the loss, computa-

¹That said, several approaches have been developed to facilitate inference in the parametric case, including smoothing (Horowitz, 1992), subsampling (Delgado, Rodriguez-Poo, and Wolf, 2001; Seo and Otsu, 2018), the m -out-of- n bootstrap (Lee and Pun, 2006; Bickel, Götze, and van Zwet, 2011), and modified bootstrap procedures (Cattaneo, Jansson, and Nagasawa, 2020; Cheng and Yang, 2024).

tional difficulty remains a major obstacle to empirical implementation in both parametric and nonparametric settings, particularly when the covariate dimension is large.

This paper proposes a novel surrogate-loss approach that offers advantages along all three dimensions discussed above. On identification, our strictly convex surrogate loss uniquely selects a representative function from the generally infinite set of optimal classifiers, without relying on approximation or regularization. This representative induces the same classification boundary almost surely as every other optimal classifier and can therefore be viewed as a “point-identified” functional parameter, characterized as the unique minimizer of the surrogate loss. On computation, the empirical representative classification function can be obtained conveniently by minimizing a smooth and strictly convex empirical surrogate loss using standard optimization routines. For inference, we show that the empirical representative classification function converges in distribution to a Gaussian limit at the standard nonparametric rate. More importantly, by using the estimated representative classification function as an intermediate device, we establish conditions under which the original optimal policy value can be estimated at the root- n rate, and valid Gaussian inference can be conducted.

One of the central contributions of this paper is the “identification” of a representative one from the generally infinite set of optimal classification functions by replacing the 0–1 loss with a strictly convex surrogate loss. The use of surrogate losses has been extensively studied in the statistical learning literature (e.g., [Lugosi and Vayatis, 2004](#); [Zhang, 2004](#); [Steinwart, 2005](#); [Bartlett, Jordan, and McAuliffe, 2006](#); [Zhao, Zeng, Rush, and Kosorok, 2012](#)), and more recently in the econometrics literature ([Kitagawa, Sakaguchi, and Tetenov, 2023](#); [Babii, Chen, Ghysels, and Kumar, 2025](#)). However, the motivation of the existing literature differs from ours. Existing work primarily uses surrogate losses to facilitate computation by convexifying the objective while preserving risk equivalence, whereas our objective is to achieve “identification” of a representative optimal classification function. This distinction has substantive implications for the choice of surrogate loss. In particular, although some existing studies advocate the hinge loss for their respective purposes, the hinge loss is not strictly convex and therefore does not guarantee the unique representative optimal classifier. So, we explicitly rule it out for ours because it does not deliver the uniqueness required for identification. In this sense, both our choice of surrogate loss and our objective are distinct from those of the seemingly related literature.

Our proposed inference procedure is straightforward to implement and therefore practical,

despite the inherent difficulty of the original classification problem. The sample analog of the surrogate loss, which we refer to as the empirical surrogate loss, remains smooth and strictly convex in the classification function. The empirical representative classification function can be computed by directly minimizing the empirical surrogate loss over the sieve space using standard optimization routines. Our procedure is also convenient in that it does not require data-dependent tuning parameters. For example, unlike the smoothed maximum score estimator of [Horowitz \(1992\)](#), our approach does not require the choice of a bandwidth for smoothing the indicator function. Moreover, because our inference is based on Gaussian limiting distributions, it admits a straightforward bootstrap implementation and avoids the choice of a subsample size required by subsampling methods such as [Delgado et al. \(2001\)](#).

As noted above, we estimate the representative classification function using a sieve approach based on minimization of the empirical surrogate loss. Our analysis goes beyond standard sieve M-estimation by allowing for infinite-dimensional nuisance parameters, such as the propensity score and conditional mean functions that arise in augmented inverse propensity score weighting. In this setting, we establish the standard nonparametric rate of convergence for the estimated representative function. We further obtain a strong Gaussian approximation by applying Yurinskii’s coupling inequality ([Pollard, 2002](#), Theorem 10), which provides the basis for Gaussian inference on the representative classification function.

Although the representative function need not itself be the primary parameter of interest, it induces the same classification boundary as any optimal classification function in the original problem. This allows us to estimate the optimal policy value by evaluating its (cross-fitted) empirical analog at the estimated representative function. We establish root- n consistency and asymptotic normality of the resulting estimator. The argument combines several features of our setting and is related in spirit to [Chen, Linton, and Van Keilegom \(2003\)](#), who derive root- n semiparametric results in settings involving nonsmooth objective functions. A key feature of our approach is that the representative classification function is not merely an approximation to an optimal rule: it remains exactly optimal for the original population problem. This exact optimality yields a first-order condition that controls the effect of estimating the classification function and ultimately permits root- n Gaussian inference under standard regularity conditions. Because our framework also accommodates infinite-dimensional nuisance parameters in the estimation of a finite-dimensional target, it bears some resemblance to double machine learning (DML, [Chernozhukov, Chetverikov, Demirer, Duflo, Hansen, Newey, and Robins, 2018](#)). The

Table 1: Relation to the econometric literature on cost-sensitive binary classification.

(A) Maximum Score Estimation

Function Class	Estimation	Inference
Parametric	Manski (1975, 1985) , etc.	Kim and Pollard (1990) & Horowitz (1992) , etc.
Nonparametric	Matzkin (1992)	–This Paper–

(B) Empirical Welfare Maximization

Function Class	Estimation	Inference
Parametric	Kitagawa and Tetenov (2018) & Athey and Wager (2021) , etc.	Rai (2018) , etc.
Nonparametric	Mbakop and Tabord-Meehan (2021)	–This Paper–

Panel (A) summarizes the gaps this paper fills in the maximum score estimation literature, and Panel (B) summarizes the gaps it fills in the empirical welfare maximization literature.

asymptotic analysis is nevertheless distinct from standard DML because it additionally involves a nonparametrically estimated functional optimizer, namely the classification policy.

Our paper contributes to and complements the existing econometric literature on binary classification. In the maximum score literature, most existing work focuses on linear-index rules with a finite number of covariates. For example, [Kim and Pollard \(1990\)](#) and [Horowitz \(1992\)](#) derive the limiting distributions of the original maximum score estimator and the smoothed maximum score estimator, respectively. [Chen and Lee \(2018\)](#) construct a best-subset maximum score prediction rule that is linear in covariates by maximizing a penalized objective function. More recently, [Chen, Gao, and Wen \(2025\)](#) reformulate maximum score estimation using rectified linear unit (ReLU) functions to improve computational tractability and obtain a convergence rate faster than the cube-root- n rate. Similarly, within the empirical welfare maximization framework, [Rai \(2018\)](#) develops inference procedures for optimal policy functions and maximized welfare under finite VC dimension. [Chernozhukov, Lee, Rosen, and Sun \(2025\)](#) introduce a framework for selecting policies by maximizing a penalized empirical welfare criterion, where the penalty is proportional to a consistent estimator of the risk associated with each candidate policy. Their main analysis focuses on policy classes with finite cardinality, with some discussion of extensions to infinite classes. We contribute to both strands of the literature by allowing the VC dimension of the class of classification policy functions to diverge, thereby accommodating genuinely nonparametric policy rules.

We are, of course, not the first to consider nonparametric binary classification. [Matzkin \(1992\)](#) and [Mbakop and Tabord-Meehan \(2021\)](#) introduce nonparametric classification policy functions in the contexts of maximum score estimation and empirical welfare maximization, respectively. Our contribution is to introduce a surrogate-loss formulation that not only facilitates computation but also enables the development of large-sample inference procedures, an issue not addressed in these papers. [Table 1](#) summarizes the gaps in the existing literature that are filled by the present paper.

In treatment assignment settings, several plug-in approaches to inference have been developed that exploit directly estimable nonparametric nuisance functions, such as the conditional average treatment effect (e.g., [Bhattacharya and Dupas, 2012](#); [Armstrong and Shen, 2023](#); [Park, 2025](#); [Chen, Chen, and Gao, 2025](#)). In contrast, our approach is aligned with empirical welfare maximization, viewed as a special case of cost-sensitive binary classification, and is therefore complementary to these plug-in methods.² More broadly, our binary classification framework is not limited to treatment assignment problems, but also encompasses maximum score estimation and utility-maximization settings.

Finally, it is worth noting several related results from the statistics and biostatistics literature. [Luedtke and Van Der Laan \(2016\)](#) develop root- n rate confidence intervals for the optimal welfare even when the parameter is not pathwise differentiable. [Wu and Wang \(2021\)](#) propose a smoothing-based procedure for model-free inference on the optimal treatment. [Liang, Choi, Ning, Smith, and Zhao \(2022\)](#) employ surrogate convex losses to conduct inference on low-dimensional parameters. [Wu, Wang, and Fu \(2023\)](#) develop a uniform inference method by quantifying the unknown link function, thereby enabling flexible modeling of interactions between treatment and covariates. The present paper, which adopts a surrogate-loss approach to uniform inference for nonparametric policies, complements specific aspects of each of these contributions.

The paper is organized as follows. [Section 2](#) introduces the setup and presents motivating examples. [Section 3](#) discusses the key issues arising from the discontinuous nature of the loss function. [Section 4](#) develops the “identification” of the representative classification policy

²See [Hirano and Porter \(2020, Section 2.3\)](#) for a discussion of the conceptual distinction between plug-in approaches and empirical welfare maximization in statistical decision problems. Our framework is also conceptually distinct from [Xu \(2025\)](#), who studies optimal point decisions over a continuum of alternatives in limit experiments.

function. Section 5 presents estimation and inference procedures for the representative classification policy function, and Section 6 establishes estimation and inference results for the optimal policy value. Section 7 examines the finite-sample performance of the proposed methods through simulation studies. Section 8 provides an empirical analysis that demonstrate the application of our method. Section 9 concludes. Proofs and additional details are provided in the supplementary appendix.

2 Setup and Examples

Let Z_i and X_i , $i = 1, 2, \dots, n$, denote random vectors of unit i observed by researchers or policymakers. Some variables may be common to both Z_i and X_i . Let $\mathcal{G}$ denote a class of functions. With these notations, a broad class of econometric frameworks can be represented as a cost-sensitive binary classification problem:

$$\begin{aligned} \min_{g \in \mathcal{G}} L(g), \quad \text{where} \\ L(g) = \mathbb{E}[\psi^+(Z_i, \eta_0) \mathbf{1}\{g(X_i) \geq 0\} + \psi^-(Z_i, \eta_0) \mathbf{1}\{g(X_i) < 0\}]. \end{aligned} \quad (2.1)$$

Here, ψ^+ and ψ^- denote non-negative functions that may depend on a possibly infinite-dimensional nuisance parameter η , whose true value is denoted by η_0 . This general risk-minimization framework is motivated by three examples. Sections 2.1, 2.2, and 2.3 below introduce the maximum score estimator, utility maximization problem, and welfare maximization framework, respectively.

2.1 Example I: Maximum Score

Let Y_i be a binary random variable and X_i a random vector such that

$$Y_i = \mathbf{1}\{g(X_i) + v_i \geq 0\},$$

where the error term v_i satisfies $\text{Median}(v_i|X_i) = 0$ almost surely. The maximum score estimator (Manski, 1975, 1985) for g is defined as the sample analogue of

$$\arg \max_{g \in \mathcal{G}} \mathbb{E}[Y_i \cdot \mathbf{1}\{g(X_i) \geq 0\} + (1 - Y_i) \cdot \mathbf{1}\{g(X_i) < 0\}],$$

for some function class $\mathcal{G}$. This can be equivalently reformulated as

$$\arg \min_{g \in \mathcal{G}} \mathbb{E} [\psi^+(Z_i, \eta) \cdot \mathbf{1}\{g(X_i) \geq 0\} + \psi^-(Z_i, \eta) \cdot \mathbf{1}\{g(X_i) < 0\}],$$

where $Z_i := Y_i$, $\psi^+(z, \eta) := 1 - z$, and $\psi^-(z, \eta) := z$. Therefore, the maximum score estimation can be viewed as a special case of the general risk-minimization framework with the risk given in (2.1). $\blacktriangle$

2.2 Example II: Expected Utility Maximization with Binary Actions

[Elliott and Lieli \(2013\)](#) consider the problem of maximizing the expected utility:

$$\max_{a(\cdot)} \mathbb{E} [U(a(X_i), Y_i, X_i)]$$

under a $\{-1, 1\}$ -valued binary outcome Y_i with respect to a binary action $a(x) \in \{-1, 1\}$ for each x . Imposing the condition that $U(1, 1, x) > U(-1, 1, x)$ and $U(-1, -1, x) > U(1, -1, x)$ for all x in the support of X to ensure correct classification yields higher utility, [Elliott and Lieli \(2013\)](#) show that the problem can be reformulated as $\max_{g \in \mathcal{G}} V(g)$, where

$$V(g) = \mathbb{E}[b(X_i)[Y_i + 1 - 2c(X_i)] \cdot \mathbf{1}\{g(X_i) \geq 0\} - b(X_i)[Y_i + 1 - 2c(X_i)] \cdot \mathbf{1}\{g(X_i) < 0\}],$$

$b(x) = U(1, 1, x) - U(-1, 1, x) + U(-1, -1, x) - U(1, -1, x)$, and $c(x) = [U(-1, -1, x) - U(1, -1, x)]/b(x)$. The optimal action $a^*(x)$ can then be recovered from the optimal classifier g^* as $a^*(x) = \text{sign}(g^*(x))$ for each x . Like Example I (Section 2.1), the above maximization problem can be further reformulated as a special case of the general risk-minimization framework with the risk given in (2.1) by taking $Z_i := (Y_i, X_i')'$, $\psi^+(z, \eta) := |b(x)(y + 1 - 2c(x))| - b(x)(y + 1 - 2c(x))$, and $\psi^-(z, \eta) := |b(x)(y + 1 - 2c(x))| + b(x)(y + 1 - 2c(x))$. $\blacktriangle$

2.3 Example III: Welfare Maximization

Let $Z_i = (Y_i, A_i, X_i')'$, where Y_i denotes a scalar outcome, A_i a binary treatment indicator, and X_i a vector of covariates supported on $\mathcal{X}$. Let $Y_i(1)$ and $Y_i(0)$ denote the potential outcomes under treatment and control, respectively. The observed outcome is generated according to

$$Y_i = Y_i(1) \cdot A_i + Y_i(0) \cdot (1 - A_i).$$

Define the propensity score function as $\pi(x) = \mathbb{P}(A_i = 1 \mid X_i = x)$.

A policymaker is interested in selecting a policy function g from $\mathcal{G}$ that maximizes welfare, measured by the mean outcome:

$$\mathbb{E}[Y_i(1) \cdot \mathbf{1}\{g(X_i) \geq 0\} + Y_i(0) \cdot \mathbf{1}\{g(X_i) < 0\}]. \quad (2.2)$$

To analyze this objective, the literature commonly imposes the following set of assumptions.

(WM.i) **Unconfoundedness:** $(Y_i(1), Y_i(0)) \perp\!\!\!\perp A_i \mid X_i$;

(WM.ii) **Moments:** $\mathbb{E}[|Y_i(a)|] < \infty$ for each $a = 0, 1$ and;

(WM.iii) **Overlap:** There exists $\underline{C} \in (0, 1/2)$ such that $\mathbb{P}(\underline{C} \leq \pi_0(x) \leq 1 - \underline{C}) = 1$ for all $x \in \mathcal{X}$.

Under these assumptions (WM.i)–(WM.iii), the mean outcome (2.2) can be identified by the inverse propensity score weighting (IPW) as in [Kitagawa and Tetenov \(2018\)](#), which in turn may be rewritten in the augmented-IPW (AIPW) as in [Athey and Wager \(2021\)](#):

$$V(g) := \mathbb{E}[Y_i(0)] + \mathbb{E}\left[\left(\left(\frac{A_i(Y_i - \mu(1, X_i))}{\pi(X_i)} + \mu(1, X_i)\right) - \left(\frac{(1 - A_i)(Y_i - \mu(0, X_i))}{1 - \pi(X_i)} + \mu(0, X_i)\right)\right) \cdot \mathbf{1}\{g(X_i) \geq 0\}\right],$$

where $\mu(l, x) = \mathbb{E}[Y_i | A_i = l, X_i = x]$ for $l \in \{0, 1\}$. Collecting the nuisance parameters $\eta := (\mu(1, \cdot), \mu(0, \cdot), \pi(\cdot))$, we express the g -relevant component of the identified objective, $V(g)$, as

$$V(g) = \mathbb{E}[\psi_0(Z_i, \eta)] + \mathbb{E}[(\psi_1(Z_i, \eta) - \psi_0(Z_i, \eta)) \cdot \mathbf{1}\{g(X_i) \geq 0\}], \quad \text{where} \quad (2.3)$$

$$\psi_1(Z_i, \eta) := \frac{A_i(Y_i - \mu(1, X_i))}{\pi(X_i)} + \mu(1, X_i) \text{ and } \psi_0(Z_i, \eta) := \frac{(1 - A_i)(Y_i - \mu(0, X_i))}{1 - \pi(X_i)} + \mu(0, X_i). \quad (2.4)$$

Denote the positive components of $\psi_1(Z_i, \eta)$ and $\psi_0(Z_i, \eta)$ by $\psi_1^+(Z_i, \eta)$ and $\psi_0^+(Z_i, \eta)$, respectively, and similarly denote the negative parts of $\psi_1(Z_i, \eta)$ and $\psi_0(Z_i, \eta)$ by $\psi_1^-(Z_i, \eta)$ and $\psi_0^-(Z_i, \eta)$, respectively. Then, the welfare (2.3) can be rewritten as

$$\begin{aligned} V(g) &= \mathbb{E}[\psi_1(Z_i, \eta) \mathbf{1}\{g(X_i) \geq 0\} + \psi_0(Z_i, \eta) \mathbf{1}\{g(X_i) < 0\}] \\ &= \mathbb{E}[(\psi_1^+(Z_i, \eta) - \psi_1^-(Z_i, \eta)) \mathbf{1}\{g(X_i) \geq 0\} + (\psi_0^+(Z_i, \eta) - \psi_0^-(Z_i, \eta)) \mathbf{1}\{g(X_i) < 0\}] \\ &= \mathbb{E}[\psi_1^+(Z_i, \eta) (1 - \mathbf{1}\{g(X_i) < 0\}) - \psi_1^-(Z_i, \eta) \mathbf{1}\{g(X_i) \geq 0\}] \end{aligned}$$

$$\begin{aligned}
& +\psi_0^+(Z_i, \eta) (1 - \mathbf{1}\{g(X_i) \geq 0\}) - \psi_0^-(Z_i, \eta) \mathbf{1}\{g(X_i) < 0\}] \\
& = \mathbb{E} [\psi_1^+(Z_i, \eta) + \psi_0^+(Z_i, \eta)] - \underbrace{\mathbb{E} [\psi^+(Z_i, \eta) \cdot \mathbf{1}\{g(X_i) \geq 0\} + \psi^-(Z_i, \eta) \cdot \mathbf{1}\{g(X_i) < 0\}]}_{=(2.1)},
\end{aligned} \tag{2.5}$$

where $\psi^+(Z_i, \eta) := \psi_1^-(Z_i, \eta) + \psi_0^+(Z_i, \eta)$ and $\psi^-(z, \eta) := \psi_1^+(Z_i, \eta) + \psi_0^-(Z_i, \eta)$. Note that the second expectation in the last line takes the form of (2.1). Thus, the welfare maximization problem can be viewed as a special case of the general risk-minimization framework with a risk of the form (2.1). $\blacktriangle$

3 Issues

As Sections 2.1–2.3 demonstrate, the cost-sensitive binary classification problem $\min_{g \in \mathcal{G}} L(g)$ with the risk function given by (2.1) encompasses several important econometric frameworks. However, this problem is more challenging to address than standard econometric problems. The difficulty becomes even more pronounced when allowing for a nonparametric class $\mathcal{G}$ of policy functions.

The primary challenge arises from the discontinuous nature in the sample analog of the objective function. In practice, this discontinuity induces computational difficulties: the sample objective function is non-convex, making numerical optimization difficult, if not infeasible. Beyond these practical concerns, the discontinuity also has adverse theoretical implications, giving rise to non-standard limiting behavior and non-identification, as detailed in the following two subsections.

3.1 Non-Standard Limit under Identifying Normalizations

To simplify the discussion, suppose for the moment that g belongs to a parametric class specified as $g(x) = x'\gamma$, with appropriate normalizations. In this case, the problem $\min_{g \in \mathcal{G}} L(g)$ reduces to an M -estimation of γ with a discontinuous sample objective function. In this setting, Kim and Pollard (1990) show that the resulting M -estimator converges at the cube-root- n rate and admits a non-standard asymptotic distribution. Consequently, naive bootstrap procedures are invalid in this context (Abrevaya and Huang, 2005; Léger and MacGibbon, 2006).

3.2 Non-Identification in General

Although Section 3.1 focuses on a simple parametric class, it already highlights the inherent difficulty of the problem. When $\mathcal{G}$ is a nonparametric class, the challenge becomes even more severe, as the optimal policy function g is no longer point-identified even after normalization. This is because only the sign of g affects the risk in (2.1). In particular, if $g^* \in \mathcal{G}$ is a solution, then any $g^{**} \in \mathcal{G}$ such that $\text{sign}(g^{**}(x)) = \text{sign}(g^*(x))$ for all x is also a solution. In a nonparametric class $\mathcal{G}$, one can readily construct infinitely many such functions g^{**} that are sign-equivalent to g^* , thereby rendering the optimal policy function unidentifiable. Theoretical analysis of these partially identified nonparametric models poses even greater challenges than those encountered in the parametric case of Section 3.1.

4 Representative Identification

The difficulties associated with (2.1), as discussed in the previous section, stem from the discontinuity of the 0–1 loss function. We remedy them by replacing the 0–1 loss with a strictly convex surrogate loss. This substitution not only yields a continuous and convex objective to ease computation, but also ensures point identification of the optimal policy function even when $\mathcal{G}$ is nonparametric.

Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be a surrogate loss function, and consider the following modification of (2.1):

$$L^*(g) := \mathbb{E} [\psi^+(Z_i, \eta_0) \cdot \phi(-g(X_i)) + \psi^-(Z_i, \eta_0) \cdot \phi(g(X_i))] . \quad (4.1)$$

The following theorem presents the utility of using this function $\phi(\cdot)$ in place of the original 0–1 loss.

Theorem 4.1 (Identification): *Suppose that $\mathbb{E}[\psi^+(Z_i, \eta_0)|X_i] - \mathbb{E}[\psi^-(Z_i, \eta_0)|X_i] \neq 0$ almost surely and $\mathcal{G}$ includes all measurable functions. Let $\phi(\cdot)$ be differentiable at zero and $\partial\phi(0) < 0$.*

- (i) *If $\phi(\cdot)$ is convex, then any $g_* \in \arg \min_{g \in \mathcal{G}} L^*(g)$ satisfies $g_* \in \arg \min_{g \in \mathcal{G}} L(g)$.*
- (ii) *If $\phi(\cdot)$ is strictly convex, $\mathcal{G}$ is convex, and L^* attains its infimum over $\mathcal{G}$, then there is unique g_* such that $g_* = \arg \min_{g \in \mathcal{G}} L^*(g)$.³*

³The uniqueness is up to the equivalence class in $\mathcal{G}$ identified by the underlying probability measure.

Theorem 4.1(i) guarantees that minimizing the surrogate risk L^* yields an optimal policy function with respect to the original risk L , as established in the literature (Bartlett et al., 2006). Furthermore, Theorem 4.1(ii) ensures that the surrogate risk admits a unique minimizer (up to L_1 -equivalence). In other words, by employing a strictly convex surrogate function $\phi(\cdot)$, we achieve *point identification* of a nonparametric solution g_* to the original problem $\min_{g \in \mathcal{G}} L(g)$. This, in turn, facilitates the application of econometric methods based on point estimation and their associated asymptotic properties, which are typically more tractable than those relying on set identification.

While the original binary classification problem admits infinitely many optimal classification functions, all these functions share the identical classification boundaries and signs almost surely, and therefore induce exactly the same optimal classification rule. Our strictly convex surrogate loss uniquely selects one representative, g_* , from this infinite set of optimal classification functions. This *point-identification* result ensures that statistical inference based on g_* targets *exactly* the same optimal classification rule as that obtained from the original population objective, rather than an approximate, regularized, or alternative classification rule. In addition, as formally shown later, the uniqueness and smooth characterization of g_* provide a tractable foundation for point estimation and standard Gaussian-based inference on the optimal binary classification rule and its associated policy value.

The surrogate loss approach has been explored in the literature (e.g., Bartlett et al., 2006; Zhao et al., 2012; Kitagawa et al., 2023); however, the motivation behind our use of it is fundamentally different. While existing studies primarily leverage surrogate losses to enable numerical optimization by convexifying the originally non-convex objective, our primary aim is to establish *identification* of the optimal policy function. Moreover, while Zhao et al. (2012) and Kitagawa et al. (2023) advocate the hinge loss for their purposes, we explicitly rule out the hinge loss in part (ii) of our Theorem 4.1 and instead promote alternative loss functions that better align with our identification objective – see below for a concrete list. In this respect, both our goals and the methodological approaches are largely orthogonal to those in the literature. To the best of our knowledge, the use of surrogate losses for the purpose of identification (or representation), rather than computational convenience, is novel.

Although we identify only one representative solution g_* , this suffices for conducting inference on the optimal classification policy. Indeed, g_* is sign-equivalent to any solution of $\min_{g \in \mathcal{G}} L(g)$ up to a null set. Accordingly, the optimal classification rule defined by $x \mapsto \mathbf{1}\{x \in$

$\mathcal{X} \mid g_*(x) \geq 0$ delivers the same policy decisions as the rule $x \mapsto \mathbf{1}\{x \in \mathcal{X} \mid g(x) \geq 0\}$ associated with any solution $g \in \arg \min_{g \in \mathcal{G}} L(g)$, possibly except on a set of measure zero. Thus, we may perform inference on the optimal classification policy and the corresponding optimal policy value (e.g., welfare) using the representative, point-identified representative policy function g_* , effectively treating it as the unique solution.

Various choices for the function ϕ have been proposed in the literature: the hinge loss, the exponential loss, the logistic loss, the squared loss, and the sigmoid loss, among others. While part (i) of Theorem 4.1 accommodates all of these examples, part (ii) rules out the hinge loss and sigmoid loss.

The use of different loss functions leads to different representers g_* . However, as argued above, these representers yield the same optimal classification rule, since they agree in sign except on a null set. As such, the choice of loss function, and hence the resulting representer, may be viewed as analogous to selecting different parameter normalizations proposed by [Manski \(1975, 1985\)](#) in the context of the maximum score framework (Example I) introduced in Section 2.1.

The assumption that

$$\mathbb{E}[\psi^+(Z_i, \eta) \mid X_i] - \mathbb{E}[\psi^-(Z_i, \eta) \mid X_i] \neq 0 \quad \text{a.s.}$$

plays a crucial role. If there exists an event with positive probability on which

$$\mathbb{E}[\psi^+(Z_i, \eta) \mid X_i] - \mathbb{E}[\psi^-(Z_i, \eta) \mid X_i] = 0,$$

then the function g could take either positive or negative values on that event, thereby failing to point-identify the solution. This condition is essential not only for point identification but also for the inference on the optimal policy value; see, for example, [Luedtke and Van Der Laan \(2016, Theorem 1\)](#) for related discussions.

5 Estimation and Inference for the Representative Policy Function

Given the ‘identification’ result from Section 4, one can obtain a point estimate of the representative classification policy function g_* . Although g_* is only one among infinitely many

solutions, it suffices for constructing the optimal classification rule $x \mapsto \mathbf{1}\{g_*(x) \geq 0\}$, since only the sign of g_* matters and any other solution has an equivalent sign almost everywhere. Consequently, the optimal classification policy and the optimal policy value (e.g., welfare) can be estimated.

5.1 Estimation of the Representative Policy Function

Let $\mathcal{X}$ denote the support of X_i , and $\{p_1(\cdot), p_2(\cdot), \dots\}$ be a sieve basis consisting of measurable functions $p_k : \mathcal{X} \rightarrow \mathbb{R}$, $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, let $\mathcal{G}_k = \{x \mapsto p_{1:k}(x)' \beta : \beta \in \mathbb{R}^k\}$ denote the sieve space, where $p_{1:k}(\cdot) := (p_1(\cdot), \dots, p_k(\cdot))'$. Let $g_k = \arg \min_{g \in \mathcal{G}_k} L^*(g)$, and let $r_g = g_* - g_k$ be its approximation error. Denote the complexity measure by $\xi_k := \sup_{x \in \mathcal{X}} \|p_{1:k}(x)\|$.

We now introduce the cross-fitting method for estimating g_* . Partition the index set $[n] = \{1, \dots, n\}$ into m folds $I_1, \dots, I_m$, and let $I_{(\ell)}^c := [n] \setminus I_{(\ell)}$ denote the complement of the ℓ -th fold. Assume that the folds are approximately balanced, so that $n_{(1)} \asymp \dots \asymp n_{(m)}$, where $n_{(\ell)} := |I_{(\ell)}|$, and the number of folds m is fixed.

For a given fold $\ell \in [m] = \{1, \dots, m\}$, obtain an estimator $\hat{\eta}_{(\ell)}$ of the nuisance parameter η_0 using the complementary subsample $I_{(\ell)}^c$. Then, estimate the representative policy function g_* on the held-out fold $I_{(\ell)}$. Specifically, the fold-wise estimator of g_* is given by

$$\begin{aligned} \hat{g}_{(\ell)}(x) &:= p_{1:k}(x)' \hat{\beta}_{(\ell)}, \quad \text{where} \\ \hat{\beta}_{(\ell)} &:= \arg \min_{b \in \mathbb{R}^k} \mathbb{E}_{n,(\ell)} \left[\hat{\psi}_{(\ell)}^+(Z_i) \phi(-p_{1:k}(X_i)'b) + \hat{\psi}_{(\ell)}^-(Z_i) \phi(p_{1:k}(X_i)'b) \right], \end{aligned}$$

$p_{1:k}(\cdot) := (p_1(\cdot), \dots, p_k(\cdot))'$, $\hat{\psi}_{(\ell)}^\pm(\cdot) := \psi^\pm(\cdot, \hat{\eta}_{(\ell)})$ for shorthand, and $\mathbb{E}_{n,(\ell)}[f] := \frac{1}{|I_{(\ell)}|} \sum_{i \in I_{(\ell)}} f(Z_i)$ denotes the sample mean over the ℓ -th fold.

Finally, aggregate the m fold-wise estimators to obtain the cross-fitted estimator

$$\hat{g}(x) := \frac{1}{m} \sum_{\ell=1}^m \hat{g}_{(\ell)}(x). \quad (5.1)$$

For later use, define $\hat{\eta}$ as the full-sample estimator of the nuisance parameter η_0 , and let $\hat{\psi}^\pm(\cdot) := \psi^\pm(\cdot, \hat{\eta})$. This estimator will be employed for variance estimation and bootstrap inference. The conditions required for both the cross-fitted and full-sample estimators are presented in Section 5.2.

5.2 Large Sample Theory for the Representative Policy Function Estimation

We next establish the uniform convergence rate of $\hat{g}$ toward g_* and prove a Gaussian coupling principle that characterizes the Gaussian approximation of its estimation error.

We now introduce a set of regularity conditions.

Assumption 5.1 (Sieves): *Let the following conditions be satisfied:*

- (i) **(Bounded Moments)** $\mathbb{E}[|\psi^+(Z_i, \eta)|^3 | X_i = x] < \infty$ and $\mathbb{E}[|\psi^-(Z_i, \eta)|^3 | X_i = x] < \infty$ uniformly for all $x \in \mathcal{X}$.
- (ii) **(Eigenvalues)** For $Q := \mathbb{E}[p_{1:k}(X_i)(\psi^+(Z_i) \cdot \partial^2 \phi(-g_*(X_i)) + \psi^-(Z_i) \cdot \partial^2 \phi(g_*(X_i)))p_{1:k}(X_i)']$ and $\Sigma := \mathbb{E}[p_{1:k}(X_i)(-\psi^+(Z_i) \cdot \partial \phi(-g_*(X_i)) + \psi^-(Z_i) \cdot \partial \phi(g_*(X_i)))^2 p_{1:k}(X_i)']$, there exist constant values $C_{\min}, C_{\max} \in (0, \infty)$ such that $C_{\min} < \lambda_{\min}(Q) < \lambda_{\max}(Q) < C_{\max}$ and $C_{\min} < \lambda_{\min}(\Sigma) < \lambda_{\max}(\Sigma) < C_{\max}$ with $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ indicating the minimum and maximum eigenvalues.
- (iii) **(Growth Condition)** The complexity measure ξ_k grows sufficiently slowly such that it satisfies $\sqrt{\xi_k^2 \log(k)/n} \rightarrow 0$ and $\log(\xi_k) \lesssim \log(k)$
- (iv) **(Approximation Error)** There is a sequence $\{B_k\}_k$ of bounds on approximation errors associated with the sequence $\{\mathcal{G}_k\}_k$, such that $\sqrt{n}B_k \log(k) \rightarrow 0$ and

$$\|r_g\|_{\mathbb{P}, \infty} := \sup_{x \in \mathcal{X}} |r_g(x)| \leq B_k,$$

Assumption 5.1 collects standard regularity conditions commonly used in the series-based nonparametric estimation literature and adapts them to our cost-sensitive binary classification problem. Assumption 5.1(i) requires the conditional moments of the weights to be bounded and serves multiple purposes. In particular, the existence of the third moment is needed to establish a coupling inequality. Assumption 5.1(ii), which rules out collinearity in the population signal matrix, is a conventional requirement in nonparametric analysis (e.g., Newey, 1997). Assumptions 5.1(iii) and (iv) control the complexity of the function class and the nonparametric approximation error, respectively.

To facilitate our subsequent writings, we define some short-hand notations:

$$\Psi(Z_i, \eta, g) := \psi^+(Z_i, \eta)\phi(-g(X_i)) + \psi^-(Z_i, \eta)\phi(g(X_i)),$$

$$\begin{aligned}\Psi_1(Z_i, \eta, g) &:= -\psi^+(Z_i, \eta) \cdot \partial\phi(-g(X_i)) + \psi^-(Z_i, \eta) \cdot \partial\phi(g(X_i)), \text{ and} \\ \Psi_2(Z_i, \eta, g_1, g_2) &:= \psi^+(Z_i, \eta) \cdot \partial^2\phi(-g_1(X_i)) + \psi^-(Z_i, \eta) \cdot \partial^2\phi(g_2(X_i)),\end{aligned}$$

where $\partial\phi(g)$ and $\partial^2\phi(g)$ denote the first- and second-order derivatives of ϕ , respectively. With these notations, we state the following conditions on the estimators of the nuisance parameter that enter the weights.

Assumption 5.2 (Nuisance Parameter Estimators): *There exists a sequence $\mathcal{S}_n$ of shrinking subsets such that $g_*, g_k, \hat{g} \in \mathcal{S}_n$ with probability at least $1 - \Delta_n$ such that $\Delta_n \rightarrow 0$ and the following conditions are satisfied.*

(i) **(Cross-Fitting)** *For each $\ell \in [m]$ and any $g_1, g_2 \in \mathcal{S}_n$, the cross-fitted nuisance parameter estimator satisfies*

$$\|\mathbb{E}_{n,(\ell)} [(\Psi_1(Z_i, \hat{\eta}_{(\ell)}, g_1) - \Psi_1(Z_i, \eta_0, g_*)) p_{1:k}(X_i)]\| = o_p(\sqrt{1/(n \log(k)^2)}), \quad (5.2)$$

$$\|\mathbb{E}_{n,(\ell)} [(\psi^\pm(Z_i, \hat{\eta}_{(\ell)}) - \psi^\pm(Z_i, \eta_0)) p_{1:k}(X_i) p_{1:k}(X_i)']\| = o_p(\xi_k \sqrt{\log(k)/n}), \quad (5.3)$$

$$\|\mathbb{E}_{n,(\ell)} [(\Psi_2(Z_i, \hat{\eta}_{(\ell)}, g_1, g_2) - \Psi_2(Z_i, \eta_0, g_*, g_*)) p_{1:k}(X_i) p_{1:k}(X_i)']\| = o_p(\xi_k \sqrt{\log(k)/n}), \quad (5.4)$$

$$\|\mathbb{E}_{n,(\ell)} [(\Psi_1(Z_i, \hat{\eta}_{(\ell)}, g_1)^2 - \Psi_1(Z_i, \eta_0, g_*)^2) p_{1:k}(X_i) p_{1:k}(X_i)']\| = o_p(\xi_k \sqrt{\log(k)/n}). \quad (5.5)$$

(ii) **(Full-Sample)** *For any $g_1, g_2 \in \mathcal{S}_n$ of g_* , the full-sample nuisance parameter estimator satisfies*

$$\|\mathbb{E}_n [(\Psi_1(Z_i, \hat{\eta}, g_1) - \Psi_1(Z_i, \eta_0, g_*)) p_{1:k}(X_i)]\| = o_p(\sqrt{1/(n \log(k)^2)}), \quad (5.6)$$

$$\|\mathbb{E}_n [(\psi^\pm(Z_i, \hat{\eta}) - \psi^\pm(Z_i, \eta_0)) p_{1:k}(X_i) p_{1:k}(X_i)']\| = o_p(\xi_k \sqrt{\log(k)/n}), \quad (5.7)$$

$$\|\mathbb{E}_n [(\Psi_2(Z_i, \hat{\eta}, g_1, g_2) - \Psi_2(Z_i, \eta_0, g_*, g_*)) p_{1:k}(X_i) p_{1:k}(X_i)']\| = o_p(\xi_k \sqrt{\log(k)/n}), \quad (5.8)$$

$$\|\mathbb{E}_n [(\Psi_1(Z_i, \hat{\eta}, g_1)^2 - \Psi_1(Z_i, \eta_0, g_*)^2) p_{1:k}(X_i) p_{1:k}(X_i)']\| = o_p(\xi_k \sqrt{\log(k)/n}). \quad (5.9)$$

Assumption 5.2 imposes rate restrictions on the estimation error of the subsample and full-sample estimators for η_0 . Conditions (5.2) and (5.6) are used for the Gaussian coupling of large-dimensional score vectors, Conditions (5.3)–(5.4) and (5.7)–(5.8) ensure the convergence rate of the sample Gram matrix, and Conditions (5.5) and (5.9) guarantee the convergence rate of the variance estimator.⁴

⁴The sample Gram matrix and the variance estimator will be introduced later in (5.10) and (5.11) respectively.

In the context of Example I (Maximum Score; Section 2.1) and Example II (Expected Utility Maximization; Section 2.2), there is no nuisance parameter η , so Assumption 5.2 is trivially satisfied. In the context of Example III (Welfare Maximization; Section 2.3), one can adopt alternative estimators subject to suitable low-level conditions to ensure that Assumption 5.2 holds. Such examples include the ℓ_1 -penalized and related methods in sparse models (Belloni, Chernozhukov, and Hansen, 2013) and other machine learning methods under appropriate conditions. For the current work, we focus on ℓ_1 -penalized methods for estimating our nuisance parameters using full-sample and sample-splitting techniques.

Finally, we impose a smoothness condition on the surrogate loss and the representative policy function to facilitate the derivation of their asymptotic properties.

Assumption 5.3 (Surrogate Loss): *The surrogate loss function ϕ is twice continuously differentiable.*

Assumption 5.4 (Representative Policy): *The representative classification policy function g_* is uniformly bounded on compact $\mathcal{X}$ and belongs to a Hölder ball $\Lambda^h(L)$ of smoothness order $h > 1/2$ and radius $L \in (0, \infty)$.⁵*

For standardization, we introduce the variance process $\sigma^2(x) = p_{1:k}(x)'Q^{-1}\Sigma Q^{-1}p_{1:k}(x)$, where

$$Q := \mathbb{E}[(\psi^+(Z_i) \cdot \partial^2 \phi(-g_*(X_i)) + \psi^-(Z_i) \cdot \partial^2 \phi(g_*(X_i)))p_{1:k}(X_i)p_{1:k}(X_i)'] \quad \text{and} \\ \Sigma := \mathbb{E}[(-\psi^+(Z_i) \cdot \partial \phi(-g_*(X_i)) + \psi^-(Z_i) \cdot \partial \phi(g_*(X_i)))^2 p_{1:k}(X_i)p_{1:k}(X_i)'].$$

We may estimate it by the sample counterpart $\hat{\sigma}^2(x) = n^{-1} \cdot_{1:k}(x)' \hat{Q}^{-1} \hat{\Sigma} \hat{Q}^{-1} p_{1:k}(x)$ where

$$\hat{Q} := \mathbb{E}_n[(\hat{\psi}^+(Z_i) \cdot \partial^2 \phi(-\hat{g}(X_i)) + \hat{\psi}^-(Z_i) \cdot \partial^2 \phi(\hat{g}(X_i)))p_{1:k}(X_i)p_{1:k}(X_i)'] \quad \text{and} \quad (5.10)$$

$$\hat{\Sigma} := \mathbb{E}_n[(-\hat{\psi}^+(Z_i) \cdot \partial \phi(-\hat{g}(X_i)) + \hat{\psi}^-(Z_i) \cdot \partial \phi(\hat{g}(X_i)))^2 p_{1:k}(X_i)p_{1:k}(X_i)']. \quad (5.11)$$

With the above definitions and notation in place, the following theorem establishes the uniform convergence rate and Gaussian coupling for the representative policy function estimator $\hat{g}$.

⁵See (B.1) in the appendix for the definition of the Hölder ball $\Lambda^h(L)$ of smoothness order h and radius L .

Theorem 5.1: *In addition to the conditions for Theorem 4.1 (ii), suppose that Assumptions 5.1, 5.2, 5.3 and 5.4 hold. Then, we have*

$$\sup_{x \in \mathcal{X}} |\widehat{g}(x) - g_*(x)| \lesssim_p \xi_k \sqrt{\log(k)/n} + B_k.$$

In addition, letting $\mathcal{N}_k \sim \mathcal{N}(\mathbf{0}_k, I_{k \times k})$, we have

$$\sqrt{n} \frac{(\widehat{g}(x) - g_*(x))}{\|\widehat{\sigma}(x)\|} =_d \frac{\sigma(x)'}{\|\sigma(x)\|} \mathcal{N}_k + o(1/\log(k)) \text{ uniformly for all } x \in \mathcal{X}, \quad (5.12)$$

provided that $\sup_{x \in \mathcal{X}} \sqrt{n} |r_g(x)| / \|\sigma(x)\| = o(1/\log(k))$ and $\log(k)^6 k^4 \xi_k^2 \log(n)^2 / n \rightarrow 0$.

A few remarks are in order. First, the point identification result from Section 4 is the key enabler of our asymptotic analysis and plays a crucial role in constructing valid inference procedures. Second, as a consequence, our estimator achieves the standard nonparametric rate of convergence established in the sieve literature, despite the originally non-standard problem involving partial identification. Third, in addition to the requirements stated in Theorem 4.1(ii), Assumption 5.3 requires the surrogate loss to be twice continuously differentiable. This condition still accommodates all of the aforementioned options, again except the hinge loss and sigmoid loss.

Since our nonparametric policy function estimator is strongly approximated by a Gaussian process whose distribution depends on the data only through the covariance, it is feasible to conduct uniform nonparametric inference using bootstrapped critical values. The details of this inference procedure for $g(\cdot)$ are presented in Section 5.3.

5.3 Bootstrap Inference for the Representative Policy Function

In this section, we discuss how to make statistical inference on the representative function g_* . While g_* may not be the primary parameter of interest, it shares the same classification boundary as any optimal classification functions of the original problem, therefore providing valuable information on, for instance, who should be treated. Furthermore, constructing uniform, not just pointwise, confidence bands is important because uncertainty in the classification function directly translates into uncertainty about treatment assignments in practice. When $g_*(x)$ is close to zero, binary classification (e.g., recommended treatment) will be sensitive to the estimation error for individuals with characteristics x . The uniform confidence bands in Theorems 5.2 and 5.3 control this uncertainty over the entire covariate space and distinguish

regions in which classification decisions are statistically supported from those in which the optimal classification remains uncertain.

Given Theorem 5.1, it is natural to consider the following t -statistic process:

$$t(x) := \frac{\widehat{g}(x) - g_*(x)}{\widehat{\sigma}(x)}, \quad x \in \mathcal{X}. \quad (5.13)$$

The sieve score bootstrap (Chen and Pouzo, 2015) can be employed to compute the relevant critical values. Specifically, let $\{\omega_i\}_i$ be i.i.d. standard Gaussian random variables, independent of the data. Consider the score-bootstrap t -statistic process

$$t_g^b(x) := \frac{p_{1:k}(x)' \widehat{Q}^{-1} \left\{ \mathbb{G}_n[\omega_i (-\widehat{\psi}^+(Z_i) \partial \phi(-\widehat{g}(X_i)) + \widehat{\psi}^-(Z_i) \partial \phi(\widehat{g}(X_i))) p_{1:k}(X_i)] \right\}}{\widehat{\sigma}(x)} \quad (5.14)$$

for $x \in \mathcal{X}$, where $\widehat{\psi}^\pm(\cdot)$ is defined in Section 5.1, $\widehat{Q}$ and $\widehat{\sigma}$ are defined in Section 5.2, and $\mathbb{G}_n[f] = \sqrt{n}(\mathbb{E}_n[f] - \mathbb{E}[f])$ denotes the empirical process under the i.i.d. sampling scheme.

To compute the critical value $cv_n(1 - \alpha)$ of the supremum statistic $\sup_{x \in \mathcal{X}} |t(x)|$ in (5.13), one can evaluate $\sup_{x \in \mathcal{X}} |t_g^b(x)|$ based on a large number of independent draws of $\{\omega_i\}_i$. The critical value $cv_n(1 - \alpha)$ is then approximated as

$$cv_n^b(1 - \alpha) := (1 - \alpha)\text{-quantile of } \sup_{x \in \mathcal{X}} |t_g^b(x)| \text{ conditional on the data.} \quad (5.15)$$

Construct the uniform confidence band (UCB) for the representative policy function g_* as

$$[\widehat{g}_l^b(x), \widehat{g}_u^b(x)] := \left[\widehat{g}(x) - cv_n^b(1 - \alpha) \cdot \widehat{\sigma}(x), \widehat{g}(x) + cv_n^b(1 - \alpha) \cdot \widehat{\sigma}(x) \right], \quad x \in \mathcal{X}, \quad (5.16)$$

where $cv_n^b(1 - \alpha)$ guarantees that $g_*(x) \in [\widehat{g}_l^b(x), \widehat{g}_u^b(x)]$ uniformly for all $x \in \mathcal{X}$ with asymptotic confidence level $100(1 - \alpha)\%$, as formally stated in the following theorem.

We introduce the notation $\Psi_1^*(Z_i, \omega_i, \eta, g) := \omega_i(-\psi^+(Z_i, \eta) \cdot \partial \phi(-g(X_i)) + \psi^-(Z_i, \eta) \cdot \partial \phi(g(X_i)))$, and the following assumption on the nuisance parameter estimation analogously to Assumption 5.2:

Assumption 5.5: *For any g belonging to a shrinking neighborhood $\mathcal{S}_n$ of g_* , the full-sample nuisance parameter estimator satisfies*

$$\|\mathbb{E}_n[(\Psi_1^*(Z_i, \omega_i, \widehat{\eta}, g) - \Psi_1^*(Z_i, \omega_i, \eta_0, g_*)) p_{1:k}(X_i)]\| = o_p(\sqrt{1/(n \log(k)^2)}).$$

The following theorem establishes the uniform probability coverage.

Theorem 5.2 (Uniform Inference via Two-Sided UCB): *In addition to the conditions of Theorem 5.1, suppose that Assumption 5.5 holds. With the critical value $cv_n^b(1 - \alpha)$ constructed in (5.15), the uniform confidence band (UCB) defined in (5.16) satisfies*

$$\mathbb{P} \{g_*(x) \in [\hat{g}_l^b(x), \hat{g}_u^b(x)] \text{ for all } x \in \mathcal{X}\} = 1 - \alpha + o(1). \quad (5.17)$$

There has been progress in developing inference methods for the parametric policy functions (Wu and Wang, 2021; Liang et al., 2022; Cheng and Yang, 2024). However, to the best of our knowledge, inference for nonparametric policy functions is first addressed in this paper, thereby filling an important gap in the literature.

To compute the one-sided critical values, one can evaluate $\sup_{x \in \mathcal{X}} t_g^b(x)$ and $\inf_{x \in \mathcal{X}} t_g^b(x)$ based on a large number of independent draws of $\{\omega_i\}_i$, as before. The upper and lower critical values, $cv_n^{1,b}(1 - \alpha)$ and $cv_n^{1,b}(\alpha)$, are then computed as

$$cv_n^{1,b}(1 - \alpha) := (1 - \alpha)\text{-quantile of } \sup_{x \in \mathcal{X}} t_g^b(x) \text{ conditional on the data and } \{\omega_i\} \quad \text{and} \quad (5.18)$$

$$cv_n^{1,b}(\alpha) := \alpha\text{-quantile of } \inf_{x \in \mathcal{X}} t_g^b(x) \text{ conditional on the data and } \{\omega_i\}. \quad (5.19)$$

We construct the one-sided uniform confidence bands (UCBs) for testing the uniform null hypotheses $\mathcal{H}_0 : g_*(x) \leq 0 \ \forall x \in \mathcal{X}$ or $\mathcal{H}_0 : g_*(x) \geq 0 \ \forall x \in \mathcal{X}$:

$$\hat{g}_{1,l}^b(x) := \hat{g}(x) - cv_n^{1,b}(1 - \alpha) \cdot \hat{\sigma}(x), \quad x \in \mathcal{X}, \quad (5.20)$$

$$\hat{g}_{1,u}^b(x) := \hat{g}(x) - cv_n^{1,b}(\alpha) \cdot \hat{\sigma}(x), \quad x \in \mathcal{X}. \quad (5.21)$$

Here, (5.18) and (5.20) ensure that $\sup_{x \in \mathcal{X}} \hat{g}_{1,l}^b(x) \leq 0$ with asymptotic confidence level $100(1 - \alpha)\%$ under $\mathcal{H}_0 : g_*(x) \leq 0 \ \forall x \in \mathcal{X}$. Similarly, (5.19) and (5.21) ensure that $\inf_{x \in \mathcal{X}} \hat{g}_{1,u}^b(x) \geq 0$ with asymptotic confidence level $100(1 - \alpha)\%$ under $\mathcal{H}_0 : g_*(x) \geq 0 \ \forall x \in \mathcal{X}$. This result is formalized in the theorem below.

Theorem 5.3 (Uniform Inference via One-Sided UCB): *In addition to the conditions of Theorem 5.1, suppose that Assumption 5.5 holds. With the critical values $cv_n^{1,b}(1 - \alpha)$ constructed in (5.18), the one-sided lower UCB defined in (5.20) satisfies*

$$\mathbb{P} \{\hat{g}_{1,l}^b(x) \leq 0 \text{ for all } x \in \mathcal{X}\} \geq 1 - \alpha + o(1) \quad \text{under } \mathcal{H}_0 : g_*(x) \leq 0 \text{ for all } x \in \mathcal{X}. \quad (5.22)$$

Similarly, with the critical values $cv_n^{1,b}(\alpha)$ constructed in (5.19), the one-sided upper UCB defined in (5.21) satisfies

$$\mathbb{P} \{\hat{g}_{1,u}^b(x) \geq 0 \text{ for all } x \in \mathcal{X}\} \geq 1 - \alpha + o(1) \quad \text{under } \mathcal{H}_0 : g_*(x) \geq 0 \text{ for all } x \in \mathcal{X}. \quad (5.23)$$

6 Estimation and Inference for the Optimal Policy Value

Suppose that a policymaker is interested in making inference on the optimal policy value of the form

$$V_0 := \sup_{g \in \mathcal{G}} V(g), \quad \text{where}$$

$$V(g) := \mathbb{E}[\psi_1(Z_i, \eta_0) \mathbf{1}\{g(X_i) \geq 0\} + \psi_0(Z_i, \eta_0) \mathbf{1}\{g(X_i) < 0\}],$$

with $\psi_1(z, \eta) := \bar{\psi}(z, \eta) - \psi^+(z, \eta)$ and $\psi_0(z, \eta) := \bar{\psi}(z, \eta) - \psi^-(z, \eta)$ for some function $\bar{\psi}$.

Note that under this setup, the welfare function V can be rewritten as

$$V(g) = \mathbb{E}[\bar{\psi}(Z_i, \eta_0)] - L(g),$$

where L is defined in (2.1). Therefore, the representative classification policy function g_* identified via the surrogate loss approach attains the maximum level of welfare within $\mathcal{G}$; that is, $V_0 = V(g_*)$. Although the representative policy function g_* is obtained by minimizing the surrogate risk L^* in (4.1) to ensure point identification, this surrogate risk L^* does not coincide with the *value* of the *original* risk L in (2.1). To estimate the optimal value V_0 , we need to evaluate the *original* risk L , rather than the surrogate risk L^* , at an estimate of g_* .

For instance, in the context of Example III discussed in Section 2.3, V_0 corresponds to the optimal welfare value in (2.5), with the functions ψ_1 and ψ_0 defined in (2.4) and $\bar{\psi}(z, \eta) = \psi_1^+(z, \eta) + \psi_0^-(z, \eta)$.

6.1 Estimation of the Optimal Policy Value

Let $\hat{g}$ denote the estimator of the representative classification policy function g_* , as obtained in Section 5.1. To account for the estimation of η , we employ a cross-fitting procedure to estimate the optimal policy value. Partition $[n] = \{1, \dots, n\}$ into m folds $I_1, \dots, I_m$. Assume that the folds are approximately balanced, i.e., $n_{(1)} \asymp \dots \asymp n_{(m)}$, where $n_{(\ell)} := |I_{(\ell)}|$, and that m is fixed. For a given fold $\ell \in [m] = \{1, \dots, m\}$, obtain an estimator $\hat{\eta}_{(\ell)}$ of η_0 using the complementary subsample $I_{(\ell)}^c$. We then estimate the optimal policy value V_0 via the plug-in cross-fitted estimator

$$\hat{V}_n(\hat{g}) := \frac{1}{n} \sum_{\ell=1}^m \sum_{i \in I_{(\ell)}} \left[\hat{\psi}_{1,(\ell)}(Z_i) \mathbf{1}\{-\hat{g}(X_i) \leq 0\} + \hat{\psi}_{0,(\ell)}(Z_i) \mathbf{1}\{\hat{g}(X_i) < 0\} \right], \quad (6.1)$$

where $\widehat{\psi}_{1,(\ell)}(Z_i) := \psi_1(Z_i, \widehat{\eta}_{(\ell)})$ and $\widehat{\psi}_{0,(\ell)}(Z_i) := \psi_0(Z_i, \widehat{\eta}_{(\ell)})$ for all $i \in I_{(\ell)}$ and each $\ell \in [m]$.

In the context of Example III in Section 2.3, the estimated weights take the form

$$\begin{aligned}\widehat{\psi}_{1,(\ell)}(Z_i) &= \frac{A_i(Y_i - \widehat{\mu}_{(\ell)}(1, X_i))}{\widehat{\pi}_{(\ell)}(X_i)} + \widehat{\mu}_{(\ell)}(1, X_i) \quad \text{and} \\ \widehat{\psi}_{0,(\ell)}(Z_i) &= \frac{(1 - A_i)(Y_i - \widehat{\mu}_{(\ell)}(0, X_i))}{1 - \widehat{\pi}_{(\ell)}(X_i)} + \widehat{\mu}_{(\ell)}(0, X_i),\end{aligned}$$

where $\widehat{\eta}_{(\ell)} = (\widehat{\mu}_{(\ell)}(1, \cdot), \widehat{\mu}_{(\ell)}(0, \cdot), \widehat{\pi}_{(\ell)}(\cdot))$ denotes the estimator of the conditional mean functions $\mu(1, \cdot)$ and $\mu(0, \cdot)$, as well as the propensity score function $\pi(\cdot)$, constructed using the complementary subsample $I_{(\ell)}^c$ for all $i \in I_{(\ell)}$.

6.2 Asymptotic Properties for the Optimal Policy Value

The optimal policy value estimator (6.1) is constructed via a plug-in rule using our estimator $\widehat{g}$ of the representative policy function g_* , as given in Section 5.1. For valid inference, $\widehat{g}$ must satisfy conditions ensuring that both its estimation error and its surrogate misspecification error are dominated and hence do not affect the limiting distribution of the policy value estimator. Theorem 5.1 shows that our estimator $\widehat{g}$ converges at the rate $\xi_k \sqrt{\log(k)/n}$. We verify below that, under a set of mild conditions, this nonparametric rate suffices to ensure its impact on the distribution of the policy value estimator is asymptotically negligible.

Assumption 6.1 (Margin Condition): *The random variable $U_i := \mathbb{E}[\psi_1(Z_i, \eta_0)|X_i] - \mathbb{E}[\psi_0(Z_i, \eta_0)|X_i]$ admits a continuous probability density function f_U over a small neighborhood $[-\delta, \delta]$ of zero.*

Assumption 6.1 is a standard margin condition in the policy learning literature; see, for example, Kitagawa and Tetenov (2018).⁶ Continuity of f_U around zero implies

$$\mathbb{P}(|U_i| < t) = O(t) \quad \text{and} \quad \mathbb{E}[|U_i| \cdot \mathbf{1}_{|U_i| < t}] = O(t^2),$$

for sufficiently small $t > 0$. We use this condition only as a convenient regularity condition in our analysis, as it is often considered innocuous.

⁶Ponomarev and Semenova (2024) show that, when the margin condition fails, lower confidence bounds based on suitably chosen suboptimal policies may first-order dominate those based on efficient estimators of optimal welfare. In this paper, we focus on the computational advantage of the surrogate approach under standard conditions and will further explore such trade-off in future research.

For convenience in stating the next assumption on nuisance parameter estimation, define the $L_2(\mathbb{P})$ norm by $\|f\|_{\mathbb{P},2} := (\mathbb{E}[\|f(X)\|^2])^{1/2}$ where the norm $\|\cdot\|$ inside the expectation denotes the Euclidean norm.

Assumption 6.2: (i) (**Nuisance Parameter Estimation**) $\|\hat{\eta}_{(\ell)} - \eta_0\|_{\mathbb{P},2} = o_p(n^{-1/4})$ for each $\ell \in [m]$. Furthermore, with probability at least $1 - \epsilon_n$ for $\epsilon_n = o(1)$, $\hat{\eta}_{(\ell)}$ belongs to a shrinking neighborhood $\mathcal{T}_n$ of η_0 for each $\ell \in [m]$, such that the following convergence results hold:

$$\begin{aligned} \sqrt{n} \sup_{\eta \in \mathcal{T}_n} |\mathbb{E}[(\psi_1(Z_i, \eta) - \psi_1(Z_i, \eta_0)) \mathbf{1}(-g_k(X_i) < 0)]| &= o(1), \\ \sqrt{n} \sup_{\eta \in \mathcal{T}_n} |\mathbb{E}[(\psi_0(Z_i, \eta) - \psi_0(Z_i, \eta_0)) \mathbf{1}(g_k(X_i) < 0)]| &= o(1), \\ \sqrt{n} \sup_{\eta \in \mathcal{T}_n} (\mathbb{E}[|(\psi_1(Z_i, \eta) - \psi_1(Z_i, \eta_0)) \mathbf{1}(-g_k(X_i) < 0)|^2])^{1/2} &= o(1), \text{ and} \\ \sqrt{n} \sup_{\eta \in \mathcal{T}_n} (\mathbb{E}[|(\psi_0(Z_i, \eta) - \psi_0(Z_i, \eta_0)) \mathbf{1}(g_k(X_i) < 0)|^2])^{1/2} &= o(1). \end{aligned}$$

(ii) (**Boundedness**) For $j = 0, 1$, and uniformly over the shrinking neighborhood $\mathcal{T}_n$ of η_0 , $\mathbb{E}[|\psi_j(Z_i, \eta)|^q]$ is bounded for some $q > 4$, and $\mathbb{E}[\psi_j(Z_i, \eta)|X_i = x]$ is bounded over the support of X .

Assumption 6.2(i) requires that the cross-fitted estimator $\hat{\eta}$ of the nuisance parameter induces asymptotically negligible estimation errors. This condition has been also employed by [Athey and Wager \(2021, Assumption 2\)](#) in the context of Example III, and can be achieved by employing a Neyman-orthogonal score, which ensures that first-step estimation errors are asymptotically negligible provided that $\hat{\eta}_{(\ell)}$ converges at the $n^{-1/4}$ rate, rather than the parametric $n^{-1/2}$ rate. Such convergence rates are attainable with a variety of machine learning methods. For example, the Lasso achieves the $n^{-1/4}$ rate under approximate sparsity and restricted eigenvalue conditions ([Bickel, Ritov, and Tsybakov, 2009](#); [Belloni, Chernozhukov, Chetverikov, and Kato, 2015](#)). Random forests and boosted trees are valid provided the regression function is sufficiently smooth and sample splitting is employed ([Bühlmann and Yu, 2002](#); [Wager and Athey, 2018](#)). Deep neural networks with controlled depth and bounded weight norms can approximate smooth functions at nearly minimax rates ([Chen and White, 1999](#); [Yarotsky, 2017](#); [Schmidt-Hieber, 2020](#)), among others. Assumption 6.2(ii) further imposes uniform bounds on the (conditional) expectations of weight functions, ψ_1 and ψ_0 . Note that

this condition allows for unbounded outcome variable Y in the welfare maximization (Example III) and is also satisfied in the expected utility maximization with binary action (Example II) under the Condition 1 of [Elliott and Lieli \(2013\)](#). It holds trivially in the maximum score estimation (Example I).

Under these rate conditions and the boundedness conditions, we obtain the following asymptotic normality result for our estimator of the optimal policy value.

Theorem 6.1: *In addition to the conditions for Theorem 5.1, suppose that Assumptions 6.1 and 6.2 hold, and the sieve order k satisfies $k^{4+\varepsilon}/n \lesssim 1$ for some $\varepsilon > 0$. Then, for any sequence of data generating processes $\mathbb{P}_n$, we have*

$$\sqrt{n}\{\widehat{V}_n(\widehat{g}) - V_0\} \rightarrow_d \mathcal{N}(0, \sigma_v^2),$$

where $\sigma_v^2 = \mathbb{E}[s_L(Z_i, \eta_0, g_*)^2]$ is the variance of the score

$$s_L(Z_i, \eta_0, g_*) = \psi_1(Z_i, \eta_0) \cdot \mathbf{1}(g_*(X_i) \geq 0) + \psi_0(Z_i, \eta_0) \cdot \mathbf{1}(g_*(X_i) < 0) - V_0.$$

The asymptotic Gaussianity established in Theorem 6.1 enables inference on the optimal policy value V_0 in the standard manner, i.e., using normal critical values such as 1.96. To approximate the asymptotic variance σ_v^2 in practice, we introduce a bootstrap procedure in Section 6.3.

While the proof in the appendix formally establishes the results of Theorem 6.1, we provide the intuition here, particularly illustrating why the estimation effect of our classification policy function estimator $\widehat{g}$ as given in Section 5.1 is asymptotically negligible despite the fact that it appears in the indicator function and converges at a slower rate than the root- n rate. Consider the decomposition:

$$\sqrt{n}(\widehat{V}_n(\widehat{g}) - V_0) = \sqrt{n}(\widehat{V}_n(\widehat{g}) - \widehat{V}_n(g_k)) \tag{6.2}$$

$$+ \sqrt{n}(\widehat{V}_n(g_k) - V(g_k)) \tag{6.3}$$

$$+ \sqrt{n}(V(g_k) - V(g_*)) \tag{6.4}$$

$$+ \sqrt{n}(V(g_*) - V_0), \tag{6.5}$$

where the term in (6.2) captures the estimation error of $\widehat{g}$, the term in (6.3) collects the sampling error from $\mathbb{E}_n$ and the estimation error of $\widehat{\eta}$, the term (6.4) indicates the sieve approximation error of g_k for g_* , and the term in (6.5) measures the identification error. Theorem 4.1 implies

that the identification error in (6.5) is zero. The component in (6.3) corresponds to the standard semiparametric part and is asymptotically Gaussian under standard assumptions, in particular under ours. The remaining task is to show that the first component in (6.2) is asymptotically negligible. Since this argument is not entirely obvious, we provide the intuition here in the main text, while leaving the formal proof to the appendix.

The component in (6.2) in question can be further decomposed as

$$(6.2) = \sqrt{n} \left[\widehat{V}_n(\widehat{g}) - \widehat{V}_n(g_k) - \{V(\widehat{g}) - V(g_k)\} \right] + \sqrt{n}[V(g_*) - V(g_k)] \quad (6.6)$$

$$+ \sqrt{n}\{V(\widehat{g}) - V(g_*)\}. \quad (6.7)$$

The two bracket groups on the right-hand side of (6.6) can be shown to be asymptotically negligible by applying the maximal inequality for VC classes (Massart and Nédélec, 2006, Lemma A.3), Sauer’s Lemma (Lugosi, 2002), and the dominated sieve approximation errors. The term in (6.7) is more delicate, since $\widehat{g}$ converges only at the nonparametric rate as obtained in Theorem 5.1. Nevertheless, this term is also negligible because its first-order effect vanishes due to the first-order condition for the optimality of g_* with respect to V . As a result, the second-order term becomes the leading component, and its existence is guaranteed by the margin condition (Assumption 6.1). We further illustrate this point through numerical simulations in Section 7.2. In particular, we show that the plug-in estimator $\widehat{V}_n(\widehat{g})$ and the oracle estimator $\widehat{V}_n(g_*)$ yield nearly identical confidence intervals.

In summary, the component in (6.3) drives the asymptotic normality in the same way as in standard semiparametric analysis, while all other components are asymptotically negligible. See Appendix A.3–A.4 for further details. A couple of remarks are in order.

Remark 1: *Although the plug-in value estimator in (6.1) is constructed using the estimator $\widehat{g}$ described in Section 5.1, our inferential results do not rely on the specific form of this estimator. What is required for valid second-stage inference is that the first-stage policy function estimator converges sufficiently fast so that its estimation error becomes asymptotically negligible. Indeed, with slight modifications on the proof, we can show that any estimator $\widetilde{g}$ would be admissible if it converges uniformly over $\mathcal{X}$ to a limit g_{**} satisfying $V(g_{**}) = V_0$ at a rate faster than $o_p(n^{-1/4})$. That is, our framework accommodates a broad class of first-stage methods, including ℓ_1 -penalized estimators (Athey and Wager, 2021), parametric approaches based on the 0–1 loss (e.g., Kitagawa and Tetenov, 2018), and nonparametric CATE-based methods*

(e.g., [Bhattacharya and Dupas, 2012](#); [Armstrong and Shen, 2023](#); [Park, 2025](#)), provided they achieve the uniform convergence rate $o_p(n^{-1/4})$ to $\mathbb{E}[\psi_1(Z, \eta_0) \mid X = x] - \mathbb{E}[\psi_0(Z, \eta_0) \mid X = x]$. Thus, while our construction employs the estimator $\hat{g}$ for the representative classification policy g_* , the resulting limit theory supports valid inference for a wide range of first-stage estimators used in the policy learning literature.

Remark 2: Our optimal policy value estimation is related to semiparametric estimation with a non-smooth criterion ([Chen et al., 2003](#)). Our framework extends this by allowing not only non-smooth but also discontinuous criteria. Even when the loss, and hence the sample criterion, is discontinuous, the asymptotically Gaussian term dominates provided that the population criterion is smooth, as illustrated in the intuition above. In particular, the first-order condition for the optimality of g_* plays a crucial role in ensuring that the potentially problematic term in (6.7) is in fact negligible. $\blacktriangle$

Remark 3: [Hirano and Porter \(2012, Section 2.4\)](#) highlight that non-differentiability in a population criterion function prevents regular estimation. In particular, such an impossibility phenomenon arises in estimating $\phi(\theta) = \max\{\theta_1, \theta_2\}$, which is non-differentiable at the kink point $\theta_1 = \theta_2$. This kink causes the plug-in estimator $\max\{\hat{\theta}_1, \hat{\theta}_2\}$ to have a non-Gaussian distribution when $\theta_1 = \theta_2$, even if $(\hat{\theta}_1, \hat{\theta}_2)$ were Gaussian and centered at (θ_1, θ_2) .

In contrast, our welfare criterion can be rewritten as

$$\begin{aligned} \sup_{g \in \mathcal{G}} V(g) &= \sup_{g \in \mathcal{G}} \mathbb{E}[\mathbb{E}[\psi_1(Z_i, \eta_0) \mid X_i] \cdot \mathbf{1}\{g(X_i) \geq 0\} + \mathbb{E}[\psi_0(Z_i, \eta_0) \mid X_i] \cdot \mathbf{1}\{g(X_i) < 0\}] \\ &= \mathbb{E}[\max\{\vartheta_1, \vartheta_2\}], \end{aligned}$$

rather than $\max\{\theta_1, \theta_2\}$, where $\vartheta_1 := \mathbb{E}[\psi_1(Z_i, \eta_0) \mid X_i]$ and $\vartheta_2 := \mathbb{E}[\psi_0(Z_i, \eta_0) \mid X_i]$ are random variables. Hence, the two problems differ mathematically. As emphasized in [Remark 2](#), given the smoothness of the population objective, asymptotic normality is attainable even with a non-smooth sample criterion, with the first-order condition for the optimality of g_* playing the crucial role. $\blacktriangle$

6.3 Bootstrap Inference for the Optimal Policy Value

Theorem [6.1](#) facilitates standard inference based on Gaussian critical values, e.g., ≈ 1.96 . We test the null hypothesis $\mathcal{H}_0 : V_0 = v_0$ using the statistic

$$T_{g^*, L} := \sqrt{n}(\hat{V}_n(\hat{g}) - v_0).$$

To approximate the variance σ_v^2 of its Gaussian limit, we employ the bootstrap counterpart

$$\tilde{Z}_n := \mathbb{G}_n [\hat{s}_L(Z_i, \hat{\eta}_{(\ell(i))}, \hat{g}) \cdot \delta_i],$$

where the estimated score takes the form of

$$\hat{s}_L(Z_i, \hat{\eta}_{(\ell(i))}, \hat{g}) = \psi_1(Z_i, \hat{\eta}_{(\ell(i))}) \cdot \mathbf{1}(\hat{g}(X_i) \geq 0) + \psi_0(Z_i, \hat{\eta}_{(\ell(i))}) \cdot \mathbf{1}(\hat{g}(X_i) < 0) - \hat{V}_n(\hat{g}),$$

$\hat{\eta}_{(\ell(i))}$ denotes the nuisance parameter estimator based on the complementary subsample $I_{(\ell(i))}^c$ of the fold $I_{(\ell(i))}$ that contains observation i , and the perturbation process $\{\delta_i\}_{i \in [n]}$ is a sequence of i.i.d. standard Gaussian bootstrap weights independent of the data.

The following theorem establishes the consistency of the score bootstrap method for approximating the limiting Gaussian distribution of the estimated optimal policy value.

Theorem 6.2: *In addition to the conditions of Theorem 6.1, suppose that the perturbation process $\{\delta_i\}_{i \in [n]}$ is a sequence of i.i.d. standard Gaussian bootstrap weights independent of the data. Given the data generating process $\mathbb{P}_n$,*

$$\sup_{t \in \mathbb{R}} \left| \mathbb{P}(\tilde{Z}_n \leq t \mid Z_1, \dots, Z_n) - \Phi\left(\frac{t}{\sigma_v}\right) \right| \rightarrow_p 0,$$

in which Φ denotes the standard normal distribution function and σ_v^2 was defined in Theorem 6.1. Thus, conditional on the observed data, the distribution of $\tilde{Z}_n$ consistently approximates $\mathcal{N}(0, \sigma_v^2)$.

For a fixed $\alpha \in (0, 1)$, let $\hat{c}_{1-\alpha}$ denote the $(1 - \alpha)$ -quantile of $\tilde{Z}_n$. A consequence of Theorem 6.2 is the asymptotic coverage property

$$\liminf_{n \rightarrow \infty} \mathbb{P}\left(\sqrt{n}(\hat{V}_n(\hat{g}) - v_0) \leq \hat{c}_{1-\alpha}\right) \geq 1 - \alpha,$$

under the null hypothesis $\mathcal{H}_0 : V_0 \leq v_0$ against alternative hypothesis $\mathcal{H}_1 : V_0 > v_0$.

7 Simulation Studies

In this section, we use numerical simulations to evaluate the finite-sample performance of our proposed inference methods for cost-sensitive binary classification. As an illustrative example, we focus on the welfare maximization problem (Example III) introduced in Section 2.3. Within

this context, we examine both the performance of our proposed uniform inference procedure for the optimal treatment assignment (classification) policy and the inference method for the optimal welfare (policy value).

The outcome is modeled as $Y_i = A_i \cdot \Delta(X_i) + S(X_i) + u_i$, where the idiosyncratic error term is generated as $u_i \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(0, 1)$. The binary treatment assignment is given by $A_i = 2 \cdot \text{Bernoulli}(\pi(X_i)) - 1$, so that $A_i \in \{-1, 1\}$. Half of the conditional average treatment effect is specified as $\Delta(X_i) = \tanh((1, X_i')\gamma)$. The baseline function is specified as $S(X_i) = \sin(X_i'\beta_S)$, and the propensity score as $\pi(X_i) = \frac{\exp(X_i'\beta_\pi)}{1 + \exp(X_i'\beta_\pi)}$.

We consider a nonlinear design featuring two covariates. The first covariate, $X_{i1} \sim \text{Uniform}[0, 1]$, represents normalized pre-treatment income. The second covariate, X_{i2} , is constructed by rescaling a discrete variable that mimics years of education. Specifically, for X_{i2} , we draw a categorical variable taking values in $\{7, \dots, 18\}$ with probabilities calibrated to the empirical distribution of education levels in the JTPA data, and divide by 18 to ensure support on the unit interval.

The parameter values of γ are set to be consistent with the null hypotheses under consideration. For the one-sided null hypothesis $\mathcal{H}_0 : g_*(x) \leq 0 \ \forall x \in \mathcal{X}$, we consider the setting $\gamma = (0, -1/\sqrt{n}, -1/\sqrt{n})'$, with $\beta_S = (-1, 1)'$ and $\beta_\pi = (1, -1)'$. For the null $\mathcal{H}_0 : g_*(x) \geq 0 \ \forall x \in \mathcal{X}$, on the other hand, we instead use the setting $\gamma = (0, 1/\sqrt{n}, 1/\sqrt{n})'$, with $\beta_S = (-1, 1)'$ and $\beta_\pi = (1, -1)'$. Extensive simulations are conducted with sample sizes $n \in \{250, 500, 1000\}$, and each configuration is replicated $S = 1,000$ times through Monte Carlo iterations. The number of cross-fitting folds is set to $m = 2$ for estimating both the policy and welfare.

For nuisance parameters, the simulation design incorporates the conditional mean outcome functions $\mu(a; x)$ for $a \in \{-1, 1\}$, estimated via Lasso, and the propensity score function $\pi(x)$, estimated via ℓ_1 -penalized logistic regression. The tuning parameters for both procedures are chosen by five-fold cross-validation.

The representative classification policy function is estimated using a sieve approximation based on a tensor-product Legendre polynomial basis. Specifically, for $k \in \{2, 3, 4\}$, we employ the tensor-product basis constructed from the orthonormal Legendre polynomials $\{p_j(x)\}_{j=0}^\infty$, evaluated directly at the rescaled covariate values in $[0, 1]$. The sieve basis is given by $p_{0:k}(x_1, x_2) = \{p_j(x_1)p_l(x_2) : j, l \in \{0, \dots, k\}\}$, and the policy function is approximated as

$$g_k(x_1, x_2) = \sum_{j=0}^k \sum_{l=0}^k \beta_{jl} p_j(x_1) p_l(x_2),$$

(I) $\mathcal{H}_0: g_*(x) \leq 0$ for all $x \in \mathcal{C}$ with $\gamma = (0, -1/\sqrt{n}, -1/\sqrt{n})'$									
Sample Size	Logistic Loss			Exponential Loss			Squared Loss		
	$k = 2$	$k = 3$	$k = 4$	$k = 2$	$k = 3$	$k = 4$	$k = 2$	$k = 3$	$k = 4$
250	0.977	0.992	0.994	0.971	0.979	0.975	0.965	0.983	0.981
500	0.987	0.995	0.999	0.977	0.994	0.996	0.972	0.994	0.995
1,000	0.993	0.998	0.999	0.988	0.997	0.995	0.987	0.996	0.993

(II) $\mathcal{H}_0: g_*(x) \geq 0$ for all $x \in \mathcal{C}$ with $\gamma = (0, 1/\sqrt{n}, 1/\sqrt{n})'$									
Sample Size	Logistic Loss			Exponential Loss			Squared Loss		
	$k = 2$	$k = 3$	$k = 4$	$k = 2$	$k = 3$	$k = 4$	$k = 2$	$k = 3$	$k = 4$
250	0.977	0.989	0.995	0.965	0.971	0.974	0.959	0.977	0.972
500	0.992	0.995	0.996	0.985	0.988	0.992	0.976	0.987	0.989
1,000	0.992	0.997	0.998	0.991	0.997	0.995	0.983	0.996	0.991

Table 2: Empirical coverage frequencies for the one-sided uniform tests of the optimal policy rule.

or, equivalently, $g_k(x_1, x_2) = p_{0:k}(x_1, x_2)'\beta$.

To remain consistent with the theoretical conditions required for the asymptotic analysis, the simulation study focuses exclusively on twice continuously differentiable surrogate loss functions. We consider the following three examples:

(a) Logistic loss: $\phi(x) = \log(1 + e^{-x})$;

(b) Exponential loss: $\phi(x) = e^{-x}$; and

(c) Squared loss: $\phi(x) = (1 - x)^2$.

7.1 Uniform Inference for Optimal Policy

This subsection presents the finite-sample performance of our proposed uniform inference method for the optimal policy rule. We consider the two null hypotheses: $\mathcal{H}_0 : g_*(x) \leq 0 \ \forall x \in \mathcal{C}$ and $\mathcal{H}_0 : g_*(x) \geq 0 \ \forall x \in \mathcal{C}$, where the set $\mathcal{C}$ is defined as $\mathcal{C} := [0.05, 0.95] \times \{10/18\}$.

Table 2 reports finite-sample coverage frequencies for the one-sided uniform tests across sample sizes $n \in \{250, 500, 1000\}$, sieve orders $k \in \{2, 3, 4\}$, and the three surrogate loss

functions. The top panel (I) corresponds to the null hypothesis $\mathcal{H}_0 : g_*(x) \leq 0 \ \forall x \in \mathcal{C}$, while the bottom panel (II) corresponds to the null hypothesis $\mathcal{H}_0 : g_*(x) \geq 0 \ \forall x \in \mathcal{C}$.

The results confirm that our bootstrap uniform confidence bands successfully control the test size. The size performance is slightly conservative, which is attributable to the fact that the true data-generating functions lie in the interior of the null, far from the least favorable boundary points of the null.

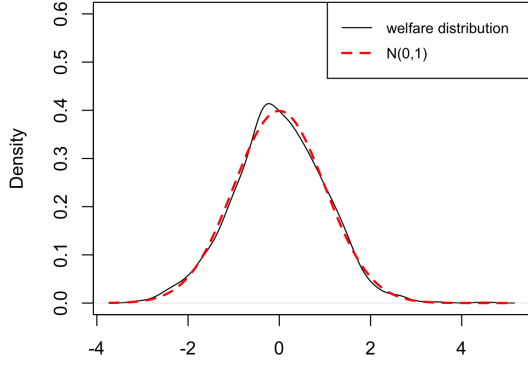
7.2 Gaussian Limit and Inference for the Optimal Welfare

This subsection demonstrates the Gaussian limit of our welfare estimator via numerical studies. In addition to (a) the plug-in estimated optimal policy rule, we also study the Gaussian limit theory of welfare under three comparative benchmarks: (b) an oracle treatment assignment rule, (c) a treat-everyone policy, and (d) a random-design policy. The random-design policy (d) assigns treatment according to an independent Bernoulli(0.5) distribution. The welfare experiments use $\gamma = (0, -1, 1)'$, so that $\Delta(x) = \tanh(-x_1 + x_2)$. The other components of the DGP remain as described above.

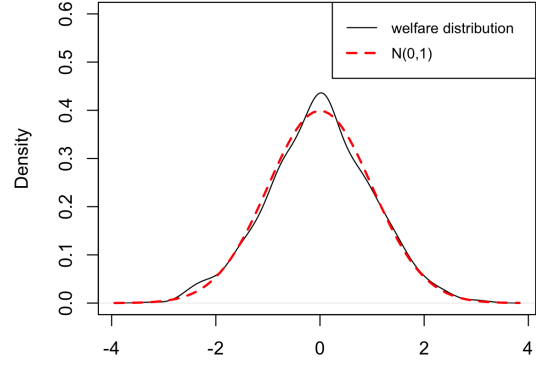
For each of the four scenarios (a)–(d), we normalize the estimated optimal welfare using the Monte Carlo mean and standard deviation estimate computed across 1,000 simulation replications. We then compare the simulated densities of these normalized estimates with that of the standard normal distribution for each scenario. The sample size is fixed at $n = 500$ throughout. Figure 1 displays the simulation density of the estimated optimal welfare in solid black, alongside the standard normal density in dashed red.

Each plotted density from (a) to (d) closely resembles the standard normal distribution. In particular, the result in panel (a) for the plug-in estimated optimal welfare corroborates the theoretical property of Gaussian limit behavior established in Theorem 6.2. Panel (a) is comparable to panel (b) for the oracle counterpart, and this observation is consistent with the asymptotic negligibility of the preliminary representative classification policy estimation emphasized in Section 6.2. Although we report one set of results based on the logistic loss, these findings hold robustly across different surrogate loss functions.

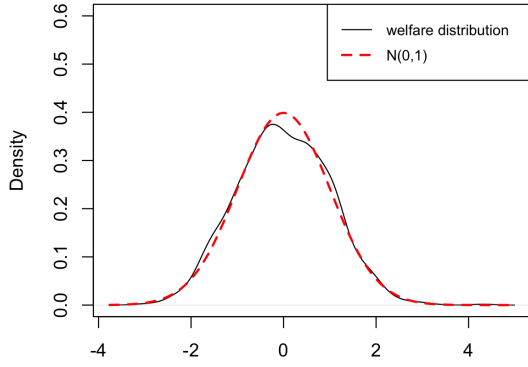
Figure 2 reports the median, interquartile, and interdecile ranges of empirical welfare estimates under four treatment assignment rules: (a) the plug-in optimal policy estimated using the logistic loss (*Est. Optimal*); (b) the oracle assignment rule (*Oracle Optimal*); (c) the treat-



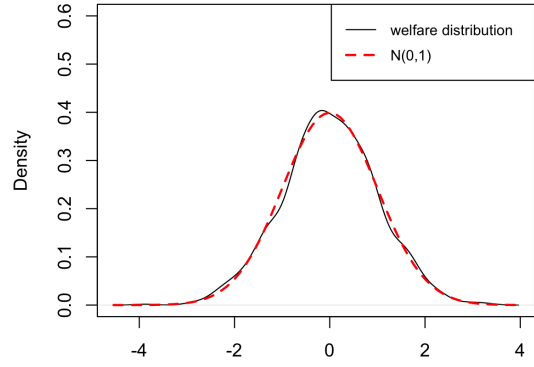
(a) Estimated optimal welfare



(b) Oracle optimal welfare



(c) Welfare with everyone treated



(d) Welfare with random assignment

Figure 1: Empirical densities of normalized welfare and standard normal density ($n = 500$ and $d_x = 2$).

everyone rule (*Everyone*); and (d) the random assignment rule (*Random Assign*). The results are based on $n = 500$ observations and $S = 1,000$ Monte Carlo replications. The distribution under the estimated optimal policy closely resembles that under the oracle rule, corroborating the asymptotic negligibility of $\hat{g}$ for $\sqrt{n}(\hat{V}_n(\hat{g}) - V_0)$ as argued in Section 6.2. Moreover, the welfare estimates under both the estimated optimal policy and the oracle policy substantially

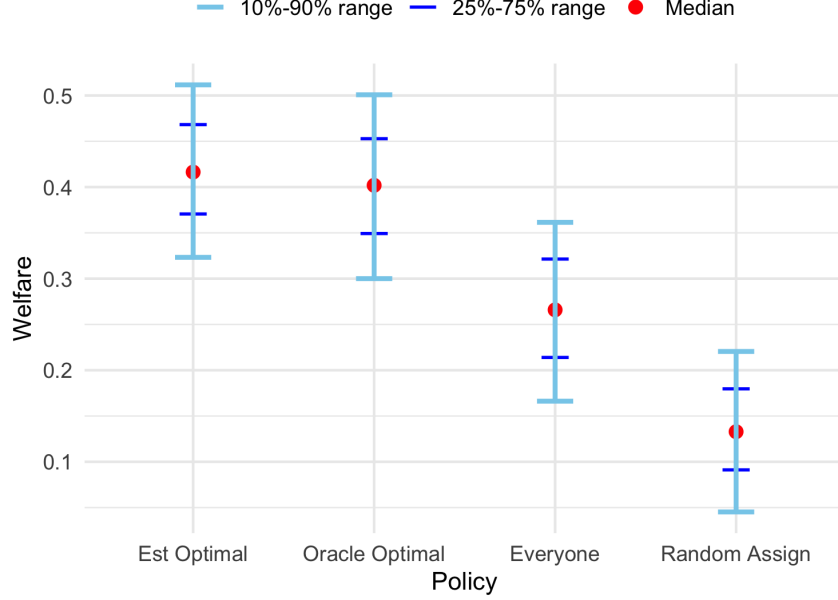


Figure 2: Median, interquartile range, and interdecile range of the empirical welfare estimates under the estimated optimal, oracle optimal, treat-all, and random assignment rules.

exceed those attained under the treat-everyone and random assignment rules.

Finally, we conduct hypothesis tests to compare the welfare achieved under the estimated optimal policy to that under the benchmark policies $g_{\dagger}$.⁷ Specifically, we perform a two-sided test of $\mathcal{H}_0 : V_0 = V(g_{\dagger})$ against $\mathcal{H}_1 : V_0 \neq V(g_{\dagger})$, as well as one-sided tests of $\mathcal{H}_0 : V_0 \leq V(g_{\dagger})$ versus $\mathcal{H}_1 : V_0 > V(g_{\dagger})$ and $\mathcal{H}_0 : V_0 \geq V(g_{\dagger})$ versus $\mathcal{H}_1 : V_0 < V(g_{\dagger})$. Table 3 reports the Monte Carlo rejection frequencies. The two-sided and right-sided tests reject their corresponding null hypotheses in most Monte Carlo draws, indicating that the estimated optimal policy delivers a significantly higher level of welfare.

8 Empirical Analysis

With the “identification”, estimation, and inference methods developed in this paper, we now revisit the problem of policy learning using the experimental data from the National JTPA Study. While prior studies such as Kitagawa and Tetenov (2018) and Mbakop and Tabord-Meehan (2021) have analyzed this problem, they did not conduct formal inference for the

⁷See Appendix C for details on this testing procedure.

	Benchmark policy $g_{\dagger}$	
	Treat Everyone	Random Treat
Two-sided test	0.790	0.990
Left-sided test	0.000	0.000
Right-sided test	0.881	0.996

Table 3: Monte Carlo rejection frequencies from hypothesis tests comparing the welfare under the optimal policy to that under benchmark policies.

optimal treatment assignment (classification) policy or its associated welfare (policy value). Our goal is to build on this literature and provide inference for these important features.

The JTPA study randomly assigned applicants to be eligible for a job training program for a period of 18 months. It collected background information on the applicants prior to random assignment, as well as administrative and survey data on their earnings over the 30 months following the assignment. The dataset consists of 11,008 observations from the adult sample (ages 22 and older).

Following benchmark studies in the literature, we adopt the following variable definitions. The outcome variable, Y_i , is defined in two ways: (i) total individual earnings during the 30 months following program assignment, or (ii) total individual earnings during the 30 months following program assignment minus the average cost of services per treatment assignment (\$774) for individuals assigned to treatment. The covariates, X_i , used to define the policy class are *years of education* and *pre-program earnings*.⁸

We consider two types of policy functions: (a) a linear rule $g(x) = \gamma'x$; and (b) a class of nonlinear functions defined as follows: Let $\mathcal{X}_1$ denote the covariate space of *years of education*, and let $\mathcal{X}_2$ denote the covariate space of *pre-program earnings*. The set of nonlinear assignment rules is given by

$$\mathcal{G} = \{ g(x_1, x_2) = f(x_1) - x_2 \mid f : \mathcal{X}_1 \rightarrow \mathcal{X}_2 \}.$$

Linear and nonlinear rules have been studied in the literature (Kitagawa and Tetenov, 2018; Mbakop and Tabord-Meehan, 2021). In particular, while the nonlinear class follows Mbakop and Tabord-Meehan (2021), we allow for a more general specification without imposing their sign restriction ($f(x_1) \geq x_2$). We can accommodate this generality by leveraging our method’s

⁸Years of education are divided by their sample maximum, while pre-program earnings are rescaled by their sample range as described below.

ability to achieve nonparametric point “identification,” along with the associated estimation and inference procedures. That said, the main contribution of this paper lies in its inference component, which has not been addressed by these preceding papers.

The optimal policy function is approximated using a sieve expansion based on tensor-product Legendre polynomial basis functions with sieve order $k = 3$. The number of folds is set to $m = 2$. The nuisance functions are estimated via machine learning methods: the propensity score is obtained using ℓ_1 -penalized logistic regression, and the conditional mean outcome is estimated using Lasso. In both cases, tuning parameters are selected via cross-validation. For inference, we implement a score-based bootstrap procedure with 1,000 repetitions to construct confidence bands for the optimal policy and a confidence interval for its corresponding welfare.

To ensure that the approximation error of Legendre polynomials diminishes in the L_∞ space, we normalize both covariates to lie within the unit interval $[0, 1]$. Specifically, we define x_1 as the *years of education* divided by its maximum value, and x_2 as normalized pre-treatment earnings, computed as

$$x_2 = \frac{\text{pre-treatment earnings} - \min(\text{pre-treatment earnings})}{\max(\text{pre-treatment earnings}) - \min(\text{pre-treatment earnings})}.$$

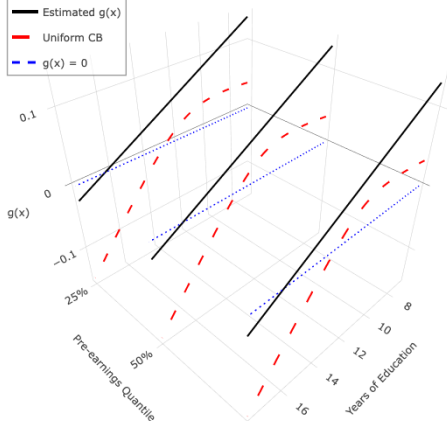
The surrogate loss function used in our analysis is the logistic loss. Our objective is to construct a left-sided uniform confidence band for (5.20) to test the null hypothesis

$$\mathcal{H}_0 : g_*(x_1, x_2) \leq 0,$$

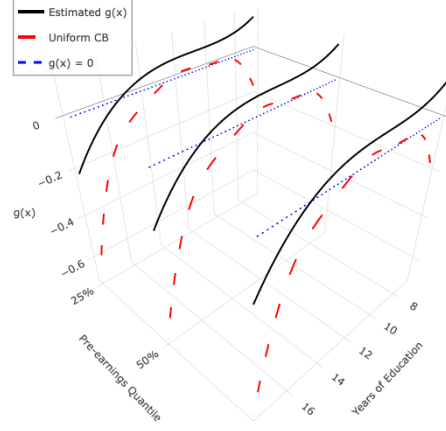
over $x_1 \in [7/18, 1]$, with x_2 fixed at the 25th, 50th, and 75th quantiles of its empirical distribution.

First, we present inference results for the optimal policy rule. We compute the one-sided uniform confidence bands of (5.20) using the critical values obtained from the score bootstrap method described in Section 5. The sieve-based nonparametric estimators and their corresponding uniform confidence bands are computed for the four cases described above: (i) no training cost with a linear policy rule, (ii) no training cost with a nonlinear policy rule, (iii) training cost with a linear policy rule, and (iv) training cost with a nonlinear policy rule. For each case, we construct one-sided uniform confidence bands over *years of education*, holding *pre-program earnings* fixed at the 25th, 50th, and 75th quantiles.

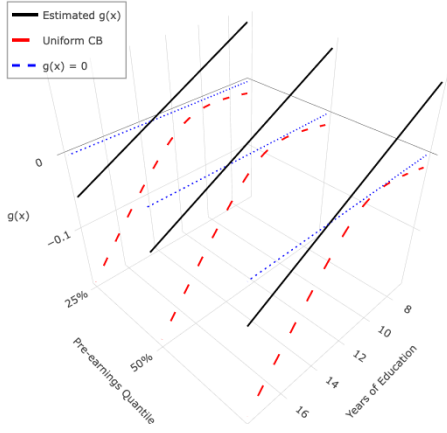
Figure 3 presents the estimation and inference results for the optimal policy. When training costs are not considered and *years of education* are below 12, Figures 3(a) and 3(b) show that



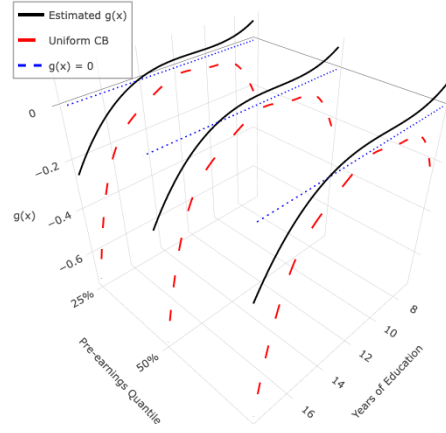
(a) No Training Cost & Linear Rule



(b) No Training Cost & Nonlinear Rule



(c) Training Cost & Linear Rule



(d) Training Cost & Nonlinear Rule

Figure 3: 95% uniform confidence bands for optimal policy rule $g_*(x)$. The estimated policy functions are plotted in black, and the bootstrap uniform confidence bands are shown as red dotted lines.

		Outcome Variable: 30-Month Post-Program Earnings		
		Optimal treat	Treat everyone	Random treat
No Treatment Cost				
Linear	Estimated Welfare Gain	\$1,285	\$1,247	\$502
	95% Confidence Interval	(\$901, \$1,668)	(\$867, \$1,626)	(\$16, \$988)
Nonlinear	Estimated Welfare Gain	\$1,338	\$1,247	\$499
	95% Confidence Interval	(\$943, \$1,733)	(\$867, \$1,626)	(\$9, \$990)
\$774 Cost for Each Assigned Treatment				
Linear	Estimated Welfare Gain	\$533	\$473	\$111
	95% Confidence Interval	(\$120, \$946)	(\$93, \$852)	(-\$376, \$598)
Nonlinear	Estimated Welfare Gain	\$566	\$473	\$109
	95% Confidence Interval	(\$160, \$972)	(\$93, \$852)	(-\$381, \$598)

Table 4: 95% confidence intervals of estimated welfare gain of treatment assignments

the 95% uniform confidence bands allow us to reject the uniform null hypothesis, indicating that assigning individuals in this range to the treatment group is statistically significant at the 5% level. In contrast, when training costs are taken into account, this statistical significance disappears, as shown in Figures 3(c) and 3(d). The economic intuition behind this finding is straightforward: as the cost of the National JTPA program increases, the net benefit of participation declines, reducing its appeal and thereby lowering the expected welfare for the representative individual. Overall, our inference method for the optimal policy rule complements the existing findings (e.g., Kitagawa and Tetenov, 2018; Mbakop and Tabord-Meehan, 2021) by providing formal statistical inference as well as more general nonlinearity.

We now report the estimation and inference results for (optimal) welfare. Table 4 presents the 95% confidence intervals (CIs) for the estimated welfare under three different treatment assignment rules: (i) the estimated optimal policy rule based on the logistic surrogate loss, (ii) the policy that assigns all individuals to the treatment group, and (iii) the random assignment rule following a Bernoulli(0.5) distribution. Since Theorem 6.1 establishes that the empirical welfare of a suitable plug-in treatment assignment rule is asymptotically normal, constructing CIs for welfare functions only requires appropriate variance estimates. Specifically, for the estimated optimal policy rule, we obtain the variance estimate using our score bootstrap procedure (see Section 6.3), while for the other two cases, we use the variance formula provided in Theorem

	$k = 1$		$k = 3$	
	Treat Everyone	Random Assign	Treat Everyone	Random Assign
No Treatment Cost				
Two-sided	0.514	0.000	0.282	0.000
Left-sided	0.766	1.000	0.877	1.000
Right-sided	0.234	0.000	0.123	0.000
\$774 Treatment Cost				
Two-sided	0.669	0.048	0.339	0.042
Left-sided	0.691	0.978	0.862	0.979
Right-sided	0.309	0.022	0.138	0.021

Table 5: P-values from the hypothesis tests comparing the welfare under the optimal policy to that under benchmark policies.

6.1.

When linear rules are considered, the estimated welfare of the estimated optimal rule is \$1,285 in the case of no training cost and \$533 in the case of training costs. These findings are very similar to those reported by [Kitagawa and Tetenov \(2018, Table 1\)](#), who obtain estimated welfare values of \$1,180 and \$404, respectively. While they do not develop a formal theory for inference, [Kitagawa and Tetenov \(2018, Table 1\)](#) also report confidence intervals based on bootstrap for this application. Our confidence intervals for the two aforementioned cases, (\$901, \$1,668) and (\$120, \$946), are comparable to theirs but notably narrower, thereby offering greater precision and enhancing the reliability of statistical decisions.

Additionally, Table 4 compares the estimated optimal rule, the “treat-everyone” rule, and the random assignment rule across the cases. The estimated welfare of the random assignment rule lies outside the 95% confidence intervals for the estimated optimal rule, clearly indicating that our estimated rule achieves higher welfare than the random assignment rule, which has been widely used in empirical economic research. Moreover, the estimated welfare of the optimal rule is consistently higher than that of the “treat-everyone” rule, further demonstrating the practical usefulness of our policy learning method.

Finally, we conduct hypothesis tests to compare the welfare achieved under the estimated optimal policy with that under the benchmark policies $g_{\dagger} \in \{ \text{“Treat Everyone”}, \text{“Random Assign”} \}$.⁹

⁹See Appendix C for details on this testing procedure.

Specifically, we perform a two-sided test of $\mathcal{H}_0 : V_0 = V(g_{\dagger})$ against $\mathcal{H}_1 : V_0 \neq V(g_{\dagger})$, as well as one-sided tests of $\mathcal{H}_0 : V_0 \leq V(g_{\dagger})$ versus $\mathcal{H}_1 : V_0 > V(g_{\dagger})$ and $\mathcal{H}_0 : V_0 \geq V(g_{\dagger})$ versus $\mathcal{H}_1 : V_0 < V(g_{\dagger})$. We consider two settings, depending on whether treatment costs are incorporated. Table 5 summarizes the resulting p -values.

Several findings emerge. First, when comparing against the random assignment rule, both the two-sided and right-sided tests are rejected at the 5% significance level, providing strong empirical evidence that the optimal policy rule achieves higher welfare than the random assignment rule. Second, when compared with the “treat-everyone” rule, the optimal policy rule does not provide sufficiently strong evidence to demonstrate higher welfare. This result holds robustly across the cases with and without treatment costs. This finding is unsurprising, as assigning all individuals to treatment can raise the overall level of welfare, especially among less educated individuals. However, when highly educated individuals represent only a small share of the population, the difference between the optimal treatment rule and the “treat-everyone” rule can be small, making it difficult to reject the null hypothesis that treats the two rules as equivalent.

9 Conclusion

In this paper, we have developed a toolkit for nonparametric uniform inference in cost-sensitive binary classification, a broad framework that encompasses maximum score estimation, utility-maximizing choice prediction, policy learning, and related problems. These settings are well known for slow convergence and nonstandard limiting behavior under parametric classification rules. Furthermore, nonparametric formulations of classification policy functions introduce the additional challenge of failure of identification.

To address these difficulties, we propose a strictly convex surrogate loss that point-identifies a representative binary classification function within the set of all the optimal classification functions. While this set is generally infinite, the point-identified function has the same classification boundaries and signs as any other optimal function, and therefore induces the same classification rule and policy value. The object being estimated is not an approximation or an alternative to the original object, but is *exactly* the true optimal classification rule. This equivalence provides a unified route to Gaussian inference for both the optimal classification rule and its associated policy value.

Simulation evidence corroborates the theoretical results, showing that the proposed methods perform well in finite samples. Finally, our application to the National JTPA Study demonstrates how the procedure can be used to recover an optimal treatment assignment rule and to evaluate its welfare implications relative to alternative policies at hand.

While the primary contributions of this paper lie in developing inference methods for non-parametric classification policies, our approach also provides, as a by-product, practical benefits for estimating these policies. This improvement arises from two sources. First, the use of a strictly convex surrogate loss facilitates routine numerical optimization for sieve estimation. Second, and more importantly, our point identification of the representative policy function enables a straightforward estimation of the resulting classification rules.

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Supplementary Material for

“Nonparametric Surrogate Binary Classification”

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Abstract

This supplementary material provides details that are omitted from the main text. Appendix [A](#) contains the mathematical proofs of the results presented in the main paper. Appendix [B](#) collects auxiliary lemmas from the literature that are used in our analysis. Appendix [C](#) presents additional testing procedures employed in the simulation and empirical analyses of the main text.

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Appendix

This appendix contains technical details. We use the same notations as in the main text unless otherwise noted.

A Proofs

A.1 Proof for Section 4: Representative Identification

Proof of Theorem 4.1: The proof consists of two parts, (i) and (ii).

Proof of Part (i): We are going to prove that any $g_* \in \arg \min_{g \in \mathcal{G}} L^*(g)$ satisfies $g_* \in \arg \min_{g \in \mathcal{G}} L(g)$. If a policy g minimizes $L(\cdot)$ in (2.1), it also minimizes the conditional risk

$$R(g(x); x) = C_1(x)\mathbf{1}\{-g(x) \leq 0\} + C_0(x)\mathbf{1}\{g(x) < 0\}$$

pointwisely for every x possibly except on a set of probability measure zero, where

$$C_1(x) = \mathbb{E}[\psi^+(Z_i, \eta_0)|X_i = x] \quad \text{and} \quad C_0(x) = \mathbb{E}[\psi^-(Z_i, \eta_0)|X_i = x].$$

When $g(x) > 0$, $R(g(x)) = C_1(x)$. Likewise, when $g(x) < 0$, $R(g(x)) = C_0(x)$. Hence, any element $g^\dagger$ in the set of optimal policy functions satisfies

$$g^\dagger(x) = \begin{cases} \text{any positive value or } +\infty, & \text{if } C_0(x) > C_1(x), \\ \text{any negative value or } -\infty, & \text{if } C_0(x) < C_1(x), \\ \text{any value or } \pm\infty, & \text{if } C_0(x) = C_1(x), \end{cases}$$

possibly except on a set of probability measure zero. The last case with $C_0(X_i) = C_1(X_i)$ is an event of probability measure zero by the statement of the theorem.

Now, consider the surrogate risk minimization problem

$$\min_{g \in \mathcal{G}} \mathbb{E}[\psi^+(Z_i, \eta) \cdot \phi(-g(X_i)) + \psi^-(Z_i, \eta) \cdot \phi(g(X_i))].$$

Similarly to the indicator case, we consider the conditional counterpart:

$$R^*(g(x); x) = C_1(x) \cdot \phi(-g(x)) + C_0(x) \cdot \phi(g(x)).$$

Since ϕ is differentiable at 0 in the statement of the theorem, we can take the derivative

$$\left. \frac{\partial R^*(\bar{g}; x)}{\partial \bar{g}} \right|_{\bar{g}=0} = \phi'(0)[-C_1(x) + C_0(x)].$$

Since $\phi'(0) < 0$ in the statement of the theorem, the sign of this derivative equals

$$C_1(x) - C_0(x).$$

Thus, if $C_0(x) > C_1(x)$, $R^*(g(x); x)$ decreases as $g(x)$ increases in a neighborhood of 0. It follows from the convexity of $\phi(\cdot)$ that $R^*(g(x); x)$ is minimized by some $g(x) \in (0, \infty]$ at any $x \in \mathcal{X}$ with $C_0(x) > C_1(x)$. To see this, note that for any $x \in \mathcal{X}$ with $C_0(x) > C_1(x)$, there exists some $\check{g}(x) > 0$ such that

$$R^*(\check{g}(x); x) \leq R^*(0; x) + \frac{\check{g}(x)}{2} \cdot \left. \frac{\partial R^*(\bar{g}; x)}{\partial \bar{g}} \right|_{\bar{g}=0}.$$

Furthermore, the convexity of ϕ implies that for all $g(x)$,

$$R^*(g(x); x) \geq R^*(0; x) + g(x) \cdot \left. \frac{\partial R^*(\bar{g}; x)}{\partial \bar{g}} \right|_{\bar{g}=0}.$$

In particular, if $g(x) \leq \check{g}(x)/4$, then

$$R^*(g(x); x) \geq R^*(0; x) + \frac{\check{g}(x)}{4} \cdot \left. \frac{\partial R^*(\bar{g}; x)}{\partial \bar{g}} \right|_{\bar{g}=0} > R^*(0; x) + \frac{\check{g}(x)}{2} \cdot \left. \frac{\partial R^*(\bar{g}; x)}{\partial \bar{g}} \right|_{\bar{g}=0} \geq R^*(\check{g}(x); x).$$

Hence, a minimizer $g_*(x)$ must be greater than $\check{g}(x)/4$, therefore should be positive as is $g^\dagger(x)$.

If $C_1(x) > C_0(x)$, on the other hand, then $R^*(g; x)$ increases as g increases in a neighborhood of 0. By a similar argument as above, a minimizer $g^*(x)$ should be negative as is $g^\dagger(x)$.

From the above analysis, the sign of the minimizer $g^*(x)$ of the conditional surrogate risk $R^*(g(x))$ stays consistent with that of conditional risk $R(g(x))$ possibly except a set of probability measure zero. Therefore, it follows that any $g_* \in \arg \min L^*(g)$ satisfies $g_* \in \arg \min_{g \in \mathcal{G}} L(g)$. The proof is then complete.

Proof of Part (ii): First, we prove that the solution to $\min_{g \in \mathcal{G}} L^*(g)$ is unique up to the equivalence class in $\mathcal{G}$ identified by the underlying probability measure, when ϕ is strictly convex. Suppose that $g_1, g_2 \in \arg \min_{g \in \mathcal{G}} L^*(g)$ and $g_1 \neq g_2$ (up to the equivalence). For $\lambda \in (0, 1)$, define the strict linear combination

$$g_\lambda = \lambda g_1 + (1 - \lambda) g_2.$$

Note that $g_\lambda \in \mathcal{G}$ because $\mathcal{G}$ is convex. Since ϕ is strictly convex, $\mathbb{E}[\psi^+(Z_i, \eta_0)|X_i] > 0$ a.s., and $\mathbb{E}[\psi^-(Z_i, \eta_0)|X_i] > 0$ a.s.,

$$\begin{aligned} & \mathbb{E}[\psi^+(Z_i, \eta_0)|X_i] \cdot \phi(-g_\lambda(X_i)) + \mathbb{E}[\psi^-(Z_i, \eta_0)|X_i] \cdot \phi(g_\lambda(X_i)) \\ & \leq \lambda[\mathbb{E}[\psi^+(Z_i, \eta_0)|X_i] \cdot \phi(-g_1(X_i)) + \mathbb{E}[\psi^-(Z_i, \eta_0)|X_i] \cdot \phi(g_1(X_i))] \\ & \quad + (1 - \lambda)[\mathbb{E}[\psi^+(Z_i, \eta_0)|X_i] \cdot \phi(-g_2(X_i)) + \mathbb{E}[\psi^-(Z_i, \eta_0)|X_i] \cdot \phi(g_2(X_i))], \end{aligned}$$

where the inequality ‘ $\leq$ ’ is satisfied with the strict inequality ‘ $<$ ’ with probability greater than one as $g_1 \neq g_2$ (up to the equivalence). Thus, integrating both sides and applying the law of iterated expectations give

$$L^*(g_\lambda) < \lambda \cdot L^*(g_1) + (1 - \lambda)L^*(g_2),$$

showing a contradiction with $g_1, g_2 \in \arg \min_{g \in \mathcal{G}} L^*(g)$. This implies the uniqueness (up to the equivalence class) stated in part (ii) of the statement of the theorem. ■

A.2 Proofs for Section 5: Estimation & Inference for the Representative Policy Function

Proof of Theorem 5.1: We first derive the convergence rate, and then establish the strong Guassian approximation.

Convergence Rate: For simplicity of writings, we focus on the case with $m = 2$. Thus, we simplify the notations, $\widehat{\psi}_{(\ell)}^+(Z_i)$, $\widehat{\psi}_{(\ell)}^-(Z_i)$ and $\mathbb{E}_{n,(\ell)}[\cdot]$, as $\widehat{\psi}_i^+$, $\widehat{\psi}_i^-$, and $\mathbb{E}_n[\cdot]$, respectively. We also simplify the notation of $\widehat{\eta}_{(\ell)}$, that use the complementary subsample $I_{(\ell)}^c$, as $\widehat{\eta}$. The subscripts of $\widehat{\beta}_{(\ell)}$ and $\widehat{g}_{(\ell)}$ are omitted similarly as $\widehat{\beta}$ and $\widehat{g}$, respectively.

We split the proof into two steps: (a) we first apply [Chen \(2007, Theorem 3.2\)](#) and [Nair, Pereverzev, and Tautenhahn \(2005\)](#) to prove a rough convergence rate; and (b) we then sharpen this rate to obtain a sharper bound.

Recall the short-hand notation $\Psi(Z_i, \eta, g) := \psi_i^+ \phi(-g(X_i)) + \psi_i^- \phi(g(X_i))$. First, when $\|g - g_*\|_{\mathbb{P}, 2} \leq \delta$ with all small enough $\delta > 0$, by [Chen and Shen \(1998, Theorem 3\)](#), there exist some proxy functions $g_{\delta,1}$ and $g_{\delta,2}$ on the line segment connecting g and g_* such that

$$\begin{aligned} & \text{Var}(\Psi(Z_i, \eta, g) - \Psi(Z_i, \eta, g_*)) \\ & \leq 4\mathbb{E}[(\psi_i^+)^2 \partial \phi(-g_{\delta,1}(X_i))^2 (g(X_i) - g_*(X_i))^2] + 4\mathbb{E}[(\psi_i^-)^2 \partial \phi(g_{\delta,2}(X_i))^2 (g(X_i) - g_*(X_i))^2] \end{aligned}$$

$$\begin{aligned}
&\leq 4\mathbb{E}[\mathbb{E}[(\psi_i^+)^2|X_i](g(X_i) - g_*(X_i))^2] \left(\sup_x \partial\phi(-g_{\delta,1}(x)) \right)^2 \\
&\quad + 4\mathbb{E}[\mathbb{E}[(\psi_i^-)^2|X_i](g(X_i) - g_*(X_i))^2] \left(\sup_x \partial\phi(g_{\delta,2}(x)) \right)^2 \\
&\leq C_+ \|g - g_*\|_{\mathbb{P},2} + C_- \|g - g_*\|_{\mathbb{P},2} \leq C_1 \delta^2,
\end{aligned} \tag{A.1}$$

where the first inequality is due to the functional mean value expansion in Banach space (Zeidler, 1995, Chapter 4) around g_* and the twice continuously Fréchet differentiable functional $g \mapsto \mathbb{E}[(\Psi(Z_i, \eta, g) - \Psi(Z_i, \eta, g_*))^2]$ (Zeidler, 1995) with Assumptions 5.3 and 5.4, the second inequality uses the law of iterated expectations, and the last inequality follows from (i) $\mathbb{E}[(\psi_i^+)^2|X_i = x], \mathbb{E}[(\psi_i^-)^2|X_i = x] < \infty$ uniformly for all $x \in \mathcal{X}$ under Assumption 5.1.(i) and (ii) $\sup_{x \in \mathcal{X}} \partial\phi(g_*(x)) < \infty$, which in turn follows from Assumptions 5.3 and 5.4. Here, C_+ , C_- and C_1 are finite constants independent of n and k . Specifically, the boundedness for $\sup_x \partial\phi(-g_{\delta,1}(x)) < \infty$ and $\sup_x \partial\phi(g_{\delta,2}(x)) < \infty$ in (A.1) holds by Assumption 5.3 and since

$$\|g_{\delta,1}\|_{\mathbb{P},2} \leq t\|g_* - g_{\delta,1}\|_{\mathbb{P},2} + \|g_*\|_{\mathbb{P},2} < \infty \quad \text{and} \quad \|g_{\delta,2}\|_{\mathbb{P},2} \leq t\|g_* - g_{\delta,2}\|_{\mathbb{P},2} + \|g_*\|_{\mathbb{P},2} < \infty$$

for some $t \in [0, 1]$, which accompanied by Lemma B.1 further yield $\|g_{\delta,1}\|_{\mathbb{P},\infty} < \infty$ and $\|g_{\delta,2}\|_{\mathbb{P},\infty} < \infty$.

Hence, Condition 3.7 of Chen (2007) is satisfied.

Similarly, when $\|g - g_*\|_{\mathbb{P},2} \leq \delta \in (0, \infty)$, we have

$$\begin{aligned}
|\Psi(Z_i, \eta, g) - \Psi(Z_i, \eta, g_*)| &\leq \|g - g_*\|_{\mathbb{P},\infty} \cdot 2|(-\psi_i^+ \partial\phi(-\check{g}^{(1)}(X_i)) + \psi_i^- \partial\phi(\check{g}^{(2)}(X_i)))| \\
&\leq \|g - g_*\|_{\mathbb{P},2}^{2h/(2h+d_x)} \cdot 2|(-\psi_i^+ \partial\phi(-\check{g}^{(1)}(X_i)) + \psi_i^- \partial\phi(\check{g}^{(2)}(X_i)))|,
\end{aligned}$$

where the first inequality is due to the definition of $\|\cdot\|_{\mathbb{P},\infty}$ -norm and the second inequality follows by Lemma B.1. Since $\mathbb{E}[|(-\psi_i^+ \partial\phi(-\check{g}^{(1)}(X_i)) + \psi_i^- \partial\phi(\check{g}^{(2)}(X_i)))|^2] \leq C_2 \in (0, \infty)$ follows similarly to (A.1) and $(2h/(2h+d_x)) \in (0, 2)$, Condition 3.8 of Chen (2007) is satisfied.

Define the $L_2(\mathbb{P})$ covering number with bracketing, $\mathcal{N}_{[\cdot]}(\mathcal{F}_n, \|\cdot\|_{\mathbb{P},2}, \varepsilon)$, for the class

$$\mathcal{F}_n = \{(\psi_i^+ \phi(-g(X_i)) + \psi_i^- \phi(g(X_i))) - (\psi_i^+ \phi(-g_*(X_i)) + \psi_i^- \phi(g_*(X_i))) : \|g - g_*\|_{\mathbb{P},2} \leq \delta, g \in \mathcal{G}_k\}.$$

For a constant $b > 0$, the rate δ_n solves

$$\frac{1}{\sqrt{n}\delta_n^2} \int_{b\delta_n^2}^{\delta_n} \sqrt{\log(\mathcal{N}_{[\cdot]}(\mathcal{F}_n, \|\cdot\|_{\mathbb{P},2}, \varepsilon))} d\varepsilon \leq \frac{1}{\sqrt{n}\delta_n^2} \int_{b\delta_n^2}^{\delta_n} \sqrt{k \cdot \log\left(1 + \frac{k}{\varepsilon}\right)} d\varepsilon \tag{A.2}$$

$$\leq \frac{1}{\sqrt{n}\delta_n^2} \sqrt{k \log(k)} \cdot \delta_n \leq \text{constant},$$

where $\log \mathcal{N}(\mathcal{F}_n, \|\cdot\|_{\mathbb{P},2}, \varepsilon) \lesssim \dim(\mathcal{G}_k) \log(\dim(\mathcal{G}_k)/\varepsilon)$ for the finite-dimensional linear sieves (Chen and Shen, 1998, Page 311, Cases 1.1, 1.2 and 1.3).

The solution is given by $\delta_n \asymp \sqrt{k \log(k)/n}$. Theorem 3.2 of Chen (2007) along with Lemma B.1 yields $\|\hat{g} - g_k\|_{\mathbb{P},2} \asymp \sqrt{k \log(k)/n}$ and $\|\hat{g} - g_k\|_{\mathbb{P},\infty} \asymp (k \log(k)/n)^{h/(2h+d_x)}$. Thus, a rough convergence rate is obtained as

$$\|\hat{g} - g_*\|_{\mathbb{P},\infty} = (k \log(k)/n)^{h/(2h+d_x)} + B_k.$$

Next, we sharpen the convergence rate. The first-order condition for the sieve estimator yields

$$\mathbb{E}_n \left[\left(-\hat{\psi}_i^+ \partial \phi(-p_{1:k}(X_i)' \hat{\beta}) + \hat{\psi}_i^- \partial \phi(p_{1:k}(X_i)' \hat{\beta}) \right) p_{1:k}(X_i) \right] = \mathbf{0}_{k \times 1}.$$

Then, the mean value theorem yields the coordinate-wise asymptotic representation

$$\begin{aligned} \mathbf{0}_{k \times 1} &= \mathbb{E}_n [(-\hat{\psi}_i^+ \partial \phi(-p_{1:k}(X_i)' \beta_0) + \hat{\psi}_i^- \partial \phi(p_{1:k}(X_i)' \beta_0)) p_{1:k}(X_i)] \\ &\quad + \mathbb{E}_n [(\hat{\psi}_i^+ \partial^2 \phi(-p_{1:k}(X_i)' \check{\beta}^{(1)}) + \hat{\psi}_i^- \partial^2 \phi(p_{1:k}(X_i)' \check{\beta}^{(2)})) p_{1:k}(X_i) p_{1:k}(X_i)'] (\hat{\beta} - \beta_0), \end{aligned}$$

where $\check{\beta}^{(s)}$ (with $s = 1, 2$) lies between $\hat{\beta}$ and β_0 . (These values vary by coordinates, but we use the single notation for simplicity of writing.)

For the score vector, the following decomposition holds:

$$\begin{aligned} &\mathbb{E}_n [(-\hat{\psi}_i^+ \partial \phi(-p_{1:k}(X_i)' \beta_0) + \hat{\psi}_i^- \partial \phi(p_{1:k}(X_i)' \beta_0)) p_{1:k}(X_i)] \\ &= \mathbb{E}_n [(-\psi_i^+ \partial \phi(-p_{1:k}(X_i)' \beta_0) + \psi_i^- \partial \phi(p_{1:k}(X_i)' \beta_0)) p_{1:k}(X_i)] \end{aligned} \quad (\text{A.3})$$

$$+ \mathbb{E}_n [(-(\hat{\psi}_i^+ - \psi_i^+) \partial \phi(-p_{1:k}(X_i)' \beta_0) + (\hat{\psi}_i^- - \psi_i^-) \partial \phi(p_{1:k}(X_i)' \beta_0)) p_{1:k}(X_i)]. \quad (\text{A.4})$$

We are going to find bounds for each of the two terms, (A.3) and (A.4), on the right-hand side. Note the fact that $\{[-\psi_i^+ \partial \phi(-p_{1:k}(X_i)' \beta_0) + \psi_i^- \partial \phi(p_{1:k}(X_i)' \beta_0)] p_{1:k}(X_i)\}_{i \in [n]}$ are mean-zero k -dimensional vectors. Consider the function set $\mathcal{F} = \{f_1, \dots, f_k\}$ where

$$f_k(Z_i; X_i) = [-\psi_i^+ \partial \phi(-p_{1:k}(X_i)' \beta_0) + \psi_i^- \partial \phi(p_{1:k}(X_i)' \beta_0)] p_s(X_i)$$

for $s \in [k]$ with the envelope

$$F_i = \max_{s \in [k]} |[-\psi_i^+ \partial \phi(-p_{1:k}(X_i)' \beta_0) + \psi_i^- \partial \phi(p_{1:k}(X_i)' \beta_0)] p_s(X_i)| \quad \text{and}$$

$$M = \max_{s \in [k], i \in [n]} |[-\psi_i^+ \partial \phi(-p_{1:k}(X_i)' \beta_0) + \psi_i^- \partial \phi(p_{1:k}(X_i)' \beta_0)] p_s(X_i)|.$$

Then, $\mathcal{N}(\mathcal{F}, e_Q, \varepsilon \|F\|_{Q,2}) \leq k$ and the uniform entropy satisfies

$$J(\delta, \mathcal{F}, F) = \int_0^\delta \sup_Q \sqrt{1 + \log(\mathcal{N}(\mathcal{F}, e_Q, \varepsilon \|F\|_{Q,2}))} d\varepsilon \lesssim \sqrt{\log(k)} \delta.$$

Note that $\max_{s \in [K]} \mathbb{E}\{[-\psi_i^+ \partial \phi(-p_{1:k}(X_i)' \beta_0) + \psi_i^- \partial \phi(p_{1:k}(X_i)' \beta_0)] p_s(X_i)\}^2 \leq C^2$ and $\|M\|_{\mathbb{P},2} \leq (nk)^{1/q}$. Define $\delta = C/\|F\|_{\mathbb{P},2}$ and apply the maximal inequality ([Chernozhukov, Chetverikov, and Kato, 2014](#), Theorem 5.2) to obtain the upper bound of the first term (A.3) as

$$\begin{aligned} \mathbb{E} \left\| \mathbb{E}_n \left[[-\psi_i^+ \partial \phi(-p_{1:k}(X_i)' \beta_0) + \psi_i^- \partial \phi(p_{1:k}(X_i)' \beta_0)] p_{1:k}(X_i) \right] \right\|_\infty \\ \lesssim \sqrt{\log(k)/n} + \frac{\|M\|_{\mathbb{P},2} \log(k)}{n} \lesssim \sqrt{\log(k)/n}, \end{aligned} \quad (\text{A.5})$$

where we use the fact that, as $q > 4$

$$\frac{\|M\|_{\mathbb{P},2} \log(k)}{n} \lesssim \frac{(nk)^{1/q} \log(k)}{n} \lesssim \sqrt{\log(k)/n}.$$

By Assumption 5.2(i), the second term (A.4) can be bounded above as

$$\begin{aligned} \left\| \mathbb{E}_n \left[[-(\widehat{\psi}_i^+ - \psi_i^+) \partial \phi(-p_{1:k}(X_i)' \beta_0) + (\widehat{\psi}_i^- - \psi_i^-) \partial \phi(p_{1:k}(X_i)' \beta_0)] p_{1:k}(X_i) \right] \right\|_\infty \\ = o_p(\sqrt{1/(n \log(k))}). \end{aligned} \quad (\text{A.6})$$

For the sample Gram matrix, we have a similar decomposition:

$$\begin{aligned} & \|\mathbb{E}_n[(\widehat{\psi}_i^+ \partial^2 \phi(-p_{1:k}(X_i)' \check{\beta}^{(1)}) + \widehat{\psi}_i^- \partial^2 \phi(p_{1:k}(X_i)' \check{\beta}^{(2)})) p_{1:k}(X_i) p_{1:k}(X_i)']\| \\ & \lesssim_p \|\mathbb{E}_n[(\widehat{\psi}_i^+ + \widehat{\psi}_i^-) p_{1:k}(X_i) p_{1:k}(X_i)']\| \\ & = \|\mathbb{E}_n[(\psi_i^+ + \psi_i^-) p_{1:k}(X_i) p_{1:k}(X_i)']\| + \|\mathbb{E}_n[(\widehat{\psi}_i^+ - \psi_i^+ + \psi_i^- - \widehat{\psi}_i^-) p_{1:k}(X_i) p_{1:k}(X_i)']\|. \end{aligned}$$

The first inequality is from the fact $\sup_{x \in \mathcal{X}} \partial^2 \phi(p_{1:k}(x)' \check{\beta}^{(s)})$ is bounded a.s. (with $s \in \{1, 2\}$) since

$$\sup_{x \in \mathcal{X}} |p_{1:k}(x)' \check{\beta}^{(s)}| \lesssim_p \sup_{x \in \mathcal{X}} |g_k(x)| + \sup_{x \in \mathcal{X}} |p_{1:k}(x) \cdot (\widehat{\beta} - \beta_0)| \lesssim_p 2 \sup_{x \in \mathcal{X}} |g_k(x)| < \infty,$$

where $\sqrt{k^2 \log(k)/n} = o(1)$ and $\widehat{\psi}_i^+$ and $\widehat{\psi}_i^-$ are non-negative. Rudelson's LLN for Matrices given in [Belloni, Chernozhukov, Chetverikov, and Kato \(2015, Lemma 6.2\)](#), with Assumptions 5.2-5.3, yields

$$\|(\mathbb{E}_n - \mathbb{E})[(\psi_i^+ + \psi_i^-) p_{1:k}(X_i) p_{1:k}(X_i)']\| \lesssim_p \sqrt{\xi_k^2 \log(k)/n} \quad \text{and} \quad (\text{A.7})$$

$$\left\| (\mathbb{E}_n - \mathbb{E})[(\widehat{\psi}_i^+ - \psi_i^+ + \widehat{\psi}_i^- - \psi_i^-)p_{1:k}(X_i)p_{1:k}(X_i)'] \right\| \lesssim_p \xi_k^2 \log(k)/n. \quad (\text{A.8})$$

Combining the results of the score vector and sample matrix, we have

$$\|\widehat{\beta} - \beta_0\|_\infty \lesssim_p \sqrt{\log(k)/n}, \quad (\text{A.9})$$

$$\|\widehat{\beta} - \beta_0\|_2 \lesssim_p \sqrt{k \log(k)/n}, \text{ and } \|\widehat{\beta} - \beta_0\|_1 \lesssim_p \sqrt{k^2 \log(k)/n}.$$

Consider a copy of the first-order approximation to our score vector:

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n \xi_i, \quad \xi_i := \Sigma^{-1/2} [(-\psi_i^+ \partial \phi(-p_{1:k}(X_i)' \beta_0) + \psi_i^- \partial \phi(p_{1:k}(X_i)' \beta_0)) p_{1:k}(X_i)].$$

Observe that the upper bound of the third moment is given by

$$\begin{aligned} \mathbb{E} \|\xi_i\|^3 &\lesssim \mathbb{E} \left\| (-\psi_i^+ \partial \phi(-p_{1:k}(X_i)' \beta_0) + \psi_i^- \partial \phi(p_{1:k}(X_i)' \beta_0)) p_{1:k}(X_i) \right\|^3 \\ &\leq 4 \mathbb{E} \left[(|\psi_i^+ \partial \phi(-g_k(X_i))|^3 + |\psi_i^- \partial \phi(g_k(X_i))|^3) \|p_{1:k}(X_i)\|^3 \right] \\ &\leq 4 \sup_{x \in \mathcal{X}} (|\partial \phi(-g_k(x))|^3 \vee |\partial \phi(g_k(x))|^3) \cdot \mathbb{E} \left[\mathbb{E} [|\psi_i^+|^3 + |\psi_i^-|^3 | X_i] \cdot \|p_{1:k}(X_i)\|^3 \right] \\ &\lesssim \mathbb{E} \|p_{1:k}(X_i)\|^3 \lesssim k \xi_k, \end{aligned} \quad (\text{A.10})$$

where the first inequality uses Assumption 5.1(ii), the second inequality follows from the triangle inequality and the arithmetic inequality $(a + b)^3 \leq 4(a^3 + b^3)$ for any non-negative a and b , the third inequality employs the law of iterated expectations, the fourth inequality holds by Assumption 5.1(i) and the bounds, $\sup_x |\partial \phi(g_k(x))| \lesssim 1$ and $\sup_x |\partial \phi(-g_k(x))| \lesssim 1$, and the last equality is due to the definition of ξ_k . In particular, $\sup_x |\partial \phi(g_k(x))| \lesssim 1$ holds since

$$\begin{aligned} \sup_x |\partial \phi(g_k(x))| &\leq \sup_x |\partial \phi(g_*(x))| + \sup_x |\partial^2 \phi(g_*(x))| \cdot \|g_k - g_*\|_{\mathbb{P}, \infty} + o(B_k), \\ &\lesssim \sup_x |\partial \phi(g_*(x))| + C_0 \sup_x |\partial^2 \phi(g_*(x))| < \infty \end{aligned}$$

holds by the Taylor theorem in Banach space (Zeidler, 1995, Chapter 4) and Assumptions 5.1(iv), 5.3, and 5.4. Similarly, we have $\sup_x |\partial \phi(-g_k(x))| \lesssim 1$.

Therefore, Lemma B.4 yields

$$\mathbb{P} \left\{ \left\| \frac{\sum_{i=1}^n \xi_i}{\sqrt{n}} - \mathcal{N}_k \right\| \geq 3\delta(\log(k))^{-1} \right\} \lesssim \frac{\log(k)^3 n k^2 \xi_k}{(\delta \sqrt{n})^3} \left(1 + \frac{\log(k^2 \xi_k)}{k} \right) \quad (\text{A.11})$$

$$\lesssim \frac{\log(k)^3 k^2 \xi_k}{\delta^3 \sqrt{n}} \left(1 + \frac{\log(n)}{k} \right) \rightarrow 0, \quad (\text{A.12})$$

where the convergence follows from the rate restriction $\log(k)^6 k^4 \xi_k^2 \log^2 n/n \rightarrow 0$ stated in the theorem.

Next, we are going to bound the errors of the covariance and moment estimators. Observe that

$$\begin{aligned} & \|\widehat{Q} - Q\| \\ & \leq \|\mathbb{E}_n[(\widehat{\psi}_i^+ \partial^2 \phi(-\widehat{g}(X_i)) + \widehat{\psi}_i^- \partial^2 \phi(\widehat{g}(X_i))) - (\psi_i^+ \partial^2 \phi(-g_*(X_i)) + \psi_i^- \partial^2 \phi(g_*(X_i)))] p_{1:k}(X_i) p_{1:k}(X_i)'\| \end{aligned} \quad (\text{A.13})$$

$$+ \|(\mathbb{E}_n - \mathbb{E})[(\psi_i^+ \partial^2 \phi(-g_*(X_i)) + \psi_i^- \partial^2 \phi(g_*(X_i))) p_{1:k}(X_i) p_{1:k}(X_i)']\|. \quad (\text{A.14})$$

The second term (A.14) on the right-hand side can be bounded by Rudelson's LLN for Matrices (Belloni et al., 2015, Lemma 6.2) as

$$\begin{aligned} & \mathbb{E}\|(\mathbb{E}_n - \mathbb{E})[(\psi_i^+ \partial^2 \phi(-g_*(X_i)) + \psi_i^- \partial^2 \phi(g_*(X_i))) p_{1:k}(X_i) p_{1:k}(X_i)']\| \\ & \lesssim \frac{\xi_k^2 \log(k)}{n} + \sqrt{\frac{\xi_k^2 \|Q\| \log(k)}{n}} \lesssim (1/\log(k)). \end{aligned}$$

Hence, Markov's inequality yields that

$$\begin{aligned} & \|(\mathbb{E}_n - \mathbb{E})[(\psi_i^+ \partial^2 \phi(-g_*(X_i)) + \psi_i^- \partial^2 \phi(g_*(X_i))) p_{1:k}(X_i) p_{1:k}(X_i)']\| \\ & \lesssim_p \frac{\xi_k^2 \log(k)}{n} + \sqrt{\frac{\xi_k^2 \|Q\| \log(k)}{n}} \lesssim (1/\log(k)). \end{aligned}$$

The first term (A.13) can be bounded with (5.8) of Assumption 5.2 as

$$\begin{aligned} & \|\mathbb{E}_n[(\widehat{\psi}_i^+ \partial^2 \phi(-\widehat{g}(X_i)) + \widehat{\psi}_i^- \partial^2 \phi(\widehat{g}(X_i))) - (\psi_i^+ \partial^2 \phi(-g_*(X_i)) + \psi_i^- \partial^2 \phi(g_*(X_i)))] p_{1:k}(X_i) p_{1:k}(X_i)'\| \\ & = o_p(\xi_k \sqrt{\log(k)}/n^{1/2}). \end{aligned}$$

Combining the results, we obtain $\|\widehat{Q} - Q\| \lesssim_p (1/\log(k))$. By using Rudelson's LLN for Matrices and (5.9) of Assumption 5.2, we similarly obtain $\mathbb{E}\|\widehat{\Sigma} - \Sigma\| = o(1/\log(k))$. Thus, it follows that

$$\begin{aligned} & \|\widehat{Q}^{-1} \widehat{\Sigma} \widehat{Q}^{-1} - Q^{-1} \Sigma Q^{-1}\| \\ & \lesssim_p \|(\widehat{Q}^{-1} - Q^{-1}) \widehat{\Sigma} \widehat{Q}^{-1}\| + \|Q^{-1} (\widehat{\Sigma} - \Sigma) \widehat{Q}^{-1}\| + \|Q^{-1} \Sigma (\widehat{Q}^{-1} - Q^{-1})\| = o_p(1/\log(k)). \end{aligned}$$

All eigenvalues of $Q^{-1} \Sigma Q^{-1}$ are bounded away from zero by Assumption 5.1(ii). Therefore, we have

$$\left| \frac{\widehat{\sigma}(x)}{\sigma(x)} - 1 \right| \leq \left| \frac{\widehat{\sigma}(x)^2}{\sigma(x)^2} - 1 \right| = \frac{\|p_{1:k}(x)' (\widehat{Q}^{-1} \widehat{\Sigma} \widehat{Q}^{-1} - Q^{-1} \Sigma Q^{-1}) p_{1:k}(x)\|}{\|p_{1:k}(x)' (Q^{-1} \Sigma Q^{-1}) p_{1:k}(x)\|}$$

$$\lesssim_p \|\widehat{Q}^{-1}\widehat{\Sigma}\widehat{Q}^{-1} - Q^{-1}\Sigma Q^{-1}\| \lesssim_p o_p(1/\log(k)). \quad (\text{A.15})$$

Define $\alpha_{1:k}(x) = p_{1:k}(x)/\|p_{1:k}(x)\|$, and then we have $\sup_x \|\alpha(x)\| \leq 1$. Combining (A.6)–(A.8), (A.11), (A.15) and Assumption 5.1.(iv), we obtain

$$\begin{aligned} & \left\| \sqrt{n}\alpha(x)'(\widehat{\beta} - \beta) - \alpha(x)'(Q^{-1}\Sigma Q^{-1})^{1/2}\mathcal{N}_k \right\| \\ & \leq \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^n \alpha(x)'(Q^{-1}\Sigma Q^{-1})^{1/2}\xi_i - \alpha(x)'(Q^{-1}\Sigma Q^{-1})^{1/2}\mathcal{N}_k \right\| + o_p(1/\log(k)) + O_p(\sqrt{n}B_k) \\ & = o_p(1/\log(k)) \end{aligned}$$

uniformly over $x \in \mathcal{X}$. Since $\|\alpha(x)'(Q^{-1}\Sigma Q^{-1})^{1/2}\|$ is bounded from below by Assumption 5.1.(ii), we conclude that the Gaussian coupling principle (5.12) holds. ■

Proof of Theorem 5.2: Similarly to the proof of Theorem 5.1, we apply Yurinskii's coupling (Belloni et al., 2015, Theorem 4.4) and strongly approximate the leading term of the score vector.

Consider the bootstrap counterpart of the first-order approximation to our score vector:

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n \widetilde{\xi}_i, \quad \widetilde{\xi}_i := \Sigma^{-1/2}[\omega_i(-\psi_i^+ \partial\phi(-p_{1:k}(X_i)'\beta_0) + \psi_i^- \partial\phi(p_{1:k}(X_i)'\beta_0))p_{1:k}(X_i)].$$

Observe that the upper bound of the third moment is given by

$$\mathbb{E}\|\widetilde{\xi}_i\|^3 \lesssim \mathbb{E}\left\|[\omega_i(-\psi_i^+ \partial\phi(-p_{1:k}(X_i)'\beta_0) + \psi_i^- \partial\phi(p_{1:k}(X_i)'\beta_0))p_{1:k}(X_i)]\right\|^3 \quad (\text{A.16})$$

$$\lesssim \mathbb{E}\left\|[(\psi_i^+ \partial\phi(-p_{1:k}(X_i)'\beta_0) + \psi_i^- \partial\phi(p_{1:k}(X_i)'\beta_0))p_{1:k}(X_i)]\right\|^3 \quad (\text{A.17})$$

$$\lesssim \mathbb{E}\|p_{1:k}(X_i)\|^3 \lesssim k\xi_k, \quad (\text{A.18})$$

where the line (A.16) holds by the bounded eigenvalues of Σ under Assumption 5.1(ii), the line (A.17) is due to the independence between ω_i and Z_i , and the line (A.18) can be proved identically to (A.10).

Therefore, Yurinskii's coupling principle yields

$$\begin{aligned} \mathbb{P}\left\{\left\|\frac{\sum_{i=1}^n \widetilde{\xi}_i}{\sqrt{n}} - \mathcal{N}_k\right\| \geq 3\delta \log(k)^{-1}\right\} & \lesssim \frac{\log(k)^3 n k^2 \xi_k}{(\delta \sqrt{n})^3} \left(1 + \frac{\log(k^2 \xi_k)}{k}\right) \\ & \lesssim \frac{\log(k)^3 k^2 \xi_k}{\delta^3 \sqrt{n}} \left(1 + \frac{\log(n)}{k}\right) \rightarrow 0, \end{aligned} \quad (\text{A.19})$$

where the convergence follows from the rate restriction $\log(k)^6 k^4 \xi_k^2 \log^2 n/n \rightarrow 0$. Therefore,

$$\begin{aligned}
& \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^n \Sigma^{-1/2} [\omega_i (-\widehat{\psi}_i^+ \partial \phi(-\widehat{g}(X_i)) + \widehat{\psi}_i^- \partial \phi(\widehat{g}(X_i))) p_{1:k}(X_i)] - \mathcal{N}_k \right\| \\
& \leq \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^n \Sigma^{-1/2} [\omega_i (-\psi_i^+ \partial \phi(-g_k(X_i)) + \psi_i^- \partial \phi(g_k(X_i))) p_{1:k}(X_i)] - \mathcal{N}_k \right\| \\
& + \sqrt{n} \left\| \Sigma^{-1/2} \mathbb{E}_n [(\Psi_1^*(Z_i, \omega_i, \widehat{\eta}, g_1) - \Psi_1^*(Z_i, \omega_i, \eta_0, g_*)) p_{1:k}(X_i)] \right\| \\
& = o_p(1/\log(k))
\end{aligned} \tag{A.20}$$

for the numerator of the bootstrap process, where the last line is due to Assumption 5.5, (A.19), and the bounded eigenvalues of Σ under Assumption 5.1(ii). We thus obtain the strong Gaussian approximation

$$\left\| \frac{1}{\sqrt{n}} \sum_{i=1}^n [\omega_i (-\widehat{\psi}_i^+ \partial \phi(-\widehat{g}(X_i)) + \widehat{\psi}_i^- \partial \phi(\widehat{g}(X_i))) p_{1:k}(X_i)] - \Sigma^{1/2} \mathcal{N}_k \right\| = o_p(1/\log(k)),$$

for the bootstrapping statistic process. It follows that

$$\sup_{x \in \mathcal{X}} \left| t_g^b(x) - \frac{p_{1:k}(x)' \widehat{Q}^{-1} \cdot \Sigma^{1/2} \mathcal{N}_k}{\widehat{\sigma}(x)} \right| = o_p(1/\log(k)). \tag{A.21}$$

Second, define the centered Gaussian process $x \mapsto t_g^{b,*}(x)$ by

$$t_g^{b,*}(x) := \frac{p_{1:k}(x)' Q^{-1} \cdot \Sigma^{1/2} \mathcal{N}_k}{\sigma(x)}.$$

By Theorem 5.1, which shows the consistency of $\widehat{\Sigma}$ and $\widehat{Q}$, we obtain

$$\begin{aligned}
\sup_{x \in \mathcal{X}} \left| t_g^{b,*}(x) - \frac{p_{1:k}(x)' \widehat{Q}^{-1} \cdot \Sigma^{1/2} \mathcal{N}_k}{\widehat{\sigma}(x)} \right| &= \sup_{x \in \mathcal{X}} \left| \frac{p_{1:k}(x)' Q^{-1} \cdot \Sigma^{1/2} \mathcal{N}_k}{\sigma(x)} - \frac{p_{1:k}(x)' \widehat{Q}^{-1} \cdot \Sigma^{1/2} \mathcal{N}_k}{\widehat{\sigma}(x)} \right| \\
&= o_p(1/\log(k)).
\end{aligned} \tag{A.22}$$

Third, combining (A.21) and (A.22), we have

$$\begin{aligned}
\sup_{x \in \mathcal{X}} |t_g^b(x) - t_g^{b,*}(x)| &\leq \sup_{x \in \mathcal{X}} \left| t_g^b(x) - \frac{p_{1:k}(x)' \widehat{Q}^{-1} \cdot \Sigma^{1/2} \mathcal{N}_k}{\widehat{\sigma}(x)} \right| + \sup_{x \in \mathcal{X}} \left| t_g^{b,*}(x) - \frac{p_{1:k}(x)' \widehat{Q}^{-1} \cdot \Sigma^{1/2} \mathcal{N}_k}{\widehat{\sigma}(x)} \right| \\
&= o_p(1/\log(k)).
\end{aligned} \tag{A.23}$$

Based on (A.23) and the proof of Belloni et al. (2015, Theorem 5.5), we obtain

$$\sup_{s \in \mathbb{R}} \left| \mathbb{P} \left(\sup_{x \in \mathcal{X}} |t_g^b(x)| \leq s \right) - \mathbb{P} \left(\sup_{x \in \mathcal{X}} |t_g^{b,*}(x)| \leq s \right) \right| = o(1). \quad (\text{A.24})$$

Also, observe that

$$\mathbb{P} \left(\sup_{x \in \mathcal{X}} |t_g^{b,*}(x)| \leq s \right) - \mathbb{P} \left(\sup_{x \in \mathcal{X}} |t(x)| \leq s \right) = o(1) \quad (\text{A.25})$$

as $n \rightarrow \infty$ for any $s \in \mathbb{R}$.

Combining (A.24) and (A.25), we obtain

$$\begin{aligned} & \sup_{s \in \mathbb{R}} \left| \mathbb{P} \left(\sup_{x \in \mathcal{X}} |t(x)| \leq s \right) - \mathbb{P} \left(\sup_{x \in \mathcal{X}} |t_g^b(x)| \leq s \right) \right| \\ & \leq \sup_{s \in \mathbb{R}} \left| \mathbb{P} \left(\sup_{x \in \mathcal{X}} |t(x)| \leq s \right) - \mathbb{P} \left(\sup_{x \in \mathcal{X}} |t_g^{b,*}(x)| \leq s \right) \right| + \sup_{s \in \mathbb{R}} \left| \mathbb{P} \left(\sup_{x \in \mathcal{X}} |t_g^b(x)| \leq s \right) - \mathbb{P} \left(\sup_{x \in \mathcal{X}} |t_g^{b,*}(x)| \leq s \right) \right| \\ & = o_p(1), \end{aligned}$$

and prove the claim. ■

A.3 An Auxiliary Result Related to Section 6

To facilitate our discussions, we write

$$C_1(x) = \mathbb{E}[\psi^+(Z_i, \eta_0)|X_i = x] \quad \text{and} \quad C_0(x) = \mathbb{E}[\psi^-(Z_i, \eta_0)|X_i = x].$$

Lemma A.1: *Suppose Assumptions 5.3, 6.1, 6.2(ii), and conditions of Theorem 4.1(ii) hold. Then, there exists small δ such that for any $\varepsilon < \delta$, $\mathbf{1}\{|g_*(x)| \leq \varepsilon\} \leq \mathbf{1}\{|C_1(x) - C_0(x)| \leq K_1 \varepsilon\}$ for a positive K_1 that does not depends on x and ε .*

Proof of Lemma A.1: Fix an arbitrary $x \in \mathcal{X}$. By Assumption of Theorem 4.1, we can ignore the case in which $C_1(x) = C_0(x) = 0$.

Suppose first $C_0(x) \neq 0$. From the proof of Theorem 4.1, we know that for a given choice of smooth function ϕ , the first order condition implies

$$H(g_*(x)) = \frac{C_1(x)}{C_0(x)},$$

where $H(g) := \frac{\phi'(g)}{\phi'(-g)}$ is differentiable and a strictly decreasing in g , with $H(0) = 1$. Let $K_\delta = \max_{t \in [-\delta, \delta]} |H'(t)|$. By Mean Value Theorem, it follows that $|g_*(x)| \leq \varepsilon$ implies that $|\frac{C_1(x)}{C_0(x)} - 1| \leq K_\delta \varepsilon$ or $|C_1(x) - C_0(x)| \leq K_\delta |C_0(x)| \varepsilon$.

If $C_0(x) = 0$, then $C_1(x) \neq 0$ (since we have excluded the case that both are zeros), then we can define $\tilde{H}(g) = \frac{\phi'(-g)}{\phi'(g)}$. Following similar logic, we can conclude that $|g_*(x)| < \varepsilon$ implies that $|C_1(x) - C_0(x)| < \tilde{K}_\delta |C_1(x)| \varepsilon$, where $\tilde{K}_\delta = \max_{t \in [-\delta, \delta]} |\tilde{H}'(t)|$.

Finally, let $K_1 = \sup_{x \in \mathcal{X}} \max\{K_\delta C_1(x), \tilde{K}_\delta C_0(x)\}$, then we can conclude that $|g_*(x)| \leq \varepsilon$ implies that $|C_1(x) - C_0(x)| \leq K_1 \varepsilon$, where K_1 does not depends on x or ε . ■

The next proposition bounds the estimation error of the estimated welfare (6.1) and establishes its oracle inequality.

Proposition 1: *In addition to the conditions for Theorem 5.1, suppose that that Assumptions 6.1 and 6.2 hold and the sieve order for the estimator $\hat{g}$ satisfies $k^{4+\varepsilon}/n \lesssim 1$ for some $\varepsilon > 0$. Then, we have*

$$\mathbb{E} \left[\left| \hat{V}_n(\hat{g}) - \sup_{g \in \mathcal{G}} V(g) \right| \right] \lesssim \inf_k \left[|V_0 - V(g_k)| + C_v \sqrt{\frac{\log(k)}{n}} \right],$$

where C_v is a finite universal constant and $g_k := \arg \min_{g \in \mathcal{G}_k} L^*(g)$.

Proof of Proposition 1: The estimated welfare follows the decomposition:

$$\begin{aligned} \hat{V}_n(\hat{g}) - V_0 &= (\hat{V}_n(\hat{g}) - \hat{V}_n(g_k)) + (\hat{V}_n(g_k) - V(g_k)) + (V(g_k) - V(g_*)) + \underbrace{(V(g_*) - V_0)}_{=0} \\ &= \underbrace{(\hat{V}_n(\hat{g}) - \hat{V}_n(g_k))}_{\text{term A}} + \underbrace{(\hat{V}_n(g_k) - V(g_k))}_{\text{term B}} + \underbrace{(V(g_k) - V(g_*))}_{\text{term C}}. \end{aligned}$$

In what follows, we will bound each of the three terms, A, B, and C, in the above expression. For the convenience of proofs, we define

$$V_n(g) := \mathbb{E}_n[\psi_1(Z_i, \eta_0) \cdot \mathbf{1}\{g(X_i) \geq 0\} + \psi_0(Z_i, \eta_0) \cdot \mathbf{1}\{g(X_i) < 0\}].$$

For term A, we adopt the further decomposition:

$$\begin{aligned} \hat{V}_n(\hat{g}) - \hat{V}_n(g_k) &= [V_n(\hat{g}) - V_n(g_k) - (V(\hat{g}) - V(g_k))] + (V(\hat{g}) - V(g_k)) \\ &\quad + [\hat{V}_n(\hat{g}) - \hat{V}_n(g_k) - (V_n(\hat{g}) - V_n(g_k))] \\ &= [V_n(\hat{g}) - V_n(g_k) - (V(\hat{g}) - V(g_k))] + (V(\hat{g}) - V(g_k)) + o_p(1/\sqrt{n}), \end{aligned}$$

where the last right-hand side term is $o_p(1/\sqrt{n})$ by Assumption 6.2.

We now analyze the second term on the right-hand side. Note $|V(\hat{g}) - V(g_k)| \leq |V(\hat{g}) - V(g_*)| + |V(g_*) - V(g_k)|$. Let $U_i := \mathbb{E}[\psi_1(Z_i, \eta_0)|X_i] - \mathbb{E}[\psi_0(Z_i, \eta_0)|X_i]$. Then, with some positive constant K_1 ,

$$\begin{aligned}
& |V(\hat{g}) - V(g_*)| \\
& \leq \mathbb{E}[|U_i| \mathbf{1}\{|g_*(X_i)| \leq |\hat{g}(X_i) - g_*(X_i)|\}] \\
& \leq \mathbb{E}[|U_i| \mathbf{1}\{|g_*(X_i)| \leq \sup_x |\hat{g}(x) - g_*(x)|\}] \\
& \leq \mathbb{E}[|U_i| \mathbf{1}\{|U_i| \leq K_1 \sup_x |\hat{g}(x) - g_*(x)|\}] \\
& \lesssim_p K_1 \sup_x |\hat{g}(x) - g_*(x)| \mathbb{P}(|U_i| \leq \sup_x |\hat{g}(x) - g_*(x)|) \\
& \lesssim_p K_1 \left(\sup_x |\hat{g}(x) - g_*(x)| \right)^2 = o_p\left(n^{-\frac{1}{2}}\right), \tag{A.26}
\end{aligned}$$

where the first inequality holds because $\hat{g}(X_i)$ and $g_*(X_i)$ disagree implies that $|g_*(X_i)| \leq |\hat{g}(X_i) - g_*(X_i)|$, and when they disagree, the absolute difference $|V(\hat{g}) - V(g_*)|$ is bounded by $|U_i|$. The third inequality holds by Lemma A.1, and the fourth inequality holds by the margin condition in Assumption 6.1. The last equality holds by Theorem 5.1 because the condition $k^{4+\varepsilon}/n \lesssim 1$ implies that the rate restriction $\xi_k \sqrt{\log(k)/n} \lesssim 1/n^{1/4}$. Also, Assumption 5.1(iv) implies $B_k \lesssim (\log(k))^{-1} n^{-1/2} = o(n^{-1/2})$. Similarly, using g_k in the place of $\hat{g}$, we can show that $|V(g_k) - V(g_*)| = o_p(n^{-1/2})$ because $\sup_x |g_k(x) - g_*(x)| \leq B_k = o(n^{-1/2})$ again by Assumption 5.1(iv). Therefore, we can conclude $|V(\hat{g}) - V(g_k)| \leq |V(\hat{g}) - V(g_*)| + |V(g_*) - V(g_k)| = o_p(n^{-1/2})$.

Since the smoothness of $V(\cdot)$ yields the upper bound of the envelope function $\sigma^2 \lesssim (1/n^{1/4})$, the maximal inequality of VC-class (Lemma B.3), Sauer's Lemma (Lemma B.2), and Kitagawa and Tetenov (2018, Lemma A.1) yield the upper bound:

$$\begin{aligned}
& [V_n(\hat{g}) - V_n(g_k) - \{V(\hat{g}) - V(g_k)\}] \\
& \leq \mathbb{E} \left[\sup_{\substack{g \in \mathcal{G} \\ |g(x) - g_k(x)| \leq \check{c}_2/n^{1/4}}} |V_n(g) - V_n(g_k) - \{V(g) - V(g_k)\}| \right] \\
& \leq C(1/\sqrt{n}) \sqrt{(1/n^{1/4}) k \log(n/k)} \lesssim 1/n^{1/2}, \tag{A.27}
\end{aligned}$$

where $\check{c}_2$ is a generic constant such that $\sup_x |\hat{g}(x) - g_k(x)| \leq \check{c}_2 \cdot (1/n)^{1/4}$. Combine both (A.26) and (A.27), we can conclude that the “term A” is of order $o_p(n^{-1/2})$.

For term B, we adopt the further decomposition:

$$\widehat{V}_n(g_k) - V(g_k) = (\widehat{V}_n(g_k) - V_n(g_k)) + (V_n(g_k) - V(g_k)), \quad (\text{A.28})$$

whose first term can be in turn decomposed as:

$$\begin{aligned} & (\widehat{V}_n(g_k) - V_n(g_k)) \\ &= -\frac{1}{n} \sum_{\ell=1}^m \sum_{i \in I(\ell)} [(\psi_1(Z_i, \widehat{\eta}_{(\ell)}) - \psi_1(Z_i, \widehat{\eta}_0)) \mathbf{1}\{g_k(X_i) \geq 0\}] \\ & \quad \underbrace{\hspace{10em}}_{\text{term (R1)}} \\ & \quad - \frac{1}{n} \sum_{\ell=1}^m \sum_{i \in I(\ell)} [(\psi_0(Z_i, \widehat{\eta}_{(\ell)}) - \psi_0(Z_i, \widehat{\eta}_0)) \mathbf{1}\{g_k(X_i) < 0\}], \\ & \quad \underbrace{\hspace{10em}}_{\text{term (R2)}} \end{aligned}$$

where each of the two terms, (R1) and (R2), converges in probability to zero uniformly as

$$\sup_{g \in \mathcal{G}_k} |(\text{R1})| = o_p((\log(k)/n)^{1/2}) \quad \text{and} \quad \sup_{g \in \mathcal{G}_k} |(\text{R2})| = o_p((\log(k)/n)^{1/2}),$$

under Assumption 6.2.

To bound the second term of (A.28), consider the function class

$$\mathcal{F} := \{f_k \mid f_k(Z_i) := \psi_1(Z_i, \eta_0) \cdot \mathbf{1}(g_k(X_i) \geq 0) + \psi_0(Z_i, \eta_0) \cdot \mathbf{1}(g_k(X_i) < 0)\}$$

with envelope $F_i = \max_{g_k \in \mathcal{G}_k} |f_k(Z_i)|$ and $M = \max_{i \in [n], g_k \in \mathcal{G}_k} |f_k(Z_i)|$. We have

$$\begin{aligned} M &\leq \max_{i \in [n], g_k \in \mathcal{G}_k} |\psi_1(Z_i, \eta_0) \cdot \mathbf{1}(g_k(X_i) \geq 0) + \psi_0(Z_i, \eta_0) \cdot \mathbf{1}(g_k(X_i) < 0)| \\ &\leq \max_{i \in [n], g_k \in \mathcal{G}_k} |\psi_1(Z_i, \eta_0) \cdot \mathbf{1}(g_k(X_i) \geq 0) - \psi_1(Z_i, \eta_0) \cdot \mathbf{1}(g_k(X_i) < 0)| \\ &\quad + \max_{i \in [n], g_k \in \mathcal{G}_k} |\psi_1(Z_i, \eta_0) \cdot \mathbf{1}(g_k(X_i) < 0) - \psi_0(Z_i, \eta_0) \cdot \mathbf{1}(g_k(X_i) < 0)| \\ &\lesssim_p n^{1/q}. \end{aligned}$$

Similarly, we have $\|F_i\|_{\mathbb{P}_n, 2} \lesssim 1$ and $\sigma_n^2 := \sup_{g_k \in \mathcal{G}_k} (1/n) \sum_{i \in [n]} f_k(Z_i)^2 \lesssim_p 1$ under Assumption 6.1. The entropy integral inequality (van der Vaart and Wellner, 1996, Theorem 2.14.1) yields

$$\begin{aligned} & \sqrt{n} \mathbb{E} \left[\sup_{g_k \in \mathcal{G}_k} |(\mathbb{E}_n - \mathbb{E}) [\psi_1(Z_i, \eta_0) \cdot \mathbf{1}(g_k(X_i) \geq 0) + \psi_0(Z_i, \eta_0) \cdot \mathbf{1}(g_k(X_i) < 0)]| \right] \\ & \lesssim \int_0^{\sigma_n} \sqrt{1 + \log \mathcal{N}(\mathcal{F}, e_{\mathbb{P}_n}, \varepsilon)} d\varepsilon \lesssim \|F_i\|_{\mathbb{P}_n, 2} \int_0^{\sigma_n / \|F_i\|_{\mathbb{P}_n, 2}} \sqrt{1 + \log \mathcal{N}(\mathcal{F}, e_{\mathbb{P}_n}, \|F_i\|_{\mathbb{P}_n, 2} \varepsilon)} d\varepsilon \end{aligned}$$

$$\lesssim \sqrt{\log(k)},$$

where $e_{\mathbb{P}_n}(f_k^{(1)}, f_k^{(2)}) := \|f_k^{(1)} - f_k^{(2)}\|_{\mathbb{P}_n, 2}$.

Finally, for term C, we have

$$|V(g_k) - V(g_*)| = \left| V(g_k) - \sup_{g \in \mathcal{G}} V(g) \right|,$$

where $g_k = \arg \min_{g \in \mathcal{G}_k} L^*(g)$.

Combining the three terms together proves the claimed statement ■

A.4 Proofs for Section 6: Estimation and Inference for the Optimal Policy Value

Proof of Theorem 6.1: As in Proposition 1, the difference between $\widehat{V}_n(\widehat{g})$ and V_0 can be decomposed as

$$\sqrt{n}(\widehat{V}_n(\widehat{g}) - V_0) = \sqrt{n}(\widehat{V}_n(\widehat{g}) - \widehat{V}_n(g_k)) \tag{A.29}$$

$$+ \sqrt{n}(\widehat{V}_n(g_k) - V(g_k)) \tag{A.30}$$

$$+ \sqrt{n}(V(g_k) - V(g_*)) \tag{A.31}$$

$$+ \sqrt{n}(V(g_*) - V_0), \tag{A.32}$$

where (A.29) measures the estimation error of the approximate policy; (A.30) contains the estimation error of nuisance parameters and sample moments, (A.31) measures sieve approximation error; and (A.32) is the identification error by the surrogate loss.

Since the identified representative classification policy function g_* belongs to the set of optimal policy functions by Theorem 4.1, the identification error (A.32) of the surrogate loss approach is exactly zero. (A.29) and (A.31) are negligible by the analysis on term A in Proposition 1, respectively. It remains to further analyze the second right-hand side term (A.30), which can be decomposed as

$$\begin{aligned} \sqrt{n}\{\widehat{V}_n(g_k) - V(g_k)\} &= \sqrt{n}\{\widehat{V}_n(g_k) - V_n(g_k) + V_n(g_k) - V(g_k)\} \\ &= o_p(1) + \sqrt{n}\{V_n(g_k) - V(g_k)\}, \end{aligned} \tag{A.33}$$

where the second inequality follows from Assumption 6.2.(i).

Write

$$\sqrt{n}(V_n(g_k) - V(g_k)) = \frac{1}{\sqrt{n}} \sum_{i=1}^n s_L(Z_i, \eta_0, g_k), \quad (\text{A.34})$$

where

$$s_L(Z_i, \eta_0, g_k) = \psi_1(Z_i, \eta_0) \mathbf{1}(g_k(X_i) \geq 0) + \psi_0(Z_i, \eta_0) \mathbf{1}(g_k(X_i) < 0) - V(g_k).$$

Thus, the summands in (A.34) are i.i.d., so they form a row-wise i.i.d. triangular array.

We first establish convergence of their variance. Since g_* induces the optimal policy and

$$U_i = \mathbb{E}[\psi_1(Z_i, \eta_0) \mid X_i] - \mathbb{E}[\psi_0(Z_i, \eta_0) \mid X_i],$$

we have

$$V(g_*) - V(g_k) = \mathbb{E} [|U_i| \mathbf{1}(\mathbf{1}(g_k(X_i) \geq 0) \neq \mathbf{1}(g_*(X_i) \geq 0))]. \quad (\text{A.35})$$

Consequently, for every $t > 0$,

$$\begin{aligned} & \mathbb{P}(\mathbf{1}(g_k(X_i) \geq 0) \neq \mathbf{1}(g_*(X_i) \geq 0)) \\ & \leq \mathbb{P}(|U_i| \leq t) + \mathbb{P}(|U_i| > t, \mathbf{1}(g_k(X_i) \geq 0) \neq \mathbf{1}(g_*(X_i) \geq 0)) \\ & \leq Ct + t^{-1}(V(g_*) - V(g_k)), \end{aligned} \quad (\text{A.36})$$

in which the last inequality follows from Assumption 6.1 and (A.35). The sieve approximation result gives $V(g_*) - V(g_k) = o(1)$. Taking $t = (V(g_*) - V(g_k))^{1/2}$ whenever $V(g_*) - V(g_k) > 0$ yields

$$\mathbb{P}(\mathbf{1}(g_k(X_i) \geq 0) \neq \mathbf{1}(g_*(X_i) \geq 0)) = o(1). \quad (\text{A.37})$$

If $V(g_*) - V(g_k) = 0$, the same result follows directly from (A.35) and $\mathbb{P}(U_i = 0) = 0$.

Moreover, Assumption 6.2.(ii), Hölder's inequality and (A.37) imply

$$\begin{aligned} & \left\| (\psi_1(Z_i, \eta_0) - \psi_0(Z_i, \eta_0)) [\mathbf{1}(g_k(X_i) \geq 0) - \mathbf{1}(g_*(X_i) \geq 0)] \right\|_{\mathbb{P},2}^2 \\ & \leq (\mathbb{E} |\psi_1(Z_i, \eta_0) - \psi_0(Z_i, \eta_0)|^q)^{2/q} \mathbb{P}(\mathbf{1}(g_k(X_i) \geq 0) \neq \mathbf{1}(g_*(X_i) \geq 0))^{1-2/q} = o(1). \end{aligned} \quad (\text{A.38})$$

After accounting for centering, (A.38) gives

$$\|s_L(Z_i, \eta_0, g_k) - s_L(Z_i, \eta_0, g_*)\|_{\mathbb{P},2} = o(1). \quad (\text{A.39})$$

It follows from the reverse triangle inequality that

$$\left| \left(\mathbb{E} [s_L(Z_i, \eta_0, g_k)^2] \right)^{1/2} - \sigma_v \right| \leq \|s_L(Z_i, \eta_0, g_k) - s_L(Z_i, \eta_0, g_*)\|_{\mathbb{P}, 2} = o(1). \quad (\text{A.40})$$

Assumption 6.2.(ii) also ensures that

$$\sup_n \mathbb{E} |s_L(Z_i, \eta_0, g_k)|^q < \infty.$$

If $\sigma_v^2 > 0$, (A.40) therefore implies the Lyapunov condition

$$\frac{\sum_{i=1}^n \mathbb{E} |s_L(Z_i, \eta_0, g_k)|^q}{\left(\sum_{i=1}^n \mathbb{E} [s_L(Z_i, \eta_0, g_k)^2] \right)^{q/2}} = \frac{\mathbb{E} |s_L(Z_i, \eta_0, g_k)|^q}{n^{q/2-1} \left(\mathbb{E} [s_L(Z_i, \eta_0, g_k)^2] \right)^{q/2}} \rightarrow 0.$$

The Lyapunov central limit theorem for the row-wise i.i.d. triangular arrays (Billingsley, 1995, Theorem 27.3, p. 362), together with (A.40), yields

$$\sqrt{n}(V_n(g_k) - V(g_k)) \rightarrow_d \mathcal{N}(0, \sigma_v^2). \quad (\text{A.41})$$

If $\sigma_v^2 = 0$, (A.40) and Chebyshev's inequality instead imply that the left-hand side of (A.41) converges in probability to zero, which gives the same result with a degenerate Gaussian limit. Combining all these components proves the claimed statement. ■

Proof of Theorem 6.2: The conditions of Theorem 6.1 imply

$$\mathbb{E}_n[(\widehat{s}_L(Z_i, \widehat{\eta}^{(\ell(i))}, \widehat{g}) - s_L(Z_i, \eta_0, g_k))^2] \rightarrow_p 0.$$

Conditional on $Z_1, \dots, Z_n$, the difference between $\widetilde{Z}_n$ and $\mathbb{G}_n[s_L(Z_i, \eta_0, g_k)\delta_i]$ has mean zero and variance equal to this empirical second moment. Then, for every $\varepsilon > 0$, conditional Chebyshev's inequality gives

$$\mathbb{P} \left(|\widetilde{Z}_n - \mathbb{G}_n[s_L(Z_i, \eta_0, g_k)\delta_i]| > \varepsilon \mid Z_1, \dots, Z_n \right) \rightarrow_p 0.$$

Because the weights are Gaussian and independent of data, we also have, conditionally on data,

$$\mathbb{G}_n[s_L(Z_i, \eta_0, g_k)\delta_i] \mid Z_1, \dots, Z_n \sim \mathcal{N}(0, \mathbb{E}_n[s_L(Z_i, \eta_0, g_k)^2]).$$

The law of large numbers and diminishing error of sieve approximation also yield

$$\mathbb{E}_n[s_L(Z_i, \eta_0, g_k)^2] \rightarrow_p \mathbb{E}[s_L(Z_i, \eta_0, g_*)^2] = \sigma_v^2.$$

Since σ_v^2 is bounded away from zero, conditional Slutsky's theorem and continuity of Φ imply the stated uniform convergence of the conditional distribution functions. The proof is complete. ■

B Auxiliary Results from the Literature

Lemma B.1 (Chen and Shen, 1998, Lemma 2): *Let $h = s + \gamma$ where $s \in \mathbb{Z}_+$ and $\gamma \in (0, 1]$. Define the Hölder space with smoothness order $p > 1/2$ and radius $L \in (0, \infty)$ by*

$$\Lambda^h(L) = \left\{ f : \mathcal{X} \rightarrow \mathbb{R}, \|f\|_{\mathcal{H}} = \sup_{x, y \in \mathcal{X}, x \neq y} \frac{|f^{(s)}(x) - f^{(s)}(y)|}{|x - y|^\gamma} \leq L \right\}, \quad (\text{B.1})$$

where $\mathcal{X} \subseteq \mathbb{R}^{d_x}$ is compact. Then, for any $f \in \Lambda^h(L)$, we have

$$\|f\|_{\mathbb{P}, \infty} \leq 2\|f\|_{\mathbb{P}, 2}^c L^{1-c},$$

where $c = 2(s + \gamma)/(2(s + \gamma) + d_x)$.

Consider a class $\mathcal{A}$ of subsets of $\mathbb{R}^d$, and let $x_1, \dots, x_n \in \mathbb{R}^d$ be arbitrary points. Recall that the finite set $\mathcal{A}(x_1^n) \subset \{0, 1\}^n$ defined by

$$\mathcal{A}(x_1^n) = \{b = (b_1, \dots, b_n) \in \{0, 1\}^n : b_i = \mathbb{I}_{[x_i \in A]}, i = 1, \dots, n \text{ for some } A \in \mathcal{A}\}$$

plays an essential role in bounding uniform deviations of empirical measures. In particular, the maximal cardinality of $\mathcal{A}(x_1^n)$ denoted by

$$\mathbb{S}_{\mathcal{A}}(n) = \max_{x_1, \dots, x_n \in \mathbb{R}^d} |\mathcal{A}(x_1^n)|,$$

i.e., the shatter coefficient, yields simple bounds via the Vapnik-Chervonenkis inequality.

Lemma B.2 (Sauer's Lemma; Lugosi, 2002, Corollary 1.3): *Let $\mathcal{A}$ be a class of sets with VC dimension $V < \infty$. Then, for all n ,*

$$\mathbb{S}_{\mathcal{A}}(n) \leq (n + 1)^V,$$

and for all $n \geq V$,

$$\mathbb{S}_{\mathcal{A}}(n) \leq \left(\frac{ne}{V}\right)^V.$$

Lemma B.3 (Maximal Inequality of VC Class; Massart and Nédélec, 2006, Lemma A.3):

Let $\mathcal{F}$ be a countable VC-class with dimension not larger than $V \geq 1$ and assume that $\sigma > 0$ is such that $\mathbb{E}(f(Z_i)) \leq \sigma^2$ for every $f \in \mathcal{F}$. Let

$$W_{\mathcal{F}}^+ = \sup_{f \in \mathcal{F}} \mathbb{G}_n(f) \quad \text{and} \quad W_{\mathcal{F}}^- = \sup_{f \in \mathcal{F}} -\mathbb{G}_n(f),$$

and $H_{\mathcal{F}} = \log \mathbb{S}_{\mathcal{F}}(n)$. Then, there exists an absolute constant K such that

$$\sqrt{n}(\mathbb{E}[W_{\mathcal{F}}^-] \vee \mathbb{E}[W_{\mathcal{F}}^+]) \leq \frac{K}{2} \sigma \sqrt{\mathbb{E}[H_{\mathcal{F}}]},$$

provided that $\sigma \geq K \sqrt{\mathbb{E}[H_{\mathcal{F}}]}/n$.

Lemma B.4 (Pollard, 2002, Theorem 10): *Let $\zeta_1, \dots, \zeta_n$ be independent k -vectors with $\mathbb{E}[\zeta_i] = 0$ for each i , and $\Delta := \sum_{i=1}^n \mathbb{E}\|\zeta_i\|^3$ finite. Let S denote a copy of $\zeta_1 + \dots + \zeta_n$ on a sufficiently rich probability space $(\Omega, \mathcal{A}, \mathbb{P})$. For each $\delta > 0$, there exists a random vector T in this space with a $\mathcal{N}(0, \text{var}(S))$ distribution such that*

$$\mathbb{P}\{\|S - T\| > 3\delta\} \leq C_y B \left(1 + \frac{|\log(1/B)|}{k}\right) \text{ where } B := \Delta k \delta^{-3},$$

for some generic constant $C_y \in (0, \infty)$.

C Testing Optimal Welfare Against Benchmark Designs

This section proposes tests that compare the welfare of the estimated optimal policy to that of benchmark policies. We consider two benchmark designs for $g_{\dagger}$: (i) treating everyone; and (ii) assigning treatment at random. Formally, we consider the null hypothesis

$$\mathcal{H}_0 : V_0 = V(g_{\dagger}),$$

where $V(g_{\dagger})$ denotes the welfare associated with the benchmark policy $g_{\dagger}$. The test statistic is defined as

$$T_{g_*, g_{\dagger}} := \sqrt{n} (\widehat{V}_n(\widehat{g}) - \widehat{V}_n(g_{\dagger})), \quad (\text{C.1})$$

where $\widehat{g}$ is the estimated optimal policy and $\widehat{V}_n(\cdot)$ is defined in (6.1).

We partition $[n]$ into m approximately balanced folds $I_1, \dots, I_m$, so that $n_{(1)} \asymp \dots \asymp n_{(m)}$. For each fold $\ell \in [m] = \{1, \dots, m\}$, we have a nuisance parameter estimator $\widehat{\eta}_{(\ell)}$ of η_0 using the complementary subsample $I_{(\ell)}^c$. The critical values of $T_{g_*, g_{\dagger}}$ are obtained via the score bootstrap,

$$\widetilde{Z}_n := \mathbb{G}_n \left[\widehat{s}_L(Z_i, \widehat{\eta}_{(\ell(i))}, \widehat{g}, g_{\dagger}) \cdot \delta_i \right], \quad (\text{C.2})$$

where $\{\delta_i\}_{i \in [n]}$ are i.i.d. $\mathcal{N}(0, 1)$ random variables independent of the observed data and $\hat{\eta}_{(\ell(i))}$ denotes the nuisance estimator calculated on the complementary subsample $I_{(\ell(i))}^c$. The score function is

$$\begin{aligned} & \hat{s}_L(Z_i, \hat{\eta}_{(\ell(i))}, \hat{g}, g_{\dagger}) \\ &= (\psi_1(Z_i, \hat{\eta}_{(\ell(i))}) - \psi_0(Z_i, \hat{\eta}_{(\ell(i))})) \left(\mathbf{1}\{\hat{g}(X_i) \geq 0\} - \mathbf{1}\{g_{\dagger}(X_i) \geq 0\} \right) - \left(\hat{V}_n(\hat{g}) - \hat{V}_n(g_{\dagger}) \right). \end{aligned} \quad (\text{C.3})$$

Thus, the indicator in the feasible bootstrap score uses the estimated policy $\hat{g}$, and the score is centered by the estimated welfare difference appearing in the test statistic.

For a given significance level $\alpha \in (0, 1)$, let $\hat{c}_{1-\alpha}$ denote the $(1 - \alpha)$ quantile of $|\tilde{Z}_n|$. Then, under the null hypothesis $\mathcal{H}_0 : V_0 = V(g_{\dagger})$, we have

$$\lim_{n \rightarrow \infty} \mathbb{P}(|T_{g^*, g_{\dagger}}| \leq \hat{c}_{1-\alpha}) = 1 - \alpha,$$

and the theoretical validity of this result follows by arguments analogous to those used in the proof of Theorem 6.2.

The implementation details for conducting this test are summarized as follows:

Step (i) **Test statistic.** Construct the statistic

$$T_{g^*, g_{\dagger}} := \sqrt{n} \left(\hat{V}_n(\hat{g}) - \hat{V}_n(g_{\dagger}) \right),$$

where $\hat{g}$ is the estimated policy and $\hat{V}_n(\cdot)$ is defined in (6.1).

Step (ii) **Bootstrap process.** Generate the score bootstrap process

$$\tilde{Z}_n := \mathbb{G}_n \left[\hat{s}_L(Z_i, \hat{\eta}_{(\ell(i))}, \hat{g}, g_{\dagger}) \cdot \delta_i \right],$$

following (C.2) and (C.3), where $\{\delta_i\}_{i \in [n]}$ are i.i.d. $\mathcal{N}(0, 1)$ variates independent of the data.

Step (iii) **Critical value.** For a given level $\alpha \in (0, 1)$, let $\hat{c}_{1-\alpha}$ denote the $(1 - \alpha)$ quantile of $|\tilde{Z}_n|$.

Step (iv) **Decision rule.** Reject the null hypothesis $\mathcal{H}_0 : V_0 = V(g_{\dagger})$ whenever

$$|T_{g^*, g_{\dagger}}| > \hat{c}_{1-\alpha}.$$

Then, under $\mathcal{H}_0 : V_0 = V(g_{\dagger})$,

$$\lim_{n \rightarrow \infty} \mathbb{P}(|T_{g^*, g_{\dagger}}| \leq \hat{c}_{1-\alpha}) = 1 - \alpha.$$

Step (v) **Bootstrap P-value.** In practice, the P-value can be computed directly from the bootstrap distribution as

$$\text{P-value} = \mathbb{P}^* \left(|\tilde{Z}_n| \geq |T_{g_*, g_{\dagger}}| \mid \{Z_i\}_{i \in [n]} \right),$$

where $\mathbb{P}^*$ denotes the probability conditional on data. Equivalently, P-value is estimated as the proportion of bootstrap draws for which $|\tilde{Z}_n|$ exceeds $|T_{g_*, g_{\dagger}}|$.

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