Tracking Changes in Bilateral Trading Patterns in the Absence of PPP *†

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Abstract

When the 'law of one price' fails to hold between trading partners, they may find it difficult to measure the balance of trade due to the inherent currency discrepancy. This may be of particular interest when the nature of the goods and services trade change in their composition over time. In this paper we show how to define trade indexes capable of tracking changing trade patterns in the absence of purchasing power parity and proportional price movement. We find that the natural indicator of the change in trade is not a number but a more general matrix valued index of Divisia type. This matrix shows that the best current period analog of a country's past exports is naturally a mixture of both nation's current exports being exchanged, and that the components of the 2×2 matrix allow one to express this via a formula borrowed from differential geometry and quantum theory.

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1 Introduction

Consider two nations (randomly assigned the labels $N_1,\ N_2$), which export baskets of goods $q_1, q_2 \in V$ to each other in a given period t. Assume that the space of goods V is unchanging during the period in question and that these baskets are valued by the pricing systems of the two countries $p_1, p_2 \in V^*$[1]. We now have a number of problems to confront if we are to engage in growth accounting and discuss the trading relationship and its evolution in quantitative terms.

For example the bilateral question of whether to compare the export vectors via a quantity index using the first country's currency and prices p_1 in a Laspeyres type formula or a Paasche type formula using the second country's currency p_2, is analogous to the intertemporal question in the one country situation where the subscripts refer to base time and current time. Of course if it so happens that an idealized trade situation exists so that the law of one price holds exactly, then there will exist a positive number $c \in \mathbb{R}_+$ such that $p_1 = c\ p_2$ resulting in the equality

$$\frac{p_1 \cdot q_2}{p_1 \cdot q_1} = \frac{c\ p_2 \cdot q_2}{c\ p_2 \cdot q_1} = \frac{p_2 \cdot q_2}{p_2 \cdot q_1} \tag{1}$$

of the two index numbers. Thus in pure theory, there is some justification for thinking that the trade problem is easier than the dual intertemporal problem. It is, however, generally accepted that the law of one price does not hold strictly enough to give us this resolution. We therefore have to once again deal with the issue of an index number problem.

As we have said before, without further data our methods offer little guidance in this single period bilateral trade problem and we shall not clutter this paper to discuss the pure instantaneous index number problem further. However, in the trade situation, while we can separate out the geographic from the temporal issue, they are in fact related. Thus what appears to start off as a pure bilateral problem does, in fact, *lead* directly to an intertemporal problem which is even more intricate than the 1-country problem of indexing GDP: how does one measure intertemporal **changes** in trade when the (relative) pricing systems of the two currencies and compositions of the trade vectors are changing? Here one has twice as many time dependent quantity vectors and price vectors as in the single country intertemporal growth problem. The questions arising from this doubling of data are of some interest as it may cause two countries to disagree on whether a perceived shift in the balance of trade is due to changing patterns of imports and exports or shifts in the relative purchasing power of the two currencies.

In this paper we respond to this challenge by showing that our techniques, when generalized to the bilateral context, define new natural indexes for changing trade patterns. By extending the framework developed in [M-W][2] we can develop a generalized Divisia 'type' matrix-valued index for trade which tracks changes in trade vectors *without* assuming the existence of Purchasing Power Parity by taking into account both the

[1]Technically we are not concerned with vectors of prices as only the relative prices of each country are of interest as can be seen in the various formulae.

[2]In this paper we will refer to the program initiated in the three papers in the Bibliography authored by Malaney and Weinstein collectively as [M-W].

vectors $p_1(t), p_2(t)$ representing *both* nations pricing systems on an equal footing, and thus resolving the consistency problem between the two countries.

While this index can be viewed from the point of view of differential geometry to be a generalization of the single nation Divisia quantity index, this paper translates the geometric techniques into calculus and linear algebra so that the paper develops the techniques from scratch using methods already in use in neo-classical economics. With that said, the construction is considerably more intricate than that giving the familiar scalar valued Divisia formulae and seems not to be easily accessible from the perspective of traditional techniques. In fact this index is not so much a generalization of the Divisia concept as the Divisia concept is a particularly drastic simplification of the general technique peculiar to the special case of '1×1' matrices.

Even without access to the geometric intuition, there is a major new feature that is elucidated: the intertemporal two country trade indexes of prices and quantities should be matrix valued when viewed in their natural setting. While one can always extract scalars from a matrix (e.g. via the determinant or trace operations familiar from linear algebra), this is different than the traditional Divisia index in two main ways. Firstly, the non-commutativity of 2×2 matrices complicates the derivation and construction of the trade index and, at least at first blush, obscures its relationship to the Divisia index so that even with the formula in hand it may not be obvious that there is a connection. Secondly, the interpretation of a matrix index may need to be dealt with at the level of the individual matrix entries. Since this differs from the standard bilateral theory we take this opportunity to explain the significance of the matrix index component by component in a simplified situation.

Let us imagine that in a simplified model, there are only 2 items q^a, q^b traded between two countries (e.g. Canada and the United States) which we will refer to as goods and services respectively; we also assume that they are measured in suitable units appropriate to each tradable chosen once and for all. For simplicity we imagine that their currencies have perfect PPP with 1 Canadian dollar equal to 1 U.S. dollar and that goods and services are both priced at 1 dollar per unit throughout the period of inquiry.

A country's relative change in its export profile can be viewed as occurring in two ways. Firstly, it can change the nature of the goods which it exports; in particular, it can shift its exports towards or away from its trading partner's traditional export profile. Secondly, it can increase or decrease the quantity of its traditional exports. Since the accounting identity governing bilateral trade says that one country's exports are the other's imports, dual statements apply to imports as well.

With this in mind, we imagine that at time $t = 0$, country 1 exports only goods to country 2 in the amount of 60 units and imports only services in the amount of 60 units. At a later time we imagine that country 1 exports 120 units of goods and 20 units of services while importing 140 units of services and no goods.

While it is fairly clear in this contrived example that trade has grown according to most any reasonable measure, there is more going on than simple growth. Country 2 has not merely increased its level of exports but has partially shifted its export basket into a sector (services) previously dominated by country 1.

In such a trivial 2-good example, we can easily construct a natural 2×2 matrix-valued index of trade between times 0 and t by

$$\Lambda = \begin{pmatrix} \lambda_{11}(t) & \lambda_{12}(t) \\ \lambda_{21}(t) & \lambda_{22}(t) \end{pmatrix} = \begin{pmatrix} \frac{120}{60} & 0 \\ \frac{20}{60} & \frac{140}{60} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ \frac{1}{3} & \frac{7}{3} \end{pmatrix} \tag{2}$$

and interpret the components as follows. The matrix entries Q_{11} and Q_{22} give the growth or shrinkage in each country's exports relative to its old export profile. The off-diagonal entries Q_{ij} $i \neq j$ give the competitive shift; as an example Q_{21} gives a measure of the extent to which country 1's exports have shifted towards country 2's traditional export profile. What we see therefore, is that if we could extend this simple minded example in a natural way to a more general situation, component by component analysis of such a matrix-valued quantity index of trade would enable us to separate out changes in a country's exports in terms of growth in its traditional export niche from shifts into its trading partner's competitive niche.

All of this discussion is fairly straightforward in the case of two goods under purchasing power parity and does not require any significant technique to speak of. We may then ask why such a matrix is not a more common feature of bilateral trade discussions. It is at this point that we notice that things change considerably when we assume that the number of tradeables increases beyond the magical value of 2 (or that the currencies are not in agreement on relative valuations).

The 2×2 nature of the matrix, together with its component by component interpretation, reflects the fact that we are looking at a bilateral trade problem between 2 nations; in particular the '2' in '2×2' refers to there being 2 countries and should have nothing to do with the number of goods being artificially limited at 2. However, if we had added a third good (e.g. raw materials) there would be no guarantee that the new export vectors would be expressible as a linear combination of the old export vectors; yet we would still want a 2×2 matrix of trade just as we ultimately use a scalar valued (i.e. a 1×1 matrix) index of productivity in a single country no matter how many goods are present. Thus naively, this matrix valued index appears to be valid only in the artificial case of 2 goods (where linear algebra guarantees that the old linearly independent vectors span the space of goods).

The approach we present in this paper automatically modernizes the baskets of trade under consideration. This means, for example, that should a developing country experience industrialization, yesterday's trade in baskets tilted towards agricultural commodities and raw materials will be re-expressed as a kind of present value analogue basket weighted towards services, manufactured goods, and the like. Further, we are able to perform this present value modernization at all times in a manner which is symmetrical with respect to both currencies, even though PPP is assumed to have failed. This is accomplished by locating substitution effects which, even in the absence of PPP, are recognized by both currencies.

In previous work [M-W], the 'Index Number Problems' arising in the measurement of inflation and growth were treated for a single economy or agent; the techniques developed therein, among other things, dealt with the discrepancies between various index number formulas such as those for the Laspeyres, Paasche and Fisher

$$Q_L = \frac{p_1 \cdot q_2}{p_1 \cdot q_1} \qquad Q_P = \frac{p_2 \cdot q_2}{p_2 \cdot q_1} \qquad Q_F = \sqrt{\frac{p_1 \cdot q_2}{p_1 \cdot q_1} \frac{p_2 \cdot q_2}{p_2 \cdot q_1}} \tag{3}$$

indices.

In an intertemporal context, the formulas in line (3) are all ‘fixed basket’ formulas. As we have argued previously, such formulas can be seen as implicitly defining ‘fixed’ or ‘constant’ relative to vector-valued functions annihilated by the usual total derivative learned from calculus. Unfortunately, this derivative operator can be seen to confuse income and substitution effects. In the inter-temporal case it is possible to resolve the index number problem by replacing the ordinary derivative with a distinguished derivative operator adapted exclusively for the economics of this situation. This adapted derivative is distinguished in that it is the unique natural differential operator which correctly separates income from substitution effects within the confines of the available economic data. This is unfortunately not possible to do in the single period geographic situation, as any kind of marginal analysis fails in the absence of a smooth path between the two baskets q_1 and q_2. The intertemporal marginal techniques developed in [M-W] were found to recover the Divisia index as a geometric scaling factor in the one country case then under consideration. This lead to the appreciation of the Divisia index as the universal primitive formula independent index consistent with the geometry implicitly defined by separating income and substitution effects.

In this paper we show that the geometric perspective goes beyond elucidating the role of the Divisia index. To the best of our knowledge, the matrix valued generalized Divisia type index has not been discovered within the bilateral trade theory literature. This is not totally surprising as its construction is much less obvious than the traditional Divisia integrals.

In section 2 of this paper we sketch how the usual Divisia index can be derived as a linear scaling factor which allowed us to translate an initial period vector of goods through time so that it changes only by substitution effects while keeping its components up to date (i.e. the vector is kept parallel to the actual vector of goods being produced or consumed). This construction appears not to be much discussed and turns out to generalize to our trade (and other) problem(s). In section 3 we set about constructing the mathematical gadgetry (e.g. projection maps, distinguished sub-spaces) defined by our trade problem. We define an ‘economic derivative’ operator customized to our trade problem which tells us which time dependent quantity vectors of imports or exports should be considered ‘economically constant’ by sending those vector valued functions to zero. This allows us to move antiquated vectors in the initial period through time to give analogues of what those vectors would have looked like had they been part of today’s trading regime. This is more difficult in the case of trade than it was in the single country situation where we first introduced these techniques: when products of scalars are generalized to matrices, it is not clear in what order they should be multiplied. In section 3.5 we introduce time ordered products to deal with this non-commutativity of the matrix valued integrands. The technique is now standard in quantum theory and differential geometry but may be new within economics.

In section 4 we construct our candidate for the natural trade index. We can see from an inspection of the formula that it is a kind of non-obvious matrix valued analogue of the Divisia indexes. To further this claim, in section 5 we then show that it is possible to construct natural but slightly subtle matrix valued analogues of the Paasche and Laspeyres indexes which use the same incorrect notion of constancy which conflates income and substitution effects. We then show that, just as in our earlier work, this Divisia Trade index solves the expected index number problem between these inappropriate matrix valued bilateral indexes. In section 6 we summarize and give some concluding remarks.

2 The Geometric Interpretation of the Divisia Quantity Index.

In [M-W] it was shown that all common index numbers can be viewed as an implicit combination of two separate constituents:

1. An *explicit* choice of algebraic formula
2. An *implicit* choice of a notion of constancy (i.e. a derivative operator so that the constant functions of the theory are those annihilated by the operator).

There has been an implicit use of the ordinary (total) derivative operator in all bilateral index number formulas despite the fact that the economics of income and substitution effects points to a second distinguished derivative operator which has gone unmentioned in the literature. When this implicit choice of the ordinary total derivative is made explicit, and replaced with the choice of the derivative operator capable of separating income from substitution effects, it is found that all index numbers converge to the Divisia index independent of the choice of formula. In particular, this argument does not depend on chaining procedures and so can be interpreted to shed light on the controversial (path-dependent) chaining arguments.

What the geometry showed was that the Divisia index Q_D could be viewed as a scaling factor which allowed for the original basket $q(t)$ to change by 'barters'[3] only, thereby separating out the income from the substitution effects. We recapitulate and ask the reader to refer to [M-W] for the details:

Proposition 1 *Let $\alpha : \mathbb{R} \longrightarrow V \times V^*$ be a path representing the economic history for a basket $q(t)$ relative to a pricing system $p(t)$[4] and let $Q_D(t)$ be the Divisia quantity index for α relative to a choice of initial time s. Then $\tilde{q}(t) = q(t)(Q_D(t))^{-1}$ changes only by barters $\forall t$. That is*

$$p(t) \cdot \frac{d\tilde{q}(t)}{dt} = 0 \tag{4}$$

Proof:

$$p(t) \cdot \frac{d\tilde{q}(t)}{dt} = p(t) \cdot (\frac{dq(t)}{dt}(Q_D(t))^{-1} + q(t)\frac{d(Q_D(t))^{-1}}{dt}) \tag{5}$$

$$= p(t) \cdot \frac{dq(t)}{dt}(e^{-\int_s^t \frac{p(r)\cdot\frac{dq(r)}{dr}}{p(r)\cdot q(r)}dr}) + p(t) \cdot q(t)(\frac{de^{-\int_s^t \frac{p(r)\cdot\frac{dq(r)}{dr}}{p(r)\cdot q(r)}dr}}{dt}) \tag{6}$$

$$= p(t) \cdot \frac{dq(t)}{dt}(e^{-\int_s^t \frac{p(r)\cdot\frac{dq(r)}{dr}}{p(r)\cdot q(r)}dr}) - p(t) \cdot q(t)(e^{-\int_s^t \frac{p(r)\cdot\frac{dq(r)}{dr}}{p(r)\cdot q(r)}dr})\frac{p(t) \cdot \frac{dq(t)}{dt}}{p(t) \cdot q(t)} \tag{7}$$

$$= p(t) \cdot \frac{dq(t)}{dt}(e^{-\int_s^t \frac{p(r)\cdot\frac{dq(r)}{dr}}{p(r)\cdot q(r)}dr}) - p(t) \cdot \frac{dq(t)}{dt}(e^{-\int_s^t \frac{p(r)\cdot\frac{dq(r)}{dr}}{p(r)\cdot q(r)}dr}) = 0 \tag{8}$$

[3] By a barter at time t, we mean a vector $x \in V$ in the space of goods V which is valueless at time t so that $p_t \cdot x_t = 0$.

[4] We assume here that the pricing system $p(t)$ never sees the basket $q(t)$ as valueless at any point in the history α.

QED

Part of the value in rethinking the Divisia index from this perspective is that this point of view is the one that generalizes naturally to the rather more intricate multi-basket setting. In this case, the Divisia scaling factor is replaced by a matrix valued function whose construction will form the focus of this paper.

3 Set-up.

3.1 Identification of the Fundamental Sub-spaces.

Consider two countries $N_1,\ N_2$ which produce and/or consume a collection of n-goods given by a vector space V^n. Assume that they are trading partners which use two currencies C_1, C_2 which determine two separate pricing systems $p_1(t),\ p_2(t) \in V^*$ linked by a floating exchange rate. Assuming that PPP doesn't hold perfectly the currencies will span a two dimensional currency subspace $[\mathcal{P}(t)] \subset V^*$. Let the exports from country i to country j be given by $q_i \in V$. From these vectors of exports, we form the $n \times 2$ matrix $\mathcal{Q}(t)$ given as

$$\mathcal{Q}(t) = \begin{pmatrix} | & | \\ q_1(t) & q_2(t) \\ | & | \end{pmatrix} \tag{9}$$

which allows us to write N_1's exports, imports, total trade and trade balance relative to N_2 in matrix notation via

$$\mathcal{Q}(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix},\quad \mathcal{Q}(t)\begin{pmatrix} 0 \\ 1 \end{pmatrix},\quad \mathcal{Q}(t)\begin{pmatrix} 1 \\ 1 \end{pmatrix},\quad \mathcal{Q}(t)\begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{10}$$

respectively.

Assuming that a country's imports are not a multiple of its exports, $q_1(t)$ and $q_2(t)$ span a two dimensional 'trade plane' $[\mathcal{Q}(t)] \subset V$ given by

$$[\mathcal{Q}(t)] = \mathcal{R}(\mathcal{Q}(t)) \tag{11}$$

where $\mathcal{R}$ denotes the column space of a matrix.

As in [M-W] we will denote by $\beta_{[p_i(t)]}(t)$ the subspace of the space of goods viewed by the currency C_i under price vector $p_i(t)$ as valueless. Since each currency will determine an $n-1$ dimensional hyperplane of such barters, the collection of baskets of goods which both currencies view as valueless will generically be a $n-2$ dimensional space as the intersection of $n-m$ dimensional planes in general position in an n dimensional ambient space will be a plane of dimension $n-2m$ if $m \geq \frac{n}{2}$. We will notate this plane of mutual barters by $\beta_{[\mathcal{P}(t)]}$.

3.2 Construction of the Space of Economic Possibilities.

In [M-W] we considered the index number problem involved in tracking a basket of goods $q(t)$ over time in the pricing $p(t)$ system of a single currency. Let $v(t) = p(t) \cdot q(t)$ be the time dependent value function of the basket. We defined the economic history $\alpha(t)$ to be a map

$$\alpha : \mathbb{R} \longrightarrow \Xi \tag{12}$$

where $\Xi = V \times V^* - v^{-1}(0)$ was referred to as the space of possible economic states. While this definition cannot be carried over directly into the present two currency situation, differential geometry suggests the generalization which we present here in terms of linear algebra.

First, to make the transition from single currency problems to a dual currency situation, we must generalize the familiar scalar valued 'value' function: $v(t) = p(t) \cdot q(t)$. To this end, we define the 2×2 matrix valued trade-value function $\mathbf{v(t)}$ by

$$\mathbf{v}(t) = \mathcal{P}(t) \cdot \mathcal{Q}(t) = \begin{pmatrix} p_1(t) \cdot q_1(t) & p_1(t) \cdot q_2(t) \\ \\ p_2(t) \cdot q_1(t) & p_2(t) \cdot q_2(t) \end{pmatrix} \tag{13}$$

With this defined we can now mimic our earlier construction. Where before our space of single currency economic states was given set theoretically as

$$\Xi_0 = V \times V^* - v^{-1}(0) \tag{14}$$

Thus in our new situation we can define the space

$$\Xi_T = V \times V \times V^* \times V^* - D \tag{15}$$

where

$$D = \{(q_1, q_2, p_1, p_2) \in V \times V \times V^* \times V^* \ s.t. \begin{vmatrix} p_1 \cdot q_1 & p_1 \cdot q_2 \\ p_2 \cdot q_1 & p_2 \cdot q_2 \end{vmatrix} = 0\}. \tag{16}$$

We will refer to the space $\Xi_T = \Xi$ as the space of trading states.

3.3 Construction of the Economic Derivative.

The key point behind constructing an 'economic derivative' is that the derivative should not confuse income and substitution effects. Thus changes in a basket of goods by the substitution of other goods possessing equal market value should not be confused for growth. While this is clearly spelled out in [M-W] this paper adds a new twist. It may be in the case of two currencies that what looks like a substitution effect in one currency appears to be an income effect in the other currency. It is for this reason that we have introduced the space of mutual barters in this paper so that we can separate out those substitution effects which are common to both currencies. Thus the dual currency analogue of the derivative operator introduced in [M-W] is given by:

$$\nabla^a = \Pi_{[\mathcal{Q}(t)]} \nabla^o \Pi_{[\mathcal{Q}(t)]} + \Pi_{\beta_{[\mathcal{P}(t)]}} \nabla^o \Pi_{\beta_{[\mathcal{P}(t)]}} \tag{17}$$

Once the conceptual change is made, there is little difference in the form of the object. The explicit formulas for the projection maps are straightforward if slightly more intricate and we now turn ourselves to their explicit construction.

3.4 Construction of the Projection maps.

We begin by defining the projection map $\Pi_{[\mathcal{Q}(t)]} : V \longrightarrow [\mathcal{Q}(t)] \subset V$.

It should be noted that the definition of the projection map is somewhat subtle. While we know from linear algebra that one can define projection onto the column space of a matrix M with linearly independent columns by the formula

$$\Pi_{\mathcal{R}(M)} = M(M^T M)^{-1} M^T, \tag{18}$$

it is *not* valid to do this for our matrix $\mathcal{Q}(t)$. If we were to use this formula, it would give us the formula for the projection matrix onto $[\mathcal{Q}(t)]$ relative to its orthogonal complement. The orthogonal complement however has no economic significance as it depends on the choice of units in which our goods are measured.

To get valid economic results we must project onto $[\mathcal{Q}(t)]$ relative to the space $\beta_{[\mathcal{P}]}$ of mutual barters.

Proposition 2 *The projection maps*

$$\Pi_{[\mathcal{Q}]} : V \longrightarrow [\mathcal{Q}] \tag{19}$$

and

$$\Pi_{\beta_{[\mathcal{P}]}} : V \longrightarrow [\beta_{[\mathcal{P}]}] \tag{20}$$

are determined at the level of matrices by the formulas:

$$\Pi_{[\mathcal{Q}]} = \mathcal{Q}(\mathcal{P}\mathcal{Q})^{-1}\mathcal{P} \qquad \Pi_{\beta_{[\mathcal{P}]}} = Id - \Pi_{[\mathcal{Q}]} \tag{21}$$

Proof:

The formula

$$\Pi_{\beta_{[\mathcal{P}]}} = Id - \Pi_{[\mathcal{Q}]} \tag{22}$$

is essentially a tautological consequence of the decomposition of a vector relative to a pair of complementary subspaces so it suffices to prove the formula for the projection $\Pi_{[\mathcal{Q}]}$.

To prove that

$$\Pi_{[\mathcal{Q}]}(v) = \mathcal{Q}(\mathcal{P}\mathcal{Q})^{-1}\mathcal{P}v \tag{23}$$

for all $v \in V$, it is sufficient to establish two things:

1. $\mathcal{Q}(\mathcal{P}\mathcal{Q})^{-1}\mathcal{P}v \in [\mathcal{Q}]$.
2. $v - \mathcal{Q}(\mathcal{P}\mathcal{Q})^{-1}\mathcal{P}v \in \beta_{[\mathcal{P}]}$.

The first point is immediate as the $\mathcal{Q}$ matrix on the left guarantees that $\mathcal{Q}(\mathcal{P}\mathcal{Q})^{-1}\mathcal{P}v$ will be a linear combination of q_1 and q_2 for any $v \in V$.

To establish the second point we see that:

$$\mathcal{P}(v - \mathcal{Q}(\mathcal{P}\mathcal{Q})^{-1}\mathcal{P}v) = \mathcal{P}v - \mathcal{P}\mathcal{Q}(\mathcal{P}\mathcal{Q})^{-1}\mathcal{P}v \tag{24}$$

$$= \mathcal{P}v - \mathcal{P}v = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{25}$$

which is the condition that $v - \mathcal{Q}(\mathcal{P}\mathcal{Q})^{-1}\mathcal{P}v \in \beta_{[\mathcal{P}]}$.

QED

3.5 Time Ordered Products.

The answer which we will obtain in the end for our trade index is a direct analog of the one basket Divisia index. However, without the technique of time ordered products the relationship is obscured. Time ordered products[5] are unknown in the usual Divisia literature because the Divisia index is the exponential of the integral of an ordinary function. In our situation however the function is replaced by a time dependent (2×2) matrix valued function. Seen from this vantage point, the key feature is that in the simpler one basket case we are integrating a 1×1 matrix (a function) and its powers. This means that since all 1×1 matrices commute there is no question of specifying the orders of their products. In the 2-basket situation the non-commutative nature of the matrices must be handled with care.

Let $\mathcal{M}$ be the space of real valued $m \times m$ matrices and let $M : \mathbb{R} \longrightarrow \mathcal{M}$ be a matrix valued function. Let $\{t_i\}_{i=1}^{l}$ be a collection of times which the permutation σ arranges in decreasing chronological order

$$\sigma(\{t_i\}_{i=1}^{l}) = \{\tilde{t}_i\}_{i=1}^{l} \text{ with } \ \tilde{t}_l \geq \tilde{t}_{l-1} \geq \tilde{t}_{l-2} ... \tag{26}$$

We define the time ordered product operator TOP to be the operator given by

$$TOP(M(t_1)M(t_2)M(t_3)...M(t_l)) = M(\tilde{t}_l)M(\tilde{t}_{l-1})M(\tilde{t}_{l-2})...M(\tilde{t}_1) \tag{27}$$

and by extension we define the time ordered product of the k^{th} power of the matrix integral

$$\int_a^b M(t) \ dt \tag{28}$$

to be the k-fold integral with time-ordered matrix integrand

$$TOP((\int_a^b M(t) \ dt)^k) \tag{29}$$

$$= \int_a^b dt_k \int_a^b dt_{k-1} ... \int_a^b dt_1 \ TOP(M(t_1)M(t_2)...M(t_k)) \tag{30}$$

$$= k! \int_a^b dt_k \int_a^{t_k} dt_{k-1} \int_a^{t_{k-1}} ... \int_a^{t_2} dt_1 \ M(t_k)M(t_{k-1})M(t_{k-2})...M(t_1) \tag{31}$$

[5] The technique appears to date back at least to Freeman Dyson.

4 Construction of the Trade Index.

When discussing whether or not the trade between two nations has changed, we are faced with two basic problems; one is familiar from the simpler situation of changing domestic production with the other the result of pricing discrepancies between the two currencies. The major result of this section (and this paper) is the explicit construction of a matrix valued index which accomplishes both tasks by relating the later trade relationship to the earlier trade patterns.

In the single currency situation where one is interested in measuring the growth in GDP, the problem arises that one must allow for variation in outputs. It is clear that even amongst products common to two eras, goods are almost always manufactured in different proportions. How then are we to measure the evolution of trade between two nations which start off trading their natural resources or agricultural products but which gradually move away from these sectors and towards services and manufactured goods?

Further, it may be that the currencies are not perfectly in equilibrium. Perhaps one reason for their trade is that the currencies do not value all commodities equally and thus there is in part a straightforward opportunity for profit from arbitrage. How then can we measure the change in trade so that both countries agree to the magnitude of the shifts?

Theorem 3 *Let $\mathcal{P}$, $\mathcal{Q}$ be as before with $s \in \mathbb{R}$ an instant in time. Let $\Lambda(t)$ be the 2×2 matrix given by*

$$\Lambda_s(t) = TOP(e^{-\int_{\alpha(s)}^{\alpha(t)} (\mathcal{P}\mathcal{Q})^{-1}\mathcal{P}d\mathcal{Q}}) \tag{32}$$

Then if

$$\tau(s) = aq_1(s) + bq_2(s) = \mathcal{Q}(s)\begin{pmatrix} a \\ b \end{pmatrix} = \mathcal{Q}(s)v \tag{33}$$

is an arbitrary trade vector at time s,

1. *There exists a unique extension $\tilde{\tau}(t)$ of $\tau(s)$ to all time t so that:*

   (a) *We have agreement at time s:*

   $$\tilde{\tau}(s) = \tau(s). \tag{34}$$

   (b) *The value of $\tilde{\tau}(t)$ is a trade vector in the contemporary trade space for all time: $\tilde{\tau}(t) \in [\mathcal{Q}(t)]\ \forall t$.*

   (c) *The change in $\tilde{\tau}(t)$ is a barter in both currencies for all time. That is all shifts are by* **mutual substitution effects** *so that*

   $$\nabla^a_{\dot{\alpha}(t)}\tilde{\tau}(t) = 0 \tag{35}$$

   .

2. *$\tilde{\tau}(t)$ is given explicitly by the formula*

$$\tilde{\tau}(t) = \mathcal{Q}(t)\Lambda_s(t)v \tag{36}$$

.

Proof:

$$\nabla^a_{\dot\alpha(t)}(\mathcal{Q}(t)\Lambda_s(t)v) \tag{37}$$

$$= (\Pi_{[\mathcal{Q}(t)]}\nabla^o_{\dot\alpha(t)}\Pi_{[\mathcal{Q}(t)]} + \Pi_{\beta_{[\mathcal{P}(t)]}}\nabla^o_{\dot\alpha(t)}\Pi_{\beta_{[\mathcal{P}(t)]}})(\mathcal{Q}(t)\Lambda_s(t)v) \tag{38}$$

$$= \Pi_{[\mathcal{Q}(t)]}\nabla^o_{\dot\alpha(t)}\Pi_{[\mathcal{Q}(t)]}(\mathcal{Q}(t)\Lambda_s(t)v) \tag{39}$$

$$= \Pi_{[\mathcal{Q}(t)]}\nabla^o_{\dot\alpha(t)}(\mathcal{Q}(t)\Lambda_s(t)v) \tag{40}$$

$$= \Pi_{[\mathcal{Q}(t)]}(\mathcal{Q}'(t)\Lambda_s(t) + \mathcal{Q}(t)\Lambda'_s(t))v \tag{41}$$

But

$$\Pi_{[\mathcal{Q}(t)]} = \mathcal{Q}(\mathcal{P}\mathcal{Q})^{-1}\mathcal{P} \tag{42}$$

and it is clear that the null space $\mathcal{N}$ of the matrix $\mathcal{Q}$ is trivial

$$\mathcal{N}(\mathcal{Q}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{43}$$

implying

$$\mathcal{N}(\mathcal{Q}(\mathcal{P}\mathcal{Q})^{-1}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{44}$$

which further implies

$$\mathcal{N}(\Pi_{[\mathcal{Q}(t)]}) = \mathcal{N}(\mathcal{Q}(\mathcal{P}\mathcal{Q})^{-1}\mathcal{P}) = \mathcal{N}(\mathcal{P}). \tag{45}$$

This shows

$$\Pi_{[\mathcal{Q}(t)]}(\mathcal{Q}'(t)\Lambda_s(t) + \mathcal{Q}(t)\Lambda'_s(t))v = 0 \tag{46}$$

if and only if

$$\mathcal{P}(t)(\mathcal{Q}'(t)\Lambda_s(t) + \mathcal{Q}(t)\Lambda'_s(t))v = 0 \tag{47}$$

In order to proceed it is necessary to compute the formula giving the derivative for the time ordered product $\Lambda_s(t)$. In order to make things notationally simpler we compute the derivative of the exponential of the integral of a time ordered product for an arbitrary 2×2 matrix valued function $\mathcal{A}(t)$. Thus

$$\frac{d}{dt}TOP(e^{\int_s^t \mathcal{A}(r)dr}) \tag{48}$$

$$= \frac{d}{dt}TOP(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{i=1}^{\infty}\frac{1}{i!}(\int_s^t \mathcal{A}(r)dr)^i) \tag{49}$$

$$= \frac{d}{dt} TOP(\sum_{i=1}^{\infty} \frac{1}{i!} (\int_{\alpha(s)}^{\alpha(t)} \mathcal{A}(r)dr)^i) \tag{50}$$

$$= \frac{d}{dt} \sum_{i=1}^{\infty} \frac{1}{i!} (\int_s^t dt_i \int_s^t dt_{i-1} ... \int_s^t dt_1 \ TOP(\mathcal{A}(t_1)\mathcal{A}(t_2)...\mathcal{A}(t_i))) \tag{51}$$

$$= \frac{d}{dt} \sum_{i=1}^{\infty} (\int_s^t dt_i \int_s^{t_i} dt_{i-1} \int_s^{t_{i-1}} ... \int_s^{t_2} dt_1 \ \mathcal{A}(t_i)\mathcal{A}(t_{i-1})\mathcal{A}(t_{i-2})...\mathcal{A}(t_1)) \tag{52}$$

$$= \mathcal{A}(t)(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{53}$$

$$+ \sum_{i=1}^{\infty} (\int_s^t dt_i \int_s^{t_i} dt_{i-1} \int_s^{t_{i-1}} ... \int_s^{t_2} dt_1 \ \mathcal{A}(t_i)\mathcal{A}(t_{i-1})\mathcal{A}(t_{i-2})...\mathcal{A}(t_1)) \tag{54}$$

$$\mathcal{A}(t)(TOP(e^{\int_s^t \mathcal{A}(r)dr})) \tag{55}$$

The above formula shows the significance of the time ordered product. It is the device which allows us to generalize the familiar properties of the derivative of the exponential to the realm of non-commutative matrices.

Thus we compute:

$$\mathcal{P}(t)(\mathcal{Q}'(t)\Lambda_s(t) + \mathcal{Q}(t)\Lambda_s'(t))v \tag{56}$$

$$= \mathcal{P}(t)\mathcal{Q}'(t)\Lambda_s(t)v + \mathcal{P}(t)\mathcal{Q}(t)\Lambda_s'(t)v \tag{57}$$

$$= \mathcal{P}(t)\mathcal{Q}'(t)TOP(e^{-\int_{\alpha(s)}^{\alpha(t)} (\mathcal{P}\mathcal{Q})^{-1}\mathcal{P}d\mathcal{Q}})v + \mathcal{P}(t)\mathcal{Q}(t)\frac{d}{dt}TOP(e^{-\int_{\alpha(s)}^{\alpha(t)} (\mathcal{P}\mathcal{Q})^{-1}\mathcal{P}d\mathcal{Q}})v \tag{58}$$

$$= \mathcal{P}(t)\mathcal{Q}'(t)TOP(e^{-\int_{\alpha(s)}^{\alpha(t)} (\mathcal{P}\mathcal{Q})^{-1}\mathcal{P}d\mathcal{Q}})v \tag{59}$$

$$-\mathcal{P}(t)\mathcal{Q}(t)(\mathcal{P}(t)\mathcal{Q}(t))^{-1}\mathcal{P}(t)\mathcal{Q}'(t)TOP(e^{-\int_{\alpha(s)}^{\alpha(t)} (\mathcal{P}\mathcal{Q})^{-1}\mathcal{P}d\mathcal{Q}})v \tag{60}$$

$$= \mathcal{P}(t)\mathcal{Q}'(t)TOP(e^{-\int_{\alpha(s)}^{\alpha(t)} (\mathcal{P}\mathcal{Q})^{-1}\mathcal{P}d\mathcal{Q}})v - \mathcal{P}(t)\mathcal{Q}'(t)TOP(e^{-\int_{\alpha(s)}^{\alpha(t)} (\mathcal{P}\mathcal{Q})^{-1}\mathcal{P}d\mathcal{Q}})v \tag{61}$$

$$= 0 \tag{62}$$

QED

5 Trade Index Number Problem

In the case of a single basket varying over time, the traditional index number problem

$$\frac{p(0) \cdot q(1)}{p(0) \cdot q(0)} \neq \frac{p(1) \cdot q(1)}{p(1) \cdot q(0)} \tag{63}$$

is the inherent discrepancy between the Laspeyres and Paasche quantity indices above. If we interpret the denominators of the above two formulas as inverses

$$(p(0) \cdot q(0))^{-1} p(0) \cdot q(1) \neq (p(1) \cdot q(0))^{-1} p(1) \cdot q(1) \tag{64}$$

we may rewrite the index number problem without any change in content. The reason for performing this cosmetic change is that by changing the denominators to inverses we may generalize these well known formulas in the single basket case to matrix formulas in the 2-basket case. We thus define the Laspeyres trade index to be given as

$$\text{``}\frac{\mathcal{P}_0 \cdot \mathcal{Q}_1}{\mathcal{P}_0 \cdot \mathcal{Q}_0}\text{''} = (\mathcal{P}_0 \cdot \mathcal{Q}_0)^{-1} \mathcal{P}_0 \cdot \mathcal{Q}_1 \tag{65}$$

with the Paasche formula given by

$$\text{``}\frac{\mathcal{P}_1 \cdot \mathcal{Q}_1}{\mathcal{P}_1 \cdot \mathcal{Q}_0}\text{''} = (\mathcal{P}_1 \cdot \mathcal{Q}_0)^{-1} \mathcal{P}_1 \cdot \mathcal{Q}_1 \tag{66}$$

Theorem 4 *While the trade analogs of the traditional Laspeyres and Paasche indices*

$$\text{``}\frac{\mathcal{P}_0 \cdot \mathcal{Q}_1}{\mathcal{P}_0 \cdot \mathcal{Q}_0}\text{''} = (\mathcal{P}_0 \cdot \mathcal{Q}_0)^{-1} \mathcal{P}_0 \cdot \mathcal{Q}_1 \neq (\mathcal{P}_1 \cdot \mathcal{Q}_0)^{-1} \mathcal{P}_1 \cdot \mathcal{Q}_1 = \text{``}\frac{\mathcal{P}_1 \cdot \mathcal{Q}_1}{\mathcal{P}_1 \cdot \mathcal{Q}_0}\text{''} \tag{67}$$

suffer from the same index number problem, the adapted Laspeyres and Paasche trade indices

$$\text{``}\frac{\mathcal{P}_0 \cdot \tilde{\mathcal{Q}}_1}{\mathcal{P}_0 \cdot \mathcal{Q}_0}\text{''} = (\mathcal{P}_0 \cdot \mathcal{Q}_0)^{-1} \mathcal{P}_0 \cdot \tilde{\mathcal{Q}}_1 = (\mathcal{P}_1 \cdot \tilde{\mathcal{Q}}_0)^{-1} \mathcal{P}_1 \cdot \mathcal{Q}_1 = \text{``}\frac{\mathcal{P}_1 \cdot \mathcal{Q}_1}{\mathcal{P}_1 \cdot \tilde{\mathcal{Q}}_0}\text{''} \tag{68}$$

are equivalent for $\forall t$.

Proof:

$$\text{``}\frac{\mathcal{P}_0 \cdot \tilde{\mathcal{Q}}_1}{\mathcal{P}_0 \cdot \mathcal{Q}_0}\text{''} = (\mathcal{P}_0 \cdot \mathcal{Q}_0)^{-1} \cdot \mathcal{P}_0 \cdot \tilde{\mathcal{Q}}_1 \tag{69}$$

$$= (\mathcal{P}_0 \cdot \mathcal{Q}_0)^{-1} \cdot \mathcal{P}_0 \cdot \mathcal{Q}_0 \cdot \Lambda_1(0) = \Lambda_1(0) \tag{70}$$

but

$$\Lambda_1(0) \tag{71}$$

$$= TOP(e^{-\int_{\alpha(1)}^{\alpha(0)}(\mathcal{P}\mathcal{Q})^{-1}\mathcal{P}d\mathcal{Q}}) = TOP(e^{\int_{\alpha(0)}^{\alpha(1)}(\mathcal{P}\mathcal{Q})^{-1}\mathcal{P}d\mathcal{Q}}) \tag{72}$$

$$= (\Lambda_0(1))^{-1} \tag{73}$$

while

$$\text{“}\frac{\mathcal{P}_1 \cdot \mathcal{Q}_1}{\mathcal{P}_1 \cdot \tilde{\mathcal{Q}}_0}\text{”} = (\mathcal{P}_1 \cdot \tilde{\mathcal{Q}}_0)^{-1} \cdot \mathcal{P}_1 \cdot \mathcal{Q}_1 \tag{74}$$

$$= (\mathcal{P}_1 \cdot \mathcal{Q}_1 \cdot \Lambda_0(1))^{-1} \cdot \mathcal{P}_1 \cdot \mathcal{Q}_1 \tag{75}$$

$$= (\Lambda_0(1))^{-1}(\mathcal{P}_1 \cdot \mathcal{Q}_1)^{-1} \cdot \mathcal{P}_1 \cdot \mathcal{Q}_1 \tag{76}$$

$$= (\Lambda_0(1))^{-1} \tag{77}$$

As 0 and 1 are chosen arbitrarily, equality holds for all time.

QED

6 Corresponding Bilateral Price Index for Bilateral Trade

We also note the answer to a question that has likely already occured to the alert reader. By duality between prices and quantities, isn't there a corresponding price index for bilateral trading partners who do not necessarily even share a currency? We give the answer to this question as:

$$\Lambda_s^P(t) = \text{TOP}(e^{-\int_{\alpha(s)}^{\alpha(t)}(d\mathcal{P})\cdot\mathcal{Q}\cdot(\mathcal{P}\cdot\mathcal{Q})^{-1}}) \tag{78}$$

which we point out is mathematically no more demanding than the bilateral quantity index. We have not focused on it in this note not because it doesn't exist or isn't as important, but instead because it is slightly more difficult to talk about the analog of 'mutual substitution effects' for a pair of currencies. Beyond that, it is really simply a matter of repeating the constructions for the projection maps that decompose a pricing vector $v^* \in V^*$ according to

$$v^* = \underbrace{v^*\mathcal{Q}(\mathcal{P}\mathcal{Q})^{-1}\mathcal{P}}_{\text{Dual 'Income Effect'}} + \underbrace{(v^* - v^*\mathcal{Q}(\mathcal{P}\mathcal{Q})^{-1}\mathcal{P})}_{\text{Dual 'Substitution Effect'}} \tag{79}$$

for all $v \in V$, where we again establish two things:

1. $v^*\mathcal{Q}(\mathcal{P}\mathcal{Q})^{-1}\mathcal{P} \in [\mathcal{P}]$.
2. $v^* - v^*\mathcal{Q}(\mathcal{P}\mathcal{Q})^{-1}\mathcal{P} \in \beta_{[\mathcal{Q}]}$.

to ensure that everything goes through just as before.

7 Conclusion

We have defined a generalized quantity index

$$\Lambda_s(t)^Q = \mathrm{TOP}(e^{-\int_{\alpha(s)}^{\alpha(t)} (\mathcal{P}\cdot\mathcal{Q})^{-1}\cdot\mathcal{P}\cdot d\mathcal{Q}}) \in \Omega(\mathrm{GL}(2,\mathbb{R})) \tag{80}$$

in the path space $\Omega(\mathrm{Gl}(2,\mathbb{R}))$ for tracking changes in bilateral trading patterns in the absence of purchasing power parity, which so far as we are aware is new to the literature. Likewise, we also introduced a corresponding price index for a pair of trading partner currencies which stimulate trade by violating the law of one price:

$$\Lambda_s^P(t) = \mathrm{TOP}(e^{-\int_{\alpha(s)}^{\alpha(t)} (d\mathcal{P})\cdot\mathcal{Q}\cdot(\mathcal{P}\cdot\mathcal{Q})^{-1}}) \in \Omega(\mathrm{Gl}(2,\mathbb{R})) \tag{81}$$

also in the same path space.

The representation of the index as the exponential of an integral introduces the time-ordered-product operator TOP from quantum physics and differential geometry and has several novel features which may be of some interest.

- The fact that the natural object is matrix rather than scalar valued.
- The fact that the geometric framework leads one through the maze of possible generalizations to an otherwise non-obvious construct of generalized Divisia type.
- The fact that non-commutativity present in the matrix valued integrands requires care when generalizing from the scalar case where the order of terms is irrelevant (e.g. the introduction of time ordered products which are 'invisible' in the special commutative case of the scalar valued Divisia index).
- The fact that this construction solves a non-commutative index number problem just as in the single currency case.

While our first foray into index number theory solved the bilateral index number problem by revealing the geometric relationship of all major bilateral index numbers to the Divisia index, the Divisia index itself was already 70 years old. This paper, like our treatment of changing preference welfare theory, adds an apparently new construction and formula which we have been unable to find elsewhere. While it should be clear that the above formula can be seen to reduce to the Divisia formula when the 2 currency price and quantity matrices $\mathcal{P},\ \mathcal{Q}$ are replaced by the single currency vectors q and co-vectors p, it should be nearly as plain to see that its precise form would scarcely be guessed directly from the scalar Divisia index:

$$Q_D = e^{\int_{\alpha(t_0)}^{\alpha(t_1)} \frac{p_s\cdot dq_s}{p_s\cdot q_s}} \tag{82}$$

thus it is perhaps not so surprising that the more general formula would go un-noticed. To wit, both of these formulae are actually special cases of a more general construction requiring knowledge of Lie groups and universal enveloping algebras.

While the construction here has been presented so that it can be verified within the toolkit of neo-classical analysis, the results only become fully natural within the context of gauge theoretic differential geometry. In particular, like the special case of the Divisia index, the reason for the path-dependence of our trade index becomes natural and desirable as a consequence of the geometry created from the need to separate income and substitution effects. This should be suggestive of further study.

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