AN EXTENSION OF INTERTEMPORAL ORDINAL WELFARE TO CHANGING TASTES: ECONOMICS AS GAUGE THEORY (DRAFT FOR UNIVERSITY OF CHICAGO MONEY AND BANKING WORKSHOP. IN PREPARATION.)

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Abstract. Economic utility theory is given the structure of a Principal G-bundle of function spaces $\bar{P}_G$. Parallel translation under a connection is shown to induce a consistent welfare structure (here termed 'mirth') on changing consumer preferences which extends the ordinal concept of 'util' by reducing to ordinal utility under imposition of static preferences. In the presence of market pricing, a distinguished connection is constructed on the augmented bundle P_G, thereby showing that consistent welfare systems are in 1-1 correspondence with torsion tensors on P_G. This appears to offer an answer to the research program of Weizsäcker by disproving a long-standing claims of Fisher and Shell referenced recently by Nesmith of the Federal Reserve. This in turn obviates the analytic need for the inscrutable claims of Becker and Stigler that human preferences are both unchanging and homogeneous between agents.

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"It turns out that the structure Einstein was seeking was the gauge field. ... It was only in 1975, when [Jim] Simons gave a series of talks to us at the Institute for Theoretical Physics at Stony Brook, that I finally understood the basic ideas of fiber bundles and connections on fiber bundles ... it was not just joy. There was something more, something deeper: After all, what could be more mysterious, what could be more awe-inspiring, than to find that the basic structure of the physical world is intimately tied to deep mathematical concepts, concepts which were developed out of considerations rooted only in logic and in the beauty of form? ... I found it amazing that gauge theory are [sic] exactly connections on fibre bundles, which the mathematicians developed without reference to the physical world."

-Nobel Laureate C.N. Yang, "Einstein's Impact on Theoretical Physics"

1. Introduction

The purpose of this paper is to solve a well-known problem at the core of economic theory posed most recently in a paper of von Weizsäcker:

> "Traditional neo-classical economics has worked with the assumption that preferences of agents in the economy are fixed. This assumption has always been disputed, and, indeed, in the social sciences outside of neoclassical economics the assumption has never been accepted by anyone. ... The obstacle is the lack of an answer to the question: how can you do welfare economics, if preferences change endogenously? After all, preferences of individual agents are the basic measuring rod of economic welfare, of the performance generated in an economic system. How can we evaluate an economic system with a measuring rod that itself changes with the system?"
>
> -Carl Christian von Weizsäcker, 2005

We show here in the most direct terms that economics, when considered geometrically as a gauge theory, can be shown to accommodate dynamic ordinal welfare.

2. Spaces

Let us begin by discussing the various spaces which will be relevant to us. We start with a Euclidean vector space $V = \mathbb{R}^n$ with explict set of basis vectors $\{e_i\}_{i=1}^n$ where the positive orthant V_+ represents baskets $q = \sum_{i=1}^n q_i \cdot e_i$ of quantities of n-goods ordered as $q_i \cdot e_i$ measurable in continuous units (e.g. q_1 might be a real number giving liters or liters per second, kilograms or kilos per second, etc.). Our goal is to analyze the consumer welfare of an agent with evolving ordinal preferences $\mathbf{O}_t$ facing market prices $P_t : V_+ \longrightarrow \mathbb{R}_+$ given at all times as an increasing function on V_+.

We define the space $\mathcal{C}$ of cardinal utility functions to be the function space of smooth strictly increasing surjective maps

$$\mathcal{C} = \{ \mathbf{C} : V_+ \longrightarrow \mathbb{R}_+ \mid \text{Image}(\mathbf{C}) = \mathbb{R}_+,\ \forall x \in \mathbb{R}_+\ \mathbf{C}^{-1}(x) \text{ is complete and convex to } 0\} \tag{2.1}$$

from the positive orthant V_+ of our vector space $V = \mathbb{R}^n$ onto the positive reals $\mathbb{R}_+$ so that the inverse images under any such function (i.e. level hypersurfaces of this function) foliate V_+ with complete smooth hypersurfaces which are everywhere convex to the origin $0 \in V$.

For every such function $\mathbf{C}$, the underlying foliation is referred to as the **ordinal preference map** $\mathbf{O} = \mathbf{O}_\mathbf{C}$. Denote by $\mathcal{O}$ the space of all ordinal preference maps.

Remark 1. *Note that different functions* $\mathbf{C}_1$ *and* $\mathbf{C}_2$ *may well give rise to the same foliation* $\mathbf{O} = \mathbf{O}_{\mathbf{C}_1} = \mathbf{O}_{\mathbf{C}_2}$ *(see below for more details). In what follows, we will use the notation* $\mathbf{O}$ *for an arbitrary foliation satisfying the above conditions. If the foliation consists of the level hypersurfaces of a specific function* $\mathbf{C}$*, we will denote this foliation by* $\mathbf{O}_\mathbf{C}$*.* □

The leaves of the ordinal preference map $\mathbf{O}$ define an equivalence relationship $I = I_\mathbf{O}$ called '**indifference**' whose 1-dimensional quotient space $V_+/I_\mathbf{O} = U_\mathbf{O}$ is (non-canonically) diffeomorphic to $\mathbb{R}_+$ and is referred to as the space of **ordinal utils**.

For a given ordinal preference map $\mathbf{O}$, its associated **ordinal utility function**

$$V_+ \longrightarrow U_\mathbf{O} \tag{2.2}$$

which, by slight abuse of notation, we will also denote by $\mathbf{O}$ is simply the surjective quotient construction gotten from sending every basket $q \in V_+$ to its indifference equivalence class under the $\mathbf{O}$ foliation. As such, the image of the mapping is the natural indexing set for the leaves of the foliation $\mathbf{O}$.

For every cardinal utility function $\mathbf{C} : V_+ \longrightarrow \mathbb{R}_+$ there is a unique factorization into an ordinal utility function $\mathbf{O}_\mathbf{C}$, together with a cardinalization diffeomorphism $\mathbf{k}_\mathbf{C}$ such that the diagram:

$$\begin{array}{ccccc} & & U_{\mathbf{O}_\mathbf{C}} & & \\ & \overset{\mathbf{O}_\mathbf{C}}{\nearrow} & & \overset{\mathbf{k}_\mathbf{C}}{\searrow} & \\ V_+ & & \overset{\mathbf{C}}{\longrightarrow} & & \mathbb{R}_+ \end{array} \tag{2.3}$$

commutes.

A cardinal cross-section $\mathbf{X}_\mathbf{C}$ is a path

$$\mathbf{X}_\mathbf{C} : \mathbb{R}_+ \longrightarrow V_+ \tag{2.4}$$

that forms a right-inverse to $\mathbf{C}$

$$\mathbf{C} \circ \mathbf{X}_\mathbf{C} = \mathrm{Id} \tag{2.5}$$

giving a representative basket for each indifference surface.

For every such cardinal cross-section $\mathbf{X}_\mathbf{C}$ there exists a unique ordinal cross-section $\mathbf{X}_{\mathbf{O}_\mathbf{C}}$ and a cross-section $\mathbf{k}_\mathbf{C}^{-1}$ such that the diagram:

$$\begin{array}{ccccc} & & U_{\mathbf{O}_\mathbf{C}} & & \\ & \overset{\mathbf{X}_{\mathbf{O}_\mathbf{C}}}{\swarrow} & & \overset{\mathbf{k}_\mathbf{C}^{-1}}{\nwarrow} & \\ V_+ & & \overset{\mathbf{X}_\mathbf{C}}{\longleftarrow} & & \mathbb{R}_+ \end{array} \tag{2.6}$$

commutes.

3. Gauge Theory as the Natural Economic Calculus for Welfare

In welfare theory, the mathematical representation of preferences is as complementary marginal subspaces of indifference and utility. The key argument we will advance here is that the standard ordinal consumer welfare theory, when properly constructed, can be re-envisioned as an infinite dimensional principal $\mathcal{G}$-bundle with distinguished connection, requiring no additional assumptions to those usually made in neo-classical theory.

We begin by defining the base space $\mathcal{B} = \mathcal{B}_{\mathcal{OX}}$ of the main principal bundle to be the space of ordinal preference maps with cross-section

$$\mathcal{B}_{\mathcal{OX}} = \{(\mathbf{O}, \mathbf{X_O}) \mid \mathbf{O} \circ \mathbf{X_O} = \mathrm{Id} \qquad \mathbf{X_O} : U_\mathbf{O} \longrightarrow V_+, \quad \mathbf{O} : V_+ \longrightarrow U_\mathbf{O}\} \tag{3.1}$$

while the total space $\mathcal{T} = \mathcal{T}_{\mathcal{CX}}$ is given as the space of cardinal utility functions with cross-section:

$$\mathcal{T}_{\mathcal{CX}} = \{(\mathbf{C}, \mathbf{X_C}) \mid \mathbf{C} \circ \mathbf{X_C} = \mathrm{Id} \qquad \mathbf{X_C} : \mathbb{R}_+ \longrightarrow V_+, \quad \mathbf{C} : V_+ \longrightarrow \mathbb{R}_+\} \tag{3.2}$$

The group $\mathcal{G} = \mathrm{Diff}_0(\mathbb{R}_+)$ has a natural right action on $\mathcal{T}_{\mathcal{CX}}$ given by:

$$\mathcal{G} : \mathcal{T}_{\mathcal{CX}} \longrightarrow \mathcal{T}_{\mathcal{CX}} \qquad (\mathbf{C},\ \mathbf{X_C}) \cdot \mathbf{G} = (\mathbf{G}^{-1} \circ \mathbf{C},\ \mathbf{X_C} \circ \mathbf{G}) \tag{3.3}$$

The projection map $\Pi^{\mathcal{T}} : \mathcal{T}_{\mathcal{CX}} \longrightarrow \mathcal{B}_{\mathcal{OX}} = \mathcal{T}_{\mathcal{CX}}/\mathcal{G}$ is then given by:

$$\Pi^{\mathcal{T}} : \mathcal{T}_{\mathcal{CX}} \longrightarrow \mathcal{B}_{\mathcal{OX}} \qquad \Pi^{\mathcal{T}}((\mathbf{C},\ \mathbf{X_C})) = (\mathbf{O_C},\ \mathbf{X_C} \circ \mathbf{k_C}) \tag{3.4}$$

4. Cost Function as a section

For each $\mathbf{O} \in \mathcal{O}$ we construct a map

$$N_\mathbf{O} : V_+ \longrightarrow \mathrm{Gr}_{n-1}(\mathbb{R}^n) \tag{4.1}$$

which associates to each $v \in V_+$ the tangent hyperplane $W_v^\mathbf{O}$ to the indifference surface S_v from the foliation $\mathbf{O}$ passing through v (at the point v). In other words,

$$W_v^\mathbf{O} = \mathrm{Ker}(d\mathbf{O}_v) \subset T_v V_+ \tag{4.2}$$

where $\mathrm{Ker}(d\mathbf{O}_v)$ stands for $\mathrm{Ker}(d\mathbf{C}_v)$ for any cardinal function $\mathbf{C}$ such that $\mathbf{O} = \mathbf{O_C}$ and $d\mathbf{C}_v$ is the value of its differential at v (its kernel does not depend on the choice of $\mathbf{C}$). We view $W_v^\mathbf{O}$ as a point in $\mathrm{Gr}_{n-1}(\mathbb{R}^n)$, the Grassmannian of co-dimension-1 subspaces (i.e. hyperplanes) in $\mathbb{R}^n$.

Since the indifference surfaces of $\mathbf{O}$ are complete and everywhere convex to the origin, the restriction of $N_\mathbf{O}$ to a given indifference surface $S \subset V_+$ has as image all hyperplanes in $\mathbb{R}^n$ which are orthogonal to the lines of the form $\mathrm{span}\{w\}, w \in V_+$, with each hyperplane occurring exactly once (otherwise the convexity requirement on $\mathbf{O}$ would be violated).

Given $\mathbf{P} \in \mathcal{P}$ where $\mathcal{P}$ is the space of all pricing functions[1], we may similarly define a map

$$N_\mathbf{P} : V_+ \longrightarrow \mathrm{Gr}_{n-1}(\mathbb{R}^n) \tag{4.3}$$

given by

$$v \mapsto \mathrm{Ker}(d\mathbf{P}_v) \subset T_v V_+ \tag{4.4}$$

considered as a point of $\mathrm{Gr}_{n-1}(\mathbb{R}^n)$.

[1] A pricing function $\mathbf{P}$ may be taken as linear and given by a positive co-vector $p \in V_+^*$ for simplicity. More generally, however, it need only exhibit a unique income expansion path $\mathbf{X_O^P}$ against all preferences $\mathbf{O}$ for the constructions used in this paper.

Since we take pricing functions to be linear and positive, $N_{\mathbf{P}}$ has image the unique co-dimension-1 subspace

$$\ker(d\mathbf{P}_v) \in \mathrm{Gr}_{n-1}(\mathbb{R}^n) \quad \text{with} \quad \mathbb{R}^n = \ker(d\mathbf{P}_v) \oplus \mathrm{Span}(v) \tag{4.5}$$

for all non-zero $v \in V_+$.

It follows that if we have a pair

$$(\mathbf{O}, \mathbf{P}) \in \mathcal{O} \times \mathcal{P} \tag{4.6}$$

we may define a map

$$N_{(\mathbf{O},\mathbf{P})} : V_+ \longrightarrow \mathrm{Gr}_{n-1}(\mathbb{R}^n) \times \mathrm{Gr}_{n-1}(\mathbb{R}^n) \tag{4.7}$$

which is given by

$$N_{(\mathbf{O},\mathbf{P})}(v) = (N_{\mathbf{O}}(v), N_{\mathbf{P}}(v)). \tag{4.8}$$

Since for every indifference surface S we have that $\mathrm{Im}\big(\, N_{\mathbf{O}}\big|_S \,\big)$ is the set of co-dimension-1 subspaces of $\mathbb{R}^n$ which are orthogonal to the spans of non-zero vectors in V_+ and since $N_{\mathbf{P}}$ is a constant map whose image is precisely one such subspace, there is a unique $v \in V_+$ with

$$v = (\, N_{(\mathbf{O},\mathbf{P})}\big|_S)^{-1}\Big(\mathrm{Im}\big(\, N_{(\mathbf{O},\mathbf{P})}\big|_S\,\big) \cap \mathrm{Im}\big(\, \Delta\big|_S\,\big)\Big) \tag{4.9}$$

where Δ is the diagonal map

$$\Delta : \mathrm{Gr}_{n-1}(\mathbb{R}^n) \longrightarrow \mathrm{Gr}_{n-1}(\mathbb{R}^n) \times \mathrm{Gr}_{n-1}(\mathbb{R}^n) \tag{4.10}$$

given explicitly by

$$\Delta(W) = (W, W). \tag{4.11}$$

Whence, by the positivity and linearity of $\mathbf{P} \in \mathcal{P}$ we have a well-defined map

$$m : \mathcal{O} \times \mathcal{P} \longrightarrow \mathcal{T}_{\mathcal{CX}} \tag{4.12}$$

given by

$$(\mathbf{O}, \mathbf{P}) \overset{m}{\rightsquigarrow} (\mathbf{P} \circ \mathbf{X}^{\mathbf{P}}_{\mathbf{O}} \circ \mathbf{O}, \mathbf{X}^{\mathbf{P}}_{\mathbf{O}} \circ \mathbf{k}^{-1}_{\mathbf{P}\circ\mathbf{X}^{\mathbf{P}}_{\mathbf{O}}\circ\mathbf{O}}) \tag{4.13}$$

where for an ordinal level of utility $u \in U_{\mathbf{O}}$,

$$\mathbf{X}^{\mathbf{P}}_{\mathbf{O}}(u) = \big(\, N_{(\mathbf{O},\mathbf{P})}\big|_{\mathbf{O}^{-1}(u)}\,\big)^{-1}\Big(\mathrm{Im}\big(\, N_{(\mathbf{O},\mathbf{P})}\big|_{\mathbf{O}^{-1}(u)}\,\big) \cap \mathrm{Im}\big(\, \Delta\big|_{\mathbf{O}^{-1}(u)}\,\big)\Big) \tag{4.14}$$

and for $r \in \mathbb{R}_+$,

$$\mathbf{k}^{-1}_{\mathbf{P}\circ\mathbf{X}^{\mathbf{P}}_{\mathbf{O}}\circ\mathbf{O}}(r) = u_r \tag{4.15}$$

where u_r is the unique element of $U_{\mathbf{O}}$ such that

$$(\mathbf{P} \circ \mathbf{X}^{\mathbf{P}}_{\mathbf{O}})(u_r) = r. \tag{4.16}$$

That is, $\mathbf{X}^{\mathbf{P}}_{\mathbf{O}}$ takes an ordinal level of utility x to the *unique* basket v in the indifference surface

$$\mathbf{O}^{-1}(x) \quad \text{s.t.} \quad \mathrm{Ker}(d\mathbf{P}_v) = \mathrm{Ker}(d\mathbf{O}_v) \tag{4.17}$$

and $\mathbf{k}^{-1}_{\mathbf{P}\circ\mathbf{X}^{\mathbf{P}}_{\mathbf{O}}\circ\mathbf{O}}$ is the de-cardinalization map induced by $\mathbf{P} \circ \mathbf{X}^{\mathbf{P}}_{\mathbf{O}} \circ \mathbf{O}$.

In other words, the map m attaches to the pair $(\mathbf{O}, \mathbf{P})$, first of all, the cardinal function $\mathbf{C}_{\mathbf{O},\mathbf{P}}$ whose level hypersurface foliation coincides with $\mathbf{O}$ and whose value on the hypersurface S (i.e. indifference surface) from $\mathbf{O}$ is the value of $\mathbf{P}$ at the unique point $v(S)$ of S where the tangent plane to S is orthogonal to $\mathbf{P}$. And second, it attaches to $(\mathbf{O}, \mathbf{P})$ the path traversed by the points $v(S)$ as S varies within the foliation $\mathbf{O}$.

Thus, for every dynamic ordinal consumer with market history

$$\hat{\alpha} : [t_0, t_1] \longrightarrow \mathcal{O} \times \mathcal{P} \tag{4.18}$$

a path is given in the base space

$$\alpha : [t_0, t_1] \longrightarrow \mathcal{B}_{\mathcal{O}\mathcal{X}} \qquad \alpha = \Pi^{\mathcal{T}} \circ m \circ \hat{\alpha} \tag{4.19}$$

together with a lift $\tilde{\alpha}$

$$\tilde{\alpha} : [t_0, t_1] \longrightarrow \mathcal{T}_{\mathcal{C}\mathcal{X}} \qquad \tilde{\alpha} = m \circ \hat{\alpha} \tag{4.20}$$

where for this particular lift we will write

$$\tilde{\alpha} = (\mathbf{C}_t, \mathbf{X}_t) = (\mathcal{E}_t, \mathcal{E}_t^{-1_R}) \tag{4.21}$$

where $\mathcal{E}_t$ is called the cost function and $\mathcal{E}_t^{-1_R}$ is the income expansion path.

5. Tangent Spaces

The spaces of greatest interest in this discussion are function spaces and, while we are not being fastidious about the particulars of their manifold structure, we do need to identify the spaces containing the tangent bundles to our principal bundle.

Perhaps the first thing to note is that as function spaces, the tangents to mappings are usually best thought of as sections of bundles pulled back from the target spaces under the various mappings. In the case of the space $\mathcal{O}$ of ordinal utility maps $\mathbf{O}$, the tangent bundle is contained within a quotient

$$T_{\mathbf{O}}\mathcal{O} \ \subset \ \Gamma^{\infty}(\mathbf{O}^*(TU_{\mathbf{O}}))/\mathbf{O}^*(\Gamma^{\infty}(TU_{\mathbf{O}})) \tag{5.1}$$

as flowing globally along the one dimensional target space $U_{\mathbf{O}}$ does not change the preference structure as it would in cardinal utility theory where we have:

$$T_{\mathbf{C}}\mathcal{C} \ \subset \ \Gamma^{\infty}(\mathbf{C}^*(T\mathbb{R}_+)) \tag{5.2}$$

leaving aside the issue of which sections are integrable in the above.

The Lie algebra of the group $\mathcal{G}$ exhibits the containment

$$T_{\mathbf{e}}\mathcal{G} \ \subset \ \mathfrak{X}(\mathbb{R}_+) \tag{5.3}$$

and can be identified with the vector fields on the circle S^1 vanishing at the so-called 'point at infinity' $0 = \infty$, if one wishes to identify the space exactly.

Finally, most important for our purposes will be the tangent bundle to the total space $\mathcal{T}_{\mathcal{C}\mathcal{X}}$ of the principal fibration where the containment relationship looks like:

$$T_{(\mathbf{C}, \mathbf{X}_{\mathbf{C}})}\mathcal{T}_{\mathcal{C}\mathcal{X}} \subset \Gamma^{\infty}(\mathbf{C}^*(T\mathbb{R}_+)) \oplus \Gamma^{\infty}(\mathbf{X}^*_{\mathbf{C}}(TV_+)) \tag{5.4}$$

6. The Economic Welfare Connection

The key benefit to this construction is that if one views the economic data from this perspective, it is possible to construct a canonical welfare connection $\mathbf{A}^W$ on $\mathcal{T}_{\mathcal{C}\mathcal{X}}$.

We define the canonical connection $\mathbf{A}^W$ as a decomposition of the tangent bundle to the total space $T\mathcal{T}_{\mathcal{C}\mathcal{X}}$ into vertical and horizontal subspaces

$$T_{(\mathbf{C}, \mathbf{X})}\mathcal{T}_{\mathcal{C}\mathcal{X}} = \text{Vert}_{(\mathbf{C}, \mathbf{X})}\mathcal{T}_{\mathcal{C}\mathcal{X}} \oplus \text{Horiz}_{(\mathbf{C}, \mathbf{X})}\mathcal{T}_{\mathcal{C}\mathcal{X}} \tag{6.1}$$

A tangent vector $\mu \in T_{(\mathbf{C},\mathbf{X})}\mathcal{T}_{\mathcal{C}\mathcal{X}}$ to the principal bundle is given by a pair of sections $\mu = (\gamma, \nu)$ where

$$\gamma \in \Gamma^{\infty}(\mathbf{C}^*(T\mathbb{R}_+)) \qquad \nu \in \Gamma^{\infty}(\mathbf{X}^*(TV_+)) \tag{6.2}$$

give the infinitesimal changes in the ranges of the mappings $\mathbf{C}$, $\mathbf{X}$ respectively.

Along the image of $\mathbf{X} = \mathbf{X}_{\mathbf{C}}$ for each $r \in \mathbb{R}_+$, there is a splitting

$$T_{\mathbf{X}_{\mathbf{C}}(r)}V_+ = \{\ \lambda \cdot D\mathbf{X}_r[\frac{\partial}{\partial u}] \mid \lambda \in \mathbb{R}\} \oplus \mathrm{Ker}(d\mathbf{C}_{\mathbf{X}_{\mathbf{C}}(r)}) \tag{6.3}$$

which is given explicitly for $v \in T_{\mathbf{X}(r)}V_+$ by

$$v = \Pi^{\mathbf{C},\mathbf{X}}_{\frac{\partial \mathbf{X}}{\partial u}}(v) + \Pi^{\mathbf{C},\mathbf{X}}_{\mathrm{Ker}(d\mathbf{C})}(v) \tag{6.4}$$

where

$$\Pi^{\mathbf{C},\mathbf{X}}_{\frac{\partial \mathbf{X}}{\partial u}}(v) = d\mathbf{C}(v) \cdot D\mathbf{X}_r[\frac{\partial}{\partial u}] \qquad \Pi^{\mathbf{C},\mathbf{X}}_{\mathrm{Ker}(d\mathbf{C})}(v) = v - d\mathbf{C}(v) \cdot D\mathbf{X}_r[\frac{\partial}{\partial u}] \tag{6.5}$$

with the pull back bundle $\mathbf{X}^*_{\mathbf{C}}(TV_+)$ inheriting the decomposition.

The welfare connection $\mathbf{A}^W$ is then defined by the $\mathcal{G}$-invariant horizontal subspaces which are the image of the projection maps:

$$\Pi^{\mathrm{Horiz}}_{(\mathbf{C},\mathbf{X})}(\gamma, \nu) = \gamma - \mathbf{C}^*(\mathbf{C}_*(\gamma|_{\mathrm{Im}(\mathbf{X})}))\ + \Pi^{\mathbf{C},\mathbf{X}}_{\mathrm{Ker}(d\mathbf{C})}(\nu) \tag{6.6}$$

with the projection onto the vertical subspaces being given by

$$\Pi^{\mathrm{Vert}}_{(\mathbf{C},\mathbf{X})}(\gamma, \nu) = \mathbf{C}^*(\mathbf{C}_*(\gamma|_{\mathrm{Im}(\mathbf{X})}))\ + \Pi^{\mathbf{C},\mathbf{X}}_{\frac{\partial \mathbf{X}}{\partial u}}(\nu) \tag{6.7}$$

at the point $(\mathbf{C}, \mathbf{X})$.

7. $\mathcal{G}$-Equivariant Parallel Translation as Non-Perturbative Solution

Our solution to the changing preference problem is as follows. By the uniqueness of solutions to the path lifting problem, there exists a unique time dependent diffeomorphism, which for any value of t,

$$\Upsilon^{\tilde{\alpha}}_t : \mathbb{R}_+ \longrightarrow \mathbb{R}_+ \tag{7.1}$$

can be thought of as a group element, such that the $\Upsilon^{\tilde{\alpha}}_t$-adjusted path in the total space

$$\Pi^{\mathrm{Vert}}(\frac{d\ \ \tilde{\alpha}_t \cdot (\Upsilon^{\tilde{\alpha}}_t)}{dt}) = 0 \qquad \Upsilon^{\tilde{\alpha}}_{t_0} = \mathrm{Id} \tag{7.2}$$

is covariantly constant and agrees with $\tilde{\alpha}$ at time $t = t_0$.

The solution to the welfare problem is to use the covariantly constant lift

$$\tilde{\alpha}^{\Upsilon}_t = \tilde{\alpha}_t \cdot (\Upsilon^{\tilde{\alpha}}_t) = ({\Upsilon^{\tilde{\alpha}}_t}^{-1} \circ \mathcal{E}_t, \mathcal{E}_t^{-1_R} \circ \Upsilon^{\tilde{\alpha}}_t) \tag{7.3}$$

to do the welfare comparisons in the time interval $[t_a, t_b]$ according to:

$$({\Upsilon^{\tilde{\alpha}}_{t_b}}^{-1} \circ \mathcal{E}_{t_b})^{-1} \circ {\Upsilon^{\tilde{\alpha}}_{t_a}}^{-1} \circ \mathcal{E}_{t_a} = \mathcal{E}_{t_b}^{-1} \circ \Upsilon^{\tilde{\alpha}}_{t_b} \circ {\Upsilon^{\tilde{\alpha}}_{t_a}}^{-1} \circ \mathcal{E}_{t_a} \tag{7.4}$$

where if $[t_a, t_b] = [t_0, t]$ the comparison simplifies

$$({\Upsilon^{\tilde{\alpha}}_{t_b}}^{-1} \circ \mathcal{E}_{t_b})^{-1} \circ {\Upsilon^{\tilde{\alpha}}_{t_a}}^{-1} \circ \mathcal{E}_{t_a} = \mathcal{E}_t^{-1} \circ \Upsilon^{\tilde{\alpha}}_t \circ \mathcal{E}_{t_0} \tag{7.5}$$

which is the form to be used when a common base period is in force.

8. Samuelson Shephard Cost Function as a Section

The geometrically important feature of market prices is that they determine both a differential operator and a section to differentiate. Let:

$$\mu = \Pi^{\mathcal{T}} \circ m : \mathcal{O} \times \mathcal{P} \longrightarrow \mathcal{B}_{\mathcal{O}\mathcal{X}} \tag{8.1}$$

map price and preference data to the base space of our principal fibration allowing us to pull back the welfare connection

$$\widehat{A}^W = \mu^*(A^W) \tag{8.2}$$

to the pull back bundle over $\mathcal{O} \times \mathcal{P}$. While not all principal bundles have global sections, this pull-back bundle carries the Samuelson-Shephard cost function

$$\tilde{\alpha} = (\mathbf{C}_t, \mathbf{X}_t) = (\mathcal{E}_t, \mathcal{E}_t^{-1_R}) \tag{8.3}$$

as a naturally occurring section. This section may be covariantly differentiated using the pull back of the welfare connection

$$\nabla^{\mu^*(A^W)}\tilde{\alpha} \in \Omega^1(\mathrm{Ad}) = T_{\mu^*(A^W)}\mathcal{A}(\mathcal{O} \times \mathcal{P}) \tag{8.4}$$

giving us an Ad-valued 1-form that can be integrated along a path

$$\gamma : \mathbb{R} \longrightarrow \mathcal{O} \times \mathcal{P} \tag{8.5}$$

to give a differential equation

$$\nabla_{\dot{\gamma}}^{\mu^*(A^W)}\tilde{\alpha} = g^{-1}dg \tag{8.6}$$

for a path g in the fiber group.

9. A conjecture on uniqueness at the core of utility theory.

The foundations of economic theory have yet to evolve to a point where the artificial restriction to static preferences can be removed. To this end, a constructive conjecture is advanced concerning the uniqueness of parsimonious extensions of the theory of utility to evolving tastes.

The data needed to state the usual cost of living (COL) problem is equivalent to a cost function $\mathcal{E}_t$ together with an Engel curve given as a right inverse $\mathcal{E}_t^{-1_R}$:

$$\mathcal{E}_t : \mathbb{R}_+^n \longrightarrow \mathbb{R}_+ \qquad \mathcal{E}_t^{-1_R} : \mathbb{R}_+ \longrightarrow \mathbb{R}_+^n \qquad \mathcal{E}_t \circ \mathcal{E}_t^{-1_R} = \mathrm{Id}(\ ). \tag{9.1}$$

At an instant of time t, $\mathcal{E}_t = \mathcal{E}^{(\mathcal{O}_t,\ p_t)}$ is the cardinal function uniquely determined by an ordinal preference map $\mathcal{O}_t = \mathcal{E}_t^{-1}(\mathbb{R}_+)$ foliated by complete indifference surfaces convex to the origin and unique price (co)-vector p_t satisfying:

$$p_t \cdot q^*(\kappa) = \kappa \qquad \ker(p_t) = T_{q^*(k)}\mathcal{E}_t^{-1}(\kappa) \qquad \forall\ q^*(\kappa) = \mathcal{E}_t^{-1_R}(\kappa). \tag{9.2}$$

The 'changing preference problem' arises from lack of a welfare map $\mathcal{W}$ between indifference surfaces $\mathcal{O}_t$ extending the static ordinal theory ($\mathcal{O}_t = \mathcal{O}$). To this end:

Conjecture 1. *Let $\mathcal{W}_{t_a,t_b} : \mathcal{O}_{t_b} \longrightarrow \mathcal{O}_{t_a}$ be any smooth functional constructible from $\hat{\mathcal{E}}_{\mathbf{t}} = (\mathcal{E}_t,\ \mathcal{E}_t^{-1_R})$ alone, satisfying the requirements of a welfare map:*

(1) $\mathcal{W}_{t_a,t_b} = Id(\ )$ *if* $\mathcal{O}_t = \mathcal{O} \quad \forall t \in [t_a, t_b]$ **(Ordinality)**
(2) $\mathcal{W}_{t_a,t_b} = \mathcal{W}_{t_b,t_a}^{-1}$ **(Time Reversal Invariance)**
(3) $\mathcal{W}_{t_a,t_c} = \mathcal{W}_{t_a,t_b} \circ \mathcal{W}_{t_b,t_c}$ **(Transitivity)**

Then the map between indifference surfaces factors as:

$$\mathcal{W}_{t_a,t_b} = \mathcal{E}_{t_a}^{-1} \circ \Upsilon_{t_a,t_b}^{\hat{\mathcal{E}}_t} \circ \mathcal{E}_{t_b} \tag{9.3}$$

where $\Upsilon_{t_a,t_b}^{\hat{\mathcal{E}}_t} : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *is determined by (7.1) and gives the* <u>*unique*</u> *such extension of ordinal welfare to evolving market preferences* $\mathcal{O}_t$*.*

Conjecture 2. *The set of welfare theories possesses the structure of a vector space. That is, any welfare map which satisfies 1-3 above which can be constructed from additional information, will differ from the map determined by (7.1) by a generalized torsion tensor given as a 1-form valued in the adjoint bundle.*

Corollary 1. *The dynamic extension of the COL index is* $\frac{\Upsilon_{t_b,t_a}^{\hat{\mathcal{E}}_t}(u)}{u}$ *for* $u \in \mathbb{R}_+$ *.*

10. Conclusion

We have shown here that, counter to claims found in the economics literature, neoclassical economics *can* admit changing ordinal welfare. We have constructed such a consistent welfare comparison operator and have conjectured that such a distinguished welfare theory extending the usual theory for static preferences is suitably unique without additional data.

This effectively creates a new arena for geometric marginalism. The expansion of welfare to more realistic behavioral assumptions on taste provides a first indication of the greater biological realism that may flow, ironically, from the mathematics most familiar from mathematical physics.

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Appendix: The 'Engineering Formula' as Perturbative Solution

[2]

Our first task is to come up with an acceptable definition of an index for cardinal man's changing efficiency as a pleasure machine with respect to his dynamic Engel curves.

We define the i-simplex $\Delta_i(t)$ in Euclidean space $\mathbb{R}^i$ for $t \geq 0$ to be given as

$$\Delta_i(t) = \{\ x = (t_1, t_2, ..., t_i) \in \mathbb{R}^i \mid 0 \leq t_1 \leq t_2 \leq ... \leq t_i = t\ \} \tag{10.1}$$

and we write its element of integration as

$$\mathbf{dt_i} = dt_1\ dt_2 ... dt_i \tag{10.2}$$

and write the notation for integration over the simplex using the shorthand

$$\int_{\Delta_i(t)} = \int_0^t \int_0^{t_i \leq t} \int_0^{t_{i-1} \leq t_i \leq t} ... \int_0^{t_2 \leq ... \leq t_i \leq t} \tag{10.3}$$

[2]Note: Feel free to skip this section. This is really included as a convenience and as an attempt to simplify the previous discussion using a few tricks from Freeman Dyson's form of the perturbation series in the interaction picture. Rigorously and carefully done it would require some thinking about the universal enveloping algebra of the diffeomorphism group of the positive reals issues of convergence and details like left and right invariant vector fields. Maybe you can correct any errors (e.g. left vs. right invariance) and lack of rigor better than I.

Consider the time-dependent differential operator

$$A(t,u) = \frac{-\partial \mathbf{C}_t(\mathbf{X}_s(u))}{\partial t}|_{s=t} \frac{\partial}{\partial u} \tag{10.4}$$

acting on smooth real-valued functions $f(u)$ on the space of utils.

In $\Delta_i(t)$ we define

$$A^i_j = A(t_{i-j}, u) \tag{10.5}$$

Using this definition of $\Delta_i(t)$, we can now define a two variable family of functions

$$a_i(t,u) = \int_{\Delta_i} \left(\prod_{j=0}^{i-1} A^i_j \right) u \ \mathbf{dt_i} \tag{10.6}$$

for $i \in \mathbb{Z}_+$ where the product of differential operators in the above is to be interpreted as composition

$$\prod_{j=0}^{i-1} A^i_j = A^i_0 \circ A^i_1 \circ \cdots \circ A^i_{i-1} \circ \mathrm{Id}. \tag{10.7}$$

We extend this definition in the natural way for $i = 0$ by declaring

$$a_0(t,u) = u \tag{10.8}$$

when $i = 0$.

We then make the following definition:

Definition 1. A Market Index of Utility. *Let $\tilde{\alpha}_t = (\mathbf{C}_t,\ \mathbf{X}_t)$ be the cost-function of a market participant with cardinal preferences. We define the index of the cardinal consumer's changing efficiency as a pleasure machine as the function on the space of (cardinal) utils u.*

$$\Upsilon^{\tilde{\alpha}_t}_{t_0,t}(u) \tag{10.9}$$

where the inverse in the above is taken relative to the u variable. This gives some rigorous meaning to the concept of the psychological cardinal 'income effect' that is at least implicitly alluded to in several author's work (e.g. Gintis and Sen).