The Folk Theorem in Repeated Games with Anonymous Random Matching*

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Abstract

We prove the folk theorem for discounted repeated games with anonymous random matching. We allow non-uniform matching, include asymmetric payoffs, and place no restrictions on the stage game other than full dimensionality. No record-keeping or communication devices—including cheap talk communication and public randomization—are necessary.

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1 Introduction

In a repeated game with anonymous random matching, a finite population of players repeatedly breaks into pairs to play 2-player games. Each period, a player observes only her partner's action—not his identity, and not any other player's action. We prove the folk theorem in this environment. In particular, when the players are sufficiently patient, they can attain the same payoffs as if everyone's identity and actions were publicly observed at the end of each period.

Because players receive so little information under anonymous random matching, this environment has long been used as a benchmark against which to measure the value of various record-keeping devices and institutions, such as fiat money, merchant coalitions and guilds, credit bureaus, online rating systems, "standing" and "image scoring" in evolutionary biology, and monitoring within ethnic groups.[1] The main implication of our result is that, even in this information-poor benchmark environment, patient players can obtain any feasible and individually rational payoffs without any record-keeping devices or institutions beyond their individual memories and the ability to count periods. Thus, any role for such institutions must result from impatience of the players, or from the possibility of constructing "simpler," "more robust," or "more realistic" equilibria when more information is available.[2]

Our folk theorem thus admits both positive and negative interpretations. The positive interpretation is that a very wide range of cooperative behaviors are possible despite minimal information. The negative interpretation is that, in a finite population of patient long-run players, it is difficult to justify the value of information-sharing institutions on efficiency grounds alone. In particular, in these environments the assumptions that monitoring is decentralized and players are anonymous—which might have been expected to restrict the

[1]On money, see Kiyotaki and Wright (1989, 1993), Kocherlakota (1998), Wallace (2001), Araujo (2004), Aliprantis, Camera, and Puzzello (2007). On merchants, see Greif (1993), Greif, Milgrom, and Weingast (1994), Milgrom, North, and Weingast (1990). On credit bureaus, see Klein (1992), Padilla and Pagano (2000). On online rating systems, see Friedman and Resnick (2001). On standing and image scoring, see Sugden (1986), Nowak and Sigmund, (1998). On ethnic conflict, see Fearon and Laitin (1996).

[2]Of course, our result first fixes the population size N and then takes $\delta \to 1$. If the population is very large, the required discount factor is very close to 1. For example, if one extended our model by introducing fiat money à la Kiyotaki and Wright (1989, 1993) or Wallace (2001), our theorem would immediately imply that, for any fixed N, money is inessential for sufficiently high δ; however, for any fixed δ, for many stage games money is essential for sufficiently high N. This observation generalizes the conclusion of Araujo (2004) in the same way that our theorem generalizes the conclusions of Kandori (1992) and Ellison (1994).

set of attainable payoffs in some games—turn out to be completely payoff-irrelevant.

Our approach is to view the repeated random matching game as a single N-player repeated game with imperfect private monitoring and apply techniques from the literature on the folk theorem with private monitoring. The main obstacle to this approach is that, when viewed as a single repeated game, the random matching game fails standard statistical identifiability conditions (e.g., Fudenberg, Levine, and Maskin's (1994) pairwise identifiability) and full support conditions. To overcome this obstacle, we show that players can be given incentives to truthfully share information—despite communicating only via payoff-relevant actions—and that the aggregated information of a player's opponents always identifies her action. Our paper thus connects three literatures: repeated games with random matching, repeated games with private monitoring, and secure communication in repeated games.

Random matching Kandori (1992), Ellison (1994), and Harrington (1995) show that cooperation can be sustained in the repeated prisoners' dilemma with anonymous random matching via "contagion strategies," where a single defection triggers the breakdown of cooperation throughout the population. This approach does not generalize beyond the prisoners' dilemma, because spreading contagion may not be incentive compatible when punishing is costly. Even within the prisoners' dilemma, it cannot be used to support asymmetric equilibria, where for example a subset of players are allowed to defect while others must cooperate. In contrast, our theorem covers all games (subject to a mild full dimensionality condition) and all feasible and individually rational payoffs.

Deb (2018) proves the folk theorem for asymmetric games where players from distinct communities fill different player-roles, cheap talk communication between partners is allowed, and all players from the same community receive the same payoff. We instead consider random matching within a single population (though our approach generalizes to multiple communities), allow asymmetric payoffs, and—most importantly—disallow cheap talk.[3] Deb and González-Díaz (2019) also disallow cheap talk in the 2-community model, but they impose some conditions on the stage game, restrict attention to symmetric payoffs that Pareto dominate a Nash equilibrium (obtaining a "Nash threat" folk theorem), and require

[3]Ruling out cheap talk seems essential, as the point of our analysis is to see what outcomes are possible in the absence of record-keeping and communication devices.

the population to be sufficiently large. Their proof is completely different, as they generalize the contagion approach, while we build on the block belief-free approach introduced by Hörner and Olszewski (2006) to study repeated games with almost-perfect monitoring. We compare these two approaches below. Deb, González-Díaz, and Renault (2016) prove a general folk theorem for N-community games without discounting. Another difference from these papers is that our approach extends to non-uniform and even non-i.i.d. matching.

Other random matching models assume players directly observe some information about their partners' past play. Okuno-Fujiwara and Postlewaite (1995) and Dal Bó (2007) consider finite population models; notably, the latter paper allows asymmetric payoffs. Rosenthal (1979), Takahashi (2010), Dilmé (2016), Heller and Mohlin (2018), Bhaskar and Thomas (2018), and Clark, Fudenberg, and Wolitzky (2019) consider continuum models.

Private monitoring The literature on repeated games with imperfect private monitoring is too large to survey here. The folk theorem with public cheap talk communication is proved by Compte (1998) and Kandori and Matsushima (1998). Piccione (2002), Ely and Välimäki (2002), Matsushima (2004), Ely, Hörner and Olszewski (2005), Hörner and Olszewski (2006), and Yamamoto (2012) develop belief-free techniques that we build on. Sugaya (2019) proves a general folk theorem under identifiability and full support conditions. These conditions are violated with anonymous random matching, but we use some ideas from Sugaya's proof.[4] We explain the connection to this literature in Section 3.5.

Secure communication The most challenging part of our proof is providing incentives for secure communication with anonymous random matching, when communication can be executed only through payoff-relevant actions. As far as we know, ours is the first paper to address this problem. Incentives for secure communication have been studied in the related setting of repeated games played on fixed networks (Ben-Porath and Kahneman, 1996; Renault and Tomala, 1998; Lippert and Spagnolo, 2011; Laclau, 2012, 2014; Nava and Piccione, 2014; Wolitzky, 2015). While the technical overlap with this literature is slight, our non-uniform matching model can approximate a fixed network, as we allow the case where a player "almost always" interacts with the same partners.

[4]Fudenberg, Ishii, and Kominers (2014) also build on Hörner and Olszweski to prove a folk theorem in a setting where Sugaya's theorem does not apply, albeit a completely different one from ours.

2 Model and Folk Theorem

There is a finite set of players $I = \{1, ..., N\}$, with $N \geq 4$ even. In every period $t = 1, 2, \ldots$, players match in pairs to play a finite, symmetric 2-player game with action set A and payoff function $u : A \times A \to \mathbb{R}$, with $|A| \geq 2$. Let $a^0, a^1 \in A$ denote two arbitrary, distinct actions.

Pairs are formed as follows: (i) a *matching* μ is a partition of the population into pairs, (ii) there is an exogenous distribution p over matchings, and (iii) the period-t matching μ_t is drawn from p i.i.d. across periods.[5] We assume p has full support and let $\bar{\varepsilon} > 0$ denote the minimum of $p(\mu)$ over all matchings. As there are at least 3 possible matchings when $N \geq 4$, we have $\bar{\varepsilon} \leq \frac{1}{3}$. Let $\mu(i)$ denote player i's partner in matching μ. Let $p_{i,j} = \sum_{\mu:\mu(i)=j} p(\mu)$ denote the probability that players i and j are matched.

Players are anonymous—each player observes only the actions she faces and not her opponents' identities. Formally, letting $a_{i,t} \in A$ denote player i's period-t action, player i's observation in period t is the pair $(a_{i,t}, \omega_{i,t})$, where $\omega_{i,t} = a_{\mu_t(i),t}$. Say that a profile of observations $(a_i, \omega_i)_{i\in I}$ is *feasible* if there exists an action profile $\mathbf{a} = (a_1, \ldots a_N) \in \prod_{i\in I} A = A^N$ and a matching μ such that $\omega_i = a_{\mu(i)}$ for all $i \in I$. Player i's history at the beginning of period t is denoted $h_i^{t-1} = (a_{i,\tau}, \omega_{i,\tau})_{\tau=1}^{t-1}$, with $h_i^0 = \emptyset$. Players maximize expected discounted payoffs with common discount factor $\delta < 1$. Let $E(\delta)$ denote the sequential equilibrium payoff set with discount factor δ.[6]

For any action profile $\mathbf{a} \in A^N$, player i's expected payoff at action profile $\mathbf{a}$ is given by

$$\hat{u}_i(\mathbf{a}) = \sum_{j\neq i} p_{i,j} u(a_i, a_j).$$

Thus, the (convex hull of the) feasible payoff set in the N-player game is $F = \text{co}\left(\{\hat{\mathbf{u}}(\mathbf{a})\}_{\mathbf{a}\in A^N}\right)$, where $\hat{\mathbf{u}}(\mathbf{a}) = (\hat{u}_1(\mathbf{a}), \ldots, \hat{u}_n(\mathbf{a}))$. Let $\bar{u} = \max_{(a,a')\in A^2} |u(a, a')|$ be the greatest magnitude of any feasible payoff, and let $\underline{u} = \min_{\alpha\in\Delta(A)} \max_{a\in A} u(a, \alpha)$ be the minmax payoff. Let $\alpha^{\min} \in \text{argmin}_{\alpha\in\Delta(A)} \max_{a\in A} u(a, \alpha)$ be a minmax strategy in the 2-player game; to minmax

[5] The extension to non-i.i.d. matching is discussed in Section 4.

[6] In defining sequential equilibrium, the choice of topology on the sets of beliefs and strategies does not matter for us—for concreteness, take it to be the product topology. This is another point of contrast with the approaches in Deb (2017) and Deb and González-Díaz (2017), where choosing the product topology is essential.

player i in the N-player game, every player but i plays $\alpha^{\min}$. Denote the set of feasible and individually rational payoffs by $F^* = \{\mathbf{v} \in F : v_i \geq \underline{u} \ \forall i \in I\}$. We assume F^* has dimension N. This condition is generic: letting

$$e^i = \left(u\left(a^0, a^1\right), \left(\left(1 - p_{j,i}\right) u\left(a^1, a^1\right) + p_{j,i} u\left(a^1, a^0\right)\right)_{j \neq i}\right) \in \mathbb{R}^N$$

be the payoff vector when player i plays a^0 and all other players play a^1, the vectors $(e^i)_{i \in I}$ are linearly independent for generic values of $u\left(a^0, a^1\right)$, $u\left(a^1, a^0\right)$, and $u\left(a^1, a^1\right)$.[7]

In this setting, we establish the folk theorem:

Theorem 1 *For all $\mathbf{v} \in \operatorname{int}(F^*)$, there exists $\bar{\delta} < 1$ such that $\mathbf{v} \in E(\delta)$ for all $\delta > \bar{\delta}$.*

3 Key Ideas of the Equilibrium Construction

We provide a constructive proof of the folk theorem. The proof is deferred to the appendix. Here we describe the key ideas of the construction.

3.1 Overall Structure of the Construction

We view the repeated game as an infinite sequence of finite blocks of periods. Deviations from equilibrium play are detected as a result of communication among the players (described below) and are then punished in two ways. First, within the block where the deviation occurs, players switch to mutual minmaxing. Second, the deviator's continuation payoff at the start of the next block is reduced, while other players' continuation payoffs are adjusted to compensate them for any cost of punishing the deviator.[8]

A key feature of this block structure is that, across blocks, each player's continuation value is independent of that of her opponents. Therefore, the challenge is providing incentives

[7] Full-dimensionality of F^* and full-dimensionality of the underlying 2-player game are logically independent. If the 2-player game is a pure coordination game (with payoff dimension 1) then F^* has full dimension. Conversely, with $N = 4$ and uniform matching, the 2-player game $\begin{array}{ccc} & a^0 & a^1 \\ a^0 & 4,4 & 1,3 \\ a^1 & 3,1 & 0,0 \end{array}$ has full dimension, but F^* has dimension 1.

[8] This basic of idea of "rewarding the punishers" dates back to Fudenberg and Maskin (1986).

within each block for correct on-path play and (especially) providing incentives for truthful communication. This is unlike contagion equilibria, as in Kandori (1992), Ellison (1994), and Harrington (1995), where all players' payoffs are tied together, and so the key challenge is in providing incentives to carry out punishments.

We now describe the structure of our equilibrium. Players follow automaton strategies. In each block, each player $i \in I$ has two possible states—denoted $x_i \in \{G, B\}$, for "good" and "bad." A player's state in the current block and her history in the current block jointly determine her state in the next block. We specify each player i's block strategy in state x_i—denoted $\sigma_i(x_i)$—and the state transition rules so that (i) for every realization of the other players' states $x_{-i} \in \{G, B\}^{N-1}$, both $\sigma_i(G)$ and $\sigma_i(B)$ are optimal strategies for player i (that is, as in Hörner and Olszewski (2006), the equilibrium is *block belief-free*), and (ii) player i's equilibrium continuation payoff is completely determined by the state of player $(i-1) \pmod N$. In particular, player i's continuation payoff is high (low) if $x_{i-1} = G$ (B). Player i's state transition rule can thus be used to control player $i+1$'s continuation payoff.

Play within a block proceeds as follows. First, there is an "initial talk sub-block," where players communicate to coordinate on the state profile $x \in \{G, B\}^N$. Then, players repeat the following "play-and-talk sub-block" multiple times: they play actions that attain the target payoffs at state profile x for many periods, and then communicate to see if anyone deviated. If players detect a deviation, they switch to the minmaxing strategy starting in the next sub-block. This is followed by a "final talk sub-block," where players communicate a summary of the entire block history.

Since all communication is via payoff-relevant actions, to attain the target payoffs the players must spend most of their time in the "play" phases. In particular, they cannot take the time to communicate about every play period. Instead, when players communicate to identify deviations, player $i-1$ chooses one period at random from the preceding play phase and communicates this choice to the other players, who then share their information about that period only. Since player i does not know in advance which period player $i-1$ will choose, this scheme can provide incentives for the entire play phase.

3.2 How Communication Works

In our construction, players communicate by taking turns broadcasting information. Which player's turn it is to "talk" in each period is pre-determined.[9] We explain how a player sends a binary message $m \in \{0, 1\}$. Longer messages are sent by binary expansion.

To send message 1, the sender plays a^1 for T periods and then a^0 for another T periods, where T is a pre-determined large number. To send message 0, the order is reversed: first a^0 for T periods, then a^1. The other players—the "receivers"—play only a^0 with high probability throughout the entire $2T$-period interval. At the end of the interval, a receiver who observed a^1 during the first T periods only infers that the sender sent message 1. A receiver who observed a^1 during the last T periods only infers that the sender sent message 0. A receiver who observed any other pattern—that is, observed a^1 at least once in each half-interval, or never observed a^1 at all—receives a message of `error`.

This protocol has several desirable properties. First, if T is large, with high probability the sender matches with each receiver at least once in each T-period half-interval, and therefore the message transmits successfully when all players follow the protocol. Second, a key obstacle to communication is that, since players are anonymous, a receiver may be tempted to talk at the same time as the sender in an attempt to manipulate the message. Our protocol makes such a manipulation very unlikely to succeed: no matter what a given receiver does, every other receiver will either receive the correct message or receive `error`, so long as she meets the sender at least once in the half-interval where the sender plays a^1—a very high-probability event. Hence, to deter this attempted manipulation, it suffices to punish all players whenever anyone receives `error`.

There are however two important challenges to implementing this simple scheme.

First, in the course of communication, a receiver might learn that a low-probability realization of the matching process has occurred, at which point her expected gain from manipulation can be much larger. For example, suppose a single receiver happens to see a^1 in *all* of the first T periods—this event is very unlikely, but it is not impossible. Since only one receiver at a time sees a^1, this receiver can infer that she is the only one to have received the message successfully. This puts her at a large informational advantage over the other

[9]Here we rely on the implicit assumption that the players share a common sense of calendar time.

players, and it is difficult to predict how she may exploit this advantage in continuation play.

We address this receiver-learning problem by introducing *jamming*, a key innovation in our proof. Specifically, at the beginning of each block, with small probability each player is designated a *jamming player* for the block. (We defer the details of how this designation is determined.) Jamming players differ from regular players in that, when they are receivers, with small probability they continually play a^1 (which we refer to as *jamming communication*) rather than a^0. Clearly, communication is very unlikely to succeed when a jamming player is present and jams communication—however, since jamming players are rarely present (and rarely jam communication when they are present), this has a negligible effect on equilibrium payoffs. Moreover, even a slight possibility that communication may be jammed is enough to solve the receiver-learning problem: now, if a receiver sees a^1 repeatedly, she infers that with high probability a jamming player is present and jammed communication, rather than inferring that an low-probability match realization occurred. In the former case, it is very likely that all players inferred that communication was jammed. Thus, the possibility of jamming greatly reduces the perceived informational advantage of a receiver who repeatedly observes a^1. The resulting gain from manipulation is small enough that it can be offset by a small loss in continuation payoff at the start of the next block.

Second, when a receiver receives `error`, the subsequent punishment must be incentive compatible. How this is ensured depends on where in the block the `error` occurs. In the last communication phase in the block, it is enough to specify that, when player i receives `error`, she (costlessly) adjusts her transition probability for the next block so as to reduce player $i+1$'s continuation payoff only. In earlier communication phases, an `error` realization leads to mutual minmaxing within the block, incentivized by the promise of compensation at the start of the next block. The structure of the different communication phases within a block is described in more detail in Section 3.4.

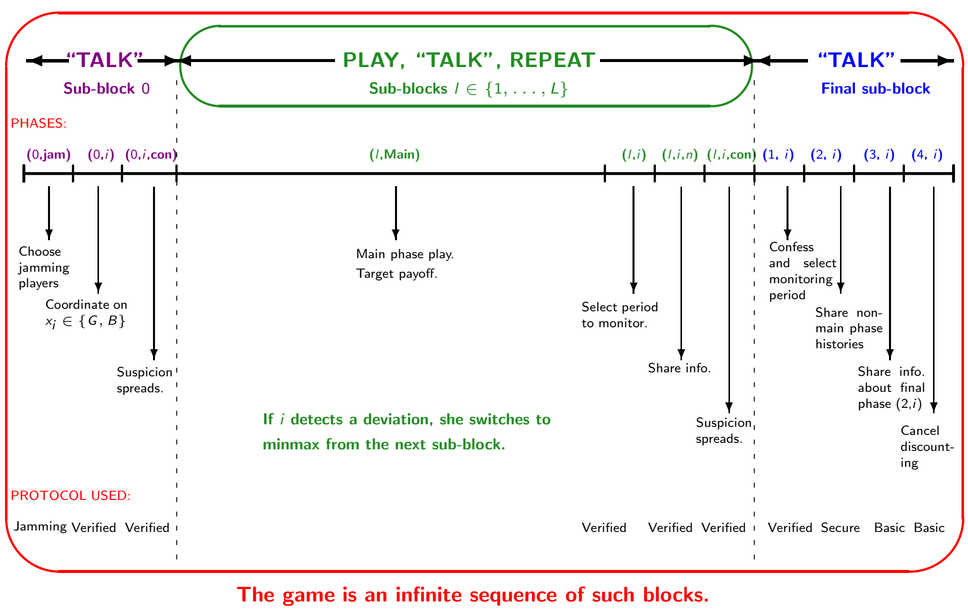
3.3 How Identification Works

Another step in the proof is that, if player i's opponents can successfully aggregate their information regarding a particular period of play, this information suffices to perfectly identify player i's action and observation in that period. This step is straightforward. Since matching occurs in pairs, the total number of players who observe the same action they play (i.e., observe $\omega_n = a_n$) is always even. Therefore, if there exists $a \in A$ such that the number of i's opponents for whom $\omega_n = a_n = a$ is odd, then $\omega_i = a_i = a$. If instead this number is even for every $a \in A$, then $a_i \neq \omega_i$. (Otherwise, the total number of players with $\omega_n = a_n = a_i$ would be odd.) In this case, there is one action a such that more of i's opponents observe $\omega_n = a$ than play $a_n = a$, and there is another action ω such that more of i's opponents play $a_n = \omega$ than observe $\omega_n = \omega$. This pair (a, ω) must then equal (a_i, ω_i). Thus, if players $-i$ can aggregate their information, they can perfectly monitor player i.[10]

3.4 A Closer Look at the Communication Sub-Blocks

Next, we provide a little more detail on the "initial talk," "play and talk," and "final talk" sub-blocks noted above. "Talk" proceeds via *communication protocols*: finite repetitions of the stage game in which players communicate via actions. Our analysis consists of stringing

Figure 1: Schematic of play within a block



[10]This perfect monitoring property is not necessary for our approach: in the working-paper version (Deb, Sugaya, and Wolitzky, 2018), we extend our proof to almost-perfect monitoring within matches. Nonetheless, perfect monitoring simplifies the proof while letting us focus on its most novel element: incentivizing truthful communication.

together analyses of different communication protocols. Since we verify incentive compatibility essentially by backwards induction, we describe the protocols backwards from the end of a block.

Figure 1 provides a schematic of play within a block. The final talk sub-block comprises four phases. In the last phase, player $i-1$ chooses one period t at random from the previous periods in the block and communicates it to the other players, who then communicate their period t information to player $i-1$: intuitively, players $-i$ "talk about" player i's play in period t. Player $i-1$ then slightly adjusts her state transition probability such that the effect of discounting in player i's payoff is cancelled out: when player $i-1$ chooses period t, she increases player i's continuation payoff by $\frac{1}{\Pr(t \text{ is chosen})}\left(1-\delta^{t-1}\right)\hat{u}_i(\mathbf{a}_t)$, where $\mathbf{a}_t$ is the period-t action profile identified from communication. This makes player i indifferent about the timing of her actions within a block. Hence, in all earlier phases, we may view the game as one without discounting, which is a substantial simplification.

Recall that player $i-1$'s state affects player i's payoff only. Thus, in the last communication phase, players $-i$ are indifferent to the outcome of communication, and are thus willing to report truthfully. Moreover, even player i has only a very small potential gain from manipulating communication when δ is large (once we fix the length of the block). Since it is always possible to provide small incentives without sacrificing much efficiency, we do not need to rely on jamming players in this phase, and a very simple communication protocol—the *basic communication protocol*, introduced in Section F.1—is sufficient.

In the penultimate and third-to-last talk phases, players $-i$ aggregate their information from all previous talk phases in the block. Player $i-1$ uses this information to adjust her state transition. As we will see, the impact of this adjustment on player i's payoff can be large, so player i may have a strong incentive to manipulate the communication. Hence, for this phase we need a communication protocol where there is no history at which player i believes she can manipulate the outcome of communication to her benefit. This requires the *secure communication protocol*, introduced in Section D.2, which relies on jamming players.

In the first talk phase of the final talk sub-block, player $i-1$ chooses one period t_l at random from each of the L main play sub-blocks and communicates it to the other players, who then communicate their period t_l information to player $i-1$. Players also confess whether they have deviated in the current block so far. Similarly, in the talk phases of the "play and talk" sub-blocks, players communicate selected periods to monitor and share information about the monitoring periods with the sub-block. Finally, talk phases in the initial talk sub-block of the block are used to determine jamming players for the block and to coordinate on the state $x \in \{G, B\}^N$. Communication in the initial sub-block and the play-and-talk sub-blocks is especially challenging. This is because these phases affect not only continuation payoffs at the end of the block but also continuation play within the block. Thus, all players (not only the one "about whom the others are talking") may have a strong incentive to manipulate communication. We therefore need a protocol that no player can profitably manipulate. We construct the *verified communication protocol* (introduced in Section H) to have this property. The key additional feature of this protocol is that each receiver communicates the message she received back to the sender. This lets all players determine whether or not they received the same message.[11]

3.5 Relation to the Private Monitoring Literature

Some readers may wish to understand how our construction relates to existing work on repeated games with private monitoring. Our goal is to construct a block belief-free equilibrium as in Hörner and Olszewski (2006). To allow accurate communication under random matching, we have players repeat actions and messages and apply a concentration inequality (Lemma 3). In this sense, our construction joins the line of research combining belief-free equilibria and review strategies, following Matsushima (2004). The closest papers in this literature are Yamamoto (2012) and Sugaya (2019).

Yamamoto shows how to combine belief-free equilibria and review strategies in general repeated games. There are several important differences with our approach, but a crucial one is that Yamamoto assumes conditional independence: player i's signal and player j's signal are independent conditional on actions. Thus, player i cannot learn player j's inference from her own signals. In contrast, with random matching signals are not conditionally independent. This is the "receiver-learning problem" noted above, which we address via the

[11] As indicated in Figure 1, we also use the verified communication protocol in the first talk phase of the final talk sub-block.

innovation of introducing jamming players.

Sugaya proves a general folk theorem by generalizing Yamamoto's construction to conditionally dependent monitoring. As in the current paper, mixed strategies are used to control incentives after erroneous histories that arise with small ex ante equilibrium probability. In particular, after observing such a history, a player believes this observation results from a rare realization of her opponents' mixed strategies. By specifying her continuation payoff to be constant after such erroneous realizations, the player is incentivized to adhere to the same continuation play as after non-erroneous histories. However, Sugaya's construction assumes pairwise identifiability (i.e., each player can unilaterally identify other players' deviations). This makes communication straightforward, as when player i "sends a message" to player j, player j can construct a statistic whose distribution depends on player i's message but is independent of unilateral deviations by players $-i$. With anonymous random matching, pairwise identifiability is robustly violated.

4 Extensions

We present three extensions: imperfect monitoring within matches, non-pairwise matching, and non-i.i.d. matching. Our goal is not to establish the most general results possible but to illustrate the broad applicability of our proof technique. To this end, in this section we allow some simplifying assumptions, such as access to public randomization and modest restrictions on the stage game. The working-paper version of the paper contains the proofs.

4.1 Almost-Perfect Within-Match Monitoring

We can allow almost-perfect monitoring within a match. This is not surprising since we build on Hörner and Olszewski (2006), who prove the folk theorem with almost-perfect monitoring.

The required modifications to our proof are relatively minor. First, we have jamming players mix over all actions, rather than just a^0 and a^1. This makes players attribute unexpected observations to randomization by jamming players rather than monitoring errors. Second, players' reward functions must be adjusted to account for monitoring errors. Third, it is useful to introduce a small probability that the block is extended to include a final "long communication phase" on which the required reward adjustments can be based.

Formally, a *within-match monitoring structure* (q, Ω) consists of a finite signal space Ω and a mapping $q : A \times A \to \Omega \times \Omega$, where $q(\omega_i, \omega_{\mu(i)}|a_i, a_{\mu(i)})$ is the probability that player i observes signal ω_i and her partner observes signal $\omega_{\mu(i)}$ when i plays a_i and her partner plays $a_{\mu(i)}$. Assume without loss of generality that q has full support. Let q_i denote the marginal distribution of q over i's signal. We say monitoring is *ϵ-perfect* if $\Omega = A$ and $q_i(a_{\mu(i)}|a_i, a_{\mu(i)}) \geq 1 - \epsilon \ \forall \left(a_i, a_{\mu(i)}\right) \in A^2$. Let $E(\delta, q)$ denote the sequential equilibrium payoff set with discount factor δ and monitoring structure q.

Theorem 2 *Suppose public randomization is available. For all $\mathbf{v} \in \text{int}(F^*)$, there exist $\bar{\delta} < 1$ and $\bar{\epsilon} > 0$ such that $\mathbf{v} \in E(\delta, q)$ for all $\delta > \bar{\delta}$ and all ϵ-perfect within-match monitoring structures q with $\epsilon \leq \bar{\epsilon}$.*

Note that Theorem 2 assumes public randomization, in contrast to both our main result and Hörner and Olszewski's folk theorem. In the proof, public randomization is used to decide when to extend the block by including a long communication phase.

4.2 Non-Pairwise Matching and Random Player-Roles

The assumption that matching is pairwise is restrictive. For example, this requires that all players "play the game" the same number of times, and thus rules out a distinction between frequent and infrequent participants. The assumption that each player has the same "role" in each match is also restrictive. It rules out games where each period one player in each match has an opportunity to do a favor for her partner, as in "monetary" models à la Kiyotaki and Wright (1989, 1993). Our approach can be extended to cover these settings, with some restrictions on the structure of the game and the target payoff set.

A matching μ is now an arbitrary partition of the population into *groups*, rather than pairs. (A group of size 1 means a player is "unmatched" in the current period.) We continue to assume that matches are drawn from a fixed i.i.d. distribution p. We also assume that there is an upper bound $M \leq N$ on the size of a group, and that any partition of the population into groups of size $\leq M$ occurs with probability at least $\bar{\varepsilon} > 0$.

Whenever $n^* \leq M$ players are matched together in a group, they play a finite game with action sets $(A_{i^*}[n^*])_{i^*=1}^{n^*}$ and payoff functions $(u^*_{i^*}[n^*])_{i^*=1}^{n^*}$, where $A_{i^*}[n^*] \geq 2\ \forall i^*$. We allow two possible structures for the n^*-player games:

1. **Symmetric stage games**: For each $n^* \leq M$, the n^*-player game is symmetric: all players have the same action set $A[n^*]$ and payoff function $u[n^*] : A[n^*]^{n^*} \to \mathbb{R}$. At the end of each period, every player observes the number of her partners who take each action $a \in A$: letting $\mu_t(i)$ denote the set of player i's period-t partners, player i's period-t signal is $\omega_{i,t} = \left(n^*(i), (\omega_{i,t}(a))_{a \in A[n^*(i)]}\right)$, where $n^*(i) = 1 + |\mu_t(i)|$ and $\omega_{i,t}(a) = |\{j \in \mu_t(i) : a_j = a\}|\ \forall a \in A[n^*(i)]$.

   Each player i's strategy in the one-shot game is a mapping from $n^*(i)$ to an element of $A[n^*(i)]$. Let $\bar{A}$ denote the pure strategy set in the one-shot game (it is the same for every player). Let $\bar{A}^{\text{mix}}$ denote the mixed strategy set.

   Let $F = \text{co}\left(\{\hat{\mathbf{u}}(\bar{\mathbf{a}})\}_{\bar{\mathbf{a}} \in \bar{A}^N}\right)$. Given a mixed strategy profile $(\bar{\alpha}_n)_{n \in I} \in \prod_{n \in I} \bar{A}^{\text{mix}}$, let $\underline{u}_i\left((\bar{\alpha}_n)_{n \in I}\right) := \max_{\bar{a}_i \in \bar{A}} \hat{u}_i\left(\bar{a}_i, (\bar{\alpha}_n)_{n \neq i}\right)$ denote the highest payoff player i can attain against $(\bar{\alpha}_n)_{n \neq i}$. Our target payoff set is

   $$F^* = \left\{\mathbf{v} \in F : \exists \left(\bar{\alpha}_n^{\min}\right)_{n \in I} \in \prod_{n \in I} \bar{A}^{\text{mix}} \text{ such that } v_i \geq \underline{u}_i\left(\left(\bar{\alpha}_n^{\min}\right)_{n \in I}\right)\ \forall i \in I\right\}.$$

   Since here punishment actions are not tailored to the player being punished, this set is smaller than the feasible and individually rational payoff set. However, it equals this set if the distribution over matches is symmetric across players. Moreover, for any match distribution, taking $\left(\bar{\alpha}_n^{\min}\right)_{n \in I}$ to be a symmetric Nash equilibrium yields a "Nash threat" folk theorem.

2. **Random player-roles**: For each $n^* \leq M$, the n^*-player game is arbitrary, but each player in I_{n^*} is randomly assigned one of the n^* player-roles. When player $i \in I_{n^*}$ is assigned player-role i^*, she has action set $A_{i^*}[n^*]$ and payoff function $u_{i^*}[n^*] : (A_{i^*}[n^*])_{i^*=1}^{n^*} \to \mathbb{R}$. Let $i^*(i)$ denote player i's assigned role. Player i's period-t signal is $\omega_{i,t} = (n^*(i), i^*(i), (a_{i^*,t}(i))_{i^*=1}^{n^*(i)})$, where $a_{i^*,t}(i)$ is the period-t action of the player assigned to role i^* in i's match.

Each player i's strategy in the one-shot game is a mapping from $(n^*(i), i^*(i))$ to an element of $A_{i^*(i)}[n^*(i)]$. Given this definition, $\bar{A}$, $\bar{A}^{\text{mix}}$, F, $\underline{u}_i\left((\bar{\alpha}_n)_{n\in I}\right)$, and F^* are defined as in the symmetric stage game specification.

Theorem 3 *With non-pairwise matching and either symmetric stage games or random player-roles, for all* $\mathbf{v} \in \text{int}(F^*)$*, there exists* $\bar{\delta} < 1$ *such that* $\mathbf{v} \in E(\delta)$ *for all* $\delta > \bar{\delta}$*.*

The required modifications to the proof are minor. For example, a player must now report her group size and player-role (if applicable) in addition to her action and observation. Given this additional information, our identification argument generalizes to non-pairwise matching.

4.3 Non-I.I.D. Matching

Our approach extends to situations where (pairwise) matching is determined by a Markov process that depends on both the current match and the current action profile. This encompasses models with endogenous match separation, such as finite population versions of Shapiro and Stiglitz (1984), Datta (1996), Kranton (1996), Carmichael and MacLeod (1997), Eeckhout (2006), Fujiwara-Greve and Okuno-Fujiwara (2009), and Peski and Szentes (2013).

Let the distribution over period-t matches $p(\cdot|\mathbf{a}_{t-1}, \mu_{t-1})$ depend on the previous action profile $\mathbf{a}_{t-1}$ and match μ_{t-1}. Assume $p(\cdot|\mathbf{a}_{t-1}, \mu_{t-1})$ has full support for each $\mathbf{a}_{t-1}, \mu_{t-1}$, and let $\bar{\varepsilon} > 0$ denote the minimum of $p(\mu_t|\mathbf{a}_{t-1}, \mu_{t-1})$ over all $\mathbf{a}_{t-1}$, μ_{t-1}, and μ_t.

We impose some identifiability conditions on $p(\cdot|\mathbf{a}_{t-1}, \mu_{t-1})$. Order the $N(N-1)/2$ pairs of distinct players $(i, j) \in I^2$, and denote the resulting sequence by C. Suppose in each period $t = 1, ..., N(N-1)/2$ players $i, j \in C_t$—the t^{th} element of C—play a^1 and other players play a^0. Call this strategy $\bar{\sigma}$. Let $y_t = 1$ denote the event that the pair of players in C_t match with each other in period t, and let $y_t = 0$ denote the complementary event. Let $y_C = (y_t)_{t=1}^{N(N-1)/2}$. We assume y_C statistically identifies the period-1 match μ_1: letting P be the matrix with dimension

$$\underbrace{\prod_{k=0}^{N/2-1}(N-2k-1)}_{\#\text{ of possible matches}} \times \underbrace{2^{N(N-1)/2}}_{\#\text{ of possible values for } y_C}$$

whose (μ, y_C)-element corresponds to the probability of y_C when $\mu_1 = \mu$ and the players follow $\bar{\sigma}$, we assume P has full row rank.

We also assume that, for each $\mathbf{a} \in A^N$, the $\prod_{k=0}^{N/2-1}(N-2k-1) \times \prod_{k=0}^{N/2-1}(N-2k-1)$ matrix $Q(\mathbf{a})$ with (μ_{t-1}, μ_t)-element $p(\mu_t|\mathbf{a}, \mu_{t-1})$ has full rank, i.e., μ_t statistically identifies μ_{t-1}.

The feasible payoff set is defined as follows: Let $F(\mu_1, \delta)$ be the set of payoffs $\mathbf{v} \in \mathbb{R}^N$ that are attained by some strategy profile in the repeated game with initial match μ_1 and discount factor δ, allowing public randomization. In the working-paper version of the paper, we show that $\lim_{\delta \to 1} F(\mu_1, \delta)$ exists and is independent of μ_1. The feasible payoff set is then defined as $F = \lim_{\delta \to 1} F(\mu_1, \delta)$ for arbitrary μ_1. We also show that $F = \lim_{\kappa \to \infty} \lim_{\delta \to 1} F^\kappa(\mu_1, \delta)$, where $F^\kappa(\mu_1, \delta)$ is the set of payoffs attainable by the infinite repetition of a strategy in the κ-period finitely repeated game with initial match μ_1, for any μ_1.[12]

The minmax payoff is the same as in the i.i.d. case: $\underline{u} = \min_{\alpha \in \Delta(A)} \max_{a \in A} u(a, \alpha)$. The set of feasible and individually rational payoffs is $F^* = \{\mathbf{v} \in F : v_i \geq \underline{u} \ \forall i \in I\}$.

Theorem 4 *With non-i.i.d. matching, for all* $\mathbf{v} \in \text{int}(F^*)$, *there exists* $\bar{\bar{\delta}} < 1$ *such that* $\mathbf{v} \in E(\delta)$ *for all* $\delta > \bar{\bar{\delta}}$.

The proof now requires substantial modification. The basic idea is to use the fact that, for large enough T, any two matches separated by T periods are almost independent. This lets us preserve the block belief-free structure.

5 Discussion

Multiple communities and player-roles: Our result can also be extended to allow multiple communities, where each community has a fixed role. For example, in a stage-game between a buyer and a seller, we can allow the case where each player is always either a buyer or a seller, and also that where each player can play different roles.

[12] Since players cannot observe the period-t match μ_t, we have a stochastic game with hidden state and private signals. Platzman (1980), Rosenberg, Solan, and Vieille (2002), and Yamamoto (2018) have shown the same result with public signals. In this case, the feasible payoff set is the solution to a single-agent partially observable Markov decision problem, and can be characterized by dynamic programming. This is no longer possible with private signals, and we use a novel argument based on the minmax theorem.

Cheap talk and public randomization: The folk theorem would be easy to prove if we allowed public cheap talk communication. This would make detecting deviations straightforward, and then cooperation could be sustained by punishing deviations through mutual minmaxing. Deb (2018) considers a setting with private (within-match) cheap talk and shows that it is possible to partially detect deviations, and then applies the perfect monitoring version of Hörner and Olszewski. On the other hand, allowing public randomization would not simplify our construction much.[13]

Incomplete information: A concern with contagion equilibria is that they are not robust to incomplete information, for instance the possibility of a few "commitment types' who always defect. Our approach of considering a single N-player game and controlling each player's continuation payoff separately should be more robust to these considerations. Incomplete information can undermine our communication protocols. Nonetheless, we conjecture that our approach combined with that in Fudenberg and Yamamoto (2010) may yield a partial folk theorem for ex post equilibria in this setting.

Low discount factors: While block belief-free strategies let us establish a folk theorem, they have the disadvantage of requiring a very high discount factor as a function of the population size. In contrast, contagion strategies are remarkably effective (in the prisoners' dilemma) even for fairly low δ.[14] Nonetheless, following Hörner and Takahashi (2016), it can be shown that the asymptotic rate of convergence of our equilibrium set to F^* is at least $(1-\delta)^{-1/2}$ for generic stage games. Formalizing and investigating performance criteria for low δ in general anonymous random matching games is an interesting future direction.

[13]In the "final talk phase" of our construction, each player i randomly chooses a set of periods to monitor and communicates this choice to her opponents. With public randomization, we could eliminate this phase by letting nature select these random periods.

[14]See in particular the calculations in Ellison (1994).

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Appendix: Proof of Theorem 1

A Overview of the Proof and Notation

Section B presents the block belief-free equilibrium conditions, to reduce the infinitely repeated game to a finitely repeated game with final-period reward functions. **Section C** defines target payoffs and presents preliminary lemmas. **Section D** defines the communication protocols. **Section E** provides an overview of the equilibrium strategies. **Sections F and G** prove *reduction lemmas* to simplify the equilibrium conditions. We reduce the game to an undiscounted game with final-period reward functions, and show that, reward functions can exhibit some dependence on other players' histories. **Section H** constructs the *verified communication module*, which augments the verified communication protocol defined in Section D with a reward function. **Section I** uses this module to further simplify the equilibrium conditions: We show that it suffices to establish optimality of a player's strategy only at histories consistent with her opponents' equilibrium strategies. **Section J** completes the description of the equilibrium strategies. **Section K** constructs the final reward function, which sums the rewards for main and non-main phases. **Sections L and M** verify the equilibrium conditions. The Supplementary Appendix contains omitted proofs.

We use different terms to refer to sets of consecutive periods that are meaningful in the construction. We define these below, from the longest (a block) to the shortest (a period).

Terminology	**Meaning**
Block	T^{**} periods, structured as in Section E.
Sub-Block	$L+2$ sub-blocks in each block: An initial talk sub-block, a final talk sub-block and L sub-blocks in between that comprise both play and talk. See Section E.
Phase	A major component of a sub-block: either a complete play of a communication protocol, or a set of periods where players take the targeted actions. See Section E.
Round	A major component of the verified protocol. See Section D.3.
Interval	$2T$ consecutive periods in the basic, secure, or verified protocol. See Section D.
Half-Interval	T consecutive periods in the basic, secure, or verified protocol.
Period	A single play of the game.

Table 1: Glossary of Terminology Describing Timing

We also collect some additional notation that will be used repeatedly in the proof.

Notation	Meaning
v_i	The target payoff.
$v_i(G)$	The lowest payoff when players coordinate on x with $x_{i-1} = G$ (see (5)).
$v_i\left(B\right)$	The highest payoff when players coordinate on x with $x_{i-1} = B$ (see (5)).
$\underline{u}$	The minmax payoff (see Section 2).
$\bar{u}$	The greatest magnitude of any feasible payoff (see Section 2).
u^G	The smallest feasible payoff (see (55)).
u^B	The largest feasible payoff (see (55)).

Table 2: Glossary of Notation for Payoffs

Notation	Meaning
$\pi_i^{\text{cancel}}(x_{i-1}, a_{-i}, \omega_{-i})$	Reward to make player indifferent over actions with payoff $v_i(x_{i-1})$ (see (7)).
$\pi_i^a\left(a_{-i}, \omega_{-i}\right)$	Reward to give payoff 0 if $a_i = a$ and -1 otherwise (see (8)).
$-\mathbf{1}_{\left\{a_{j,t} \neq a_{j,t}^*(h_{-j})\right\}}$	Reward to give payoff 0 if player follows verified protocol in checking rounds, and give payoff -1 otherwise (see (42)).
$\pi_i^{\theta=E}(x_{i-1}, a_{-i}, \omega_{-i})$	Reward to make player indifferent over actions with payoff $u^{x_{i-1}}$, while satisfying self-generation (see (56)).
$\pi_i^{v_i}(x_{i-1}, a_{-i}, \omega_{-i})$	Reward to make player indifferent over actions with payoff $v_i\left(x_{i-1}\right)$, while satisfying self-generation if all players play $a^k(x)$ (see (56)).
$\pi_i^{v_i}(x_{i-1}, a_{-i}, \omega_{-i} \vert \alpha^{\min})$	Reward to make player indifferent over actions with payoff $v_i\left(x_{i-1}\right)$ when opponents play $\alpha^{\min}$ (see (56)).

Table 3: Glossary of Notation for Reward Functions

We use standard asymptotic notation: "$f\left(T\right) = O\left(g\left(T\right)\right)$" means "$\exists C > 0, \exists \bar{T} > 0 : \forall T > \bar{T}, \left|f\left(T\right)\right| \leq C g\left(T\right)$."

B Block Belief-Free Structure

We view the repeated game as an infinite sequence of T^{**}-period blocks, with T^{**} to be specified. At the beginning of each block, each player i selects a state $x_i \in \{G, B\}$. Given x_i, player i plays a behavior strategy $\sigma_i^*\left(x_i\right)$ (her *block strategy*) within the block: in every period

$t=1,\ldots,T^{**}$ of a block, $\sigma_i^*(x_i)$ specifies a mixed action as a function of player i's *extended block history* $(\mathbb{L}_i, h_i^{t-1})$, where $\mathbb{L}_i$ encodes the result of a private randomization conducted by player i at the beginning of the block (described below), and $h_i^{t-1}=(a_{i,\tau},\omega_{i,\tau})_{\tau=1}^{t-1}\in H_i^{t-1}$. Denote player i's strategy set in the T^{**}-period game by Σ_i.

We require that player i's state x_i is determined by a *transition probability* $\rho_i(\cdot|\tilde{x}_i,\tilde{h}_i^{T^{**}})\in\triangle(\{G,B\})$ that depends only on player i's state in the previous block, $\tilde{x}_i$, and her history in the previous block, $\tilde{h}_i^{T^{**}}$. Moreover, we require that player i's payoff at the beginning of each block is determined solely by player $(i-1)$'s state, $x_{i-1}\in\{G,B\}$, and denote it by $v_i^*(x_{i-1})\in\mathbb{R}$. Hence, player i's continuation payoff at the end of a block is a function only of player $(i-1)$'s state and extended history. Denote this continuation payoff by $w_i^*(x_{i-1},h_{i-1}^{T^{**}})$.

We present conditions under which a given payoff vector $\mathbf{v}\in\mathbb{R}^N$ is attainable in a block belief-free equilibrium. These are similar to the conditions in Hörner and Olszewski (2006), with one significant difference: Hörner and Olszewski assume monitoring has full support, so in their model Nash and sequential equilibrium coincide, and there is no need to keep track of players' beliefs. In contrast, our model does not have full support, so we must introduce beliefs, verify Kreps-Wilson consistency, and—most subtly—ensure that beliefs respect the block belief-free equilibrium structure, in that sequential rationality is satisfied conditional on each possible state vector $x_{-i}\in\{G,B\}^{N-1}$. To do this, we keep track of players' beliefs conditional on each vector $x_{-i}\in\{G,B\}^{N-1}$. This approach implicitly determines a complete, unconditional belief system, but since sequential rationality is always imposed conditional on x_{-i}, these unconditional beliefs do not enter into our analysis.

Formally, an *ex post belief system* $\beta=(\beta_i)_{i\in I}$ consists of, for each player $i\in I$, opposing state vector $x_{-i}\in\{G,B\}^{N-1}$, period $t\in\{1,\ldots,T^{**}\}$, and block history $h_i^{t-1}\in H_i^{t-1}$, a probability distribution $\beta_i\left(\cdot|x_{-i},h_i^{t-1}\right)\in\Delta\left(H_{-i}^{t-1}\right)$. Together with a block strategy profile $(\sigma_i(x_i))_{i\in I,x_i\in\{G,B\}}$, an ex post belief system is *consistent* if there exists a sequence of completely mixed block strategy profiles $\left(\left(\sigma_i^k(x_i)\right)_{i\in I,x_i\in\{G,B\}}\right)_{k\in\mathbb{N}}$ converging pointwise to $(\sigma_i(x_i))_{i\in I,x_i\in\{G,B\}}$ such that, for each $i\in I$, $x_{-i}\in\{G,B\}^{N-1}$, $t\in\{1,...,T^{**}\}$, and $h^{t-1}\in H^{t-1}$, we have $\beta(h_{-i}^{t-1}|x_{-i},h_i^{t-1})=\lim_{k\to\infty}\Pr^{\left(\sigma_j^k(x_j)\right)_{j\neq i}}\left(h_{-i}^{t-1}|x_{-i},h_i^{t-1}\right)$.[15]

[15]With this definition, it is clear that, whenever an ex post belief system is consistent, the corresponding unconditional belief system is consistent in the usual Kreps-Wilson sense.

We are now ready to present the equilibrium conditions. In what follows, $\mathbb{E}^{\sigma}[\cdot]$ denotes expectation with respect to strategy profile σ, and $\mathbb{E}^{(\sigma,\beta)}[\cdot|\cdot]$ denotes conditional expectation with respect to assessment (strategy profile and beliefs) (σ, β).

For all $\mathbf{v} \in \mathbb{R}^N$ and $\delta < 1$, if there exist $T^{**} \in \mathbb{N}$, strategies $(\sigma_i^*(x_i))_{i \in I, x_i \in \{G,B\}}$, consistent ex post belief system β^*, values $(v_i^*(x_{i-1}))_{i \in I, x_{i-1} \in \{G,B\}}$, and continuation payoffs $\left(w_i^*(x_{i-1}, h_{i-1}^{T^{**}})\right)_{i \in I, x_{i-1} \in \{G,B\}, h_{i-1}^{T^{**}} \in H_{i-1}^{T^{**}}}$ such that the following conditions hold for all $i \in I$, then we have $\mathbf{v} \in E(\delta)$:

1. [Sequential Rationality] For all $x \in \{G,B\}^N$ and $h_i^{t-1} \in H_i^{t-1}$,

$$\sigma_i^*(x_i) \in \underset{\sigma_i \in \Sigma_i}{\operatorname{argmax}} \, \mathbb{E}^{\left(\left(\sigma_i, \sigma_{-i}^*(x_{-i})\right), \beta^*\right)} \left[(1-\delta) \sum_{\tau=1}^{T^{**}} \delta^{\tau-1} \hat{u}_i(\mathbf{a}_\tau) + \delta^{T^{**}} w_i^*(x_{i-1}, h_{i-1}^{T^{**}}) | x_{-i}, h_i^{t-1} \right].$$

   (Here, the sum $\sum_{\tau=1}^{T^{**}}$ could alternatively be written as $\sum_{\tau=t}^{T^{**}}$, since payoffs already incurred in h_i^{t-1} are sunk. In addition, sequential rationality is imposed for every vector $x_{-i} \in \{G,B\}^{N-1}$. This is the defining feature of a block belief-free construction.)

2. [Promise Keeping] For all $x \in \{G,B\}^N$,

$$v_i^*(x_{i-1}) = \mathbb{E}^{\sigma^*(x)} \left[(1-\delta) \sum_{t=1}^{T^{**}} \delta^{t-1} \hat{u}_i(\mathbf{a}_t) + \delta^{T^{**}} w_i^*(x_{i-1}, h_{i-1}^{T^{**}}) \right].$$

3. [Self-Generation] For all $x_{i-1} \in \{G,B\}$ and $h_{i-1}^{T^{**}}$, we have $w_i^*(x_{i-1}, h_{i-1}^{T^{**}}) \in [v_i^*(B), v_i^*(G)]$.

4. [Full Dimensionality] Player $i-1$ can randomize her initial state to deliver player i's target payoff v_i: $v_i^*(B) < v_i < v_i^*(G)$.

Defining $\pi_i^*(x_{i-1}, h_{i-1}^{T^{**}}) := \frac{\delta^{T^{**}}}{1-\delta} \left(w_i(x_{i-1}, h_{i-1}^{T^{**}}) - v_i^*(x_{i-1}) \right)$, we rewrite the conditions below:

1. [Sequential Rationality] For all $x \in \{G,B\}^N$ and $h_i^{t-1} \in H_i^{t-1}$,

$$\sigma_i^*(x_i) \in \underset{\sigma_i \in \Sigma_i}{\operatorname{argmax}} \, \mathbb{E}^{\left(\left(\sigma_i, \sigma_{-i}^*(x_{-i})\right), \beta^*\right)} \left[\sum_{\tau=1}^{T^{**}} \delta^{\tau-1} \hat{u}_i(\mathbf{a}_\tau) + \pi_i^*(x_{i-1}, h_{i-1}^{T^{**}}) | h_i^{t-1} \right]. \quad (1)$$

2. [Promise Keeping] For all $x \in \{G, B\}^N$,

$$v_i^*(x_{i-1}) = \mathbb{E}^{\sigma^*(x)} \left[\frac{1-\delta}{1-\delta^{T^{**}}} \sum_{t=1}^{T^{**}} \delta^{t-1} \hat{u}_i (\mathbf{a}_t) + \pi_i^*(x_{i-1}, h_{i-1}^{T^{**}}) \right]. \tag{2}$$

3. [Self-Generation] For all $x_{i-1} \in \{G, B\}$ and $h_{i-1}^{T^{**}}$,

$$\frac{1-\delta}{\delta^{T^{**}}} \pi_i^*(G, h_{i-1}^{T^{**}}) \leq 0, \frac{1-\delta}{\delta^{T^{**}}} \pi_i^*(B, h_{i-1}^{T^{**}}) \geq 0, \left| \frac{1-\delta}{\delta^{T^{**}}} \pi_i^*(x_{i-1}, h_{i-1}^{T^{**}}) \right| \leq v_i^*(G) - v_i^*(B). \tag{3}$$

4. [Full Dimensionality]

$$v_i^*(B) < v_i < v_i^*(G). \tag{4}$$

Lemma 1 (Hörner and Olszewski (2006)) *For all* $\mathbf{v} \in \mathbb{R}^N$ *and* $\delta \in [0, 1)$*, if there exist* $T^{**} \in \mathbb{N}$, $(\sigma_i^*(x_i))_{i \in I, x_i \in \{G, B\}}$, β^*, $(v_i^*(x_{i-1}))_{i \in I, x_{i-1} \in \{G, B\}}$, *and* $\left(\pi_i^*(x_{i-1}, h_{i-1}^{T^{**}})\right)_{i \in I, x_{i-1} \in \{G, B\}, h_{i-1}^{T^{**}} \in H_{i-1}^{T^{**}}}$ *such that Conditions (1)–(4) are satisfied, then* $\mathbf{v} \in E(\delta)$*.*

C Preliminaries

C.1 Target Payoff and Actions

Given $\mathbf{v} \in \operatorname{int}(F^*)$, there exist payoff vectors $(\bar{v}_i(x_{i-1}))_{i \in I, x_{i-1} \in \{G, B\}} \in \mathbb{R}^{2N}$ such that $(\bar{v}_i(x_{i-1}))_{i \in I} \in \operatorname{int}(F^*)$ $\forall (x_{i-1})_{i \in I} \in \{G, B\}^N$ and $\underline{u} < \bar{v}_i(B) < v_i < \bar{v}_i(G)$ $\forall i \in I$. Define

$$\varepsilon^* := \frac{1}{10} \min_i \min \left\{ \bar{v}_i(G) - v_i, v_i - \bar{v}_i(B), \bar{v}_i(B) - \underline{u} \right\}.$$

We approximate $(\bar{v}_i(x_{i-1}))_{i \in I, x_{i-1} \in \{G, B\}}$ by sequences of action profiles: for all $\varepsilon^* > 0$, there exist $K_{\mathbf{v}} \in \mathbb{N}$ and a sequence of action profiles $\left(\mathbf{a}^k(x)\right)_{k=1}^{K_{\mathbf{v}}} \in A^{NK_{\mathbf{v}}}$ $\forall x \in \{G, B\}^N$ such that, for all $i \in I$, we have $\left| \frac{1}{K_{\mathbf{v}}} \sum_{k=1}^{K_{\mathbf{v}}} \hat{u}_i\left(\mathbf{a}^k(x)\right) - \bar{v}_i(x_{i-1}) \right| < \varepsilon^*$. Let $\hat{u}_i(x) = \frac{1}{K_{\mathbf{v}}} \sum_{k=1}^{K_{\mathbf{v}}} \hat{u}_i\left(\mathbf{a}^k(x)\right)$. Next, fix $(v_i(x_{i-1}))_{i \in I, x_{i-1} \in \{G, B\}} \in \mathbb{R}^{2N}$ and sequences of action profiles $\left(\left(\mathbf{a}^k\left(x\right)\right)_{k=1}^{K_{\mathbf{v}}}\right)_{x\in\{G,B\}^N} \in A^{2^N K_{\mathbf{v}}}$ such that, for all $i \in I$,

$$v_i\left(G\right) = \min_{x:x_{i-1}=G} \hat{u}_i\left(x\right), \quad v_i\left(B\right) = \max_{x:x_{i-1}=B} \hat{u}_i\left(x\right) > \underline{u} + 9\varepsilon^*, \text{ and}$$

$$v_i\left(B\right) + 9\varepsilon^* < v_i < v_i\left(G\right) - 9\varepsilon^*. \tag{5}$$

Players will repeat the target action sequence $\left(\mathbf{a}^k\left(x\right)\right)_{k=1}^{K_{\mathbf{v}}}$ over L "sub-blocks," where

$$L := \left\lceil \frac{2\bar{u}}{\varepsilon^*} \right\rceil \left(K_{\mathbf{v}} + 1\right). \tag{6}$$

(Throughout, $\lceil\cdot\rceil$ denotes the "round-up" function.) For $l > K_{\mathbf{v}}$, let $a_i^l\left(x\right) = a_i^{l\,(\mathrm{mod}\ K_{\mathbf{v}})}\left(x\right)$.[16]

C.2 Identification

We record the observation made in Section 3.3 that the profile (a_{-i}, ω_{-i}) of i's opponents' actions and observations perfectly identifies player i's action and observation, (a_i, ω_i).

Lemma 2 *There exists a function $\varphi : A_{-i} \times A_{-i} \to A_i \times A_i$ such that, if $(a_i, \omega_i)_{i\in I}$ is feasible, then $\varphi\left(a_{-i}, \omega_{-i}\right) = (a_i, \omega_i)$.*

By Lemma 2, for each x_{i-1}, there exists a function $\pi_i^{\text{cancel}}\left(x_{i-1}, a_{-i}, \omega_{-i}\right) : A^{N-1} \times A^{N-1} \to [-2\bar{u}, 2\bar{u}]$ such that, for each $\mathbf{a} \in A^N$, we have

$$\hat{u}_i\left(\mathbf{a}\right) + \pi_i^{\text{cancel}}\left(x_{i-1}, a_{-i}, \omega_{-i}\right) = v_i(x_{i-1}). \tag{7}$$

Thus, the function $\pi_i^{\text{cancel}}\left(x_{i-1}, a_{-i,}, \omega_{-i}\right)$ cancels player i's instantaneous utility. Similarly, for each $a \in A$, there exists $\pi_i^a\left(a_{-i}, \omega_{-i}\right) : A^{N-1} \times A^{N-1} \to \mathbb{R}$ such that, for each $\mathbf{a} \in A^N$, we have

$$\pi_i^a\left(a_{-i}, \omega_{-i}\right) = \begin{cases} 0 & \text{if } a_i = a \\ -1 & \text{if } a_i \neq a \end{cases}. \tag{8}$$

[16] Hörner and Olszewski (2006) and several subsequent papers present their constructions assuming $K_{\mathbf{v}} = 1$. With random matching, this assumption is usually with loss. For example, in the prisoner's dilemma, to punish player 1 while keeping her opponents' payoffs close to $u\left(C, C\right)$, we must cycle through action profiles where player 1 and most of her opponents cooperate, while different subsets of her opponents take turns defecting. We thus present our construction for arbitrary $K_{\mathbf{v}}$.

Thus, the function $\pi_i^a(a_{-i,}, \omega_{-i})$ punishes player i for deviating from a.

C.3 A Bound on the Probability of Matches

We repeatedly use the following exponential bound on the probability that a pair of players fails to match even once during a set of T periods:

Lemma 3 *For any set of T periods $\mathbb{T} \in \mathbb{N}^T$ and any pair of distinct players $i, j \in I$, we have* $\Pr(\mu_t(i) \neq j \ \forall t \in \mathbb{T}) \leq \exp(-\bar{\varepsilon}T)$.

Proof. $\Pr(\mu_t(i) \neq j \ \forall t \in \mathbb{T}) \leq (1 - \bar{\varepsilon})^T = \exp(T \log(1 - \bar{\varepsilon})) \leq \exp(-\bar{\varepsilon}T)$. ■

Given a set of periods $\mathbb{T}$, we say the realized matching process is *erroneous over* $\mathbb{T}$ if there exists a pair of players who do not match with each other during $\mathbb{T}$.

D Communication Protocols

A basic building block of the equilibrium strategy is a *communication protocol*: a strategy profile for players to communicate via actions in a finitely repeated game. The description of a communication protocol does not include payoff functions and thus entails no claims about incentive compatibility. After constructing the equilibrium strategy, we will construct a reward function and then verify sequential rationality.

We view each protocol as a distinct, finitely-repeated game. If $\mathbb{T}$ is the set of periods comprising a protocol, a *protocol history* for player i is a vector $h_i = (a_{i,t}, \omega_{i,t})_{t \in \mathbb{T}} \in H_i$. Denote the set of protocol history profiles by $H = \prod_{i \in I} H_i$.

D.1 Basic Communication Protocol

The basic protocol lets a player $i \in I$ broadcast a message m_i from a set $M_i = \{1, \ldots, |M_i|\}$. We call player i the *sender* and call the other players *receivers*. The protocol takes $2Tb(M_i)$ periods, where $b(M_i) := \lceil \log_2 |M_i| \rceil$.[17]

[17] We sometimes abusively write $b(|M_i|)$ for $b(M_i)$.

Basic Communication Protocol for Player i to Send Message m_i with Repetition T:[18]

- Divide the $2Tb(M_i)$ periods into $b(M_i)$ *intervals* of $2T$ periods each.

- For $t \in \{1, \ldots, b(M_i)\}$,

  - If the t^{th} digit of the binary expansion of $m_i - 1$ is 0, player i plays a^0 for the first half of the t^{th} interval (i.e., the first T periods in the interval) and plays a^1 for the second half of the t^{th} interval (i.e., the last T periods in the interval).

  - If the t^{th} digit of the binary expansion of $m_i - 1$ is 1, player i plays a^1 for the first half of the t^{th} interval and plays a^0 for the second half of the t^{th} interval.

  We call a set of T periods where player i takes a constant action a *half-interval*.

- Each player $j \neq i$ plays a^0 throughout the protocol.

- At the end of the protocol, each player $j \neq i$ makes an *inference* $m_i(j) \in M_i \cup \{0\}$ as follows (based on history $(a_{j,t}, \omega_{j,t})_{t=1}^{2Tb(M_i)}$). If $m_i(j) = 0$, we say j *fails to infer a message*:

  - If, for some $t \in \{1, \ldots, b(M_i)\}$, $\omega_{j,\tau} \notin \{a^0, a^1\}$ for some period τ in the t^{th} interval, player j sets $m_i(j) = 0$.

  - If, for some $t \in \{1, \ldots, b(M_i)\}$, $\omega_{j,\tau} \neq a^1$ for every period τ in the t^{th} interval, player j sets $m_i(j) = 0$.

  - If, for some $t \in \{1, \ldots, b(M_i)\}$, $\omega_{j,\tau} = \omega_{j,\tau'} = a^1$ for some period τ in the first half of the t^{th} interval and some τ' in the second half of the t^{th} interval, player j sets $m_i(j) = 0$.

  - Otherwise, player j constructs a number $\hat{m} \in \{0, \ldots, b(M_i) - 1\}$ as follows:

    * If $\omega_{j,\tau} = a^1$ for some period τ in the first half of the t^{th} interval and $\omega_{j,\tau} = a^0$ for every period τ in the second half of the t^{th} interval, player j sets the t^{th} digit of the binary expansion of $\hat{m}$ equal to 1.

[18] In what follows, instructions of the form "play action a in period t" are to be read as unconditional on a player's past actions and observations. Thus, a communication protocol is formally a strategy profile, not just a description of on-path play.

    * If $\omega_{j,\tau} = a^1$ for some period τ in the second half of the t^{th} interval and $\omega_{j,\tau} = a^0$ for every period τ in the first half of the t^{th} interval, player j sets the t^{th} digit of the binary expansion of $\hat{m}$ equal to 0.

  – If $\hat{m} \leq |M_i| - 1$, player j sets $m_i(j) = \hat{m} + 1$. If $\hat{m} \geq |M_i|$ (which is possible if $\log_2 |M_i|$ is not an integer), player j sets $m_i(j) = 0$.

When all players follow the protocol, $m_i(j) = m_i$ if and only if player j matches with player i at least once in every T-period half-interval where player i plays a^1. Hence, by Lemma 3,

$$\Pr(m_i(j) = m_i) \geq 1 - b(M_i) \exp(-\bar{\varepsilon}T) \ \ \forall j \neq i. \tag{9}$$

Moreover, when all players follow the protocol, either j's inference is correct or j fails to infer a message: if $m_i(j) \neq m_i$ then $m_i(j) = 0$.

D.2 Secure Communication Protocol

The secure protocol is a generalization of the basic protocol that lets player i send a message so that it is harder for any receiver to manipulate. In addition to the parameters (i, m_i, and T), the secure protocol takes as given a set of players $I_{\text{jam}} \subset I \setminus \{i\}$, called *jamming players*.

Secure Communication Protocol for Player i to Send Message m_i with Repetition T and Jamming Players I_{jam}:

- Divide the $2Tb(M_i)$ periods of the protocol into $b(M_i)$ intervals of $2T$ periods each.
- Player i behaves as in the basic communication protocol.
- Each player $j \notin I_{\text{jam}} \cup \{i\}$ behaves as in the basic communication protocol (i.e., plays a^0).
- For each player $j \in I_{\text{jam}}$, in the first period of each T-period half-interval (i.e., in periods $t = kT+1$ for $k \in \{0, 1, \ldots, 2b(M_i) - 1\}$), player j plays a^0 with probability $1 - \exp(-T^{\frac{1}{2}})$ and plays a^1 with probability $\exp(-T^{\frac{1}{2}})$. She then repeats the chosen action for the remainder of the half-interval (i.e., plays $a_{j,t} = a_{j,kT+1}$ for $t \in \{kT+2, ..., (k+1)T\}$).
- At the end of the protocol, each player $j \neq i$ infers a message $m_i(j) \in M_i \cup \{0\}$ as in the basic communication protocol.

For $j \in I_{\text{jam}}$ and $k \in \{0, 1, \ldots, 2b(M_i) - 1\}$, if $a_{j,kT+1} = a^0$ we say player j *plays REG* ("regular") in the k^{th} half-interval, and if $a_{j,(k-1)T+1} \neq a^0$ we say player j *plays JAM* ("jamming") in the k^{th} half-interval. Thus, player j plays REG and JAM with probabilities $1 - \exp(-T^{\frac{1}{2}})$ and $\exp(-T^{\frac{1}{2}})$ in each half-interval, independently across each half-interval.

Denote the event that all jamming players play REG throughout the protocol by ALLREG. Conditional on ALLREG, all players behave identically in the secure and basic protocols. In particular, conditional on ALLREG, inequality (9) holds and $m_i(j) \neq 0$ implies $m_i(j) = m_i \ \forall j \neq i$. Moreover,

$$\Pr{}^{\sigma^{m_i}}\left(m_i(j) = m_i \ \forall j \neq i \cap ALLREG\right) \geq 1 - Nb(M_i)\left(\exp\left(-\bar{\varepsilon}T\right) + 2\exp(-T^{\frac{1}{2}})\right). \quad (10)$$

The key new property of the secure protocol is that, for each player $j \neq i$ with $I_{\text{jam}} \backslash \{j\} \neq \emptyset$ and every sequence of observations $(\omega_{j,t})_{t=1}^{2Tb(M_i)}$, either she believes with high probability that communication was jammed, or she believes with probability that, conditional on the event that communication was not jammed, the message is likely to have transmitted successfully. Intuitively, the former case arises when player j observes a^1 frequently, and the latter case arises when she observes a^1 less frequently. To formalize this, let

$$\bar{\eta} := \max_{\gamma \in [0,1]} \min_{i,j,j'} \left\{ \gamma \log \frac{p_{i,j} + p_{j',j}}{p_{i,j}} + (1-\gamma) \log \frac{1 - p_{i,j} - p_{j',j}}{1 - p_{i,j}}, \bar{\varepsilon}(1-\gamma) \right\} > 0, \quad (11)$$

and let $\bar{\gamma}$ be the maximizer.

Lemma 4 *For any player $j \neq i$ with $I_{\text{jam}} \backslash \{j\} \neq \emptyset$ and any sequence of observations $(\omega_{j,t})_{t=1}^{2Tb(M_i)}$ that arises with positive probability when players $-j$ follow the secure protocol,*

1. *If $\omega_{j,t} = a^1$ for at least $\bar{\gamma}T$ periods in some half-interval then, for all $(a_{j,t})_{t=1}^{2Tb(M_i)}$, we have*

$$\Pr\left(ALLREG | \left(a_{j,t}, \omega_{j,t}\right)_{t=1}^{2Tb(M_i)}\right) \leq \exp\left(-\bar{\eta}T + T^{\frac{1}{2}}\right). \quad (12)$$

2. *If $\omega_{j,t} = a^1$ for at most $\bar{\gamma}T$ periods in each half-interval, then*

(a) *For all* $(a_{j,t})_{t=1}^{2Tb(M_i)}$, *we have*

$$\Pr\left(m_i\left(j'\right)\in\{m_i,0\}\;\;\forall j'\notin\{i,j\}\,|\,\left(a_{j,t},\omega_{j,t}\right)_{t=1}^{2Tb(M_i)},ALLREG\right)\geq 1-Nb(M_i)\exp\left(-\bar{\eta}T\right) \tag{13}$$

(b) *If* $a_{j,t}=a^0$ *for all* $t\in\{1,\ldots,2Tb(M_i)\}$, *we have*

$$\Pr\left(m_i\left(j'\right)=m_i\;\forall j'\notin\{i,j\}\,|\,\left(a_{j,t},\omega_{j,t}\right)_{t=1}^{2Tb(M_i)},ALLREG\right)\geq 1-Nb(M_i)\exp\left(-\bar{\eta}T\right) \tag{14}$$

Proof. Fix $j\neq i$ with $I_{\text{jam}}\setminus\{j\}\neq\emptyset$. Suppose there is an half-interval $\mathbb{S}$ in which $\omega_{j,t}=a^1$ for γ periods, with $\gamma\geq\bar{\gamma}T$. Fix a player $j'\in I_{\text{jam}}\setminus\{j\}$. Let j'JAMS denote the event that, in half-interval $\mathbb{S}$, player j' plays JAM and all other jamming players play REG. Let $\mathbb{S}$REG denote the event that all jamming players play REG in half-interval $\mathbb{S}$. Let $\left(a_{j,t},\omega_{j,t}\right)_{t\in\mathbb{S}}$ denote the restriction of $\left(a_{j,t},\omega_{j,t}\right)_{t=1}^{2Tb(M_i)}$ to half-interval $\mathbb{S}$. Then

$$\begin{aligned}\frac{\Pr\left(\left(a_{j,t},\omega_{j,t}\right)_{t\in\mathbb{S}}|j'JAMS\right)}{\Pr\left(\left(a_{j,t},\omega_{j,t}\right)_{t\in\mathbb{S}}|\mathbb{S}REG\right)} &= \left(\frac{p_{i,j}+p_{j',j}}{p_{i,j}}\right)^{\gamma}\left(\frac{1-p_{i,j}-p_{j',j}}{1-p_{i,j}}\right)^{T-\gamma}\\ &\geq \exp\left(\left(\bar{\gamma}\log\frac{p_{i,j}+p_{j',j}}{p_{i,j}}+(1-\bar{\gamma})\log\frac{1-p_{i,j}-p_{j',j}}{1-p_{i,j}}\right)T\right),\end{aligned}$$

which is no less than $\exp\left(\bar{\eta}T\right)$. Hence, by Bayes' rule,

$$\begin{aligned}\Pr\left(\mathbb{S}REG|\left(a_{j,t},\omega_{j,t}\right)_{t\in\mathbb{S}}\right) &= \left[1+\frac{\Pr\left(j'JAMS\right)\Pr\left(\left(a_{j,t},\omega_{j,t}\right)_{t\in\mathbb{S}}|j'JAMS\right)}{\Pr\left(\mathbb{S}REG\right)\Pr\left(\left(a_{j,t},\omega_{j,t}\right)_{t\in\mathbb{S}}|ALLREG\right)}\right]^{-1}\\ &\leq \left[1+\exp(-T^{\frac{1}{2}})\frac{\Pr\left(\left(a_{j,t},\omega_{j,t}\right)_{t\in\mathbb{S}}|j'JAMS\right)}{\Pr\left(\left(a_{j,t},\omega_{j,t}\right)_{t\in\mathbb{S}}|ALLREG\right)}\right]^{-1}\\ &\leq \left[1+\exp\left(\bar{\eta}T-T^{\frac{1}{2}}\right)\right]^{-1}\leq\exp\left(-\bar{\eta}T+T^{\frac{1}{2}}\right).\end{aligned}$$

Since the event that a jamming player plays JAM is independent across half-intervals and the behavior of players $-j$ is independent of their past actions and observations, we have

$$\Pr\left(ALLREG|\left(a_{j,t},\omega_{j,t}\right)_{t=1}^{2Tb(M_i)}\right)\leq\Pr\left(\mathbb{S}REG|\left(a_{j,t},\omega_{j,t}\right)_{t=1}^{2Tb(M_i)}\right)=\Pr\left(\mathbb{S}REG|\left(a_{j,t},\omega_{j,t}\right)_{t\in\mathbb{S}}\right).$$

Combining the inequalities yields (12).

Next suppose $\omega_{j,t} = a^1$ for at most $\bar{\gamma}T$ periods in every half-interval. Then, in each half-interval where player i plays a^1, player i matches with a player other than j in at least $(1-\bar{\gamma})\,T_0$ periods. Suppose player j plays a^0 throughout the protocol. For all $j' \notin \{i,j\}$, if player i matches with player j' at least once in each half-interval where player i plays a^1, and ALLREG occurs, then $m_i\,(j') = m_i$. Hence, by Lemma 3,

$$\Pr\left(m_i\,(j') = m_i |\left(a^0, \omega_{j,t}\right)_{t=1}^{2Tb(M_i)}, ALLREG\right) \geq 1 - b(M_i)\exp\left(-\bar{\varepsilon}\,(1-\bar{\gamma})\,T\right) \geq 1 - b(M_i)\exp\left(-\bar{\eta}T\right)$$

Applying this bound repeatedly for each $j' \neq i, j$, we obtain

$$\Pr\left(m_i\,(j') = m_i \;\forall j' \notin \{i,j\}\,|\left(a^0, \omega_{j,t}\right)_{t=1}^{2Tb(M_i)}, ALLREG\right) \geq 1 - Nb(M_i)\exp\left(-\bar{\eta}T\right).$$

This establishes (14). Similarly—regardless of player j's behavior—if player i matches with player $j' \neq i, j$ in some period in each half-interval where player i plays a^1, then $m_i\,(j') \in \{m_i, 0\}$. (In particular, $m_i\,(j') = 0$ if j ever matches with j' while playing $a_j \notin \{a^0, a^1\}$, or if i and j match with j' while playing a^1 in different halves of the same interval, and $m_i\,(j') = m_i$ otherwise.) Hence, (13) also holds. ■

D.3 Verified Communication Protocol

In the verified communication protocol, player i first broadcasts a message $m_i \in M_i$ in $2b\,(M_i)$ periods using the basic communication protocol (with $T = 1$). Then, each player (including player i herself) sequentially broadcasts her actions and observations from these $2b\,(M_i)$ periods using the secure communication protocol with repetition T (with T to be specified). The verified protocol thus takes a total of $\mathcal{T}\,(M_i, T)$ periods, where

$$\mathcal{T}\,(M_i, T) := 2b\,(M_i) + 2b\left(A^{4b(M_i)}\right)NT. \tag{15}$$

Verified Communication Protocol for Player i to Send Message m_i with Repetition T:

At the beginning of the verified protocol, each player j has two possible types, denoted

$\zeta_j \in \{\text{reg}, \text{jam}\}$. A strategy in the protocol is thus a mapping from $\{\text{reg}, \text{jam}\}$ and protocol histories to actions. Let $\mathcal{I}_{\text{jam}} = \{j : \zeta_j = \text{jam}\}$. The protocol consists of $N+1$ *rounds.*

- *Message round*
  - Player i sends message $m_i \in M_i$ as in the basic communication protocol with $T = 1$.[19]
  - Each player $j \neq i$ plays a^0 throughout the round.

  Let $\mathbb{T}(\text{msg})$ denote the set of $2b(M_i)$ periods comprising the message round.

- *j-checking round, for each $j \in I$.* Each checking round consists of $b\left(A^{4b(M_i)}\right)$ intervals. Each interval consists of $2T$ periods. Let $\mathbb{T}(j)$ denote the set of $2Tb\left(A^{4b(M_i)}\right)$ periods comprising the j-checking round.
  - Player j sends message $(a_{j,t}, \omega_{j,t})_{t \in \mathbb{T}(\text{msg})} \in A^{4b(M_i)}$ as in the basic protocol.
  - Each player $n \notin \mathcal{I}_{\text{jam}} \cup \{j\}$ plays a^0 throughout the round.
  - In each half-interval, each player $n \in \mathcal{I}_{\text{jam}} \setminus \{j\}$ mixes between REG and JAM with probabilities $1 - \exp(-T^{\frac{1}{2}})$ and $\exp(-T^{\frac{1}{2}})$, as in the secure protocol.
  - Each player $n \neq j$ infers message $(a_{j,t}(n), \omega_{j,t}(n))_{t \in \mathbb{T}(\text{msg})} \in A^{4b(M_i)} \cup \{0\}$ as in the basic protocol.

- At the end of the protocol, each player $n \in I$ creates a *final inference* $m_i(n) \in M_i \cup \{0\}$ as follows:
  - If $(a_{j,t}(n), \omega_{j,t}(n))_{t \in \mathbb{T}(\text{msg})} = 0$ for some $j \neq n$, then $m_i(n) = 0$.
  - Otherwise, if the vector $(a_{j,t}(n), \omega_{j,t}(n))_{t \in \mathbb{T}(\text{msg}), j \in I}$ is not feasible—that is, for some $j' \in I$ and $t \in \mathbb{T}(\text{msg})$, $(a_{j',t}(n), \omega_{j',t}(n)) \neq \varphi((a_{j,t}(n), \omega_{j,t}(n))_{j \neq j'})$ (see Lemma 2 for the definition of φ)—then $m_i(n) = 0$.
  - If $(a_{j,t}(n), \omega_{j,t}(n))_{t \in \mathbb{T}(\text{msg}), j \in I}$ is feasible and $(a_{i,t}(n))_{t \in \mathbb{T}(\text{msg})}$ corresponds to the binary expansion of some $\hat{m}_i \in M_i$, then $m_i(n) = \hat{m}_i$.

[19] To make following the verified communication protocol sequentially rational, we will subsequently slightly modify player i's prescribed behavior after she herself deviates from the protocol. See Section H.

– If $(a_{j,t}(n), \omega_{j,t}(n))_{t\in\mathbb{T}(\text{msg}),j\in I}$ is feasible but $(a_{i,t}(n))_{t\in\mathbb{T}(\text{msg})}$ does not correspond to the binary expansion of some $\hat{m}_i \in M_i$, then $m_i(n)$ is set equal to an arbitrary, pre-determined element of M_i—for concreteness, let $m_i(n) = 1$.

In the verified protocol, we call player i the *initial sender*, and we say player $j \in I$ is *a sender in period* t if $t \in \mathbb{T}(j)$ or $[j = i$ and $t \in \mathbb{T}(\text{msg})]$. We say *players coordinate on* m_i if $m_i(n) = m_i$ for all $n \in I$.

For each $j \in I$, say that player j is *suspicious* at protocol history h_j, denoted $\text{susp}(h_j) = 1$, if $m_i(j) = 0$. Otherwise, $\text{susp}(h_j) = 0$. Note that $\text{susp}(h_j) = 1$ only if some player deviates, some jamming player plays JAM, or the realized matching process is erroneous over some half-interval. We will derive some key properties of the function $\text{susp}(\cdot)$ in Section H.

D.4 Jamming Coordination Protocol

Finally, we describe how players coordinate on the identities of the jamming players $\mathcal{I}_{\text{jam}} \subset I$.

Jamming Coordination Protocol with Parameter T:

- In each of the two periods, each player i plays a^1 with probability $\exp(-T^{\frac{1}{3}})$ and plays each $a \neq a^1$ with probability $(1 - \exp(-T^{\frac{1}{3}}))/(|A| - 1)$, independently across periods.

Given a protocol history h_i, we define $\zeta_i(h_i) = \text{jam}$ if $\omega_{i,t} = a^1$ for some $t \in \{1, 2\}$. That is, a player becomes a jamming player if she observes a^1 in either period.

Let $P_i(h_i) = \Pr(\zeta_j(h_j) = \text{jam } \forall j \neq i | h_i)$. For every protocol history h_i, the probability that all players in $I \setminus \{i, \mu_t(i)\}$ play a^1 in both periods t and $\mu_1(i) \neq \mu_2(i)$ is at least $\bar{\varepsilon} \exp(-(N-2) T^{\frac{1}{3}})$. Conditional on this event, the probability that $\zeta_j(h_j) = \text{jam } \forall j \neq i$ is 1. Hence,

$$P_i(h_i) \geq \bar{\varepsilon} \exp(-(N-2) T^{\frac{1}{3}}). \tag{16}$$

E Equilibrium Strategies: Overview

We now define the equilibrium block strategies, deferring some details to Section J. The length of a block is parameterized by a number $T_0 \in \mathbb{N}$. We fix T_0 sufficiently large such that the following three inequalities hold:

$$\begin{cases} \frac{2+\bar{u}}{\bar{\varepsilon}\min\{\varepsilon^*,1\}} 300L^2N^4 |A| \log_2 T_0 \leq (T_0)^{\frac{1}{10}}, \\ (T_0)^4 \left(\exp(-(T_0)^{\frac{1}{6}}) + \exp\left(-\bar{\varepsilon}T_0 + 2(T_0)^{\frac{5}{6}}\right)\right) \leq 1, \\ (T_0)^4 \exp(-(T_0)^{\frac{1}{3}}) \leq \frac{\varepsilon^*}{2}. \end{cases} \tag{17}$$

Below, we give a precise description of how play proceeds within a block (and an intuitive description in parentheses).

1. *Sub-block* 0: This sub-block consists of the following $2+2N$ *phases.*

(a) *Jamming coordination phase* $(0, \text{jam})$: Players play the jamming coordination protocol for 2 periods. ("The players coordinate on who will be jamming players.")

(b) *Coordination phase* $(0, i)$ (repeat for each $i = 1, \ldots, N$): Player i sends $x_i \in \{G, B\}$ using the verified communication protocol with repetition T_0. Since the message set $M_i = \{G, B\}$ has cardinality 2, this phase takes $\mathcal{T}(M_i, T) = 2b(2) + 2b\left(A^{4b(2)}\right) NT_0 \approx 4 + 16NT_0 \log_2 |A|$ periods.[20] ("The players coordinate on x.")

(c) *Contagion phase* $(0, i, \text{con})$ (repeat for $i = 1, \ldots, N$): Player i sends $\text{susp}(h_i) \in \{0, 1\}$ using the verified protocol with repetition T_0. This phase also takes $\approx 4+16NT_0 \log_2 |A|$ periods. ("If any player is suspicious, her suspicion spreads.")

2. *Sub-block* $l = 1, ..., L$: This sub-block consists of the following $1+3N$ phases.

(a) *Main phase* (l, main): This phase takes $(T_0)^3$ periods, and is described in Section J. Roughly, if player i is not suspicious, she plays $a_i^l(x(i))$ in every period; otherwise, she plays $\alpha^{\min}$ in every period.

Let $\mathbb{T}(l, \text{main})$ denote the set of $(T_0)^3$ periods in main phase (l, main). At the end of the phase, each player i selects a period $t_i(l) \in \mathbb{T}(l, \text{main})$, uniformly at random. ("Each player selects a random period to monitor.")

[20]Throughout this section, we use $\approx$ to indicate equality up to rounding up all $\log_2$ terms: formally, we write $f(x) \approx g(\log_2 y_1, \ldots, \log_2 y_m)$ if $g(\log_2 y_1, \ldots, \log_2 y_m) \leq f(x) \leq g(\lceil \log_2 y_1 \rceil, \ldots, \lceil \log_2 y_m \rceil)$.

(b) *Communication phase* (l,i) (repeat for $i=1,...,N$): Player i sends $t_i(l) \in \mathbb{T}(l,\text{main})$ using the verified protocol with repetition T_0. Since the message set has cardinality $|\mathbb{T}(l,\text{main})| = (T_0)^3$, this phase takes $2b\left((T_0)^3\right) + 2b\left(A^{4b\left((T_0)^3\right)}\right) NT_0 \approx 6\log_2 T_0 + 24NT_0 \log_2 T_0 \log_2 |A|$ periods. ("Players communicate selected monitoring periods.")

(c) *Communication phase* (l,i,n) (repeat for $i=1,...,N$ and $n=1,...,N$): Player n sends $(a_{n,t}, \omega_{n,t})$ using the verified protocol with repetition T_0, where t equals player n's inference of $t_i(l)$ in phase (l,i). Since the message set has cardinality $|A|^2$, this phase takes $2b\left(|A|^2\right) + 2b\left(A^{4b\left(|A|^2\right)}\right) NT_0 \approx 4\log_2 |A| + 16NT_0 \left(\log_2 |A|\right)^2$ periods. ("Players share information about the monitoring periods.")

(d) *Contagion phase* (l,i,con) (repeat for $i=1,...,N$): A repetition of phase $(0,i,\text{con})$, but for the current histories h_i. Again, this phase takes $\approx 4 + 16NT_0 \log_2 |A|$ periods. ("Suspicion spreads.")

Let $\mathbb{L}_i = (t_i(l))_{l=1}^L$ be the collection of random monitoring periods selected by player i.

Let T^* be the final period of the last contagion phase, phase (L,N,con). Let $\mathbb{T}^* = \{1,...,T^*\}$ be the set of periods up to period T^*. Let $\mathbb{T}'$ be the set of non-main phase periods up to period T^*:

$$\mathbb{T}' = \mathbb{T}^* \setminus \bigcup_{l=1}^{L} \mathbb{T}(l,\text{main}). \tag{18}$$

Given that T_0 satisfies (17), it can be checked that $|\mathbb{T}'| \le (T_0)^{1.1}$.[21] (In what follows, all comparisons of numbers of periods involving T_0 assume (17).)

Let $\chi_n \in \{0,1\}$ be a function of $(x_n, h_n^{T^*})$, where $\chi_n = 1$ if and only if there exists $t \in \{1,...,T^*\}$ such that $a_{n,t} \notin \text{supp}(\sigma_n^*(x_n)|_{h_n^{t-1}})$ (i.e., player n deviated from $\sigma_n^*(x_n)$ in

[21] In particular,

$$\begin{aligned}
\mathbb{T}'(T_0) &= 2 + 2N\left(2b(2) + 2b\left(A^{4b(2)}\right)NT_0\right) + L\begin{pmatrix} N\left(2b\left((T_0)^3\right) + 2b\left(A^{4b\left((T_0)^3\right)}\right)NT_0\right) \\ +N^2\left(2b\left(|A|^2\right) + 2b\left(A^{4b\left(|A|^2\right)}\right)NT_0\right) \\ +N\left(2b(2) + 2b\left(A^{4b(2)}\right)NT_0\right) \end{pmatrix} \\
&= 2 + 8N + 64\lceil \log_2 |A| \rceil N^2 T_0 + 12LN\lceil \log_2 T_0 \rceil + 96\lceil \log_2 |A| \rceil LN^2 \lceil \log_2 T_0 \rceil T_0 \\
&\quad +8LN^2\lceil \log_2 |A| \rceil + 64\lceil \log_2 |A| \rceil^2 LN^3 T_0 + 4NL + 32\lceil \log_2 |A| \rceil LN^2 T_0 \\
&\le (T_0)^{1.1}.
\end{aligned}$$

Elsewhere in the proof, similar calculations show that (17) guarantees a sufficiently high value of T_0. We omit such calculations going forward.

the first T^* periods).

3. *Final Talk Sub-block* : This sub-block consists of the following $4N$ phases.

(a) *Phase* $(\text{final}, 1, i)$ (repeat for $i = 1, ..., N$): Player $i-1$ sends the list of periods $\mathbb{L}_{i-1} \in \{1, ..., (T_0)^3\}^L$ using the verified protocol with repetition T_0. Next, sequentially, each player $n \neq i, i-1$ sends the following two messages using the secure protocol with repetition T_0: (i) $\chi_n \in \{0, 1\}$ (i.e., player n "confesses" if she deviated in the first T^* periods). (ii) $(a_{n,t}, \omega_{n,t})_{t \in \mathbb{L}_i(n)}$, where $\mathbb{L}_i(n)$ is player n's inference of $\mathbb{L}_i$. (If $\mathbb{L}_i(n) = 0$ then player n sends $(a_{n,t}, \omega_{n,t}) = (a^0, a^0)$.) ("Players confess any deviations and re-send their information about the monitoring periods."[22]) Player i's message set has cardinality $(T_0)^{3L}$ and the message set of each player $n \neq i, i-1$ has cardinality $2A^{2L}$. Hence, the length of this phase is

$$\begin{aligned} T(\text{final}, 1) &= 2b\left((T_0)^{3L}\right) + 2b\left(A^{4b\left((T_0)^{3L}\right)}\right) NT_0 + (N-2)\, 2b\left(2\,|A|^{2L}\right) T_0 \\ &\approx 6L \log_2 T_0 + 24NT_0 L \log_2 T_0 \log_2 |A| + 2(N-2)\, LT_0 \left(1 + 2\log_2 |A|\right). \end{aligned}$$

Let T_1 be the final period of phase $(\text{final}, 1, N)$. Let $\mathbb{T}_1 = \{1, ..., T_1\}$. Let

$$\mathbb{T}'' = \mathbb{T}_1 \setminus \bigcup_{l=1}^{L} \mathbb{T}(l, \text{main}). \tag{19}$$

It can be checked that $|\mathbb{T}''| \leq (T_0)^{1.1}$. Let $\mathbb{T}(\text{final}, 1, i)$ be the set of periods in phase $(\text{final}, 1, i)$.

(b) *Phase* $(\text{final}, 2, i)$ (repeat for $i = 1, ..., N$): Sequentially, each player $n \neq i, i-1$ sends x_n and $(a_{n,t}, \omega_{n,t})_{t \in \mathbb{T}''}$ using the secure protocol with repetition T_0. ("Players share their non-main phase histories.") The length of this phase is $T(\text{final}, 2) = (N-2)\, 2b\left(2A^{2\mathbb{T}''}\right) T_0 \approx 2(N-2)\, T_0 \log_2\left(2A^{2\mathbb{T}''}\right)$. Let T_2 be the final period of phase $(\text{final}, 2, N)$. Let $\mathbb{T}_2 = \{1, ..., T_2\}$. It can be checked that $T_2 \leq L(T_0)^3 + (T_0)^{2.1}$. Let $\mathbb{T}(\text{final}, 2, i)$ be the set of periods in phase $(\text{final}, 2, i)$.

[22] Confessing deviations and re-sending past messages plays a similar role here as in Hörner and Olszewski (2006) and Yamamoto (2012).

(c) *Phase* $(\text{final}, 3, i)$ (repeat for $i = 1, ..., N$): Sequentially, each player $n \neq i, i-1$ sends $(a_{n,t}, \omega_{n,t})_{t \in \cup_{j \in I} \mathbb{T}(\text{final},2,j)}$ using the basic protocol with repetition T_0. ("Players share their information about each other's non-main phase histories.") The length of this phase is $T(\text{final}, 3) = (N-2)\, 2b\,(N \times T(\text{final}, 2))\, T_0 \approx 2\,(N-2)\, T_0 \log_2 (N \times T(\text{final}, 2))$. Let T_3 be the final period of phase $(\text{final}, 3, N)$. It can be checked that $T_3 \leq L\,(T_0)^3 + (T_0)^{2.1}$.

(d) *Phase* $(\text{final}, 4, i)$ (repeat for $i = 1, ..., N$): Player $i-1$ selects a period $t_{i-1} \in \{1, ..., T_3\}$, uniformly at random. Player $i-1$ sends the realization of t_{i-1} using the basic protocol with repetition T_0. Next, sequentially, each player $n \neq i-1, i$ sends her inference $t_{i-1}(n) \in \{0, 1, ..., T_3\}$ and $\left(a_{n,t_{i-1}(n)}, \omega_{n,t_{i-1}(n)}\right)$ using the basic protocol with repetition T_0. (If $t_{i-1}(n) = 0$ then n sends $\left(a_{n,t_{i-1}(n)}, \omega_{n,t_{i-1}(n)}\right) = (a^0, a^0)$). ("Each player monitors one extra period to cancel the effects of discounting.") The length of the phase is

$$
\begin{aligned}
T(\text{final}, 4) &= 2b\,(T_3)\, T_0 + (N-2)\, 2b\left((T_3+1) \times A^2\right) T_0 \\
&\approx 2T_0 \log_2 (T_3) + (N-2)\, 2T_0 \left(\log_2 (T_3+1) + 2\log_2 |A|\right).
\end{aligned}
$$

Finally, we have $T^{**} = T_3 + T(\text{final}, 4)$. It can be checked that $T^{**} \leq L\,(T_0)^3 + (T_0)^{2.1}$.

F Reduction Lemmas: Phases $(\text{final}, 3, i)$ and $(\text{final}, 4, i)$

F.1 Basic Communication Module

We analyze the equilibrium block strategies by backwards induction. Since the basic communication protocol is used in the last phases (phases $(\text{final}, 3, i)$ and $(\text{final}, 4, i)$), we start by considering payoffs and reward functions for this protocol. We call the resulting finitely repeated game the *basic communication module.*

For each player $n \in I$, payoff functions in the module take the form

$$
\sum_{t \in \mathbb{T}} \delta^{t-1} \hat{u}_n (\mathbf{a}_t) + \pi_n (x_{n-1}, h_{n-1}) + w_n (h), \tag{20}
$$

where $\hat{u}_n$ is the stage-game payoff function; π_n is a *reward function* that depends only on player $n-1$'s state and module history (where the state vector $(x_n)_{n \in I}$ is taken as fixed and commonly known); and w_n is a *continuation payoff function* that depends on the entire module history. We wish to construct a reward function such that, when viewed as a strategy profile in this finitely repeated game, the basic protocol is a belief-free equilibrium.

Definition 1 *A strategy profile σ is a belief-free equilibrium (BFE) if, for each player i and history h_i, the continuation strategy $\sigma_i|_{h_i}$ is a best response against $\sigma_{-i}|_{h_{-i}}$ for every opposing history profile h_{-i}.*

We say that *the premise for basic communication with magnitude K is satisfied* if the following conditions hold:

1. Player i is indifferent about the result of communication: $w_i(h) = 0$ for all h.

2. For all $n \neq i$, the range of $w_n(h)$ is bounded by K: $\max_{h,\tilde{h}} \left| w_n(h) - w_n(\tilde{h}) \right| \leq K$.

Lemma 5 *For each $i \in I$, x_{i-1}, M_i, T, w, and $K \geq 2\bar{u}/\bar{\varepsilon}$ satisfying the premise for basic communication with magnitude K, there exists a family of functions $\left(\pi_n(x_{i-1}, \cdot) : H_{n-1}^{\mathbb{T}} \to \mathbb{R}\right)_{n \in I}$ such that the following hold:*

1. *With payoff functions (20), the basic protocol is a BFE for every $\delta \in [0,1]$.*

2. *For each $n \in I$ and $m_i \in M_i$, $\mathbb{E}\left[\sum_{t \in \mathbb{T}} \delta^{t-1} \hat{u}_n(\mathbf{a}_t) + \pi_n(x_{n-1}, h_{n-1})\right] = T v_n(x_{n-1})$.*

3. *For each $n \in I$ and $t \in \mathbb{T}$,*

$$\max_{h_{n-1}, \tilde{h}_{n-1}} \left| \pi_n(x_{n-1}, h_{n-1}) - \pi_n\left(\tilde{h}_{n-1}\right) \right| \leq \left(\bar{u} + 2\frac{\bar{u} + K}{\bar{\varepsilon}}\right) T. \tag{21}$$

The proof is relegated to Section N (as are all other omitted proofs). Here is a sketch: For each receiver $n \neq i$, player $n-1$ rewards player n every time she observes a^0, which incentivizes player n to play a^0 throughout the module. Although whether player i (the sender) plays a^0 or a^1 also affects the probability that player $n-1$ observes a^0 in a given period (since i and $n-1$ may match), the expected number of rewards is independent of m_i because player i plays a^0 and a^1 with the same frequency for every m_i. In addition, whether player i plays a^0 in the first or second half-interval affects player n's instantaneous utility through discounting, so we must adjust the rewards to cancel this effect.

For player i, player $i-1$ makes her indifferent between playing a^0 and a^1 in every period. This is straightforward since player $i-1$'s observations statistically identify player i's actions.

Note that Lemma 5 concerns the complete information game where the states and continuation payoff functions $(x_n, w_n)_{n\in I}$ are known. However, as the statement of the lemma holds for each realization of $(x_n, w_n)_{n\in I}$, the same argument applies for the incomplete information game where $(x_n, w_n)_{n\in I}$ is unknown but the premise for communication is satisfied for each $(x_n, w_n)_{n\in I}$. The same remark applies for Lemmas 8, 13, and 17 introduced later.

F.2 Reduction Lemma 6: Undiscounted, Finitely Repeated Game

We show that the equilibrium conditions of Lemma 1 can be replaced by corresponding undiscounted conditions:

1. [Sequential Rationality] For all $x \in \{G, B\}^N$ and $h_i^{t-1} \in H_i^{t-1}$,

$$\sigma_i^* (x_i) \in \underset{\sigma_i \in \Sigma_i}{\operatorname{argmax}} \, \mathbb{E}^{\left(\left(\sigma_i, \sigma_{-i}^*(x_{-i})\right), \beta^*\right)} \left[\sum_{\tau=1}^{T_3} \hat{u}_i \left(\mathbf{a}_\tau\right) + \pi_i^*(x_{i-1}, h_{i-1}^{T_3}) | x_{-i}, h_i^{t-1} \right]. \tag{22}$$

2. [Promise Keeping] For all $x \in \{G, B\}^N$,

$$v_i(x_{i-1}) = \frac{1}{T_3} \mathbb{E}^{\sigma^*(x)} \left[\sum_{\tau=1}^{T_3} \hat{u}_i \left(\mathbf{a}\tau\right) + \pi_i^*(x_{i-1}, h_{i-1}^{T_3}) \right]. \tag{23}$$

3. [Self-Generation] For all $x_{i-1} \in \{G, B\}$ and $h_{i-1}^{T_3} \in H_{i-1}^{T_3}$,

$$\text{sign}(x_{i-1}) \pi_i^*(x_{i-1}, h_{i-1}^{T_3}) \geq -7\varepsilon^* T_3, \tag{24}$$

where, for $x_{i-1} \in \{G, B\}$, define $\text{sign}(x_{i-1}) := \begin{cases} -1 & \text{if } x_{i-1} = G, \\ 1 & \text{if } x_{i-1} = B. \end{cases}$

Note that Condition (4) is omitted, as $v_i(x_{i-1})$ is fixed to satisfy it by (5). The third inequality in (3) (which here would be $\left| \frac{1-\delta}{\delta^{T_3}} \pi_i^*(x_{i-1}, h_{i-1}^{T_3}) \right| \leq v_i(G) - v_i(B)$) is also omitted, as we have fixed T_3, $\pi_i^*(x_{i-1}, h_{i-1}^{T_3})$, and $v_i(G) > v_i(B)$ (by (5)) and will take $\delta \to 1$.

Lemma 6 *Suppose that, in the T_3-period finitely repeated game, there exist strategies $(\sigma_i^*(x_i))_{i,x_i}$, consistent ex post belief system β^*, and reward functions $\left(\pi_i^*(x_{i-1}, h_{i-1}^{T_3})\right)_{i,x_{i-1},h_{i-1}^{T_3}}$ such that Conditions (22)–(24) are satisfied. Then there exists $\bar{\delta} < 1$ such that $\mathbf{v} \in E(\delta)$ for all $\delta > \bar{\delta}$.*

The proof shows that, for any strategies $(\sigma_i^*(x_i))_{i,x_i}$ in the T_3-period game satisfying the conditions of the lemma, the T^{**}-period game that results from concatenating these strategies with the Phase $(\text{final}, 4, i)_{i\in I}$ strategies described in Section E (in which players share information about a random past period) satisfies the equilibrium conditions of Lemma 1. To prove this, we augment the reward functions from the T_3-period game by giving each player a small reward if the newly monitored period reveals that she took an action yielding a higher payoff later in the block, so as to leave her indifferent to the timing of her actions within the first T_3 periods. Condition (22) then ensures sequential rationality for the first T_3 periods. Moreover, as $\delta \to 1$, the size of the new reward goes to 0. Hence, Lemma 5 guarantees the existence of a reward function that incentivizes players to follow the basic communication protocol in the last $T^{**} - T_3$ periods. Finally, since $(T^{**} - T_3)/T_3$ is small, communication takes a short enough time that Conditions (23) and (24) imply Conditions (2) and (3), given the slack in (5).

F.3 Lemma 7: Letting Rewards Depend on h_{-i}

Next, consider phase $(\text{final}, 3, i)$, during which players $n \neq i, i-1$ send messages $(a_{n,t}, \omega_{n,t})_{t\in\cup_j \mathbb{T}(\text{final},2,j)}$ using the basic communication protocol. Player $i-1$ then uses her history in phase $(\text{final}, 3, i)$ to compute player i's reward for phase $(\text{final}, 2, j)_{j\in I}$ so that, at the end of phase $(\text{final}, 2, N)$, player i's expected reward is equal to

$$\sum_{j\neq i} \sum_{\mathbb{T}(\text{final},2,j)} \pi_i^{\text{cancel}}(x_{i-1}, a_{-i,t}, \omega_{-i,t}) + \sum_{t\in\mathbb{T}(\text{final},2,i)} \left(\pi_i^{\text{cancel}}(x_{i-1}, a_{-i,t}, \omega_{-i,t}) + \pi_i^{a^0}(a_{-i,t}, \omega_{-i,t})\right). \tag{25}$$

Given Conditions (7) and (8), player i's expected payoff in phases $((\text{final}, 2, j))_{j\in I}$ equals

$$\sum_{t\in\mathbb{T}(\text{final},2,i)} v_i(x_{i-1}) - \sum_{t\in\mathbb{T}(\text{final},2,i)} \mathbf{1}_{a_{i,t}\neq a^0}. \tag{26}$$

Note that player i has a strict incentive to play a^0 during phase $(\text{final}, 2, i)$. Based on this construction, we further reduce the conditions for Lemma 6:

1. [Sequential Rationality] For all $x \in \{G, B\}^N$ and $h_i^{t-1} \in H_i^{t-1}$,

$$\sigma_i^*(x_i) \in \underset{\sigma_i \in \Sigma_i}{\operatorname{argmax}} \mathbb{E}^{((\sigma_i, \sigma_{-i}^*(x_{-i})), \beta^*)} \left[\begin{array}{c} \sum_{\tau=1}^{T_1} \hat{u}_i(\mathbf{a}_\tau) + \sum_{t \in \mathbb{T}(\text{final},2,i)} v_i(x_{i-1}) - \sum_{t \in \mathbb{T}(\text{final},2,i)} \mathbf{1}_{a_{i,t} \neq a^0} \\ + \pi_i^*(x_{i-1}, h_{i-1}^{T_2}) | x_{-i}, h_i^{t-1} \end{array} \right]. \tag{27}$$

2. [Promise Keeping] For all $x \in \{G, B\}^N$,

$$v_i(x_{i-1}) = \frac{1}{T_2} \mathbb{E}^{\sigma^*(x)} \left[\sum_{\tau=1}^{T_1} \hat{u}_i(\mathbf{a}_\tau) + \sum_{t \in \mathbb{T}(\text{final},2,i)} v_i(x_{i-1}) - \sum_{t \in \mathbb{T}(\text{final},2,i)} \mathbf{1}_{a_{i,t} \neq a^0} + \pi_i^*(x_{i-1}, h_{i-1}^{T_2}) \right]. \tag{28}$$

3. [Self-Generation] For all $x_{i-1} \in \{G, B\}$ and $h_{i-1}^{T_2} \in H_{i-1}^{T_2}$,

$$\text{sign}(x_{i-1}) \pi_i^*(x_{i-1}, h_{i-1}^{T_2}) \geq -6\varepsilon^* T_2. \tag{29}$$

Note that the slack in the self-generation constraint has been reduced to $6\varepsilon^* T_2$, compared to $7\varepsilon^* T_3$ in Condition (24). This is because some slack is "used up" when replacing $\pi_i^*(x_{i-1}, h_{i-1}^{T_1})$ with (25) and $\pi_i^*\left(x_{-i}, h_{i-1}^{T_2}\right)$.

Lemma 7 *Suppose that, in the T_2-period finitely repeated game, there exist strategies $(\sigma_i^*(x_i))_{i,x_i}$ consistent ex post belief system β^*, and reward functions $\left(\pi_i^*(x_{i-1}, h_{i-1}^{T_2})\right)_{i, x_{i-1}, h_{i-1}^{T_2}}$ such that Conditions (27)–(29) are satisfied. Then there exists $\bar{\delta} < 1$ such that $\mathbf{v} \in E(\delta)$ for all $\delta > \bar{\delta}$.*

The proof shows that, for any strategies $(\sigma_i^*(x_i))_{i,x_i}$ in the T_2-period game satisfying the conditions of the lemma, the T_3-period game that results from concatenating these strategies with the Phase $(\text{final}, 3, i)_{i \in I}$ strategies described in Section E satisfies the equilibrium conditions of Lemma 6. Since the Phase $(\text{final}, 3, i)_{i \in I}$ strategies are used only to compute the rewards π_i^{cancel} and $\pi_i^{a^0}$, and these rewards are of order $\bar{u}$, Lemma 5 with K of order $\bar{u}$ guarantees the existence of a reward function that incentivizes players to follow the basic communication protocol in the last $T_3 - T_2$ periods.

G Reduction Lemma: Phase $(\text{final}, 2, i)$

G.1 Secure Communication Module

In phase $(\text{final}, 2, i)$, the secure protocol is used. We consider payoffs and reward functions for this protocol. The resulting finitely repeated game is the *secure communication module*.

We need only consider the case where I_{jam} is a singleton. Fix the sender i and another player i^* with $i \neq i^*, i^* - 1$. Let $I_{\text{jam}} = \{i^* - 1\}$. Intuitively, we consider a situation where player i must communicate a message m_i to player $i^* - 1$, but player i^* may gain if player $i^* - 1$ infers some $m_i' \neq m_i$, while other players are indifferent.

For each $n \in I$, payoff functions in the secure communication module are given by

$$-\mathbf{1}_{\{n=i^*\}} \sum_{t \in \mathbb{T}} \mathbf{1}_{\{a_{n,t} \neq a_0\}} + w_n(h), \tag{30}$$

for some function $w_n : H^{\mathbb{T}} \to \mathbb{R}$. Let $(\sigma_i^{m_i}, \sigma_{-i})_{m_i \in M_i}$ denote the strategy profile in the secure protocol. Note that only the sender's strategy depends on m_i. We will give conditions on $(w_n)_{n \in I}$ under which $(\sigma_i^{m_i}, \sigma_{-i})_{m_i \in M_i}$ is an "i^*-quasi-belief-free equilibrium" of the resulting finitely repeated game. Intuitively, this means that the strategy of each player $n \neq i^*$ is sequentially rational for every opposing history profile, and player i^*'s strategy is sequentially rational for some consistent belief system. In addition, sequential rationality for player i^* is imposed ex post with respect to m_i. This ensures that the module remains incentive compatible when viewed as one part of the infinitely repeated game.

Definition 2 *A family of strategy profiles $(\sigma_i^{m_i}, \sigma_{-i})_{m_i \in M_i}$ is an* i^*-quasi-belief-free equilibrium *(*i^*-QBFE*) if (i) for each player $n \neq i^*$ and history h_n, the continuation strategy $\sigma_n|_{h_n}$ is a best response against $\sigma_{-n}|_{h_{-n}}$ for every opposing history profile h_{-n} and every possible message m_i, and (ii) for player i^*, there exists a sequence of families of completely mixed strategy profiles $\left((\sigma_i^{m_i,k}, \sigma_{-i}^k)_{m_i \in M_i}\right)_{k=1}^{\infty}$ and a corresponding family of belief systems $\beta(h_{-i^*}|m_i, h_{i^*})$ (where $\beta(h_{-i^*}|m_i, h_{i^*})$ is the limit of conditional probabilities derived from*

$\left(\left(\sigma_i^{m_i,k}, \sigma_{-i}^k\right)\right)_{k=1}^{\infty}$) *such that, for each* m_i *and* $h_{i^*}^{t-1}$,

$$\sigma_{i^*} \in \underset{\tilde{\sigma}_{i^*} \in \Sigma_{i^*}}{\operatorname{argmax}} - \sum_{t \in \mathbb{T}} \mathbf{1}_{\{a_{i^*,t} \neq a^0\}} + \mathbb{E}^{\left(\tilde{\sigma}_{i^*}, \sigma_{-i^*}^{m_i}\right)} \left[w_{i^*}(h) \,|m_i, h_{i^*}^{t-1}\right],$$

where the expectation is taken with respect to $\beta(h_{-i^*}^{t-1}|m_i, h_{i^*}^{t-1})$.

We say that *the premise for secure communication for player* i^* *with magnitude* K *is satisfied* if the following conditions hold:

1. All players but player i^* are indifferent about the result of communication: $w_n(h) = 0$ for all h and $n \neq i^*$.

2. If player $i^* - 1$ deviates from σ_{i^*-1} or $ALLREG$ does not occur,[23] then $w_{i^*}(h) = 0$ for all h.

3. If player $i^* - 1$ follows σ_{i^*-1} and $ALLREG$ occurs, then the following conditions hold:

(a) If $m_i(i^* - 1) \in M_i \cup \{0\}$ is the same at protocol histories h and $\tilde{h}$, then $w_{i^*}(h) = w_{i^*}(\tilde{h})$. Under this condition, we abuse notation and write $w_{i^*}(h) = w_{i^*}(m_i(i^* - 1))$.

(b) The range of $w_{i^*}(m_i(i^* - 1))$ is bounded by K:

$$\max_{m_i, \tilde{m}_i \in M_i \cup \{0\}} |w_{i^*}(m_i) - w_{i^*}(\tilde{m}_i)| \leq K. \tag{31}$$

(c) $w_{i^*}(0) \leq w_{i^*}(m_i(i^* - 1))$ for all $m_i(i^* - 1) \in M_i$.

We now specify player i^*'s beliefs. In particular, we specify that, after any off-path observation, she assigns probability 1 to the event that player $i^* - 1$ deviated (and hence, if the above premise holds, $w_{i^*}(h) = 0$). This belief is clearly consistent: for concreteness, define $((\sigma_i^{m_i,k}, \sigma_{-i}^k)_{m_i \in M_i})_{k=1}^{\infty}$ by letting player $i^* - 1$ tremble uniformly over all actions with probability k^{-1} at each history, and letting every other player tremble uniformly over all actions with probability k^{-k} at each history.

[23] Player $i^* - 1$ *follows* σ_{i^*-1} if, for each τ, her action $a_{i^*-1,\tau}$ is in the support of σ_{i^*-1} given $(a_{i^*-1,t}, \omega_{i^*-1,t})_{t \leq \tau-1}$. Since $i^* - 1 \neq i$, the support is independent of m_i. Player $i^* - 1$ *deviates from* σ_{i^*-1} if she does not follow σ_{i^*-1}.

Lemma 8 *For each $i^* \in I$, $i \in I \setminus \{i^*-1, i^*\}$, M_i, w, and K satisfying the premise for secure communication for player i^* with magnitude K, if*

$$b(M_i)K \exp\left(-\bar{\eta}T + T^{\frac{1}{2}}\right) \leq 1, \tag{32}$$

then with payoff functions (30) the secure communication protocol, together with the above belief system for player i, is an i^*-QBFE.

Proof. By construction, players other than i^* are indifferent over all actions throughout the module. For player i^*, fix a period $t \in \mathbb{T}$ and history $(a_{i^*,\tau}, \omega_{i^*,\tau})_{\tau \in \mathbb{T}, \tau \leq t-1}$. Suppose $\omega_{i^*,\tau} \in \{a^0, a^1\}$ for each $\tau \leq t-1$. By the same argument as for Lemma 4, for every possible continuation history $(a_{i^*,\tau}, \omega_{i^*,\tau})_{\tau \in \mathbb{T}, \tau \geq t}$, with probability at least

$$1 - b(M_i) \exp\left(-\bar{\eta}T + T^{\frac{1}{2}}\right) \tag{33}$$

conditional on $(a_{i^*,\tau}, \omega_{i^*,\tau})_{\tau \in \mathbb{T}}$, either *ALLREG* does not occur or $[m_i(i^*-1) \in \{m_i, 0\}$, and $m_i(i^*-1) = m_i$ if $a_{i^*,\tau} = a^0$ for all $\tau \in \mathbb{T}]$. Moreover, if $(\omega_{i^*,\tau})_{\tau \in \mathbb{T}}$ is such that $[m_i(i^*-1) \in \{m_i, 0\}$, and $m_i(i^*-1) = m_i$ if $a_{i^*,\tau} = a^0$ for all $\tau \in \mathbb{T}]$, then by definition of $m_i(i^*-1)$, we have $m_i(i^*-1) = m_i$ if and only if player i^* takes a^0 whenever she meets player i^*-1 in a half-interval where player i takes a^0. Hence, since $w_{i^*}(0) \leq w_{i^*}(m_i(i^*-1))$ for all $m_i(i^*-1) \in M_i$, taking $a_{i^*,\tau} = a^0$ for each $\tau \geq t$ maximizes $w_{i^*}(h)$ with probability at least (33). Given this, conditions (31) and (32) imply that the reward term $-\mathbf{1}_{\{a_{i^*,t} \neq a^0\}}$ in payoff (30) outweighs any possible benefit to player i^* from playing $a \neq a^0$ in an attempt to manipulate $m_i(i^*-1)$. If instead $\omega_{i^*,\tau} \notin \{a^0, a^1\}$ for some $\tau \leq t-1$, then by construction of the belief system player i^* believes $w_{i^*}(h) = 0$ with probability 1. Hence, player i^* maximizes the reward term $-\mathbf{1}_{\{a_{i^*,\tau} \neq a^0\}}$ in payoff (30), so playing a^0 as prescribed is optimal. ■

G.2 Reduction Lemma 9: Letting Rewards Depend on Other Players' Non-Main Phase Histories

We now use phases $((\text{final}, 2, n))_{n \in I}$ to further simplify equilibrium conditions. Player $i-1$ uses the result of this communication to construct the reward function so that the expected reward at the end of phase $(\text{final}, 1, N)$ is the same as if player $i-1$ knew the histories of players $-(i-1, i)$ for all non-main phase periods. We write the reward function as $\pi_i\left(x_{-i}, h_{i-1}^{T^*}, h_{-i}^{\mathbb{T}''}\right)$, where $\mathbb{T}''$ is the set of non-main phase periods, from (19). We wish to replace $\pi_i^*(x_{i-1}, h_{i-1}^{T_2})$ with $\pi_i^*\left(x_{-i}, h_{i-1}^{T^*}, h_{-i}^{\mathbb{T}''}\right)$ in Conditions (27)–(29), yielding the following:

1. [Range Restriction] The range of the reward function is bounded by $8\bar{u}T_1$:

$$\sup_{x_{-i}, h_{i-1}^{T^*}, h_{-i}^{\mathbb{T}''}} \left|\pi_i^*\left(x_{-i}, h_{i-1}^{T^*}, h_{-i}^{\mathbb{T}''}\right)\right| \leq 8\bar{u}T_1. \tag{34}$$

2. [Sequential Rationality] For all $x \in \{G, B\}^N$ and $h_i^{t-1} \in H_i^{t-1}$,

$$\sigma_i^*(x_i) \in \underset{\sigma_i \in \Sigma_i}{\operatorname{argmax}}\, \mathbb{E}^{\left(\left(\sigma_i, \sigma_{-i}^*(x_{-i})\right), \beta^*\right)}\left[\sum_{\tau=1}^{T_1} \hat{u}_i(\mathbf{a}_\tau) + \pi_i^*\left(x_{-i}, h_{i-1}^{T^*}, h_{-i}^{\mathbb{T}''}\right) | h_i^{t-1}\right]. \tag{35}$$

3. [Promise Keeping] For all $x \in \{G, B\}^N$,

$$v_i(x_{i-1}) = \frac{1}{T_1}\mathbb{E}^{\sigma^*(x)}\left[\sum_{t=1}^{T_1} \hat{u}_i(\mathbf{a}_t) + \pi_i^*\left(x_{-i}, h_{i-1}^{T^*}, h_{-i}^{\mathbb{T}''}\right)\right]. \tag{36}$$

4. [Self-Generation] For all x_{-i}, $h_{i-1}^{T^*}$, and $h_{-i}^{\mathbb{T}''}$,

$$\text{sign}(x_{i-1})\pi_i^*\left(x_{-i}, h_{i-1}^{T^*}, h_{-i}^{\mathbb{T}''}\right) \geq -5\varepsilon^* T_1. \tag{37}$$

Lemma 9 *Suppose that, in the T_1-period finitely repeated game, there exist strategies $(\sigma_i^*(x_i))_{i,x_i}$, consistent ex post belief system β^*, and reward functions $\left(\pi_i^*\left(x_{-i}, h_{i-1}^{T^*}, h_{-i}^{\mathbb{T}''}\right)\right)_{i, x_{-i}, h_{i-1}^{T^*}, h_{-i}^{\mathbb{T}''}}$ such that Conditions (34)–(37) are satisfied. Then there exists $\bar{\delta} < 1$ such that $\mathbf{v} \in E(\delta)$ for all $\delta > \bar{\delta}$.*

H Verified Communication Module

In phase $(\text{final}, 1, i)$ and earlier communication phases, the verified communication protocol is used. We now establish some key properties of this protocol, and then augment it with payoffs and reward functions. The resulting *verified communication module* is the most complicated of our modules.

Let $\sigma^{*,m_i} = \left(\sigma_i^{*,m_i}, \sigma_{-i}^*\right)$ denote the prescribed protocol strategy profile when player i sends message m_i. For each $j, j' \in I$, player j's equilibrium strategy in the j'-checking round is determined by $(a_{j,t}, \omega_{j,t})_{t\in\mathbb{T}(\text{msg})}$ and $\zeta_j \in \{\text{reg}, \text{jam}\}$ (independently of m_i). We say player j *follows* σ_j^* *in the* j'*-checking round* if, for each $\tau \in \mathbb{T}(j')$, her action $a_{j,\tau}$ is in the support of σ_j^* given $(a_{j,t}, \omega_{j,t})_{t\in\mathbb{T}(\text{msg})}$, $\zeta_j \in \{\text{reg}, \text{jam}\}$, and $(a_{j,t}, \omega_{j,t})_{t\in\mathbb{T}(j'),t\leq\tau-1}$. Let $H^{<j'}$ denote the set of protocol history profiles at the beginning of $\mathbb{T}(j')$ that arise with positive probability under some strategy profile σ. Given $h^{<j'} \in H^{<j'}$, let $H_j^{\mathbb{T}(j')}|_{h^{<j'}}$ denote the set of protocol history profiles during $\mathbb{T}(j')$ that are reached from $h^{<j'}$ with positive probability under some strategy profile $(\sigma_j, \sigma_{-j}^*)$ with $\sigma_j \in \Sigma_j$ (i.e., when players $-j$ follow the protocol).

H.1 Regular and Erroneous Opponents' Histories

We classify each of player j's opponents' history profiles as *regular* or *erroneous*, $\theta_j\left(h_{-j}, \zeta\right) \in \{R, E\}$. Roughly, a profile of player j's opponents' histories h_{-j} is "erroneous" if it arises whenever some jamming player plays JAM or the realized matching process is erroneous.

This classification—which will affect player j's reward function—depends on players $-j$'s protocol history h_{-j} and the type profile $\zeta = (\zeta_n)_{n\in I}$. By Lemma 9, player j's reward function can depend on her opponents' non-main phase histories. As verified communication protocol histories and jamming coordination protocol histories (which will determine ζ) are non-main phase histories, player j's reward function can depend on h_{-j} and ζ.

For $j, j' \in I$, we first define $\theta_j\left(h_{-j}, \zeta, j'\right) = E$ ("j's opponents' histories in the j'-checking round are erroneous") if and only if one or more of the following four conditions holds:

1. $\zeta_j = \text{jam}$.

2. There exists $n \in \mathcal{I}_{\text{jam}} \setminus \{j, j'\}$ who plays JAM in some half-interval in $\mathbb{T}\left(j'\right)$.

3. *[Condition FAIL]* $j \neq j'$ and there exist a half-interval $\mathbb{S}$ in $\mathbb{T}\left(j'\right)$ and a player $n \neq j'$ such that player j' plays a^1 throughout $\mathbb{S}$ but $\omega_{n,t} = a^0$ for all $t \in \mathbb{S}$. (Whether this event occurs is determined by h_{-j}, as Lemma 2 implies that h_j is uniquely determined by h_{-j}.)

4. *[Condition FAILj']* $j = j'$, player j' follows $\sigma^*_{j'}$ in the j'-checking round, and there exist a half-interval $\mathbb{S}$ in $\mathbb{T}(j')$ and a player $n \neq j'$ such that player j' plays a^1 throughout $\mathbb{S}$ but $\omega_{n,t} = a^0$ for all $t \in \mathbb{S}$. (Again, this event is determined by h_{-j}, by Lemma 2.)

(Note that $\theta_j(h_{-j}, \zeta, j')$ depends on h_{-j} only through $h_{-j}^{\mathbb{T}(j')}$ and $h_{-j}^{\mathbb{T}(\text{msg})}$, the latter because whether player j' follows $\sigma^*_{j'}$ in the j'-checking round (in [Condition FAILj']) depends on $(a_{j',t}, \omega_{j',t})_{t\in\mathbb{T}(\text{msg})}$.)

We define $\theta_j(h_{-j}, \zeta) = E$ if and only if either $\theta_j(h_{-j}, \zeta, j') = E$ for some $j' \in I$ or some player $j' \neq j$ deviates from $\sigma^*_{j'}$ in any checking round. Otherwise, define $\theta_j(h_{-j}, \zeta) = R$. In addition, for each $j' \in I$, let $JAM_{j',-j}$ denote the event that there exists $n \in \mathcal{I}_{\text{jam}} \setminus \{j, j'\}$ who plays JAM in some half-interval in $\mathbb{T}(j')$. Let $REG_{j',-j}$ denote the complementary event.

Lemma 10 *For each player $j \in I$, each type profile $\zeta \in \{\text{reg}, \text{jam}\}^N$, and each history profile $h^{<j'} \in H^{<j'}$,*

1. *If all players follow σ^* in the j'-checking round, then $\Pr\left(\theta_j(h_{-j}, \zeta, j') = E | h^{<j'}, \zeta\right)$ is the same for every $h^{<j'} \in H^{<j'}$.*

2. $\sigma^*_{j'} \in \operatorname{argmax}_{\sigma_{j'} \in \Sigma^{\mathbb{T}}_{j'}} \Pr^{\left(\sigma_{j'}, \sigma^*_{-j'}\right)}\left(\theta_{j'}(h_{-j'}, \zeta, j') = E | \zeta, h^{<j'}\right)$.

3. *If all players follow σ^* in the j'-checking round and $(a_{j',t}(n), \omega_{j',t}(n))_{t\in\mathbb{T}(\text{msg})} \neq (a_{j',t}, \omega_{j',t})_{t\in\mathbb{T}(\text{msg})}$ for some $n \in I$, then $(a_{j',t}(n), \omega_{j',t}(n))_{t\in\mathbb{T}(\text{msg})} = 0$ and $\theta_j(h_{-j}, \zeta, j') = E$.*

4. *If player j' follows $\sigma^*_{j'}$ in the j'-checking round, $(a_{j',t}(n), \omega_{j',t}(n))_{t\in\mathbb{T}(\text{msg})} \neq (a_{j',t}, \omega_{j',t})_{t\in\mathbb{T}(\text{msg})}$ for some $n \in I$, and $\theta_j(h_{-j}, \zeta, j') = R$, then $(a_{j',t}(n), \omega_{j',t}(n))_{t\in\mathbb{T}(\text{msg})} = 0$.*

5. *If $j \neq j'$, players $-j$ follow σ^*_{-j} in the j'-checking round, and $(a_{j',t}(j), \omega_{j',t}(j))_{t\in\mathbb{T}(\text{msg})} \neq (a_{j',t}, \omega_{j',t})_{t\in\mathbb{T}(\text{msg})}$, then $\theta_j(h_{-j}, \zeta, j') = E$.*

Proof.

1. For any message $(a_{j',t}, \omega_{j',t})_{t\in\mathbb{T}(\text{msg})}$, player j' plays a^1 the same number of times in each interval. Hence, the probability that FAIL (or FAILj') holds is independent of $(a_{j',t}, \omega_{j',t})_{t\in\mathbb{T}(\text{msg})}$.

2. If player j' deviates from $\sigma^*_{j'}$ then FAILj' does not hold. Moreover, Conditions 1 and 2 for $\theta_j(h_{-j}, \zeta, j') = E$ are independent of σ_j, and FAIL only applies when $j \neq j'$. Hence, the conclusion holds.

3. If $j \in \mathcal{I}_{\text{jam}}$ or a player in $\mathcal{I}_{\text{jam}} \setminus \{j, j'\}$ plays JAM in some half-interval, then $\theta_j(h_{-j}, \zeta, j') = E$ by construction. If $j \notin \mathcal{I}_{\text{jam}}$ and all players $\mathcal{I}_{\text{jam}} \setminus \{j, j'\}$ play REG in every half-interval, then $(a_{j',t}(n), \omega_{j',t}(n))_{t\in\mathbb{T}(\text{msg})} \neq (a_{j',t}, \omega_{j',t})_{t\in\mathbb{T}(\text{msg})}$ only if player n does not observe a^1 in some half-interval where player j' plays a^1. Hence, $(a_{j',t}(n), \omega_{j',t}(n))_{t\in\mathbb{T}(\text{msg})} = 0$ and FAIL or FAILj' holds.

4. If $\theta_j(h_{-j}, \zeta, j') = R$ then each $n \neq j'$ observes a^1 in each half-interval where player j' plays a^1. So, $(a_{j',t}(n), \omega_{j',t}(n))_{t\in\mathbb{T}(\text{msg})} \neq (a_{j',t}, \omega_{j',t})_{t\in\mathbb{T}(\text{msg})}$ implies $(a_{j',t}(n), \omega_{j',t}(n))_{t\in\mathbb{T}(\text{msg})} = 0$.

5. When players $-j$ follow σ^*_{-j}, $(a_{j',t}(j), \omega_{j',t}(j))_{t\in\mathbb{T}(\text{msg})} \neq (a_{j',t}, \omega_{j',t})_{t\in\mathbb{T}(\text{msg})}$ only if player j does not observe a^1 in some half-interval where player j' plays a^1. Hence, FAIL holds.

■

H.2 Statistical Properties of the Verified Protocol

Lemma 11 *Suppose that*

$$2N(N-1)\, b(A^{4b(M_i)}) \exp(-T^{\frac{1}{2}}) + N(N-1)\, b(A^{4b(M_i)}) \exp(-\bar{\varepsilon}T) \leq \exp(-T^{\frac{1}{3}}). \tag{38}$$

Then the following claims hold for every $m_i \in M_i$ and every type profile $\zeta \in \{\text{reg}, \text{jam}\}^N$:

1. *For any $j \neq i$ and any $\sigma_j \in \Sigma_j^{\mathbb{T}}$, given strategy profile $(\sigma_j, \sigma_{-j}^{*,m_i})$, either (i) $m_i(n) = m_i$ for all $n \in I$, (ii) $\text{susp}(h_n) = 1$ for some $n \neq j$, or (iii) $\theta_j(h_{-j}, \zeta) = E$. Moreover, $\text{susp}(h_j) = 1$ implies $\theta_j(h_{-j}, \zeta) = E$.*

2. *For any $\sigma_i \in \Sigma_i^{\mathbb{T}}$, given $(\sigma_i, \sigma^*_{-i})$, either (i) there exists $\hat{m}_i \in M_i$ with $m_i(n) = \hat{m}_i$ for all $n \in I$, (ii) $\text{susp}(h_n) = 1$ for some $n \neq i$, or (iii) $\theta_i(h_{-i}, \zeta) = E$. Moreover, $\text{susp}(h_i) = 1$ implies $\theta_i(h_{-i}, \zeta) = E$.*

3. *Given σ^{*,m_i}, for any $j \in I$, either (i) $m_i(n) = m_i$ and* $\text{susp}(h_n) = 0$ *for all $n \in I$, or (ii) $\theta_j(h_{-j}, \zeta) = E$.*

4. *Given σ^{*,m_i}, with probability at least $1 - \exp(-T^{\frac{1}{3}})$, all the following events occur: (i) $m_i(n) = m_i$ for all $n \in I$, (ii)* $\text{susp}(h_n) = 0$ *for all $n \in I$, and (iii) $\theta_n(h_{-n}, \zeta) = R$ for all $n \notin \mathcal{I}_{\text{jam}}$.*

5. *For any $m_i, m_i' \in M_i$ and $j \in I$, $\Pr^{\sigma^{*,m_i}}(\theta_j(h_{-j}, \zeta) = R|\zeta) = \Pr^{\sigma^{*,m_i'}}(\theta_j(h_{-j}, \zeta) = R|\zeta)$.*

The intuition is that $\theta_j(h_{-j}, \zeta) = E$ only if some player plays JAM or matching is erroneous, which is unlikely. Moreover, since the sender plays a^1 with the same frequency for all m_i, the probability of this event is independent of m_i.

The next lemma is analogous to Lemma 4. Unlike Lemmas 10 and 11, this lemma involves conditions on players' beliefs about the type profile $(\zeta_n)_{n \in I} \in \{\text{reg}, \text{jam}\}^N$. To express these conditions, we assume each player n has a prior probability distribution over $(\zeta_n)_{n \in I}$ at the beginning of the protocol. Let $\Pr_n(\cdot|\cdot)$ denote conditional probability under player n's prior.

Lemma 12 *Fix any $j \in I$, $j' \neq j$, and $h^{<j'} \in H^{<j'}$. Suppose that, for all $h_j^{\mathbb{T}(j')} \in H_j^{\mathbb{T}(j')}|_{h^{<j'}}$, we have $\Pr_j\left(\zeta_{j'} = \text{jam } \forall j' \neq j | m_i, h^{<j'}, h_j^{\mathbb{T}(j')}\right) \geq \exp(-T^{\frac{1}{2}})$. Then, for all $h_j^{\mathbb{T}(j')} \in H_j^{\mathbb{T}(j')}|_{h^{<j'}}$, at least one of the following two conditions holds:*

1. *We have*

$$\Pr_j\left(JAM_{j',-j} | m_i, h^{<j'}, h_j^{\mathbb{T}(j')}\right) \geq 1 - \exp\left(-\bar{\eta}T + 2T^{\frac{1}{2}}\right). \tag{39}$$

2. *The following two conditions hold:*

(a) For all $(a_{j,t})_{t \in \mathbb{T}(j')}$,

$$\begin{aligned} &\Pr_j\left(\begin{array}{c} (a_{j',t}(n), \omega_{j',t}(n))_{t \in \mathbb{T}(\text{msg})} \in \left\{0, (a_{j',t}, \omega_{j',t})_{t \in \mathbb{T}(\text{msg})}\right\} \ \forall n \neq j \\ | m_i, h^{<j'}, h_j^{\mathbb{T}(j')}, REG_{j',-j} \end{array}\right) \\ \geq \ & 1 - Nb(|A|^{4b(M_i)}) \exp\left(-\bar{\eta}T + 2T^{\frac{1}{2}}\right). \end{aligned} \tag{40}$$

(b) If $a_{j,t} = a^0$ for all $t \in \mathbb{T}(j')$, then

$$\begin{aligned} &\Pr_j \left(\begin{array}{c} (a_{j',t}(n), \omega_{j',t}(n))_{t \in \mathbb{T}(\mathrm{msg})} = (a_{j',t}, \omega_{j',t})_{t \in \mathbb{T}(\mathrm{msg})} \ \forall n \neq j \\ | m_i, h^{<j'}, h_j^{\mathbb{T}(j')}, REG_{j',-j} \end{array} \right) \\ \geq \ & 1 - Nb(|A|^{4b(M_i)}) \exp\left(-\bar{\eta} T + 2T^{\frac{1}{2}} \right). \end{aligned} \tag{41}$$

Proof. The same as Lemma 4, except that $2T^{\frac{1}{2}}$ replaces $T^{\frac{1}{2}}$ in the inequality (12), as now $\mathcal{I}_{\mathrm{jam}} \backslash \{j\}$ is non-empty with probability at least $\exp(-T^{\frac{1}{2}})$ rather than 1. ■

H.3 Payoffs and Incentives

Throughout this subsubsection, fix $m_i^* \in M_i$ and let $\sigma^* = \sigma^{*,m_i}$.

For each $j \in I$ and $t \in \mathbb{T}(j)$, given $(a_{j,t}, \omega_{j,t})_{t \in \mathbb{T}(\mathrm{msg})}$ identified from h_{-j} by Lemma 2, calculate the equilibrium action $a_{j,t}^*(h_{-j})$. Suppose each player j's payoff equals

$$-\mathbf{1}_{\{\zeta_j = \mathrm{reg}\}} \sum_{t \in \mathbb{T} \backslash \mathbb{T}(j)} \mathbf{1}_{\{a_{j,t} \neq a^0\}} - \sum_{t \in \mathbb{T}(j)} \mathbf{1}_{\left\{a_{j,t} \neq a_{j,t}^*(h_{-j})\right\}} + w_j(h, \zeta). \tag{42}$$

(This is similar to (30), but now player j is rewarded for following the equilibrium strategy $a_{j,t}^*(h_{-j})$ in round $\mathbb{T}(j)$.)

We say that *the premise for verified communication to send message $m_i^* \in M_i$ with magnitude K is satisfied* if there exist $\left(v_j^E\right)_{j \in I} \in \mathbb{R}^N$, and $\left(v_j^{m_i}\right)_{j \in I, m_i \in M_i \cup \{0\}} \in \mathbb{R}^N$ such that, for all $j \in I$ and $h \in H$, the following conditions hold:

1. If $\theta_j(h_{-j}, \zeta) = E$, then $w_j(h, \zeta) = v_j^E$.
2. If $\theta_j(h_{-j}, \zeta) = R$ and $\mathrm{susp}(h_n) = 1$ for some $n \neq j$, then $w_j(h, \zeta) = v_j^0$.
3. If $\theta_j(h_{-j}, \zeta) = R$, $\mathrm{susp}(h_n) = 0$ for all $n \neq j$, and $\exists \hat{m}_i \in M_i$ such that $m_i(n) = \hat{m}_i$ for all $n \in I$, then $w_j(h, \zeta) = v_j^{\hat{m}_i}$.
4. $v_j^0 \leq \min\left\{\min_{m_i \in M_i} v_j^{m_i}, v_j^E\right\}$.
5. $v_i^{m_i^*} \geq v_i^{\hat{m}_i}$ for all $\hat{m}_i \in M_i \cup \{0\}$.

6. The range of $w_j(h,\zeta)$ is bounded by K: $K \geq \max_{j\in I}\left\{\max\left\{v_j^E, \left(v_j^{m_i}\right)_{m_i\in M_i}\right\} - v_j^0\right\}$.

The interpretation is that v_j^E is player j's continuation payoff after erroneous opposing histories; v_j^0 is player j's punishment payoff (which results if $\theta_j(h_{-j},\zeta) = R$ and $\text{susp}(h_n) = 1$ for some $n \neq j$); and $v_j^{m_i}$ is j's continuation payoff after players coordinate on message m_i.

We modify player i's strategy in the message round after she herself deviates as follows: Recall that we define $m_i(n) = 1$ if player n infers some $(a_{i,t})_{t\in\mathbb{T}(\text{msg})}$ not corresponding to the binary expansion of any message. We can thus view the play of such $(a_{i,t})_{t\in\mathbb{T}(\text{msg})}$ as sending message $m_i = 1$. With this interpretation, for each h_i^{t-1}, let $M_i(h_i^{t-1}) \subset M_i$ be the (non-empty) set of messages $\tilde{m}_i$ such that $(a_{i,\tau})_{\tau=1}^{t-1}$ is consistent with the binary expansion of $\tilde{m}_i$; and let $M_i^*(h_i^{t-1}) = \text{argmax}_{m_i\in M_i(h_i^{t-1})} v_i^{m_i}$ be the elements that maximize $v_i^{m_i}$. Given h_i^{t-1}, if $m_i^* \in M_i^*(h_i^{t-1})$, player i plays $a_{i,t}$ corresponding to the binary expansion of m_i^*; otherwise, she plays $a_{i,t}$ corresponding to the binary expansion of some $m_i \in M_i^*(h_i^{t-1})$.

Call a history *σ-consistent* if it is reached with positive probability under strategy profile σ. Recall that $H^{<j'}$ is the set of module history profiles at the beginning of $\mathbb{T}(j')$ that are σ-consistent for some $\sigma \in \Sigma$, and let $H_j^{\mathbb{T}(j')}|_{h^{<j'}}$ be the set of module histories during $\mathbb{T}(j')$ that are $(\sigma_j, \sigma_{-j}^*)$-consistent for some $\sigma_j \in \Sigma_j$ given $h^{<j'}$. We assume that, for every player $j, j' \in I$, module strategy σ_j, $h^{<j'} \in H^{<j'}$, and $h_j \in H_j^{\mathbb{T}(j')}|_{h^{<j'}}$, player j believes that all other players are jamming players with probability at least $\exp(-T^{\frac{1}{2}})$:

$$\Pr_j\left(n \in \mathcal{I}_{\text{jam}}\ \forall n \neq j | h^{<j'}, h_j\right) \geq \exp(-T^{\frac{1}{2}}). \tag{43}$$

Lemma 13 *Suppose that T is sufficiently large such that*

$$KNb(A^{4b(M_i)}) \exp\left(-\bar{\eta}T + 2T^{\frac{1}{2}}\right) \leq 1. \tag{44}$$

If the premise for verified communication with magnitude K and (43) hold for each $j \in I$, then with payoff functions (42) the verified communication protocol is a sequential equilibrium. In addition, if there exists $i^* \in I \setminus \{i\}$ such that $\mathcal{I}_{\text{jam}} = I \setminus \{i^*\}$ and $v_j^E = v_j^{m_i}$ for all $j \neq i^*$ and $m_i \in M_i \cup 0$, while for player i^* the premise for verified communication and (43) hold, then with payoff functions (42) the verified communication protocol is an i^*-QBFE.

Intuitively, if the prior probability that players jam is not too low, whenever player j observes an erroneous history she believes that JAM is played and $\theta_j(h_{-j}, \zeta) = E$. Otherwise, she believes that all other players match with the sender at least once in each half-interval. Hence, if she deviates and changes some player's inference, this induces $\text{susp}(h_n) = 1$ and yields the punishment payoff v_j^0. It will be useful to remember that all the lemmas in this section hold if Conditions (38), (43), and (44) are satisfied.

I Reduction Lemmas: Phase $(\text{final}, 1, i)$

This section further simplifies Lemma 9, using phase $(\text{final}, 1, i)_{i \in I}$.

I.1 Reduction Lemma 14: Letting Rewards Depend on Other Players' Main Phase Histories

Recall that, for each main phase $l = 1, ..., L$, player i randomly selects a monitoring period $t_i(l) \in \mathbb{T}(l, \text{main})$. We show that player i's reward function in the T^*-period repeated game can be made to depend on players $-i$'s histories in periods in $\mathbb{L}_{i-1} = (t_i(l))_{l=1}^L$: that is, on

$$h_{-i}^{\mathbb{L}_{i-1}} := \left(a_{-i,t_{i-1}(l)}, \omega_{-i,t_{i-1}(l)}\right)_{l=1,...,L}. \tag{45}$$

Recall that $\mathbb{T}' := \{1, ..., T^*\} \setminus \bigcup_{l=1}^L \mathbb{T}(l, \text{main})$. The reward function takes the form $\pi_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}, \chi_{-i}\right)$, where $\chi_n \in \{0, 1\}$ was defined in Section E.[24] We wish replace $\pi_i^*(x_{i-1}, h_{i-1}^{T^*}, h_{-i}^{\mathbb{T}''})$ with $\pi_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}, \chi_{-i}\right)$ in Conditions (35)–(37). In the following conditions, we also cancel the instantaneous utilities outside of the main phases (which can be accomplished by using the reward function (7)).

1. [Range Restriction] The range of the reward function is bounded by $7\bar{u}T^*$:

$$\max_{x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}} \left| \pi_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}, \chi_{-i}\right) \right| \leq 7\bar{u}T^*. \tag{46}$$

[24] Relative to Lemma 9, the argument $h_{-i}^{\mathbb{L}}$ has been added to the reward function and the argument $h_{i-1}^{T^*}$ has been removed, as $h_{-i}^{\mathbb{L}}$ contains enough information about player $i - 1$'s main phase history to provide incentives for player i.

2. [Sequential Rationality] For all $x \in \{G, B\}^N$ and $h_i^{t-1} \in H_i^{t-1}$,

$$\sigma_i^*(x_i) \in \underset{\sigma_i \in \Sigma_i}{\operatorname{argmax}} \mathbb{E}^{\left(\left(\sigma_i, \sigma_{-i}^*(x_{-i})\right), \beta^*\right)} \left[\sum_{t \in \bigcup_{l=1}^{L} \mathbb{T}(l, \text{main})} \hat{u}_i(\mathbf{a}_t) + \pi_i^* \left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}, \chi_{-i} \right) | h_i^{t-1} \right]. \tag{47}$$

3. [Promise Keeping] For all $x \in \{G, B\}^N$,

$$v_i(x_{i-1}) - 2\varepsilon^* = \frac{1}{L(T_0)^3} \mathbb{E}^{\sigma^*(x)} \left[\sum_{t \in \bigcup_{l=1}^{L} \mathbb{T}(l, \text{main})} \hat{u}_i(\mathbf{a}_t) + \pi_i^* \left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}, \chi_{-i} \right) \right]. \tag{48}$$

4. [Self-Generation] For all x_{-i}, $h_{-i}^{\mathbb{T}'}$, and $h_{-i}^{\mathbb{L}_{i-1}}$,

$$\text{sign}(x_{i-1}) \pi_i^* \left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}, \chi_{-i} \right) \geq -2\varepsilon^* T^*. \tag{49}$$

Lemma 14 *Suppose that,in the T^*-period repeated game, there exist strategies $(\sigma_i^*(x_i))_{i,x_i}$, consistent ex post belief system β^*, and reward functions $\left(\pi_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}, \chi_{-i}\right)\right)_{i,x_{-i},h_{-i}^{\mathbb{T}'},h_{-i}^{\mathbb{L}_{i-1}},\chi_{-i}}$ such that Conditions (46)–(49) are satisfied. Then there exists $\bar{\delta} < 1$ such that $\mathbf{v} \in E(\delta)$ for all $\delta > \bar{\delta}$.*

I.2 Reduction Lemma 15: "Ignoring" Other Players' Deviations

We further simplify Lemma 14. Consider the following conditions:

1. [$t_i(l)$ Not Revealed Until End of Main Phase l] For all $x_i \in \{G, B\}$, $l \in \{1, ..., L\}$, $t \in \{1, ..., T^*\}$, $(\mathbb{L}_i, h_i^{t-1})$, and $(\tilde{\mathbb{L}}_i, \tilde{h}_i^{t-1})$, if $t \leq \tau$ for some $\tau \in \mathbb{T}(\text{main}(l))$, $t_i(\hat{l}) = \tilde{t}_i(\hat{l})$ for each $\hat{l} = 1, ..., l-1$, and $h_i^{t-1} = \tilde{h}_i^{t-1}$, then

$$\sigma_i^*(x_i)|_{(\mathbb{L}_i, h_i^{t-1})} = \sigma_i^*(x_i)|_{(\tilde{\mathbb{L}}_i, \tilde{h}_i^{t-1})}. \tag{50}$$

2. [Reward Bound]

$$\sup_{x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}} \left| \pi_i^* \left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}} \right) \right| \leq 5\bar{u} T^*. \tag{51}$$

3. [Incentive Compatibility] Let $H_i(x_{-i})$ denote the set of histories that arise with positive probability under some strategy profile $(\sigma_i, \sigma^*_{-i}(x_{-i}))$ with $\sigma_i \in \Sigma_i^{T^*}$. For all $x \in \{G, B\}^N$ and $h_i^{t-1} \in H_i(x_{-i})$,

$$\sigma_i^*(x_i) \in \underset{\sigma_i \in \Sigma_i}{\operatorname{argmax}} \mathbb{E}^{\left(\sigma_i, \sigma^*_{-i}(x_{-i})\right)} \left[\sum_{t \in \bigcup_{l=1}^{L} \mathbb{T}(l,\text{main})} \hat{u}_i(\mathbf{a}_t) + \pi_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}\right) | h_i^{t-1} \right]. \quad (52)$$

Note that we do not need to define "trembles" to define $\mathbb{E}[\cdot|\cdot]$ in (52).

4. [Promise Keeping] For all $x \in \{G, B\}^N$,

$$\left.\begin{array}{l} v_i(G) - 2\varepsilon^* \leq \\ v_i(B) + 2\varepsilon^* \geq \end{array}\right\} \frac{1}{L\left(T_0\right)^3} \mathbb{E}^{\sigma^*(x)} \left[\sum_{t \in \bigcup_{l=1}^{L} \mathbb{T}(l,\text{main})} \hat{u}_i(\mathbf{a}_t) + \pi_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}\right) \right]. \quad (53)$$

5. [Self-Generation] The same as (49).

Lemma 15 *Suppose that, in the T^*-period repeated game, there exist strategies $(\sigma_i^*(x_i))_{i,x_i}$ and reward functions $\left(\pi_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}\right)\right)_{i,x_{-i},h_{-i}^{\mathbb{T}'},h_{-i}^{\mathbb{L}_{i-1}}}$ such that Conditions (49)–(53) are satisfied. Then there exists $\bar{\delta} < 1$ such that $\mathbf{v} \in E(\delta)$ for all $\delta > \bar{\delta}$.*

As in Lemma 14, players $-i$ communicate their history profile in $\mathbb{L}_{i-1}, \chi_{-i}$. Since $\mathbb{L}_{i-1}$ is random and is not revealed until main phase l is over, by giving a reward based on the history profile in $\mathbb{L}_{i-1}$, player i can be made indifferent over actions after another player "confesses" that she deviated in or before main phase l.

J Equilibrium Strategies: Remaining Details

We now complete the construction of the equilibrium strategies $(\sigma_i^*(x_i))_{i \in I}$ in sub-block $0, ..., L$. From now on, we abbreviate "the verified communication protocol with repetition T_0" to simply "the communication protocol."Recall the different phases of each sub-block defined in Section E. We let λ represent a generic phase. That is,

$$\lambda \in \{0 \times (\{\text{jam}\} \cup I \cup (I \times \{\text{con}\}))\} \cup \left\{\{1, \ldots L\} \times \{\text{main}\} \cup I \cup I^2 \cup (I \times \{\text{con}\})\right\}.$$

In this notation, the first coordinate of λ is l throughout sub-block $l \in \{0, \ldots, L\}$. The second coordinate of λ is (i) jam for the jamming coordination phase (for $l = 0$), (ii) $i \in I$ for phase (l, i) (for $l \geq 0$), (iii) (i,con) for phase (l, i,con) (for $l \geq 0$), (iv) main for main phase l (for $l \geq 1$), or (v) (i, n) for phase (l, i, n) (for $l \geq 1$).

For $l \in \{0, \ldots, L\}$ we write $\lambda \leq l$ (resp., $\lambda < l$) if the first coordinate of λ is $\leq l$ (resp., $< l$), and similarly for $\lambda \geq l$ and $\lambda > l$. Similarly, for two phases λ and λ', we say $\lambda \leq \lambda'$ if and only if phase λ precedes or equals phase λ'.

Given λ, let h_i^λ be player i's history $(a_{i,t}, \omega_{i,t})_{t \in \mathbb{T}(\lambda)}$ within phase λ. Let $h_i^{<\lambda}$ and $h_i^{\leq\lambda}$ be player i's history at the beginning and the end of phase λ, respectively. Define $h^{<\lambda}$, $h^{\leq\lambda}$, $h_{-i}^{<\lambda}$, and $h_{-i}^{\leq\lambda}$ similarly. We now define equilibrium strategies in each phase.

J.1 Sub-Block 0

J.1.1 Jamming Coordination Phase

At the beginning of the block, player i randomly selects a period $t_i(l) \in \mathbb{T}(\text{main}(l))$ for each $l = 1, ..., L$. This is encoded in $\mathbb{L}_i$ as defined in Section I.1.

Then the jamming coordination protocol is played in phase $(0, \text{jam})$. Denote player i's protocol history by $h_i^{(0,\text{jam})} = (a_{i,t}, \omega_{i,t})_{t=1}^2$. Recall from Section D.4 that $\zeta_i(h_i^{(0,\text{jam})}) = \text{jam}$ if $\omega_{i,t} = a^1$ for some $t \in \{1, 2\}$; otherwise, $\zeta_i(h_i^{(0,\text{jam})}) = \text{reg}$. In subsequent communication protocols, let $i \in \mathcal{I}_{\text{jam}}$ if and only if $\zeta_i(h_i^{(0,\text{jam})}) = \text{jam}$.

J.1.2 Initial Communication Phase

For each $i \in I$, in phase $(0, i)$, player i sends x_i by the communication protocol. As a result, for each $j \in I$, player j's history $h_j^{(0,i)}$ in phase $(0, i)$ determines an inference $x_i(j) \in \{G, B, 0\}$ and a realization $\text{susp}(h_j^{(0,i)}) \in \{0, 1\}$. After phase $(0, i)$ is concluded for all $i \in I$, the history of each player $j \in I$ determines an inferred state profile $x(j) = (x_i(j))_{i \in I} \in \{G, B, 0\}^N$. Further, for $i \in I$, given $h^{\leq(0,i)}$, let

$$I^D(h^{\leq(0,i)}) := \left\{ j \in I : \text{susp}(h_j^\lambda) = 1 \text{ for some phase } \lambda \leq (0, i) \right\}$$

be the set of players who reach suspicious histories by the end of the phase $(0,i)$.[25]

J.1.3 Contagion Phase 0

For each $i \in I$, in phase $(0,i,\text{con})$, player i communicates whether her history is suspicious. In particular, given $I^D(h^{<(0,1,\text{con})})$ (which equals $I^D(h^{\leq(0,N)})$), in phase $(0,i,\text{con})$ player i sends $m_i^{(0,i,\text{con})} = 1$ if $i \in I^D(h^{<(0,i,\text{con})})$ and $m_i^{(0,i,\text{con})} = 0$ otherwise. For each $j \in I$, player j's history $h_j^{(0,i,\text{con})}$ determines an inference $m_i^{(0,i,\text{con})}(j) \in \{0,1\}$ and a realization $\text{susp}(h_j^{(0,i,\text{con})}) \in \{0,1\}$. For the history $h^{\leq(0,i,\text{con})}$ at the end of phase $(0,i,\text{con})$, let

$$I^D(h^{\leq(0,i,\text{con})}) := I^D(h^{<(0,i,\text{con})}) \cup \left\{ j \in I : m_i^{(0,i,\text{con})}(j) = 1 \text{ or susp}\left(h_j^{(0,i,\text{con})}\right) = 1 \right\}. \quad (54)$$

J.2 Sub-Block l

For $l = 1, \ldots, L$, strategies in sub-block l depend on the variables $I^D\left(h^{<(l,\text{main})}\right) \subset I$. We have already defined $I^D\left(h^{<(l,\text{main})}\right)$ for $l = 1$. As we will see, the outcome of sub-block l together with $I^D\left(h^{<(l,\text{main})}\right)$ determines $I^D\left(h^{<(l+1,\text{main})}\right)$. This inductively determines $I^D\left(h^{<(l,\text{main})}\right)$ for each l.

J.2.1 Main Phase l

If $i \in I^D\left(h^{<(l,\text{main})}\right)$, player i plays $\alpha^{\min}$ in every period. If $i \notin I^D\left(h^{<(l,\text{main})}\right)$, then $x_j(i) \in \{G, B\}$ for all $j \in I$, and hence the action profile $\mathbf{a}^l(x(i))$ is well-defined. In this case, in every period player i plays $a_i^l(x(i))$, the i-th component of action profile $\mathbf{a}^l(x(i))$. Given a history profile $h^{\leq(l,\text{main})}$ at the end of main phase l, let $I^D\left(h^{\leq(l,\text{main})}\right) = I^D\left(h^{<(l,\text{main})}\right)$. That is, I^D remains constant in main phase l.

J.2.2 Communication Phase l, Part 1

For each $i \in I$, player $i-1$ sends the number $t_{i-1}(l)$ by the communication protocol in phase (l,i). For each $j \in I$, player j's history $h_j^{(l,i)}$ in phase (l,i) determines $t_{i-1}(l)(j) \in \mathbb{T}(l,\text{main}) \cup \{0\}$ and $\text{susp}(h_j^{(l,i)}) \in \{0,1\}$.

[25] If $\lambda = (0,\text{jam})$, define $\text{susp}\left(h_j^{\lambda}\right) = 0$

J.2.3 Communication Phase l, Part 2

For each $i \in I$ and $n \in I$, player i sends the message $(a_{i,t_{n-1}(l)(i)}, \omega_{i,t_{n-1}(l)(i)})$ by the communication protocol in phase (l, i, n). (If $t_{n-1}(l)(i) = 0$, she sends $(a_{i,t_{n-1}(l)(i)}, \omega_{i,t_{n-1}(l)(i)}) = (a^0, a^0)$.) For each $j \in I$, player j's history $h_j^{(l,i,n)}$ in phase (l, i, n) determines an inference $\left(a_{i,t_{n-1}(l)}(j), \omega_{i,t_{n-1}(l)}(j)\right) \in A^2 \cup \{0\}$ and a realization $\text{susp}(h_j^{(l,i,n)}) \in \{0, 1\}$.

After phase (l, i, n) has concluded for each $i \in I$ and $n \in I$, the history of each player $j \in I$ determines an inferred vector of outcomes $(a_{i,t_{n-1}(l)}(j), \omega_{i,t_{n-1}(l)}(j))_{i \in I} \in \prod_{n \in I} (A^2 \cup \{0\})$.

Players identify deviations as follows: Given $n \in I$, $x \in \{G, B\}^N$, and $(\mathbf{a}, \omega) \in A^{2N}$, let $\text{dev}_n^l(x, \mathbf{a}, \omega) = 1$ denote the event that either $(a_n, \omega_n) \neq \varphi(a_{-n}, \omega_{-n})$ (Lemma 2 implies (a_n, ω_n) is infeasible given players $-n$'s history) or $a_n \neq a_n^l(x)$. In addition, let $\text{dev}_n^l\left(x(i), \mathbf{a}_{t_{n-1}(l)}(i), \omega_{t_{n-1}(l)}(i)\right) = 1$ if $x(i) \notin \{G, B\}^N$ or $\left(\mathbf{a}_{t_{n-1}(l)}(i), \omega_{t_{n-1}(l)}(i)\right) \notin A^{2N}$. Thus, $\text{dev}_n^l\left(x(i), \mathbf{a}_{t_{n-1}(l)}(i), \omega_{t_{n-1}(l)}(i)\right) = 1$ means that the outcome of the communication in phases $(l, j, n)_{j \in I}$ implies that either player n deviated in the main phase, some player deviated in the communication phase, or the players failed to coordinate on some message.

Let h be a history at the end of phase (l, i) or (l, i, n). Let $I^D(h)$ be the set of players who infer susp $= 1$ or dev $= 1$ by the end of the phase: that is, for phase (l, i), we define

$$I^D(h) := I^D(h^{\leq (l,\text{main})}) \cup \left\{ j \in I : \max_{\lambda \leq (l,i)} \text{susp}\left(h_j^\lambda\right) = 1 \right\},$$

and for phase (l, i, n), the set $I^D(h)$ is defined as

$$I^D(h^{\leq (l,\text{main})}) \cup \left\{ j \in I : \max \left\{ \begin{array}{c} \max_{\lambda \leq (l,i,n)} \text{susp}(h_j^\lambda), \\ \max_{(l,N,n') \leq (l,i,n)} \text{dev}_{n'}^l\left(x(j), \mathbf{a}_{t_{n'-1}(l)}(j), \omega_{t_{n'-1}(l)}(j)\right) \end{array} \right\} = 1 \right\}.$$

J.2.4 Contagion Phase l

For each $i \in I$, in phase (l, i, con), player i sends whether $i \in I^D(h^{<(l,i,\text{con})})$, as in phase $(0, i, \text{con})$. We define $I^D\left(h^{\leq (l,i,\text{con})}\right)$ as in phase $(0, i, \text{con})$.

Finally, for a general h, let $I_{-i}^D(h_{-i}) = I^D(h) \setminus \{i\}$. Note that I_{-i}^D is a function of players $-i$'s histories only, since whether $j \in I^D(h)$ is determined by h_j.

K Reward Function

This section constructs the reward function (ignoring for the moment the jamming coordination phase, which is addressed in Lemma 19).

K.1 Statistics Used to Construct the Reward Functions

We first define some statistics, $(\theta_i)_{i\in I}$. For phase $(0,\text{jam})$, since Lemma 2 implies that $h_{-i}^{(0,\text{jam})}$ uniquely identifies $h_i^{(0,\text{jam})}$, we can equally view $(\zeta_n)_{n\in I}$ as a function of $h_{-i}^{(0,\text{jam})}$, denoted by $\zeta(h_{-j}^{(0,\text{jam})})$. Let $\theta_i(h_{-j}^{(0,\text{jam})}) = R$ if $\zeta_i(h_{-i}^{(0,\text{jam})}) = \text{reg}$ and $\theta_i(h_{-j}^{(0,\text{jam})}) = E$ if $\zeta_i(h_{-i}^{(0,\text{jam})}) = \text{jam}$. By Lemma 14, player i's reward function can be conditioned on $\zeta(h_{-j}^{(0,\text{jam})})$ and $\theta_i(h_{-j}^{(0,\text{jam})})$.

For non-main phases $\lambda > (0,\text{jam})$, players follow the verified communication module. Define $\theta_j(h_{-j}^{\lambda}, \zeta(h_{-j}^{(0,\text{jam})})) \in \{E, R\}$ as in Section H.1. Given the history $h^{\leq\lambda}$ at the end of phase λ, define $\theta_j(h_{-j}^{\leq\lambda}) = E$ if there exists a phase $\lambda' \leq \lambda$ such that $\theta_j(h_{-j}^{\lambda'}, \zeta(h_{-j}^{(0,\text{jam})})) = E$. (If $\lambda = (0,\text{jam})$, define $\theta_j\left(h_{-j}^{\lambda}, \zeta(h_{-j}^{\text{jam}})\right) = \theta_i(h_{-j}^{(0,\text{jam})})$.) Otherwise, define $\theta_j(h_{-j}^{\leq\lambda}) = R$.

For main phase (l,main), let $\theta_j(h_{-j}^{\leq(l,\text{main})}) = \theta_j(h_{-j}^{<(l,\text{main})})$. That is, θ_j remains constant.

We make some immediate observations. For each player $i \in I$, regardless of her strategy, either all her opponents successfully infer the state x, or they all become suspicious, or $\theta_i(h_{-i}) = E$. In addition, if some player became suspicious in one sub-block, then either everyone becomes suspicious or $\theta_i(h_{-i}) = E$ in the next sub-block. Finally, a deviation by player i from $a_i(x(i))$ in period $t_{i-1}(l)$ is detected for sure.

Lemma 16 *For any $i \in I$, $x \in \{G,B\}$, $\sigma_i \in \Sigma_i$, $l \in \{1,\ldots,L\}$, $l \leq \lambda < l+1$, and $\left(\sigma_i, \sigma_{-i}^*(x_{-i})\right)$-consistent history $h^{<\lambda}$ at the beginning of phase λ, the following claims hold:*

1. *Either (i) $x(n) = x(i-1)$ $\forall n \in I$ with $x_j(n) = x_j$ for each $j \neq i$, (ii) $I_{-i}^D(h_{-i}^{<\lambda}) = I\backslash\{i\}$, or (iii) $\theta_i\left(h_{-i}^{<\lambda}\right) = E$.*

2. *If $I_{-i}^D(h_{-i}^{<(\tilde{l},\text{main})}) \neq \emptyset$ for some $\tilde{l} \leq l-1$, then either $I_{-i}^D(h_{-i}^{<\lambda}) = I\backslash\{i\}$ or $\theta_i\left(h_{-i}^{<\lambda}\right) = E$.*

3. *If $a_{i,t_{i-1}(l)} \neq a_i(x(i))$, then either $I_{-i}^D(h_{-i}^{<(l+1,\text{main})}) = I\backslash\{i\}$ or $\theta_i(h_{-i}^{<(l+1,\text{main})}) = E$.*

Proof. Claims 1 and 2: By Claims 1 and 2 of Lemma 11, either (i) $x(n) = \hat{x} \in \{G,B\}^N$ $\forall n \in I$ with $\hat{x}_j = x_j$ for each $j \neq i$, (ii) $\text{susp}_n(h_n^{(0,j)}) = 1$ for some $n \neq i$ and $j \in I$, or (iii)

$\theta_i(h_{-i}^{(0,j)}, \zeta(h_{-i}^{(0,\text{jam})})) = E$ for some $j \in I$. By the same claim applied to the contagion phase, if $I_{-i}^D(h_{-i}^{<(\tilde{l},\text{main})}) \neq \emptyset$ for some $\tilde{l} \leq l-1$, then $I_{-i}^D(h_{-i}) = I \backslash \{i\}$ or $\theta_i(h_{-i}) = E$ at the end of contagion phase $\tilde{l}$.

Claim 3: Suppose $a_{i,t_{i-1}(l)} \neq a_i(x(i))$. By Claim 1, either $a_{i,t_{i-1}(l)} \neq a_i(x(i-1))$, $I_{-i}^D(h_{-i}^{<(l,\text{main})}) = I \backslash \{i\}$, or $\theta_i(h_{-i}^{<(l,\text{main})}) = E$. If $a_{i,t_{i-1}(l)} \neq a_i(x(i-1))$, then by Claim 1 of Lemma 11, at the beginning of contagion phase l, either (i) $\text{dev}_i^l\left(x(i-1), \mathbf{a}_{t_{i-1}(l)}(i-1), \omega_{t_{i-1}(l)}(i-1)\right) = 1$, (ii) $\text{susp}_n(h_n^{\tilde{\lambda}}) = 1$ for some $n \neq i$ and $\tilde{\lambda} \in (l,i) \cup \{(l,n',i)\}_{n' \in I}$, or (iii) $\theta_i(h_{-i}) = E$. Since the former two conditions imply $I_{-i}^D(h_{-i}) \neq \{\emptyset\}$ at the beginning of contagion phase l, we have $I_{-i}^D(h_{-i}^{<(l+1,\text{main})}) = I \backslash \{i\}$ or $\theta_i(h_{-i}^{<(l+1,\text{main})}) = E$ as a result of contagion phase l by Claim 1 of Lemma 11. ■

K.2 Construction of the Reward Function

Let $u^G = \min_{(a,a') \in A^2} u(a,a')$ and $u^B = \max_{(a,a') \in A^2} u(a,a')$. By (5), for all $i \in I$, we have

$$\max\left\{v_i(G), u^B\right\} - \min\left\{u^G, v_i(B)\right\} \leq 2\bar{u}. \tag{55}$$

Recall that, by Lemma 2, the history (a_{-i}, ω_{-i}) perfectly identifies $\mathbf{a}$. So, we define $\pi_i^{\theta=E}(x_{i-1}, a_{-i}, \omega_{-i}) = u^{x_{i-1}} - \hat{u}_i(\mathbf{a})$, $\quad \pi_i^{v_i}(x_{i-1}, a_{-i}, \omega_{-i}) = v_i(x_{i-1}) - \hat{u}_i(\mathbf{a})$, and $\pi_i^{v_i}(x_{i-1}, a_{-i}, \omega_{-i}|\alpha^{\min}) = v_i(x_{i-1}) - u(a_i, \alpha^{\min})$. Given this, for each $\mathbf{a} \in A^N$, we have

$$\begin{gathered}\mathbb{E}\left[\hat{u}_i(\mathbf{a}) + \pi_i^{\text{cancel}}(x_{i-1}, a_{-i}, \omega_{-i})|\mathbf{a}\right] = u^{x_{i-1}}, \quad \mathbb{E}\left[\hat{u}_i(\mathbf{a}) + \pi_i^{v_i}(x_{i-1}, a_{-i}, \omega_{-i})|\mathbf{a}\right] = v_i(x_{i-1}) \\ \mathbb{E}\left[\hat{u}_i(\mathbf{a}) + \pi_i^{v_i}(x_{i-1}, a_{-i}, \omega_{-i}|\alpha^{\min})|a_i, \alpha_{-i}^{\min}\right] = v_i(x_{i-1}).\end{gathered} \tag{56}$$

Moreover, since $u^{x_{i-1}}$ and $v_i(x_{i-1})$ are feasible payoffs,

$$\begin{gathered}\text{sign}(x_{i-1})\pi_i^{\theta=E}(x_{i-1}, a_{-i}, \omega_{-i}) \geq 0, \\ \max_{x_{i-1}, a_{-i}, \omega_{-i}} \max\left\{\left|\pi_i^{\theta=E}(x_{i-1}, a_{-i}, \omega_{-i})\right|, \left|\pi_i^{v_i}(x_{i-1}, a_{-i}, \omega_{-i})\right|, \left|\pi_i^{v_i}(x_{i-1}, a_{-i}, \omega_{-i}|\alpha^{\min})\right|\right\} \leq 2\bar{u}.\end{gathered} \tag{57}$$

Moreover, letting $\varphi_A(a_{-i}, \omega_{-i})$ be the unique action $a_i \in A$ such that $\varphi(a_{-i}, \omega_{-i}) = (a_i, \omega_i)$

for some $\omega_i \in A$, we have, by (5),

$$\begin{array}{ll} \operatorname{sign}(x_{i-1})\frac{1}{K_v}\sum_{k=1}^{K_{\mathbf{v}}}\pi_i^{v_i}(a_{-i}^k(x),\omega_{-i,k}) \geq 0 & \text{if } \varphi_A(a_{-i}^k(x),\omega_{-i,k}) = a_i^k(x)\ \forall k \in \{1,\ldots,K_{\mathbf{v}}\}, \\ 2\bar{u} \geq \pi_i^{v_i}(x_{i-1},a_{-i},\omega_{-i}|\alpha^{\min}) \geq 0 & \text{for all } (x_{i-1},a_{-i},\omega_{-i}). \end{array} \tag{58}$$

The reward function is the sum of rewards for the main phases, π_i^{main}, and rewards for the communication and contagion phases, $\pi_i^{\text{non-main}}$. Define

$$\pi_i^{\text{non-main}}(h_{-i}^{\mathbb{T}'}) = \mathbf{1}_{\left\{\zeta_i(h_{-i}^{(0,\text{jam})})=\text{reg}\right\}}\sum_{t\in\mathbb{T}'}\pi_{i,t}(h_{-i}^{\mathbb{T}'}) \in [-|\mathbb{T}'|,|\mathbb{T}'|], \tag{59}$$

where $\pi_{i,t}(h_{-i}^{\mathbb{T}'})$ is the reward for the verified communication module in (42). Next, define

$$\pi_i^{\text{main}}(x_{-i},h_{-i}^{\mathbb{T}'},h_{-i}^{\mathbb{L}_{i-1}}) = \sum_{l=1}^{L}\pi_i^{\text{main}}(l,x_{-i},h_{-i}^{\mathbb{T}'},h_{-i}^{\mathbb{L}_{i-1}}),$$

where, for each l, we define

$$\begin{aligned} &\pi_i^{\text{main}}(l,x_{-i},h_{-i}^{\mathbb{T}'},h_{-i}^{\mathbb{L}_{i-1}}) \\ &= \sum_{t\in\mathbb{T}(\text{main}(l))}\mathbf{1}_{\{t_{i-1}(l)=t\}}(T_0)^3\begin{pmatrix} \mathbf{1}_{\left\{\theta_i(h_{-i}^{<(l,\text{main})})=E\right\}}\pi_i^{\theta=E}(x_{i-1},a_{-i,t},\omega_{-i,t}) \\ +\mathbf{1}_{\left\{\theta_i(h_{-i}^{<(l,\text{main})})=R\right\}}\mathbf{1}_{\left\{I_{-i}^D(h_{-i}^{<(l,\text{main})})\neq I\setminus\{i\}\right\}}\pi_i^{v_i}(x_{i-1},a_{-i,t},\omega_{-i,t}) \\ +\mathbf{1}_{\left\{\theta_i(h_{-i}^{<(l,\text{main})})=R\right\}}\mathbf{1}_{\left\{I_{-i}^D(h_{-i}^{<(l,\text{main})})=I\setminus\{i\}\right\}}\pi_i^{v_i}(x_{i-1},a_{-i,t},\omega_{-i,t}|\alpha^{\min}) \\ -\mathbf{1}_{\left\{\theta_i(h_{-i}^{<(l,\text{main})})=R\right\}}\mathbf{1}_{\left\{I_{-i}^D(h_{-i}^{<(l,\text{main})})\neq\emptyset\right\}}\mathbf{1}_{\{x_{i-1}=G\}}2\bar{u} \end{pmatrix}. \end{aligned} \tag{60}$$

In total, the reward function following the jamming coordination phase is defined as

$$\pi_i^{\geq 3}\left(x_{-i},h_{-i}^{\mathbb{T}'},h_{-i}^{\mathbb{L}_{i-1}}\right) = \pi_i^{\text{main}}(x_{-i},h_{-i}^{\mathbb{T}'},h_{-i}^{\mathbb{L}_{i-1}}) + \pi_i^{\text{non-main}}(h_{-i}^{\mathbb{T}'}).$$

Note that we have

$$\left|\pi_i^{\geq 3}\left(x_{-i},h_{-i}^{\mathbb{T}'},h_{-i}^{\mathbb{L}_{i-1}}\right)\right| \leq 4\bar{u}L(T_0)^3 + |\mathbb{T}'| \leq 4\bar{u}T^*. \tag{61}$$

L Reduction Lemma: Phase $(0, \text{jam})$

L.1 Jamming Coordination Module

We consider payoffs and rewards for the jamming coordination protocol. For each $i \in I$, payoff functions take the form

$$\sum_{t=1}^{2} \pi_{i,t}^{\text{indiff}}(h_{-i}) + w_i(h). \tag{62}$$

Again, as in (30), we ignore player i's instantaneous payoffs.

We say that *the premise for jamming coordination with magnitude K is satisfied* if there exist $K \geq 1$ and $(v_i(\mathcal{I}_{\text{jam}}))_{\mathcal{I}_{\text{jam}} \subset I} \in \mathbb{R}^{2^N}$ satisfying the following conditions:

1. $w_i(h) = v_i(\mathcal{I}_{\text{jam}})$ for every history h such that $\mathcal{I}_{\text{jam}} = \{n \in I : \zeta_n(h_n) = \text{jam}\}$.

2. $v_i(\mathcal{I}_{\text{jam}}) = v_i(\widetilde{\mathcal{I}_{\text{jam}}})$ for all $\mathcal{I}_{\text{jam}}$ and $\widetilde{\mathcal{I}_{\text{jam}}}$ such that $i \in \mathcal{I}_{\text{jam}} \cap \widetilde{\mathcal{I}_{\text{jam}}}$.

3. For $\mathcal{I}_{\text{jam}}$ such that $i \notin \mathcal{I}_{\text{jam}}$, the range of $v_i(\mathcal{I}_{\text{jam}})$ is at most K:

$$\max_{i \in I, \mathcal{I}_{\text{jam}}, \widetilde{\mathcal{I}_{\text{jam}}} : i \notin \mathcal{I}_{\text{jam}}, i \notin \widetilde{\mathcal{I}_{\text{jam}}}} \left| v_i(\mathcal{I}_{\text{jam}}) - v_i(\widetilde{\mathcal{I}_{\text{jam}}}) \right| \leq K. \tag{63}$$

Lemma 17 *Take $(w_i(h))_{i \in I}$ and K such that the premise for jamming coordination with magnitude K is satisfied. There exists a function $\left(\pi_{i,t}^{\text{indiff}}(h_{-i})\right)_{t \in \{1,2\}}$ such that (i) we have $\max_{h_{-i}} \left| \sum_{t=1}^{2} \pi_{i,t}^{\text{indiff}}(h_{-i}) \right| \leq 2K$ and (ii) with payoffs (62), the jamming coordination protocol is a sequential equilibrium.*

L.2 Equilibrium Condition: Final Statement

The main remaining step in the proof is verifying the equilibrium conditions given each history in the jamming coordination phase. It suffices to establish incentive compatibility and promise keeping, as self-generation is addressed in the proof of Lemma 19.

Lemma 18 *For all $i \in I$, all $x \in \{G, B\}^N$, and all jamming coordination phase histories $h_i^{(0,\text{jam})}$, we have*

1. *[Incentive Compatibility] For each $t \geq 3$ and $h_i^{t-1} \in H_i(x_{-i})$,*

$$\sigma_i^*(x_i) \in \underset{\sigma_i \in \Sigma_i}{\operatorname{argmax}} \mathbb{E}^{\left(\sigma_i, \sigma_{-i}^*(x_{-i})\right)} \left[\sum_{t \in \bigcup_{l=1}^{L} \mathbb{T}(l,\text{main})} \hat{u}_i(\mathbf{a}_t) + \pi_i^{\geq 3}\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}\right) | h_i^{(0,\text{jam})}, h_i^{t-1} \right]. \tag{64}$$

2. *[Promise Keeping after $\zeta_i(h_{-i}^{(0,\text{jam})}) = \text{reg}$] If $\zeta_i(h_{-i}^{(0,\text{jam})}) = \text{reg}$ and*

$$v_i(x_{-i}, \mathcal{I}_{\text{jam}} \backslash \{i\}) := \frac{1}{L(T_0)^3} \mathbb{E}^{\sigma^*(x)} \left[\sum_{t \in \bigcup_{l=1}^{L} \mathbb{T}(l,\text{main})} \hat{u}_i(\mathbf{a}_t) + \pi_i^{\geq 3}(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}) | \mathcal{I}_{\text{jam}} \right], \tag{65}$$

then, for all $\mathcal{I}_{\text{jam}} \backslash \{i\}, \widetilde{\mathcal{I}_{\text{jam}} \backslash \{i\}} \subset I \backslash \{i\}$, we have

$$v_i(x_{-i}, \mathcal{I}_{\text{jam}} \backslash \{i\}) \begin{cases} \geq v_i(x_{i-1}) - \varepsilon^* & \text{if } x_{i-1} = G \\ \leq v_i(x_{i-1}) + \varepsilon^* & \text{if } x_{i-1} = B \end{cases}, \text{ and} \tag{66}$$

$$\left| v_i(x_{-i}, \mathcal{I}_{\text{jam}} \backslash \{i\}) - v_i(x_{-i}, \widetilde{\mathcal{I}_{\text{jam}} \backslash \{i\}}) \right| \leq N \#_{\text{half}} \exp(-(T_0)^{\frac{1}{2}}) 2\bar{u} T^*, \tag{67}$$

where $\#_{\text{half}} = 2Nb\left(A^{4b(2)}\right) + 2Nb\left(A^{4b(2)}\right) + L\left(2Nb\left(A^{4b\left((T_0)^3\right)}\right) + 2N^2 b\left(A^{4b\left(|A|^2\right)}\right) + 2N^2 b\left(A^{4b(2)}\right)\right)$ is the number of half-intervals in sub-blocks from 0 to L.

The theorem now follows easily from Lemmas 15, 17, and 18.

Lemma 19 *Suppose Lemma 18 holds. Then there exists $\bar{\delta} < 1$ such that $\mathbf{v} \in E(\delta)$ for all $\delta > \bar{\delta}$.*

Proof. By definition of $\sigma^*(x)$ in Section J, (50) holds. Hence, putting together Lemmas 6–15, it suffices to construct reward functions π_i^* that, together with $\sigma^*(x)$, satisfy equations (49) and (51)–(53). We first construct the reward for the jamming coordination phase, denoted $\pi_i^{\text{indiff}}(x_{-i}, h_{-i}^{(0,\text{jam})})$, using Lemma 17. So, we verify the premise for jamming coordination.

The probability that any jamming player other than i plays JAM during sub-blocks $0, ..., L$ is at most $N \#_{\text{half}} \exp(-(T_0)^{\frac{1}{2}})$. (i) The range of $\pi_i^{\geq 3}$ is at most $4\bar{u}T^*$ (by (61)), (ii) once a jamming player takes a jamming strategy, the reward is bounded by $2\bar{u}T^*$, and (iii) per-period payoffs are bounded by $[-\bar{u},\bar{u}]$. Hence, we have

$$\max_{i\in I,\mathcal{I}_{\text{jam}},\widetilde{\mathcal{I}_{\text{jam}}}:i\notin\mathcal{I}_{\text{jam}},i\notin\widetilde{\mathcal{I}_{\text{jam}}}}\left|v_i\left(\mathcal{I}_{\text{jam}}\right)-v_i(\widetilde{\mathcal{I}_{\text{jam}}})\right|\leq N\#_{\text{half}}\exp(-\left(T_0\right)^{\frac{1}{2}})6\bar{u}T^*.$$

Hence, by Lemma 17, there exists $\pi_i^{\text{indiff}}(x_{-i},h_{-i}^{(0,\text{jam})})$ such that the jamming coordination protocol is incentive compatible and

$$\max_{x_{-i},h_{-i}^{(0,\text{jam})}}\left|\pi_i^{\text{indiff}}(x_{-i},h_{-i}^{(0,\text{jam})})\right|\leq N\#_{\text{half}}\exp(-\left(T_0\right)^{\frac{1}{2}})12\bar{u}T^*. \tag{68}$$

We now define the total reward function as $\pi_i(x_{-i},h_{-i}^{\mathbb{T}'},h_{-i}^{\mathbb{L}_{i-1}})=\pi_i^{\text{indiff}}(x_{-i},h_{-i}^{(0,\text{jam})})+\pi_i^{\geq 3}(x_{-i},h_{-i}^{\mathbb{T}'},h_{-i}^{\mathbb{L}_{i-1}})$. It remains to verify (49)–(53).

First, the bound (51) follows from (61) and (68), since (17) implies that $\#_{\text{half}}\leq\left(T_0\right)^{0.1}$ and $\left(T_0\right)^{0.1}\exp(-(T_0)^{\frac{1}{2}})12\bar{u}T^*\leq\varepsilon^*T^*$.

Note that, by the construction of $\pi_{i,t}(h_{-i}^{\mathbb{T}'})$ in (42), for all $x\in\{G,B\}^N$ and $h_{-i}^{\mathbb{T}'}$, we have

$$\text{sign}\left(x_{i-1}\right)\pi_i^{\text{non-main}}(h_{-i}^{\mathbb{T}'})\geq-\left|\mathbb{T}'\right|. \tag{69}$$

To derive a similar equation for π_i^{main}, if $\theta_i(h_{-i}^{<(l,\text{main})})=E$, then (57) implies that π_i^{main} is non-positive if $x_{i-1}=G$ and non-negative if $x_{i-1}=B$. If $\theta_i(h_{-i}^{<(l,\text{main})})=R$ and $I_{-i}^D(h_{-i}^{<(l,\text{main})})=I\backslash\{i\}$, then the same conclusion holds by (58).

We now show that, in all other cases, we have $\text{sign}\left(x_{i-1}\right)\pi_i^{\text{main}}(l,x_{-i},h_{-i}^{\mathbb{T}'},h_{-i}^{\mathbb{L}_{i-1}})<0$ in at most $(1+K_{\mathbf{v}})$ sub-blocks. To see this, note that if $I_{-i}^D(h_{-i}^{<(l,\text{main})})\neq\emptyset$ then Lemma 16 implies that, as a result of contagion phase $l+1$, either $I_{-i}^D(h_{-i}^{<(l+1,\text{main})})=I\backslash\{i\}$ or $\theta_i(h_{-i}^{<(l+1,\text{main})})=E$ (regardless of player i's behavior). If both $\theta_i(h_{-i}^{<(l,\text{main})})=R$ and $I_{-i}^D(h_{-i}^{<(l,\text{main})})=\emptyset$, then Lemma 16 implies that, for each $n\in I$, we have $x(n)=\hat{x}$ for some $\hat{x}\in\{G,B\}^N$ with $\hat{x}_{i-1}=x_{i-1}$. Hence, by (58),we have $\text{sign}(x_{i-1})\frac{1}{K_v}\sum_{k=1}^{K_v}\pi_i^{v_i}(x_{i-1},a_{-i,t_{i-1}(l)},\omega_{-i,t_{i-1}(l)})\geq 0$ as long as $a_i^l(x(i-1))=\varphi_A(a_{-i,t_{i-1}(l)},\omega_{-i,t_{i-1}(l)})=a_{i,t_{i-1}(l)}$. Moreover, if $a_{i,t_{i-1}(l)}\neq a_i^l(x(i-1))$, then Lemma 16 implies that either $I_{-i}^D(h_{-i}^{<(l+1,\text{main})})=I\backslash\{i\}$ or $\theta_i(h_{-i}^{<(l+1,\text{main})})=E$.

It follows that, there exists a subset $\mathcal{L}\subset\{1,...,L\}$ with $|\mathcal{L}|\geq L-(K_{\mathbf{v}}+1)$ such that

$\sum_{l\in\mathcal{L}}\text{sign}\left(x_{i-1}\right)\pi_i^{\text{main}}(l,x_{-i},h_{-i}^{\mathbb{T}'},h_{-i}^{\mathbb{L}_{i-1}})\geq 0$. Since π_i^{u} and $\pi_i^{v_i}$ are bounded by (57), we have

$$\text{sign}\left(x_{i-1}\right)\pi_i^{\text{main}}(x_{-i},h_{-i}^{\mathbb{T}'},h_{-i}^{\mathbb{L}_{i-1}})\geq -2\bar{u}\left(1+K_{\mathbf{v}}\right)\left(T_0\right)^3\geq_{\text{by (6)}}-\varepsilon^* L\left(T_0\right)^3\quad\forall x_{-i},h_{-i}^{\mathbb{T}'},h_{-i}^{\mathbb{L}_{i-1}}. \tag{70}$$

Now, by (68), (69), and (70), for all $x_{i-1},h_{-i}^{\mathbb{T}'},h_{-i}^{\mathbb{L}_{i-1}}$, we have

$$\text{sign}(x_{i-1})\left(\pi_i^{\text{indiff}}(x_{-i},h_{-i}^{(0,\text{jam})})+\pi_i^{\geq 3}(x_{-i},h_{-i}^{\mathbb{T}'},h_{-i}^{\mathbb{L}_{i-1}})\right)\geq -N\#_{\text{half}}\exp(-\left(T_0\right)^{\frac{1}{2}})12\bar{u}T^*-|\mathbb{T}'|-\varepsilon^* L\left(T_0\right)^3.$$

By (17), $\#_{\text{half}}\leq\left(T_0\right)^{0.1}$ and $N\left(T_0\right)^{0.1}\exp(-\left(T_0\right)^{\frac{1}{2}})12\bar{u}T^*+|\mathbb{T}'|+\varepsilon^* L\left(T_0\right)^3\leq 2\varepsilon^* T^*$. Combining these inequalities yields (49).

Next, Lemma 18 implies that there is no profitable deviation from $\sigma_i^*\left(x_i\right)$ after the jamming coordination phase. Given this, Lemma 17 implies that there is also no profitable deviation from $\sigma_i^*\left(x_i\right)$ during the jamming coordination phase. Hence, (52) holds.

Finally, since (i) $\mathcal{I}_{\text{jam}}\neq\emptyset$ with probability no more than $1-\left(1-\exp(-\left(T_0\right)^{\frac{1}{3}})\right)^{2N}$, (ii) $\pi_i^{\geq 3}(x_{-i},h_{-i}^{\mathbb{T}'},h_{-i}^{\mathbb{L}_{i-1}})$ is bounded by $4\bar{u}T^*$, (iii) once a jamming player takes a jamming strategy, the reward is bounded by $2\bar{u}T^*$, and (iv) $\sum_{t\in\bigcup_{l=1}^{L}\mathbb{T}(l,\text{main})}\hat{u}_i\left(\mathbf{a}_t\right)$ is bounded by $2\bar{u}L(T_0)^3$, the total payoff satisfies

$$\mathbb{E}^{\sigma^*(x)}\left[\sum_{t\in\bigcup_{l=1}^{L}\mathbb{T}(l,\text{main})}\hat{u}_i\left(\mathbf{a}_t\right)+\pi_i(x_{-i},h_{-i}^{\mathbb{T}'},h_{-i}^{\mathbb{L}_{i-1}})\right]-v_i\left(x_{-i},\emptyset\right)\leq\left(1-\left(1-\exp(-\left(T_0\right)^{\frac{1}{3}})\right)^{2N}\right)6\bar{u}T^*.$$

Since (17) implies $\left(1-\left(1-\exp(-\left(T_0\right)^{\frac{1}{3}})\right)^{2N}\right)6\bar{u}T^*\leq\varepsilon^* L(T_0)^3$, this inequality together with (65) implies (53). ■

M Proof of Lemma 18

M.1 Notation

In this section, for any strategy σ_i and history h, we assume h to be $\left(\sigma_i,\sigma_{-i}^*\left(x_{-i}\right)\right)$-consistent.

For $l\in\{0,\ldots,L\}$ and $l\leq\lambda<l+1$, let $\mathbb{L}^{\leq\lambda}:=(t_n(\tilde{l}))_{n\in I,\tilde{l}\leq l}$ be the randomizations that have been realized in phase λ. Similarly, let $\mathbb{L}^{<\lambda}:=(t_n(\tilde{l}))_{n\in I,\tilde{l}\leq l}$ if $l<\lambda$ and $\mathbb{L}^{<\lambda}:=(t_n(\tilde{l}))_{n\in I,\tilde{l}<l}$ if $\lambda=(l,\text{main})$. For each λ, at the end of phase λ, if player i knew $\mathbb{L}^{\leq\lambda}$ and

$h^{\leq\lambda}$, she could attain a continuation payoff of

$$w_i(x_{-i},\mathbb{L}^{\leq\lambda},h^{\leq\lambda}) := \max_{\sigma_i\in\Sigma_i}\mathbb{E}^{\left(\sigma_i,\sigma^*_{-i}(x_{-i})\right)}\left[\begin{array}{c}\sum_{\tilde{l}=l+1}^{L}\sum_{t\in\mathbb{T}(\tilde{l},\text{main})}\hat{u}_i\left(\mathbf{a}_t\right)+\sum_{\tilde{l}=l+1}^{L}\pi_i^{\text{main}}(\tilde{l},x_{-i},h_{-i}^{\mathbb{T}'},h_{-i}^{\mathbb{L}_{i-1}})\\ +\mathbf{1}_{\left\{\zeta_i(h_{-i}^{(0,\text{jam})})=\text{reg}\right\}}\sum_{t\in\mathbb{T}':t\succsim\lambda}\pi_{i,t}(h_{-i}^{\mathbb{T}'})\end{array}|\mathbb{L}^{\leq\lambda},h^{\leq\lambda}\right] \tag{71}$$

where $t\succsim\lambda$ means period t follows or is within phase λ. On the other hand, let $v_i(x,\mathbb{L}^{\leq\lambda},h^{\leq\lambda})$ denote player i's continuation payoff from strategy $\sigma_i^*(x_i)$. We will show that, for any phase λ and history $(\mathbb{L}^{\leq\lambda},h^{\leq\lambda})$, $w_i(x_{-i},\mathbb{L}^{\leq\lambda},h^{\leq\lambda})=v_i(x,\mathbb{L}^{\leq\lambda},h^{\leq\lambda})$.

M.2 Equilibrium Properties

First, we show that there is no instantaneous deviation gain from $\sigma_i^*(x_i)$:

Lemma 20 *For any $i\in I$, $x\in\{G,B\}^N$, $\sigma_i\in\Sigma_i$, $l\in\{1,...,L\}$, $\mathbb{L}^{<(l,\text{main})}$, and history $h^{<(l,\text{main})}$ at the beginning of phase (l,main),*

$$\begin{aligned}&\max_{\sigma_i\in\Sigma_i}\mathbb{E}^{\left(\sigma_i,\sigma^*_{-i}(x_{-i})\right)}\left[\begin{array}{c}\sum_{t\in\mathbb{T}(l,\text{main})}\hat{u}_i\left(\mathbf{a}_t\right)+\pi_i^{\text{main}}(l,x_{-i},h_{-i}^{\mathbb{T}'},h_{-i}^{\mathbb{L}_{i-1}})\\ +\mathbf{1}_{\left\{\zeta_i(h_{-i}^{(0,\text{jam})})=\text{reg}\right\}}\sum_{t\in\mathbb{T}':t\text{ in sub-block }l}\pi_{i,t}(h_{-i}^{\mathbb{T}'})\end{array}|\mathbb{L}^{<(l,\text{main})},h^{<(l,\text{main})}\right]\\ =\;&\mathbb{E}^{\sigma^*(x)}\left[\sum_{t\in\mathbb{T}(l,\text{main})}\hat{u}_i\left(\mathbf{a}_t\right)+\pi_i^{\text{main}}(l,x_{-i},h_{-i}^{\mathbb{T}'},h_{-i}^{\mathbb{L}_{i-1}})|\mathbb{L}^{<(l,\text{main})},h^{<(l,\text{main})}\right]\\ =\;&\begin{cases}(T_0)^3\left(v_i(x_{i-1})-\mathbf{1}_{\{x_{i-1}=G\}}\mathbf{1}_{\left\{I_{-i}^D(h_{-i}^{<(l,\text{main})})\neq\emptyset\right\}}2\bar{u}\right) & \text{if }\theta_i(h_{-i}^{<(l,\text{main})})=R,\\ (T_0)^3u^{x_{i-1}} & \text{if }\theta_i(h_{-i}^{<(l,\text{main})})=E.\end{cases}\end{aligned}$$

Proof. Playing $\sigma_i^*\left(x_i\right)$ yields the highest value of $\pi_{i,t}(h_{-i}^{\mathbb{T}'})$: 0. Hence, we focus on $\sum_{t\in\mathbb{T}(l,\text{main})}\hat{u}_i\left(\mathbf{a}_t\right)$ and π_i^{main}. If $\theta_i(h_{-i}^{<(l,\text{main})})=R$, then, by (60), the reward function satisfies

$$\begin{aligned}&\pi_i^{\text{main}}(l,x_{-i},h_{-i}^{\mathbb{T}'},h_{-i}^{\mathbb{L}_{i-1}})\\ =\;&(T_0)^3\times\begin{cases}\pi_i^{v_i}(x_{i-1},a_{-i,t},\omega_{-i,t})-\mathbf{1}_{\{x_{i-1}=G\}}\mathbf{1}_{\left\{I_{-i}^D(h_{-i}^{<(l,\text{main})})\neq\emptyset\right\}}2\bar{u} & \text{if }I_{-i}^D(h_{-i}^{<(l,\text{main})})\neq I\setminus\{i\},\\ \pi_i^{v_i}(x_{i-1,t},a_{-i,t},\omega_{-i,t}|\alpha^{\min})-\mathbf{1}_{\{x_{i-1}=G\}}2\bar{u} & \text{if }I_{-i}^D(h_{-i}^{<(l,\text{main})})=I\setminus\{i\}\end{cases}\end{aligned}$$

for $t=t_{i-1}(l)$ (and 0 for other t's). For each $t\in\mathbb{T}(\text{main}(l))$ and $a_{i,t}$, the random variable $t_{i-1}(l)$ equals t with probability $(T_0)^{-3}$ (recall that $\mathbb{L}^{<(l,\text{main})}$ does not include $t_{i-1}(l)$ and the condition (50) holds), and players $-i$ play $a_{-i}(x(i-1))$ when $I_{-i}^{D}(h_{-i}^{<(l,\text{main})}) = \emptyset$ (by Lemma 16) and play $\alpha^{\min}$ when $I_{-i}^{D}(h_{-i}^{<(l,\text{main})}) = I\backslash\{i\}$. Hence, the per-period expected payoff is $v_i(x_{i-1}) - \mathbf{1}_{\{x_{i-1}=G\}}\mathbf{1}_{\left\{I_{-i}^{D}(h_{-i}^{<(l,\text{main})})\neq\emptyset\right\}}2\bar{u}$, by (56). If instead $\theta_i(h_{-i}^{<(l,\text{main})}) = E$, then the result follows from (56) and (60). ■ Second, for each phase λ, if $i \in I^D(h^{\lambda})$ then $I_{-i}^{D}(h_{-i}^{<\lambda}) \neq \emptyset$ or $\theta_i(h_{-i}^{<\lambda}) = E$.

Lemma 21 *For any $i \in I$, λ, and history $h^{<\lambda}$ at the beginning of phase λ, if $i \in I^D(h^{<\lambda})$ then $I_{-i}^{D}(h_{-i}^{<\lambda}) \neq \emptyset$ or $\theta_i(h_{-i}^{<\lambda}) = E$.*

Proof. By definition, $i \in I^D(h^{<\lambda})$ only if $\text{susp}_i(h_i) = 1$ or $\text{dev}_n^l\left(x\left(i\right), \mathbf{a}_{t_{n-1}(l)}\left(i\right), \omega_{t_{n-1}(l)}\left(i\right)\right) = 1$ for some $n \in I$ as the result of communication phases preceding λ. We show that both these cases imply $I_{-i}^{D}(h_{-i}^{<\lambda}) \neq \emptyset$ or $\theta_i(h_{-i}^{<\lambda}) = E$. In each communication phase, by Claims 1 and 2 of Lemma 11, if $\text{susp}_i(h_i) = 1$ then $\theta_i(h_{-i}^{<\lambda}) = E$ for each subsequent phase. In addition, we have, either all players infer the same message, $\text{susp}_n(h_n) = 1$ for some $n \neq i$, or $\theta_i(h_{-i}) = E$. If $\text{dev}_n^l\left(x\left(i\right), \mathbf{a}_{t_{n-1}(l)}\left(i\right), \omega_{t_{n-1}(l)}\left(i\right)\right) = 1$ for some $n \in I$, then each of these three cases implies either $I_{-i}^{D}(h_{-i}^{<\lambda}) \neq \emptyset$ or $\theta_i(h_{-i}^{<\lambda}) = E$. ■ Third, the distribution of $\theta_i(h_{-i})$ is independent of the history in previous phases, and $\theta_i(h_{-i}) = E$ is rare.

Lemma 22 *For any $i \in I$, λ, and $l \geq \lambda$, there exists $p(\mathcal{I}_{\text{jam}}\backslash\{i\}, \lambda, \theta_i(h_{-i}^{\leq\lambda}), l)$ such that, for any $x \in \{G, B\}^N$, $\mathbb{L}^{\leq\lambda}$, and history $h^{\leq\lambda}$ at the end of phase λ, we have*

$$\Pr^{\sigma^*(x)}\left(\theta_i(h_{-i}^{<(l,\text{main})}) = E|\mathbb{L}^{\leq\lambda}, h^{\leq\lambda}\right) = p_i(\mathcal{I}_{\text{jam}}\backslash\{i\}, \lambda, \theta_i(h_{-i}^{\leq\lambda}), l).$$

Moreover, for $\theta_i(h_{-i}^{\leq\lambda}) = R$, we have $p_i(\mathcal{I}_{\text{jam}}\backslash\{i\}, \lambda, \theta_i(h_{-i}^{\leq\lambda}), l) \leq \exp(-(T_0)^{\frac{1}{3}})$.

Proof. By Claim 5 of Lemma 11, the distribution of θ_i in each communication phase is determined by $\mathcal{I}_{\text{jam}}\backslash\{i\}$, independent of the message sent. In addition, since $\theta_i(h_{-i}^{\leq\lambda}) = R$ implies $\zeta_i(h_{-i}^{(0,\text{jam})}) = \text{reg}$, in each communication phase the probability of $\theta_i(h_{-i}, \zeta_i(h_{-i}^{(0,\text{jam})})) = E$ is at most $\#_{\text{half}}\left(\exp(-(T_0)^{\frac{1}{2}}) + \exp\left(-\bar{\varepsilon}T_0\right)\right)$ (by Claim 4 of Lemma 11). By (17), this probability is less than $\exp(-(T_0)^{\frac{1}{3}})$. ■

M.3 Verification of Promise Keeping and Incentive Compatibility

In equilibrium, by Lemma 20, for each λ with $l \leq \lambda < l+1$, $\mathbb{L}^{\leq\lambda}$, and $h^{\leq\lambda}$, we have

$$v_i(x, \mathbb{L}^{\leq\lambda}, h^{\leq\lambda}) = \sum_{\tilde{l}\geq l+1} (T_0)^3 \left\{ \begin{array}{r} p_i(\mathcal{I}_{\text{jam}}\backslash\{i\}, \lambda, \theta_i(h_{-i}^{\leq\lambda}), \tilde{l}) u^{x_{i-1}} \\ +(1 - p_i(\mathcal{I}_{\text{jam}}\backslash\{i\}, \lambda, \theta_i(h_{-i}^{\leq\lambda}), \tilde{l})) \left(v_i(x_{i-1}) - \mathbf{1}_{\{x_{i-1}=G\}} \mathbf{1}_{\left\{I_{-i}^D(h_{-i}^{<(\tilde{l},\text{main})})\neq\emptyset\right\}} 2\bar{u} \right) \end{array} \right. \tag{72}$$

By Claim 3 of Lemma 11, the event $I_{-i}^D(h_{-i}^{<\lambda}) \neq \emptyset$ implies $\theta_i(h_{-i}^{\lambda}) = E$ on the equilibrium path. Since (17) implies $\#_{\text{half}} \leq (T_0)^{0.1}$ and $(T_0)^{0.1} \exp(-(T_0)^{\frac{1}{3}}) 3\bar{u} \leq \varepsilon^* L (T_0)^3$, with $\lambda = (0, \text{jam})$, by Lemma 22, we have (65)–(67). It thus remains to verify (64).

The proof of (64) involves verifying the premise for verified communication, which requires the following simple lower bound on the probability of JAM:

Lemma 23 *For any $i \in I$, $x_{-i} \in \{G, B\}^{N-1}$, $\mathbb{L}$, $\sigma_i \in \Sigma_i$, h_i^t, and history $h^{3:t}$ from period 3 to t, we have*

$$\Pr\left(\zeta_j(h_j^{(0,\text{jam})}) = \text{jam } \forall j \neq i | \mathbb{L}, h^{3:t}, h_i^t\right) \geq \exp(-(T_0)^{\frac{1}{2}}). \tag{73}$$

Proof. By iterated expectations, it suffices to prove the lemma for $t = T^*$. For any jamming coordination phase history $h_i^{(0,\text{jam})}$, let $p_i(h_i^{(0,\text{jam})})$ denote the conditional probability that each player $j \neq i$ observes a^1 during the jamming coordination phase. By (16), we have $p_i(h_i^{(0,\text{jam})}) \geq \bar{\varepsilon} \exp(-(N-2) T^{\frac{1}{3}})$. It remains to account for updating from $h^{3:t}$ between periods 3 and T^* (recall that the jamming coordination phase ends in period 2).

Suppose player i could perfectly observe whether her opponents play REG or JAM in every half-interval. (Note that the other information in $(\mathbb{L}, h^{3:t})$ does not update the probability of $\zeta_j(h_j^{(0,\text{jam})})$). Then $\Pr\left(\zeta_j(h_j^{(0,\text{jam})}) = \text{jam } \forall j \neq i | h_i^{T^*}\right)$ would be minimized when REG is always played. As the probability that REG is always played is at least $1 - N\#_{\text{half}} \exp(-T^{\frac{1}{2}})$ (conditional on any realization of $\left(\zeta_j(h_j^{(0,\text{jam})})\right)_{j\in I}$), we have

$$\begin{aligned} \Pr^{\sigma_{-i}^*(x_{-i})}\left(\zeta_j(h_j^{(0,\text{jam})}) = \text{jam } \forall j \neq i | h_i^{T^*}\right) &\geq \frac{\bar{\varepsilon} \exp(-(N-2)(T_0)^{\frac{1}{3}}) \left(1 - N\#_{\text{half}} \exp((T_0)^{-\frac{1}{2}})\right)}{\bar{\varepsilon} \exp(-(N-2)(T_0)^{\frac{1}{3}}) \left(1 - N\#_{\text{half}} \exp((T_0)^{-\frac{1}{2}})\right) + 1} \\ &\geq_{\text{by (17)}} \exp(-(T_0)^{\frac{1}{2}}). \end{aligned}$$

■

It will also be useful to simplify equation (72). By Lemma 22, there exists a payoff $v_i(x, \mathcal{I}_{\text{jam}} \backslash \{i\}, \lambda, \theta_i(h_{-i}^{\leq \lambda}), D)$ (where D stands for "Deviation is Detected") such that, for each $h_{-i}^{<\lambda}$ with $I_{-i}^D(h_{-i}^{<\lambda}) \neq \emptyset$, we have $v_i(x, \mathbb{L}^{\leq \lambda}, h^{\leq \lambda}) = v_i(x, \mathcal{I}_{\text{jam}} \backslash \{i\}, \lambda, \theta_i(h_{-i}^{\leq \lambda}), D)$; and for each $h_{-i}^{<\lambda}$ with $I_{-i}^D(h_{-i}^{<\lambda}) = \emptyset$, we have (since $v_i(G) - 2\bar{u} \leq u^G$ and $v_i(B) \leq u^B$ by (55))

$$v_i(x, \mathbb{L}^{\leq \lambda}, h^{\leq \lambda}) \geq v_i(x, \mathcal{I}_{\text{jam}} \backslash \{i\}, \lambda, \theta_i(h_{-i}^{\leq \lambda}), D). \tag{74}$$

In addition, on the equilibrium path, either $I_{-i}^D(h_{-i}^{<(l,\text{main})}) = \emptyset$ or $\theta_i(h_{-i}^{<(l,\text{main})}) = E$. Hence, for each λ with $l \leq \lambda < l+1$, $\mathbb{L}^{\leq \lambda}$, and $h^{\leq \lambda}$, on-path payoffs are given by

$$v_i(x, \mathbb{L}^{\leq \lambda}, h^{\leq \lambda}) = v_i(x, \mathcal{I}_{\text{jam}} \backslash \{i\}, \lambda, \theta_i(h_{-i}^{\leq \lambda}), N) := \sum_{\tilde{l} \geq l+1} (T_0)^3 \left\{ \begin{array}{c} p_i(\mathcal{I}_{\text{jam}} \backslash \{i\}, \lambda, \theta_i(h_{-i}^{\leq \lambda}), \tilde{l}) u^{x_{i-1}} \\ +(1 - p_i(\mathcal{I}_{\text{jam}} \backslash \{i\}, \lambda, \theta_i(h_{-i}^{\leq \lambda}), \tilde{l})) v_i(x_{i-1}) \end{array} \right\}.$$

M.3.1 Proof of (64) (Incentive Compatibility)

The proof is by induction. For $\lambda \geq L$, $v_i(x, \mathbb{L}^{\leq \lambda}, h^{\leq \lambda}) = w_i(x_{-i}, \mathbb{L}^{\leq \lambda}, h^{\leq \lambda}) = 0$, since there is no main phase following λ and playing $\sigma_i^*(x_i)$ yields $\pi_{i,t}(h_{-i}^{\mathbb{T}'}) = 0$. Given this observation, it suffices to establish the following claim:

Inductive hypothesis: For each x, λ, $\mathbb{L}^{<\lambda}$, and $h^{<\lambda}$, if the equilibrium continuation payoff given $(\mathbb{L}^{\leq \lambda}, h^{\leq \lambda})$ equals $v_i(x, \mathbb{L}^{\leq \lambda}, h^{\leq \lambda})$, then $\sigma_i^*(x_i)$ is sequentially rational given $(x, \mathbb{L}^{<\lambda}, h^{<\lambda})$.

If $\theta_i(h_{-i}^{<\lambda}) = E$, then the claim follows from Lemma 20 and the fact that $\theta_i(h_{-i}^{<\lambda}) = E$ implies $\theta_i(h_{-i}^{<(l,\text{main})}) = E$ for all $l \geq \lambda$. So assume $\theta_i(h_{-i}^{<\lambda}) = R$.

For communication phase λ, we use v_i^E, $(v_i^{m_i})_{m_i \in M_i}$, and v_i^0 as in Section H. By Lemma 23, we have (43). Moreover, in what follows, (17) implies (38) and (44) with relevant continuation payoffs. Hence, we focus on proving the premise. Note that (74) implies, for each $x, \mathbb{L}^{\leq \lambda}, h^{\leq \lambda}$, $v_i(x, \mathbb{L}^{\leq \lambda}, h^{\leq \lambda}) \geq v_i^0 = v_i(x, \mathcal{I}_{\text{jam}} \backslash \{i\}, \lambda, R, D)$.

Contagion Phase (l, i, con)**:** For the equilibrium message m_i (equal to 0 if $i \notin I^D(h^{<\lambda})$ and 1 if $i \in I^D(h^{<\lambda})$) and the alternative message $\hat{m}_i \in \{0,1\} \setminus \{m_i\}$, we have

- $v_i^{m_i} \geq v_i^{\hat{m}_i} = v_i^0$ if $I_{-i}^D(h_{-i}^{<\lambda}) = \emptyset$ and $i \notin I^D(h^{<\lambda})$ (by (74)),

- $v_i^{m_i} = v_i^{\hat{m}_i} = v_i^0$ if $I_{-i}^D(h_{-i}^{<\lambda}) \neq \emptyset$ or $i \in I^D(h^{<\lambda})$, and

- $K \leq 2\bar{u}$ (as $u^{x_{i-1}}$ and $v_i(x_{i-1})$ are feasible payoffs,

since the event $\{\theta_i(h_{-i}^{<\lambda}) = R$ and $i \in I^D(h^{<\lambda})\}$ implies $I_{-i}^D(h_{-i}^{<\lambda}) \neq \emptyset$ by Lemma 21. Given $v_i^E = u^{x_{i-1}}$, the premise holds. Hence, $\sigma_i^*(x_i)$ is sequentially rational.

Contagion Phase (l, j, con) **with** $j \neq i$**:** Since $v_i^{m_j} \geq v_i^0$ for all $m_j \in M_j$ by (74), the premise holds.

Communication phase (l, i, n) **with** $n \neq i$**:** In phases (l, n) and (l, j, n) with $j < i$, Claim 1 of Lemma 11 implies that either players coordinate on both $t_n(l-1)$ and $(a_{j,t_n(l-1)}, \omega_{j,t_n(l-1)})_j$, or we have $I_{-i}^D(h_{-i}^{<\lambda}) \neq \emptyset$ (given $\theta_i(h_{-i}^{<\lambda}) = R$). By the inductive hypothesis, players will follow $\sigma^*(x)$ in later phases, and therefore, by Claim 4 of Lemma 11, either players coordinate on $(a_{j,t_n(l-1)}, \omega_{j,t_n(l-1)})_{j>i}$ or $\theta_i(h_{-i}^{<(l+1,\text{main})}) = E$. If $\theta_i(h_{-i}^{\lambda}) = E$ in some later phase, then player i's payoff is independent of the message in the current phase. If $\theta_i(h_{-i}^{\lambda}) = R$ in all later phases, we have $\theta_i(h_{-i}^{<(l+1,\text{main})}) = R$. Given this event, for each message $\hat{m}_i \neq (a_{i,t_n(l-1)(i)}, \omega_{i,t_n(l-1)(i)})$, coordinating on $\hat{m}_i$ induces $\text{dev}_n = 1$. Hence, $v_i^{m_i} \geq v_i^{\hat{m}_i} = v_i^0$. Since $v_i^E = u^{x_{i-1}}$, the premise holds.

Communication phase (l, j, n) **with** $j \neq i$**:** The same as phase (l, j, con).

Communication phase (l, i)**:** If $I_{-i}^D(h_{-i}^{<\lambda}) \neq \emptyset$, then $v_i^{m_i} = v_i^0$ for each $m_i \in M_i$, so the premise holds. So assume $I_{-i}^D(h_{-i}^{<\lambda}) = \emptyset$.

Suppose first that $a_{i,t_{i-1}(l)} = a_i^l(x(i))$. Given $I_{-i}^D(h_{-i}^{<\lambda}) = \emptyset$ and $\theta_i(h_{-i}^{<\lambda}) = R$, by Claim 1 of Lemma 11, players coordinated on $t_j(l-1)$ with $j - 1 < i$. Since players will follow $\sigma^*(x)$ in later phases, Claim 4 of Lemma 11 implies that either players coordinate on the true message or $\theta_i(h_{-i}^{<(l+1,\text{main})}) = E$ in later sub-phases. Hence, for any $t \in \mathbb{T}(l, \text{main})$, as long as $t_i(l-1)(n) = t$ for each $n \in I$, we have $I_{-i}^D(h_{-i}^{<(l+1,\text{main})}) = \emptyset$ or $\theta_i(h_{-i}^{<(l+1,\text{main})}) = E$. Therefore, for each message m_i, the continuation payoff is $v_i^{m_i} = v_i(x, \mathcal{I}_{\text{jam}} \backslash \{i\}, \lambda+1, R, N) \geq v_i^0 = v_i(x, \mathcal{I}_{\text{jam}} \backslash \{i\}, \lambda+1, R, D)$, so the premise holds.

Suppose instead $a_{i,t_{i-1}(l)} \neq a_i^l(x(i))$. Then Lemma 16 implies that $I_{-i}^D(h_{-i}^{<(l+1,\text{main})}) \neq \emptyset$ or $\theta_i(h_{-i}^{<(l+1,\text{main})}) = E$, regardless of player i's behavior. Hence, for each message m_i, the continuation payoff is $v_i^{m_i} = v_i^0$. Again, the premise holds.

Communication phase (l, j) **with** $j \neq i$**:** The same as phase (l, j, con).

Main Phase: If $I_{-i}^{D}(h_{-i}^{<(l,\text{main})}) \neq \emptyset$, then the continuation payoff is independent of player i's main phase behavior, so Lemma 20 implies the result. If $I_{-i}^{D}(h_{-i}^{<\lambda}) = \emptyset$, then Lemma 20 ensures there is no instantaneous deviation gain. It remains to show that the continuation payoff decreases if player i deviates. Given history profile $(\mathbb{L}^{\leq\lambda}, h^{\leq\lambda})$ at the end of main phase l, by Lemma 16, the probability that $I_{-i}^{D}(h_{-i}^{<(l+1,\text{main})}) \neq \emptyset$ is determined by and increasing in $\frac{\left|\{t\in\mathbb{T}(\text{main}(l)):a_{i,t}\neq a_i^l(x(i))\}\right|}{(T_0)^3}$. Since the distribution of $\theta_i(h_{-i}^{<(l+1,\text{main})})$ is independent of i's behavior in main phase l by Lemma 22, continuation payoff is maximized by playing $a_{i,t} = a_i^l(x(i))$ for each t.

N Supplementary Appendix: Omitted Proofs

The omitted proofs rely on two simple lemmas, which are used to adjust the reward functions to correct for unlikely errors in communication. Let $M \subset \mathbb{N}$ be a finite set, let $F \in \mathbb{R}_{++}$, let $f : M \to [-F, F]$ be a function of $m_i \in M$, and let $\tilde{m}_i \in M \cup \{0\}$ be a random variable such that, for each $m_i \in M$, $\Pr(\tilde{m}_i = m_i | m_i) = p(m_i)$ and $\Pr(\tilde{m}_i = 0 | m_i) = 1 - p(m_i)$. Applied to the remainder of the proof, M will be a message set, f will be a reward function bounded by F, and $p(m_i)$ will be the probability that message m_i is received when message m_i is sent.

Lemma 24 *With $\hat{\varepsilon} = \max_{m_i \in M} \frac{1-p(m_i)}{p(m_i)}$, there exists a function $g : M \cup \{0\} \to [-(1+\hat{\varepsilon}) F, (1+\hat{\varepsilon}) F]$ such that $\max_{m_i \in M} |f(m_i) - g(m_i)| \leq \hat{\varepsilon} F$, and $\mathbb{E}[g(\tilde{m}_i) | m_i] = f(m_i)$ for all $m_i \in M$.*

Proof. Define $g(0) = 0$ and $g(m_i) = \frac{1}{p(m_i)} f(m_i)\ \forall m_i \in M$. The claims follow directly. ■

A similar result holds if we account for self-generation. For $x_{i-1} \in \{G, B\}$, recall that $\text{sign}(x_{i-1}) = -1$ if $x_{i-1} = G$ and $\text{sign}(x_{i-1}) = 1$ if $x_{i-1} = B$. For each $x_{i-1} \in \{G, B\}$, let $f^{x_{i-1}} : M \to [-F, F]$ be a function of $m_i \in M$ such that there exists $c \geq 0$ satisfying

$$\max_{m_i \in M, x_{i-1} \in \{G,B\}} \text{sign}(x_{i-1}) f^{x_{i-1}}(m_i) \geq -c. \tag{75}$$

Lemma 25 *With $\hat{\varepsilon} = \max_{m_i \in M} \frac{1-p(m_i)}{p(m_i)}$, for all $x_{i-1} \in \{G, B\}$, there exists a function $g^{x_{i-1}} : M \cup \{0\} \to [-(1+2\hat{\varepsilon})F, (1+2\hat{\varepsilon})F]$ such that*

(i) $\max_{x_{i-1} \in \{G,B\}, m_i \in M} |f^{x_{i-1}}(m_i) - g^{x_{i-1}}(m_i)| < \hat{\varepsilon} F$,

(ii) $\mathbb{E}[g^{x_{i-1}}(\tilde{m}_i) | m_i] = f^{x_{i-1}}(m_i)$ *for all* $m_i \in M$,

(iii) $\min_{m_i \in M} \text{sign}(x_{i-1}) g^{x_{i-1}}(m_i) \geq -(1+\hat{\varepsilon})c - \hat{\varepsilon} F$, *and*

(iv) $\min_{m_i \in M} g^{x_{i-1}}(m_i) \geq g^{x_{i-1}}(0)$.

Applied to the remainder of the proof, condition (iii) helps satisfy self-generation, and condition (iv) helps satisfy the premises for the secure and verified modules.

Proof. Without loss, assume $F \geq (1+\hat{\varepsilon}) c$ (otherwise, $F := (1+\hat{\varepsilon}) c$).[26] For $x_{i-1} = G$,

$$\text{Define } g^{x_{i-1}}(0) = -F \text{and } g^{x_{i-1}}(m_i) = \frac{1}{p(m_i)} f^{x_{i-1}}(m_i) + \frac{1-p(m_i)}{p(m_i)} F \quad \forall m_i \in M_i.$$

[26] Wherever Lemma 25 is applied, we have $F \geq (1+\hat{\varepsilon}) c$.

Then, for all m_i, we have (1) $\mathbb{E}\left[g^{x_{i-1}}(\tilde{m}_i)|m_i\right] = f^{x_{i-1}}(m_i)$, (2) $g_T^{x_{i-1}}(m_i) \in [-(1+2\hat{\varepsilon})F, (1+2\hat{\varepsilon})F]$, (3) $|f^{x_{i-1}}(m_i) - g^{x_{i-1}}(m_i)| \leq 2\hat{\varepsilon}F$, (4) $\operatorname{sign}(x_{i-1})\, g^{x_{i-1}}(\tilde{m}_i) \geq -(1+\hat{\varepsilon})\, c - \hat{\varepsilon}F$, and (5) $g^{x_{i-1}}(m_i) - g^{x_{i-1}}(0) = \frac{1}{p(m_i)}\left(f^{x_{i-1}}(m_i) + F\right) \geq 0$.

For $x_{i-1} = B$, define

$$g^{x_{i-1}}(0) = -(1+\hat{\varepsilon})\, c, \text{ and } g^{x_{i-1}}(m_i) = \frac{1}{p(m_i)} f^{x_{i-1}}(m_i) + \frac{1-p(m_i)}{p(m_i)}(1+\hat{\varepsilon})\, c \quad \forall m_i \in M_i.$$

Then, for all m_i, we have (1) $\mathbb{E}\left[g^{x_{i-1}}(\tilde{m}_i)|m_i\right] = f^{x_{i-1}}(m_i)$, (2) $g_T^{x_{i-1}}(m_i) \in [-(1+2\hat{\varepsilon})F, (1+2\hat{\varepsilon})F]$, (3) $|f^{x_{i-1}}(m_i) - g^{x_{i-1}}(m_i)| \leq 2\hat{\varepsilon}F$, (4) $\operatorname{sign}(x_{i-1})\, g^{x_{i-1}}(\tilde{m}_i) \geq -(1+\hat{\varepsilon})\, c$, and (5) $g^{x_{i-1}}(m_i) - g^{x_{i-1}}(0) = \frac{1}{p(m_i)}\left(f^{x_{i-1}}(m_i) + c\right) \geq 0$ (the last inequality follows from the condition (75)). ■

N.1 Proof of Lemma 5

Let $\mathbf{a}^1 \in A^N$ be the action profile where player i plays a^1 and all other players play a^0. Let $\mathbf{a}^0 \in A^N$ be the action profile where all players play a^0. Let $\mathbb{T}^{1\text{st}} := \bigcup_{k=1}^{b(M_i)} \{2(k-1)T + 1, ..., 2(k-1)T + T\}$ denote the set of periods in the first half of each interval. For $n \neq i$, define

$$\hat{\pi}_n(h_{n-1}) = \sum_{t \in \mathbb{T}} \frac{2K\mathbf{1}_{\{\omega_{n-1,t}=a^0\}}}{p_{n-1,n}} + \sum_{t \in \mathbb{T}^{1\text{st}}} \frac{\mathbf{1}_{\{\omega_{n-1,t}=a^1\}}\left(1-\delta^T\right)\delta^{t-1}\left(\hat{u}_n(\mathbf{a}^0) - \hat{u}_n(\mathbf{a}^1)\right)}{p_{n-1,i}}$$

and $\pi_n(x_{n-1}, h_{n-1}) = \hat{\pi}_n(h_{n-1}) + v_n(x_{n-1}) - c_n$, where c_n is a constant to be determined. We will show that, for $n \neq i$, Claims 1 and 3 of the lemma hold for any c_n, and that $\mathbb{E}\left[\sum_{t\in\mathbb{T}} \delta^{t-1}\hat{u}_n(\mathbf{a}_t) + \hat{\pi}_n(h_{n-1})\right]$ is a constant independent of m_i.
Setting $c_n = \mathbb{E}\left[\sum_{t\in\mathbb{T}} \delta^{t-1}\hat{u}_n(\mathbf{a}_t) + \hat{\pi}_n(h_{n-1})\right]$ then implies that Claim 2 also holds.

For Claim 1, we require that playing a^0 throughout the module is optimal with payoff function (20). This follows immediately from the facts that $K \geq \frac{2\bar{u}}{\bar{\varepsilon}}$ and $\max_{h,\tilde{h}} \left|w_n(h) - w_n(\tilde{h})\right| < K$, which imply that the first term of $\hat{\pi}_n(h_{n-1})$ dominates any difference in $\sum_{t\in\mathbb{T}} \delta^{t-1}\hat{u}_n(\mathbf{a}_t)$ and $w_n(h)$. Claim 3 is also immediate.

To see that $\mathbb{E}\left[\sum_{t\in\mathbb{T}} \delta^{t-1}\hat{u}_n(\mathbf{a}_t) + \hat{\pi}_n(h_{n-1})\right]$ is independent of m_i, note that player i plays a^1 the same number of times regardless of m_i. Therefore, $\mathbb{E}\left[\sum_{t\in\mathbb{T}} \frac{2K\mathbf{1}_{\{\omega_{n-1,t}=a^0\}}}{p_{n-1,n}}\right]$ is independent of m_i. It remains to show that

$$\sum_{t\in\mathbb{T}}\delta^{t-1}\hat{u}_n\left(\mathbf{a}_t\right)+\sum_{t\in\mathbb{T}^{1\mathrm{st}}}\frac{\mathbb{E}\left[\mathbf{1}_{\{\omega_{n-1,t}=a^1\}}\right]\left(1-\delta^T\right)\delta^{t-1}\left(\hat{u}_n\left(\mathbf{a}^0\right)-\hat{u}_n\left(\mathbf{a}^1\right)\right)}{p_{n-1,i}} \tag{76}$$

is independent of m_i.

We show that payoff (76) is independent of m_i for each interval, i.e., for each $k\in\{1,\dots,b(M_i)\}$, when the sums in (76) are restricted to $\tau\in\{2(k-1)T+1,\dots,2kT\}$, they are the same when player i plays a^1 in the first half of the k^{th} interval as when she plays a^1 in the second half. When player i plays a^1 in the second half of the k^{th} interval, (76) equals

$$\sum_{\tau=2(k-1)T+1}^{2(k-1)T+T}\delta^{\tau-1}\hat{u}_n\left(\mathbf{a}^0\right)+\sum_{\tau=2(k-1)T+T+1}^{2kT}\delta^{\tau-1}\hat{u}_n\left(\mathbf{a}^1\right),$$

while when player i plays a^1 in the first half of the k^{th} interval, the payoff (76) equals

$$\begin{aligned}
&\sum_{\tau=2(k-1)T+1}^{2(k-1)T+T}\delta^{\tau-1}\hat{u}_n\left(\mathbf{a}^1\right)+\sum_{\tau=2(k-1)T+T+1}^{2kT}\delta^{\tau-1}\hat{u}_n\left(\mathbf{a}^0\right)+\left(1-\delta^T\right)\sum_{\tau=2(k-1)T+1}^{2(k-1)T+T}\delta^{\tau-1}\left(\hat{u}_n\left(\mathbf{a}^0\right)-\hat{u}_n\left(\mathbf{a}^1\right)\right)\\
=&\sum_{\tau=2(k-1)T+1}^{2(k-1)T+T}\delta^{\tau-1}\hat{u}_n\left(\mathbf{a}^1\right)+\delta^T\sum_{\tau=2(k-1)T+1}^{2(k-1)T+T}\delta^{\tau-1}\hat{u}_n\left(\mathbf{a}^0\right)+\left(1-\delta^T\right)\sum_{\tau=2(k-1)T+1}^{2(k-1)T+T}\delta^{\tau-1}\left(\hat{u}_n\left(\mathbf{a}^0\right)-\hat{u}_n\left(\mathbf{a}^1\right)\right)\\
=&\ \delta^T\sum_{\tau=2(k-1)T+1}^{2(k-1)T+T}\delta^{\tau-1}\hat{u}_n\left(\mathbf{a}^1\right)+\sum_{\tau=2(k-1)T+1}^{2(k-1)T+T}\delta^{\tau-1}\hat{u}_n\left(\mathbf{a}^0\right)\\
=&\sum_{\tau=2(k-1)T+1}^{2(k-1)T+T}\delta^{\tau-1}\hat{u}_n\left(\mathbf{a}^0\right)+\sum_{\tau=2(k-1)T+T+1}^{2kT}\delta^{\tau-1}\hat{u}_n\left(\mathbf{a}^1\right).
\end{aligned}$$

Finally, for player i, define

$$\hat{\pi}_i\left(h_{i-1}\right)=\sum_{t\in\mathbb{T}}\frac{1}{p_{i-1,i}}\left(\delta^{t-1}\mathbf{1}_{\{\omega_{i-1,t}=a^1\}}\left(\hat{u}_i\left(\mathbf{a}^1\right)-\hat{u}_i\left(\mathbf{a}^0\right)\right)+\mathbf{1}_{\{\omega_{i-1,t}\in\{a^0,a^1\}\}}2\bar{u}\right).$$

The first term in the sum makes player i indifferent between playing a^0 and a^1, and the second term makes her not want to play $a\notin\{a^0,a^1\}$. Since player i is indifferent between a^0 and a^1, it follows that $c_i=\mathbb{E}\left[\sum_{t\in\mathbb{T}}\delta^{t-1}\hat{u}_i\left(\mathbf{a}_t\right)+\hat{\pi}_i\left(h_{i-1}\right)\right]$ is independent of m_i. Hence, letting $\pi_{i,t}\left(x_{i-1},h_{i-1}\right)=\hat{\pi}_{i,t}\left(h_{i-1}\right)+v_i(x_{i-1})-c_i$, Claims 1-3 of the lemma hold for $n=i$.

N.2 Proof of Lemma 6

By Lemma 1, it suffices to show that, for sufficiently large $\delta < 1$, there exist $(\sigma_i^{**}(x_i))_{i,x_i}$, β^{**}, $(v_i^{**}(x_{i-1}))_{i,x_{i-1}}$ and $\left(\pi_i^{**}(x_{i-1}, h_{i-1}^{T^{**}})\right)_{i,x_{i-1},h_{i-1}^{T^{**}}}$ such that (1)–(4) are satisfied in the T^{**}-period discounted repeated game.

Construction of $\sigma_i^{**}(x_i)$

Play within the first T_3 periods is given by $(\sigma_i^{*}(x_i))_{i\in I}$. Play from periods T_3+1 to T^{**} is given by the Phase $(\text{final}, 4, i)_{i\in I}$ strategies defined in Section E. Denote player i's strategy for periods $T_3+1, ..., T^{**}$ by $\sigma_i^{T^{**}}|_{h_i^{T_3}}$ (indicating its dependence on $h_i^{T_3}$).

At the end of phase $(\text{final}, 4, i)$, for each $n \neq i, i-1$, denote player $i-1$'s inferences of $t_{i-1}(n)$ and $h_{n,t_{i-1}(n)}$ by $t_{i-1}(n)(i-1) \in \{0, 1, ..., T_3\}$ and $h_{n,t_{i-1}(n)}(i-1) \in A^2 \cup \{0\}$, respectively. We say that *communication succeeds* if $t_{i-1}(n)(i-1) = t_{i-1}$ and $h_{n,t_{i-1}(n)}(i-1) \neq 0$ for all $n \neq i, i-1$. Denote the event that communication succeeds (resp., fails) by $s_{i-1} = 1$ (resp., $s_{i-1} = 0$). Note that, if $s_{i-1} = 1$ and all players follow $\sigma^{T^{**}}|_{h^{T_3}}$, then $h_{-i,t_{i-1}}(i-1) = h_{-i,t_{i-1}}$.

Construction of β^{**}

As will be seen, for periods $T_3+1, ..., T^{**}$, the equilibrium is belief-free. Hence, any consistent beliefs suffice. For periods $1, ..., T_3$, let $\beta^{**} = \beta^{*}$.

Construction of $\pi_i^{**}(x_{i-1}, h_{i-1}^{T^{**}})$

Since $h_{-i,t_{i-1}}$ uniquely identifies $a_{i,t_{i-1}}$ by Lemma 2, there exists $\tilde{\pi}_{i,t}^{\delta}\left(t_{i-1}, h_{-i,t_{i-1}}\right)$ such that, for all $\mathbf{a}_t \in A^N$ and $t \in \{1, ..., T_3\}$,

$$\tilde{\pi}_{i,t}^{\delta}\left(t_{i-1}, h_{-i,t_{i-1}}\right) = \mathbf{1}_{\{t_{i-1}=t\}} T_3 \left(1-\delta^{t-1}\right) \hat{u}_i(\mathbf{a}_t). \tag{77}$$

Note that

$$\lim_{\delta\to 1} \max_{t,t_{i-1},h_{-i,t_{i-1}}} \tilde{\pi}_{i,t}^{\delta}\left(t_{i-1}, h_{-i,t_{i-1}}\right) = 0. \tag{78}$$

We use Lemma 24 to adjust $\tilde{\pi}_{i,t}^{\delta}\left(t_{i-1}, h_{-i,t_{i-1}}\right)$ to account for errors in communication.

Claim 1 *There exist* $\left(\pi_{i,t}^{\delta}\left(t_{i-1}, s_{i-1}, h_{-i,t_{i-1}}(i-1)\right)\right)_{i,t,t_{i-1},s_{i-1},h_{-i,t_{i-1}}(i-1)}$ *such that*

1. *For all $i \in I$, $t_{i-1} \in \{1, \ldots, T_3\}$, and $h^{T_3} \in H^{T_3}$,*

$$\mathbb{E}\left[\pi_{i,t}^{\delta}\left(t_{i-1}, s_{i-1}, h_{-i,t_{i-1}}(i-1)\right)|h^{T_3}, t_{i-1}\right] = \tilde{\pi}_{i,t}^{\delta}\left(t_{i-1}, h_{-i,t_{i-1}}\right). \tag{79}$$

2. $\lim_{\delta \to 1} \max_{i,t,t_{i-1},s_{i-1},h_{-i,t_{i-1}}(i-1)} \pi_{i,t}^{\delta}\left(t_{i-1}, s_{i-1}, h_{-i,t_{i-1}}(i-1)\right) = 0.$

Proof. Let $\tilde{h}_{-i,t_{i-1}} = h_{-i,t_{i-1}}(i-1)$ if $s_{i-1} = 1$ and $\tilde{h}_{-i,t_{i-1}} = 0$ otherwise. Since $s_{i-1} = 1$ implies $h_{-i,t_{i-1}}(i-1) = h_{-i,t_{i-1}}$, we have $\Pr\left(\tilde{h}_{-i,t_{i-1}} = h_{-i,t_{i-1}}|t_{i-1}\right) + \Pr\left(\tilde{h}_{-i,t_{i-1}} = 0|t_{i-1}\right) = 1$. Moreover, by Lemma 3, we have
$\Pr\left(\tilde{h}_{-i,t_{i-1}} = h_{-i,t_{i-1}}|t_{i-1}\right) \geq 1 - \left(b(T_3) + (N-2)\left(b(T_3+1) + b(A^2)\right)\right)\exp(-\bar{\varepsilon}T_0)$.
The right hand side is no less than 1/2 by the definition (17). Hence, the claim follows from (77), (78), and Lemma 24 (with $\hat{\varepsilon} \leq 1$). ■

Given (77) and (79), since t_{i-1} is drawn uniformly at random from $\{1, ..., T_3\}$, we have

$$\mathbb{E}\left[\sum_{t=1}^{T_3} \pi_{i,t}^{\delta}\left(t_{i-1}, s_{i-1}, h_{-i,t_{i-1}}(i-1)\right)|h^{T_3}\right] = \sum_{\tau=1}^{T_3}\left(1 - \delta^{t-1}\right)\hat{u}_i(\mathbf{a}_t). \tag{80}$$

Let $\pi_i^{\delta}(x_{i-1}, h_{i-1}^{T^{**}}) := \sum_{t=1}^{T_3} \pi_{i,t}^{\delta}\left(t_{i-1}, s_{i-1}, h_{-i,t_{i-1}}(i-1)\right)$. Let $\mathbb{T}(\text{final}, 4) = \bigcup_{i \in I} \mathbb{T}(\text{final}, 4, i)$. Note that, for all $j \neq i$, $\pi_j^{\delta}(x_{j-1}, h_{j-1}^{T^{**}})$ does not depend on the outcome of phase $(\text{final}, 4, i)$. Hence, by Lemma 5, there exist $\left(\pi_t\left(h_{i-1}^{\mathbb{T}(\text{final},4)}\right)\right)_{i \in I}$ such that $\sigma^{T^{**}}|_{h^{T_3}}$ is a BFE in $\mathbb{T}(\text{final}, 4)$ conditional on each realized h^{T_3}, when payoffs are given by

$$\mathbb{E}\left[\sum_{n \in I} \sum_{t \in \mathbb{T}(\text{final},4,n)} \delta^{t-1}\hat{u}_i(\mathbf{a}_t) + \pi_i^{\delta}(x_{i-1}, h_{i-1}^{\mathbb{T}(\text{final},4)}) + \pi_t\left(h_{i-1}^{\mathbb{T}(\text{final},4)}\right) \mid h_i^{T_3}\right]. \tag{81}$$

Moreover, since $\lim_{\delta \to 1} \max_{x_{i-1}, h_{i-1}^{T^{**}}} \left|\pi_i^{\delta}(x_{i-1}, h_{i-1}^{T^{**}})\right| = 0$, we have

$$\lim_{\delta \to 1} \max_{h_{i-1}^{\mathbb{T}(\text{final},4)}} \left|\pi_t\left(h_{i-1}^{\mathbb{T}(\text{final},4)}\right)\right| \leq \left(\bar{u} + 2\frac{\bar{u}}{\bar{\varepsilon}}\right)(T^{**} - T_3) \leq \frac{\varepsilon^*}{2}T_3, \tag{82}$$

where the last inequality follows from (17). Finally, we define

$$\pi_i^{**}(x_{i-1}, h_{i-1}^{T^{**}}) := \pi_i^{*}(x_{i-1}, h_{i-1}^{T_3}) + \pi_i^{\delta}(x_{i-1}, h_{i-1}^{\mathbb{T}(\text{final},4)}) + \pi_t\left(h_{i-1}^{\mathbb{T}(\text{final},4)}\right) + \text{sign}(x_{i-1})8\varepsilon^* T_3. \tag{83}$$

We now verify conditions (1)–(4).

[Sequential Rationality:] Ignoring sunk payoffs and the constant term $\text{sign}(x_{i-1})8\varepsilon^* T_3$, player i maximizes the payoff (81) in $\mathbb{T}(\text{final}, 4)$. By construction of $\left(\pi_t\left(h_{i-1}^{\mathbb{T}(\text{final},4)}\right)\right)_{i\in I}$, (1) holds for all $t \in \mathbb{T}(\text{final}, 4)$ for any consistent belief system, since by Lemma 5 the basic protocol is a BFE.

Next, by Lemma 5, the expected payoff $\mathbb{E}\left[\sum_{t\in\mathbb{T}_1} \delta^{t-1}\hat{u}_i(\mathbf{a}_t) + \pi_t\left(h_{i-1}^{\mathbb{T}(\text{final},4)}\right)|h^{T_3}\right]$ does not depend on h^{T_3}. Therefore, in period $t \leq T_3$, player i maximizes

$$
\begin{aligned}
& \mathbb{E}\left[\sum_{\tau=1}^{T_3} \delta^{t-1}\hat{u}_i(\mathbf{a}_\tau) + \pi_i^*(x_{i-1}, h_{i-1}^{T_3}) + \pi_i^\delta(x_{i-1}, h_{i-1}^{\mathbb{T}(\text{final},4)})|h_i^{t-1}\right] \\
= \; & \mathbb{E}\left[\sum_{\tau=1}^{T_3} \delta^{t-1}\hat{u}_i(\mathbf{a}_\tau) + \pi_i^*(x_{i-1}, h_{i-1}^{T_3}) + \mathbb{E}\left[\pi_i^\delta(x_{i-1}, h_{i-1}^{\mathbb{T}(\text{final},4)})|h^{T_3}\right]|h_i^{t-1}\right] \\
= \; & \mathbb{E}\left[\sum_{\tau=1}^{T_3} \hat{u}_i(\mathbf{a}_\tau) + \pi_i^*(x_{i-1}, h_{i-1}^{\mathbb{T}(\text{final},4)})|h_i^{t-1}\right], \qquad (84)
\end{aligned}
$$

where the first equality follows by iterated expectation, and the second follows from (80). Since (84) equals the objective in (22), (22) implies (1).

[Promise Keeping:] Equation (2) is satisfied with $v_i^{**}(x_{i-1})$ defined by

$$
\begin{aligned}
v_i^{**}(x_{i-1}) &= \frac{1-\delta}{1-\delta^{T_1}}\mathbb{E}\left[\begin{array}{c} \sum_{t=1}^{T_1} \delta^{t-1}\hat{u}_i(\mathbf{a}_t) + \pi_i^*(x_{i-1}, h_{i-1}^{T_3}) + \pi_i^\delta(x_{i-1}, h_{i-1}^{\mathbb{T}(\text{final},4)}) \\ +\pi_t\left(h_{i-1}^{\mathbb{T}(\text{final},4)}\right) + \text{sign}(x_{i-1})8\varepsilon^* T_3 \end{array}\right] \\
&= \frac{1-\delta}{1-\delta^{T_1}}\mathbb{E}\left[\begin{array}{c} \sum_{t=1}^{T_3} \hat{u}_i(\mathbf{a}_t) + \sum_{t=T_3+1}^{T_4} v_i(x_{i-1}) \\ +\pi_i^*(x_{i-1}, h_{i-1}^{T_3}) + \text{sign}(x_{i-1})8\varepsilon^* T_3 \end{array}\right] \qquad (85)
\end{aligned}
$$

for $x_{i-1} \in \{G, B\}$, where we have used the fact that the expected value of $\sum_{t=T_3+1}^{T_4} \delta^{t-1}\hat{u}_i(\mathbf{a}_t) + \pi_t\left(h_{i-1}^{\mathbb{T}(\text{final},4)}\right)$ equals $\sum_{t=T_3+1}^{T_4} v_i(x_{i-1})$, by Lemma 5.

[Self-Generation:] Since $\lim_{\delta\to 1} \max_{x_{i-1}, h_{i-1}^{\mathbb{T}(\text{final},4)}} \left|\pi_i^\delta(x_{i-1}, h_{i-1}^{\mathbb{T}(\text{final},4)})\right| = 0$, we have

$$
\begin{aligned}
& \lim_{\delta\to 1} \text{sign}(x_{i-1})\left(\pi_i^*(x_{i-1}, h_{i-1}^{T_3}) + \pi_i^\delta(x_{i-1}, h_{i-1}^{\mathbb{T}(\text{final},4)}) + \pi_t\left(h_{i-1}^{\mathbb{T}(\text{final},4)}\right) + \text{sign}(x_{i-1})8\varepsilon^* T_3\right) \\
\geq \; & \text{sign}(x_{i-1})\pi_i^*(x_{i-1}, h_{i-1}^{T_3}) - \lim_{\delta\to 1}\left|\pi_t\left(h_{i-1}^{\mathbb{T}(\text{final},4)}\right)\right| + 8\varepsilon^* T_3 > 0.
\end{aligned}
$$

where the first inequality follows by (21), and the second by (24) and (82). Hence, for sufficiently large δ, (3) holds.

[Full Dimensionality:] Since $\frac{1-\delta}{1-\delta^{T^{**}}} \to \frac{1}{T^{**}}$ as $\delta \to 1$ and $T^{**} > T_3$, (85) implies

$$\begin{aligned}
\lim_{\delta \to 1} v_i^{**}(x_{i-1}) \quad &\to \quad \frac{1}{T^{**}} \mathbb{E}\left[\sum_{t=1}^{T_3} \hat{u}_i\left(\mathbf{a}_t\right) + \pi_i^*(x_{i-1}, h_{i-1}^{T_3}) + \text{sign}(x_{i-1}) 8\varepsilon^* T_3\right] \\
&\begin{cases} \geq \frac{T_3}{T^{**}} v_i(x_{i-1}) - 8\varepsilon^* & \text{if } x_{i-1} = G, \\ \leq \frac{T_3}{T^{**}} v_i(x_{i-1}) + 8\varepsilon^* & \text{if } x_{i-1} = B \end{cases} \\
&\begin{cases} \geq v_i(x_{i-1}) - \frac{T^{**}-T_3}{T^{**}} 2\bar{u} - 8\varepsilon^* & \text{if } x_{i-1} = G, \\ \leq v_i(x_{i-1}) + \frac{T^{**}-T_3}{T^{**}} 2\bar{u} + 8\varepsilon^* & \text{if } x_{i-1} = B. \end{cases}
\end{aligned}$$

The second line follows from (23), and the third follows from $\bar{u} \geq \max_{\mathbf{a} \in A^N} |\hat{u}\left(\mathbf{a}\right)|$ and $v(x) \in F^*$. By (5), we have $v_i\left(B\right) + 9\varepsilon^* < v_i < v_i\left(G\right) - 9\varepsilon^*$. With (17), the last line implies

$$\lim_{\delta \to 1} v_i^{**}(x_{i-1}) \begin{cases} \geq v_i(x_{i-1}) - 9\varepsilon^* & \text{if } x_{i-1} = G, \\ \leq v_i(x_{i-1}) + 9\varepsilon^* & \text{if } x_{i-1} = B. \end{cases}$$

Hence, for sufficiently large δ, we have $v_i^{**}(B) < v_i < v_i^{**}(G)$.

N.3 Proof of Lemma 7

By Lemma 6, it suffices to show that there exist $(\sigma_i^{**}\left(x_i\right))_{i,x_i}$, β^{**}, $(v_i^{**}(x_{i-1}))_{i,x_{i-1}}$ and $\left(\pi_i^{**}(x_{i-1}, h_{i-1}^{T_3})\right)_{i,x_{i-1},h_{i-1}^{T_3}}$ such that Conditions (22)–(24) are satisfied in the T_3-period discounted repeated game.

Construction of $\sigma_i^{**}(x_i)$

Play within the first T_2 periods is given by $(\sigma_i^*\left(x_i\right))_{i \in I}$. Play from periods $T_2 + 1$ to T_3 is given by the Phase $(\text{final}, 3, i)_{i \in I}$ strategies defined in Section E. Denote player $i-1$'s inference of $(a_{n,t}, \omega_{n,t})_{t \in \cup_{j \in I} \mathbb{T}(\text{final},2,j)}$ by $\left(a_{n,t}\left(i-1\right), \omega_{n,t}\left(i-1\right)\right)_{t \in \cup_{j \in I} \mathbb{T}(\text{final},2,j)}$. Note that, by (17) and Lemma 3, for each $t \in \cup_{j \in I} \mathbb{T}\left(\text{final}, 2, j\right)$, we have

$$\Pr\Big(\left(a_{n,t}\left(i-1\right), \omega_{n,t}\left(i-1\right)\right)_{n \neq i,i-1} = \left(a_{n,t}, \omega_{n,t}\right)_{n \neq i,i-1} \,|\, \left(a_{n,t}, \omega_{n,t}\right)_{n \neq i,i-1}\Big) \geq \frac{1}{2}. \tag{86}$$

Construction of β^{**}

As will be seen, for periods $T_2+1,...,T_3$, the equilibrium is belief-free. Hence, any consistent beliefs suffice. For periods $1,...,T_2$, let $\beta^{**}=\beta^*$.

Construction of $\pi_i^{**}(x_{i-1},h_{i-1}^{T_3})$

Since $(a_{-i,t},\omega_{-i,t})$ uniquely identifies $a_{i,t}$ by Lemma 2, there exists $\tilde{\pi}_{i,t}(a_{-i,t},\omega_{-i,t})$ such that, for all $\mathbf{a}_t\in A^N$ and $t\in\bigcup_{n\in I}\mathbb{T}(\text{final},2,n)$,

$$\tilde{\pi}_{i,t}(x_{i-1},a_{-i,t},\omega_{-i,t})=\begin{cases} v_i(x_{i-1})-\hat{u}_i(\mathbf{a}_t) & \text{if } t\notin\mathbb{T}(\text{final},2,n), \\ v_i(x_{i-1})-\hat{u}_i(\mathbf{a}_t)-1_{\{a_{i,t}\neq a^0\}} & \text{if } t\in\mathbb{T}(\text{final},2,n). \end{cases}.$$

We use Lemma 24 to adjust $\tilde{\pi}_{i,t}(x_{i-1},a_{-i,t},\omega_{-i,t})$ to account for errors in communication.

Claim 2 *There exist* $(\pi_{i,t}(x_{i-1},a_{-i,t}(i-1),\omega_{-i,t}(i-1)))_{i,t\in\cup_{n\in I}\mathbb{T}(\text{final},2,n),x_{i-1},a_{-i,t}(i-1),\omega_{-i,t}(i-1)}$ *such that*

1. *For all* $i\in I$, $t\in\cup_{n\in I}\mathbb{T}(\text{final},2,n)$, x_{i-1}, *and* $h^{T_2}\in H^{T_2}$,

$$\mathbb{E}\left[\pi_{i,t}(x_{i-1},a_{-i,t}(i-1),\omega_{-i,t}(i-1))\,|x_{i-1},h^{T_2}\right]=\tilde{\pi}_{i,t}(x_{i-1},a_{-i,t},\omega_{-i,t}). \tag{87}$$

2. $\max_{i,t,a_{-i,t}(i-1),\omega_{-i,t}(i-1)}|\pi_{i,t}(x_{i-1},a_{-i,t}(i-1),\omega_{-i,t}(i-1))|\leq 2(\bar{u}+1)$.

Proof. We construct $\pi_{i,t}$ from $\tilde{\pi}_{i,t}$ as we constructed $\pi_{i,t}^{\delta}$ from $\tilde{\pi}_{i,t}^{\delta}$ in Claim 1. The bound (86) and Lemma 24 imply the result. ■

Let $\pi_i^{T_3}(x_{i-1},h_{i-1}^{T_3}):=\sum_{t\in\cup_{n\in I}\mathbb{T}(\text{final},2,n)}\pi_{i,t}(x_{i-1},a_{-i,t}(i-1),\omega_{-i,t}(i-1))$. Let $\mathbb{T}(\text{final},3)$ be the set of periods in $(\text{final},3,i)_{i\in I}$. Note that, for all $j\neq i$, the reward $\pi_j^{T_3}(x_{j-1},h_{j-1}^{T_3})$ does not depend on the outcome in phase $(\text{final},3,i)$. Hence, by Lemma 5, there exist $\left(\pi_t\left(h_{i-1}^{\mathbb{T}(\text{final},3)}\right)\right)_{i\in I}$ such that σ^{T_3} is a BFE in $\mathbb{T}(\text{final},3)$ when payoffs are given by

$$\mathbb{E}\left[\sum_{t\in\mathbb{T}(\text{final},3)}\hat{u}_i(\mathbf{a}_t)+\pi_i^{T_3}(x_{i-1},h_{i-1}^{T_3})+\pi_i\left(h_{i-1}^{\mathbb{T}(\text{final},3)}\right)\mid h_i^{T_2}\right]. \tag{88}$$

Moreover, since the reward $\pi_i^{T_3}$ is additively separable across $t\in\cup_{n\in I}\mathbb{T}(\text{final},2,n)$, we have

$$\max_{i,h_{i-1}^{\mathbb{T}(\text{final},3)}}\left|\pi_i\left(h_{i-1}^{\mathbb{T}(\text{final},3)}\right)\right|\leq 2\frac{\bar{u}+2(\bar{u}+1)}{\bar{\varepsilon}}(T_3-T_2).$$

Together with Claim 2, we have

$$\begin{aligned} & \max_{i,h_{i-1}^{T_3}} \left|\pi_i^{T_3}(x_{i-1}, h_{i-1}^{T_3})\right| + \max_{i,h_{i-1}^{\mathbb{T}(\text{final},3)}} \left|\pi_i\left(h_{i-1}^{\mathbb{T}(\text{final},3)}\right)\right| \\ \leq \ & 2\left(\bar{u}+1\right)\left(T_2 - T_1\right) + 2\frac{\bar{u} + 2\left(\bar{u}+1\right)}{\bar{\varepsilon}}\left(T_3 - T_2\right) \leq \varepsilon^* T_3, \end{aligned} \tag{89}$$

where the last inequality follows from (17).

Finally, we define

$$\pi_i^{**}(x_{i-1}, h_{i-1}^{T_3}) := \pi_i^{*}(x_{i-1}, h_{i-1}^{T_2}) + \pi_i^{T_3}(x_{i-1}, h_{i-1}^{T_3}) + \pi_i\left(h_{i-1}^{\mathbb{T}(\text{final},3)}\right) + \text{sign}(x_{i-1})7\varepsilon^* T_3.$$

The verification of Conditions (1)–(4) is now the same as in Lemma 6.

N.4 Proof of Lemma 9

We construct strategies $\sigma_i^{**}(x_i)$, beliefs β^{**}, and reward functions $\pi_i^{**}\left(x_{i-1}, h_{i-1}^{T_2}\right)$ in the T_2-period game that satisfy the premise of Lemma 7.

Construction of $\sigma_i^{**}(x_i)$

Play within the first T_1 periods is given by $\left(\sigma_i^{*}(x_i)\right)_{i\in I}$. Play from periods T_1+1 to T_2 is given by the Phase $(\text{final}, 2, i)_{i\in I}$ strategies defined in Section E, with $\mathcal{I}_{\text{jam}} = \{i-1\}$ in Phase $(\text{final}, 2, i)$. For each $i \in I$ and $n \neq i, i-1$, denote player $i-1$'s inference of $m_{i-1}(n)$ by $m_{i-1}(n)(i-1)$. If $m_{i-1}(n)(i-1) = 0$ for some $n \neq i, i-1$, or if player $i-1$ plays JAM during a round where she receives a message via the secure protocol, let $s_{i-1} = 0$ ("communication fails"). Otherwise, $s_{i-1} = 1$ ("communication succeeds").

Construction of β^{**}

For periods $T_1+1, \ldots, T_2$, specify beliefs as in Lemma 8 given the sender's equilibrium message. For periods $1, ..., T_1$, let $\beta^{**} = \beta^{*}$.

Construction of $\pi_i^{**}\left(x_{i-1}, h_{i-1}^{T_2}\right)$

Fix $x_{i-1} \in \{G, B\}$ arbitrarily. If $s_{i-1} = 1$, denote player $i-1$'s inference of player n's message during phase $(\text{final}, 2, i)$ by $\left(x_n(i-1), h_n^{\mathbb{T}''}(i-1)\right)$. We first construct a function $\tilde{\pi}_i^{*}\left(x_{-i}(i-1), h_{i-1}^{T^*}, h_{-i}^{\mathbb{T}''}(i-1)\right)$ as follows: Define $\left(\tilde{x}_{-i}, \tilde{h}_{-i}^{\mathbb{T}''}\right) = \left(x_{-i}(i-1), h_{-i}^{\mathbb{T}''}(i-1)\right)$ if

$s_{i-1}=1$ and $\left(\tilde{x}_{-i},\tilde{h}_{-i}^{\mathbb{T}''}\right)=0$ otherwise. Note that (i) (10) implies

$$\min_{x_{-i},h_{-i}^{\mathbb{T}''}}\Pr\left(s_{i-1}=1|x_{-i},h_{-i}^{\mathbb{T}''}\right)\geq 1-Nb\left(2\left|A\right|^{2\left(T_1-L(T_0)^3\right)}\right)\left(\exp\left(-\bar{\varepsilon}T_0\right)+2\exp\left(-\left(T_0\right)^{\frac{1}{2}}\right)\right), \tag{90}$$

(ii) $s_{i-1}=1$ implies $\left(x_{-i}\left(i-1\right),h_{-i}^{\mathbb{T}''}\left(i-1\right)\right)=\left(x_{-i},h_{-i}^{\mathbb{T}''}\right)$, and (iii) π_i^* satisfies (37). Hence, in the notation of Lemma 25,

$$\begin{aligned}\hat{\varepsilon} &= \frac{Nb\left(2\left|A\right|^{2\left(T_1-L(T_0)^3\right)}\right)\left(\exp\left(-\bar{\varepsilon}T_0\right)+2\exp\left(-\left(T_0\right)^{\frac{1}{2}}\right)\right)}{1-Nb\left(2\left|A\right|^{2\left(T_1-L(T_0)^3\right)}\right)\left(\exp\left(-\bar{\varepsilon}T_0\right)+2\exp\left(-\left(T_0\right)^{\frac{1}{2}}\right)\right)},\\ F &= \max_{\tilde{x}_{-i},h_{i-1}^{T^*},\tilde{h}_{-i}^{\mathbb{T}''}}\left|\pi_i^*\left(\tilde{x}_{-i},h_{i-1}^{T^*},\tilde{h}_{-i}^{\mathbb{T}''}\right)\right|\leq_{\text{by (34)}} 8\bar{u}T_1,\quad c=5\varepsilon^*T_1.\end{aligned}$$

Lemma 25 implies that there exists $\tilde{\pi}_i^*\left(\tilde{x}_{-i},h_{i-1}^{T^*},\tilde{h}_{-i}^{\mathbb{T}''}\right)$ such that

$$\max_{\tilde{x}_{-i},h_{i-1}^{T^*},\tilde{h}_{-i}^{\mathbb{T}''}}\left|\tilde{\pi}_i^*\left(\tilde{x}_{-i},h_{i-1}^{T^*},\tilde{h}_{-i}^{\mathbb{T}''}\right)\right|\leq\left(1+2\hat{\varepsilon}\right)F, \tag{91}$$

$$\mathbb{E}\left[\tilde{\pi}_i^*\left(\tilde{x}_{-i},h_{i-1}^{T^*},\tilde{h}_{-i}^{\mathbb{T}''}\right)|x_{-i},h_{i-1}^{T^*},h_{-i}^{\mathbb{T}''}\right]=\pi_i^*\left(x_{-i},h_{i-1}^{T^*},h_{-i}^{\mathbb{T}''}\right), \tag{92}$$

$$\operatorname{sign}\left(x_{i-1}\right)\tilde{\pi}_i^*\left(\tilde{x}_{-i},h_{i-1}^{T^*},\tilde{h}_{-i}^{\mathbb{T}''}\right)\geq-(1+\hat{\varepsilon})c-\hat{\varepsilon}F,\ \text{and} \tag{93}$$

$$\tilde{\pi}_i^*\left(\tilde{x}_{-i},h_{i-1}^{T^*},\tilde{h}_{-i}^{\mathbb{T}''}\right)\ \text{is minimized when } s_{i-1}=0. \tag{94}$$

Finally, we define the reward function $\pi_i^{**}\left(x_{i-1},h_{i-1}^{T_2}\right)=\tilde{\pi}_i^{**}\left(x_{i-1},h_{i-1}^{T_2}\right)$. It remains to verify the premise of Lemma 7.

[Sequential Rationality:] We verify (22) for all $t=1,\ldots,T_2$ by backward induction. In phase $(\text{final},2,i)$, player i maximizes the conditional expectation of

$$-\sum_{t\in\mathbb{T}(\text{final},2,i)}1_{\{a_{i,t}\neq a^0\}}+\tilde{\pi}_i^*\left(\tilde{x}_{-i},h_{i-1}^{T^*},\tilde{h}_{-i}^{\mathbb{T}''}\right).$$

Given (91) and (94), the premise for secure communication with magnitude $\left(1+2\hat{\varepsilon}\right)F$ for player i is satisfied, for each $x\in\{G,B\}^N$. Moreover, (32) holds by inequalities (17) and (34). Hence, Lemma 8 implies (27) for $t=T_1+1,...,T_2$.

Since (92) implies that π_i^* and $\tilde{\pi}_i^*$ are equal in expectation given $\tilde{x}_{-i}, h_{i-1}^{T^*}, \tilde{h}_{-i}^{\mathbb{T}''}$ (assuming players follow σ^{**} in phases $(\text{final}, 3, i)_{i \in I}$, as we have shown is optimal), (35) implies (27).

[Promise Keeping:] Let

$$\hat{v}_i\left(x_{i-1}\right) := \frac{1}{T_2}\mathbb{E}^{\sigma^*(x)}\left[\sum_{t=1}^{T_1} \hat{u}_i\left(\mathbf{a}_t\right) + \sum_{t=T_1+1}^{T_2} v_i(x_{i-1}) - \sum_{t \in \mathbb{T}(\text{final},2,i)} 1_{\{a_{i,t} \neq a^0\}} + \tilde{\pi}_i^{**}\left(x_{-i}, h_{i-1}^{T_2}\right)\right].$$

Equation (36) implies $\hat{v}_i\left(x_{i-1}\right) = v_i\left(x_{i-1}\right)$.

[Self-Generation:] By (17), (93) implies (29).

N.5 Proof of Lemma 11

Claim 1: If $\text{susp}\left(h_n\right) = 1$ for some $n \neq j$, then (ii) holds. If $\theta_j\left(h_{-j}, \zeta, j'\right) = E$ for some $j' \in I$, then (iii) holds. So assume otherwise.

In light of the definition of FAIL, this implies that, for each $j' \neq j$ and $n \neq j'$, player n observes a^1 in each half-interval in $\mathbb{T}\left(j'\right)$ where player j' plays a^1. For $n = j$, since players $-j$ follow the equilibrium strategy and take *REG*, we have $(a_{j',t}(j), \omega_{j',t}(j))_{t \in \mathbb{T}(\text{msg})} = (a_{j',t}, \omega_{j',t})_{t \in \mathbb{T}(\text{msg})}$. Moreover, for each player $n \neq j, j'$, since $\text{susp}\left(h_n\right) = 0$, she does not observe a^1 in any other half-interval in $\mathbb{T}\left(j'\right)$ than those in which player j' takes a^1. Hence, $(a_{j',t}(n), \omega_{j',t}(n))_{t \in \mathbb{T}(\text{msg})} = (a_{j',t}, \omega_{j',t})_{t \in \mathbb{T}(\text{msg})}$. Combining these observations, we have $(a_{j',t}(n), \omega_{j',t}(n))_{t \in \mathbb{T}(\text{msg})} = (a_{j',t}, \omega_{j',t})_{t \in \mathbb{T}(\text{msg})}$ for each $j', n \in I$. Therefore, $m_i(n) = m_i\left(n'\right)$ for all $n \in I$. Finally, as player i follows the protocol, this message must equal m_i.

For the last part of the claim, consider each event that induces $\text{susp}\left(h_j\right) = 1$: If $(a_{n,t}(j), \omega_{n,t}(j))_{t \in \mathbb{T}(\text{msg})} = 0$ for some $n \neq j$, then $(a_{n,t}(j), \omega_{n,t}(j))_{t \in \mathbb{T}(\text{msg})} \neq (a_{n,t}, \omega_{n,t})_{t \in \mathbb{T}(\text{msg})}$. Hence, either some player $j' \neq n, j$ played JAM or player j did not match with player n in a half-interval in $\mathbb{T}\left(n\right)$ where player n played a^1. In either case, $\theta_j\left(h_{-j}, \zeta, n\right) = E$. If $(a_{j,t}(n), \omega_{j,t}(n))_{t \in \mathbb{T}(\text{msg}), j \in I}$ is not feasible, then again there exists $n \neq j$ with $(a_{n,t}(j), \omega_{n,t}(j))_{t \in \mathbb{T}(\text{msg})} \neq (a_{n,t}, \omega_{n,t})_{t \in \mathbb{T}(\text{msg})}$.

Claim 2: The same as Claim 1, except that the commonly inferred message $\hat{m}_i$ may differ from m_i.

Claim 3: Follows from Claim 3 of Lemma 10.

Claim 4: Given Claim 3, it suffices to show $\Pr^{\sigma^{*,m_i}}\left(\theta_j\left(h_{-j},\zeta\right)=E\right)\leq \exp(-T^{\frac{1}{3}})$. For each $j'\in I$, if no one plays JAM in $\mathbb{T}\left(j'\right)$, then $\theta_j\left(h_{-j},\zeta,j'\right)=E$ only if some player $n\neq j'$ fails to observe a^1 in a half-interval in $\mathbb{T}\left(j'\right)$ where player j' plays a^1. By Lemma 3, this event occurs with probability at most $(N-1)\,b(A^{4b(M_i)})\exp\left(-\bar{\varepsilon}T\right)$. In total, $\theta_j\left(h_{-j},\zeta\right)=E$ occurs with probability at most

$$\underbrace{2N\left(N-1\right)b(A^{4b(M_i)})\exp(-T^{\frac{1}{2}})}_{\exists j'\in I,n\neq j:\text{ player }n\text{ plays JAM in }\mathbb{T}(j')}+\underbrace{N\left(N-1\right)b(A^{4b(M_i)})\exp\left(-\bar{\varepsilon}T\right)}_{\exists j'\in I,n\neq i:n\text{ fails to observe }a^1\text{ in }\mathbb{T}(j')}. \tag{95}$$

By (38), this sum is at most $\exp(-T^{\frac{1}{3}})$.

Claim 5: Follows from Claim 1 of Lemma 10.

N.6 Proof of Lemma 13

We prove the first part of the lemma by backward induction. We assume throughout that $\zeta_j=\text{reg}$; if instead $\zeta_j=\text{jam}$, then (42) equals $w_j\left(h,\zeta\right)$ and $\theta_j\left(h_{-j},\zeta\right)=E$, so player j is indifferent over all protocol strategies by Condition 1 of the premise for communication.

Final Checking Round Let j' be the index of the final checking round. Fix $h\in H^{<j'}$. The following lemma verifies the receivers' incentives, since both $\hat{u}_j(\mathbf{a}_\tau)-\mathbf{1}_{\{a_{j,\tau}\neq a^0\}}$ and $\hat{u}_j(\mathbf{a}_\tau)-\mathbf{1}_{\left\{a_{j,\tau}\neq a^*_{j,\tau}(h_{-j})\right\}}$ for $\tau\notin\mathbb{T}\left(j'\right)$ are sunk.

Lemma 26 *Assume $j\neq j'$ and $\zeta_j=\text{reg}$. For every history $h^{<j'}\in H^{<j'}$ and h_j^{t-1} with $t\in\mathbb{T}\left(j'\right)$, and every action $a_{j,t}\in A$, when player j follows her optimal continuation strategy after taking action $a_{j,t}$, we have*

$$\mathbb{E}\left[-\sum_{\tau\in\mathbb{T}(j')}1_{\{a_{j,\tau}\neq a^0\}}+w_j\left(h,\zeta\right)|h^{<j'},h_j^{t-1},a_{j,t}=a^0\right]\geq\mathbb{E}\left[-\sum_{\tau\in\mathbb{T}(j')}1_{\{a_{j,\tau}\neq a^0\}}+w_j\left(h,\zeta\right)|h^{<j'},h_j^{t-1},a_{j,t}\neq a^0\right]. \tag{96}$$

Proof. If $\theta_j\left(h_{-j},\zeta,j''\right)=E$ for some $j''\neq j'$, the result follows immediately from (8) and (42), given $\zeta_j=\text{reg}$. So suppose $\theta_j\left(h_{-j},\zeta,j''\right)=R$ for all $j''\neq j'$. Since a deviation by any player $j''\neq j$ induces $\theta_j\left(h_{-j},\zeta\right)=E$, we also assume players $-j$ follow σ^*_{-j} in every checking round. Hence, $\theta_j\left(h_{-j},\zeta,j'\right)=E$ if and only if (i) some player $n\neq j'$ does not observe a^1 in a half-interval where player j' plays a^1 or (ii) some player $n \neq j, j'$ plays JAM in $\mathbb{T}(j')$. In particular, let $R_{j',-j}$ denote the event that each player $n \neq j, j'$ is matched with player j' in every half-interval where player j' takes a^1. Then $\Pr(\theta_j(h_{-j}, \zeta, j') = E | R_{j',-j}, h^{<j'}, h_j^{t-1})$ is independent of σ_j.

With i replaced by j', i^* replaced with j, $\mathbb{T}$ replaced with $\mathbb{T}(j')$, and Lemma 4 replaced with Lemma 12, by the same argument as for Lemma 8, with probability at least

$$1 - Nb(A^{4b(M_i)}) \exp\left(-\bar{\eta}T + 2T^{\frac{1}{2}}\right), \tag{97}$$

conditional on $(a_{j,\tau}, \omega_{j,\tau})_{\tau \in \mathbb{T}(j')}$, either $\theta_j(h_{-j}, \zeta, j') = E$ or [for each $n \neq j$, $(a_{j',t}(n), \omega_{j',t}(n))_{t \in \mathbb{T}(\text{msg})} \in \{(a_{j',t}, \omega_{j',t})_{t \in \mathbb{T}(\text{msg})}, 0\}$, and $(a_{j',t}(n), \omega_{j',t}(n))_{t \in \mathbb{T}(\text{msg})} = (a_{j',t}, \omega_{j',t})_{t \in \mathbb{T}(\text{msg})}$ if and only if $a_{j,\tau} = a^0$ for each $\tau \in \mathbb{T}$ such that $\mu_\tau(j) = n$ and τ is in a half-interval where player j' plays a^0]. The latter event implies $R_{j',-j}$.

Since $\Pr(\theta_j(h_{-j}, \zeta, j') = E | R_{j',-j}, h^{<j'}, h_j^{t-1})$ is independent of σ_j and $(a_{j',t}(n), \omega_{j',t}(n))_{t \in \mathbb{T}(\text{msg})} = 0$ induces $\text{susp}_n(h_n) = 1$, playing $a_{j,\tau} = a^0$ for each $\tau \geq t$ maximizes $w_j(h, \zeta)$ with probability at least (97). Together with (44), this implies that the reward term $-1_{\{a_{j,t} \neq a^0\}}$ outweighs any possible benefit to player j from playing $a \neq a^0$ in an attempt to manipulate $(a_{j',t}(n), \omega_{j',t}(n))_{t \in \mathbb{T}(\text{msg}), n \neq j}$. ■

We next verify the sender's incentive:

Lemma 27 *Assume $\zeta_{j'} = \text{reg}$. For every history $h^{<j'} \in H^{<j'}$ and $h_{j'}^{t-1}$ with $t \in \mathbb{T}(j')$, and every action $a_{j',t} \in A$, when player j' follows her optimal continuation strategy after taking action $a_{j',t}$, we have*

$$\begin{aligned} & \mathbb{E}\left[-\sum_{\tau \in \mathbb{T}(j')} 1_{\left\{a_{j',\tau} \neq a^*_{j',\tau}(h_{-j'})\right\}} + w_{j'}(h, \zeta) \,|h^{<j'}, h_{j'}^{t-1}, a_{j',t} = a^*_{j',t}(h_{-j'})\right] \\ \geq \; & \mathbb{E}\left[-\sum_{\tau \in \mathbb{T}(j')} 1_{\left\{a_{j',\tau} \neq a^*_{j',\tau}(h_{-j'})\right\}} + w_{j'}(h, \zeta) \,|h^{<j'}, h_{j'}^{t-1}, a_{j',t} \neq a^*_{j',t}(h_{-j'})\right]. \end{aligned}$$

Proof. Again, we assume $\theta_{j'}(h_{-j'}, \zeta, j'') = R$ for all $j'' \neq j'$ and players $-j'$ follow $\sigma^*_{-j'}$ in all checking rounds. In addition, assume $REG_{j',-j'}$, as otherwise $\theta_{j'}(h_{-j'}, \zeta, j') = E$. Given the reward $-1_{\left\{a_{j',t}\neq a^*_{j',t}(h_{-j'})\right\}}$, it suffices to show that following $\sigma^*_{j'}$ maximizes $w_{j'}(h,\zeta)$.

By Claims 4 and 5 of Lemma 10, for each $j''\neq j'$, since we have assumed $\theta_{j'}(h_{-j'},\zeta,j'')=R$, we have $(a_{j'',t}(n),\omega_{j'',t}(n))_{t\in\mathbb{T}(\text{msg})}\in\{(a_{j'',t},\omega_{j'',t})_{t\in\mathbb{T}(\text{msg})},0\}$ for all $n\in I$.

Fix $t\in\mathbb{T}(j')$, $h^{<j'}$, and $h^{t-1}_{j'}$. If $(a_{j'',t}(n),\omega_{j'',t}(n))_{t\in\mathbb{T}(\text{msg})}=0$ for some $j''\neq j'$ and $n\in I$, then Claim 1 of Lemma 11 implies that $\text{susp}_{n'}(h_{n'})=1$ for some $n'\neq j$. Hence, maximizing $w_{j'}(h,\zeta)$ is equivalent to maximizing the probability that $\theta_j(h_{-j},\zeta,j')=E$. If player j' followed $\sigma^*_{j'}$ until period $t-1$ within $\mathbb{T}(j')$, then following $\sigma^*_{j'}$ maximizes $\theta_{j'}(h_{-j'},\zeta,j')=E$, by Claim 1 of Lemma 10. Otherwise, $\theta_{j'}(h_{-j'},\zeta,j')=R$ given $REG_{j',-j'}$ and any strategy maximizes $w_{j'}(h,\zeta)$. In total, it is optimal to follow $\sigma^*_{j'}$.

Now suppose $(a_{j'',t}(n),\omega_{j'',t}(n))_{t\in\mathbb{T}(\text{msg})}=(a_{j'',t},\omega_{j'',t})_{t\in\mathbb{T}(\text{msg})}$ for each $j''\neq j'$ and $n\in I$. Suppose player j' followed $\sigma^*_{j'}$ until period $t-1$ within $\mathbb{T}(j')$. On the one hand, if player j' deviates from $\sigma^*_{j'}$ in period t, then $\theta_{j'}(h_{-j'},\zeta,j')=R$ given $REG_{j',-j'}$. Since $(a_{j',t}(n),\omega_{j',t}(n))_{t\in\mathbb{T}(\text{msg})}\neq(a_{j',t},\omega_{j',t})_{t\in\mathbb{T}(\text{msg})}$ for some $n\neq j'$ induces $\text{susp}(h_n)=1$, player j''s payoff is $P(\sigma_{j'}|h^{<j'},h^{t-1}_{j'})v^{m_i}_{j'}+(1-P(\sigma_{j'}|h^{<j'},h^{t-1}_{j'}))v^0_{j'}$, where m_i corresponds to $(a_{i,t})_{t\in\mathbb{T}(\text{msg})}$ and $P(\sigma_{j'}|h^{<j'},h^{t-1}_{j'})$ is the probability that $(a_{j',t}(n),\omega_{j',t}(n))_{t\in\mathbb{T}(\text{msg})}=(a_{j',t},\omega_{j',t})_{t\in\mathbb{T}(\text{msg})}$ for all $n\neq j'$. On the other hand, if player j' follows $\sigma^*_{j'}$ in period t, then her equilibrium payoff is $P(\sigma^*_{j'}|h^{<j'},h^{t-1}_{j'})v^{m_i}_{j'}+(1-P(\sigma^*_{j'}|h^{<j'},h^{t-1}_{j'}))v^E_{j'}$, since $(a_{j',t}(n),\omega_{j',t}(n))_{t\in\mathbb{T}(\text{msg})}\neq(a_{j',t},\omega_{j',t})_{t\in\mathbb{T}(\text{msg})}$ implies $\theta_{j'}(h_{-j'},\zeta,j')=E$. As $\min\{v^{m_i}_{j'},v^E_{j'}\}\geq v^0_{j'}$ by premise and $P(\sigma^*_{j'}|h^{<j'},h^{t-1}_{j'})\geq P(\sigma_{j'}|h^{<j'},h^{t-1}_{j'})$ by definition, it is optimal to play $\sigma^*_{j'}$.

Suppose instead player j' deviated from σ^*_j within $\mathbb{T}(j')$ before period $t-1$. Then $\theta_{j'}(h_{-j'},\zeta,j')=R$ given $REG_{j',-j'}$, so her payoff is $P(\sigma_{j'}|h^{<j'},h^{t-1}_{j'})v^{m_i}_{j'}+(1-P(\sigma_{j'}|h^{<j'},h^{t-1}_{j'}))v^0_{j'}$. Again, following $\sigma^*_{j'}$ for the rest of the round maximizes $P(\sigma_{j'}|h^{<j'},h^{t-1}_{j'})$. ■

Backward Induction: Given that players will follow σ^* in subsequent rounds and Claim 1 of Lemma 10, we can assume $\theta_j(h_{-j},\zeta,j'')=R$ for each j'' for which the j''-checking round follows the current round. Hence, the same proof as for Lemmas 26 and 27 establish each player's incentive to follow σ^* after any history.

Message Round: Again, given that players will follow σ^* in the checking rounds and Claim 1 of Lemma 10, we can assume $\theta_j(h_{-j},\zeta,j')=R$ for each $j'\in I$, and therefore assume $(a_{j',t}(n),\omega_{j',t}(n))_{t\in\mathbb{T}(\text{msg})}=(a_{j',t},\omega_{j',t})_{t\in\mathbb{T}(\text{msg})}$ and $\text{susp}_n(h_n)=0$ for all $n,j'\in I$. Given this, the strategy of each player $j \neq i$ does not affect $w_j(h, \zeta)$, so incentives are satisfied. For player i, given $(a_{j',t}(n), \omega_{j',t}(n))_{t\in\mathbb{T}(\text{msg})} = (a_{j',t}, \omega_{j',t})_{t\in\mathbb{T}(\text{msg})}$ for all $n, j' \in I$, $m_i(n)$ will equal $\hat{m}_i$ if player i plays $(a_{i,t})_{t\in\mathbb{T}(\text{msg})}$ corresponding to the binary expansion of $\hat{m}_i$ (with the interpretation that, if $(a_{i,t})_{t\in\mathbb{T}(\text{msg})}$ does not correspond to the binary expansion of any $\hat{m}_i \in M_i$, then $m_i(n) = 1$). Hence, σ_i^{*,m_i^*} is optimal after any history.

i^***-QBFE:** The last part of the lemma is immediate: Since $v_j^E = v_j^{m_i} = v_j^{\text{punish}}$ for each $m_i \in M_i$ and $j \neq i^*$, players $-i^*$'s incentives are satisfied. For player i^*, the proof of the first part of the lemma applies.

N.7 Proof of Lemma 14

We construct strategies $(\sigma_i^{**}(x_i))_{i,x_i}$ and reward functions $\left(\pi_i^{**}\left(x_{-i}, h_{-i}^{\mathbb{T}'}\right)\right)_{i,x_{-i},h_{-i}^{\mathbb{T}'}}$ in the T_1-period game that satisfy the premise of Lemma 9.

Construction of $\sigma_i^{**}(x_i)$

Play for the first T^* periods is given by $(\sigma_i^*(x_i))_i$. Play from periods T^*+1 to T_1 is given by the Phase $(\text{final}, 1, i)_{i\in I}$ strategies outlined in Section E. More precisely:

- Player $i-1 \pmod N$ sends $t_{i-1}(1), ..., t_{i-1}(L)$ using the verified protocol with repetition T_0 and $\mathcal{I}_{\text{jam}} = -i$. Each player $n \in I$ infers a message $(t_{i-1}(1)(n), ..., t_{i-1}(L)(n))$.

- Sequentially, each player $n \neq i, i-1$ sends $h_{n,t_{i-1}(l)(n)} = \left(a_{n,t_{i-1}(l)(n)}, \omega_{n,t_{i-1}(l)(n)}\right)_{l=1,\dots,L}$ and $\chi_n \in \{0,1\}$ using the secure protocol with repetition T_0 and $\mathcal{I}_{\text{jam}} = \{i-1\}$. For each $n \neq i, i-1$, player $i-1$ infers a message $\left(h_{n,t_{i-1}(l)(n)}(i-1), \chi_n(i-1)\right)$.

- If there exists a player $n \neq i$ with $\text{susp}(h_n) = 1$ or $\theta_i(h_{-i}) = E$ in the verified protocol, or if player $i-1$ infers 0 or plays JAM during a round where she receives a message in the secure protocol, let $s_{i-1} = 0$ ("communication fails"). Otherwise, $s_{i-1} = 1$ ("communication succeeds"). Note that s_{i-1} is a function of $h_{-i}^{\mathbb{T}'}$. Here, ζ_n is assumed to equal jam for each $n \neq i$ and reg for i, and so is omitted from θ_i.

Construction of β^{**}

In periods where player n sends a message via the secure protocol, specify trembles as in Lemma 8. In periods where players use the verified protocol, any consistent belief system suffices. For periods $1, ..., T^*$, let $\beta^{**} = \beta^*$.

Construction of $\pi_i^{**}\left(x_{-i}, h_{-i}^{\mathbb{T}'}\right)$

Fix x_{i-1} arbitrarily. If $s_{i-1} = 1$, we denote player $i-1$'s inference of player n's message during phase (final, $1, i$) by $h_n^{\mathbb{L}_{i-1}}(i-1)$ and $\chi_n(i-1)$. As in the proof of Lemma 9, define $\tilde{h}_{-i}^{\mathbb{L}_{i-1}} = h_{-i}^{\mathbb{L}_{i-1}}(i-1)$ and $\tilde{\chi}_{-i} = \chi_{-i}(i-1)$ if $s_{i-1} = 1$, and define $\tilde{h}_{-i}^{\mathbb{L}_{i-1}} = 0$ and $\tilde{\chi}_{-i} = 0$ otherwise. Since $M_i = \{1, ..., (T_0)^3\}^L$ for the verified communication, Condition (17) implies (38), and therefore Claim 4 of Lemma 11 holds for verified communication. In addition, (10) implies that the secure communication is successful with probability at least $(N-2)\, b(2A^{2L}) \left(\exp(-T_0) + 2\exp(-(T_0)^{\frac{1}{2}})\right)$. In total, we have

$$\begin{aligned}\min_{h_{-i}^{\mathbb{L}_{i-1}}} \Pr\left(s_{i-1} = 1 | h_{-i}^{\mathbb{L}_{i-1}}\right) \;&\geq\; 1 - \exp(-(T_0)^{\frac{1}{3}}) - (N-2)\, b(2A^{2L}) \left(\exp(-T_0) + 2\exp(-(T_0)^{\frac{1}{2}})\right) \\ : \;&=\; 1 - p_{\text{error}}^1(T_0). \qquad (98)\end{aligned}$$

Moreover, the event $s_{i-1} = 1$ implies $h_{-i}^{\mathbb{L}_{i-1}}(i-1) = h_{-i}^{\mathbb{L}_{i-1}}$ and $\chi_{-i}(i-1) = \chi_{-i}$, and the reward π_i^* satisfies the condition (49). Hence, in the notation of Lemma 25,

$$\hat{\varepsilon} = \frac{p_{\text{error}}^1(T_0)}{1 - p_{\text{error}}^1(T_0)}, \text{ and } F = \max_{x_{-i}, h_{-i}^{\mathbb{T}'}, \tilde{h}_{-i}^{\mathbb{L}_{i-1}}} \left|\pi_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}, \chi_{-i}\right)\right| \leq_{\text{by (46)}} 7\bar{u}T^*, \quad c = 2\varepsilon^* T^*. \qquad (99)$$

Therefore, Lemma 25 implies that there exists $\tilde{\pi}_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, \tilde{h}_{-i}^{\mathbb{L}_{i-1}}, \tilde{\chi}_{-i}\right)$ such that

$$\max_{x_{-i}, h_{-i}^{\mathbb{T}'}, \tilde{h}_{-i}^{\mathbb{L}_{i-1}}} \left|\tilde{\pi}_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, \tilde{h}_{-i}^{\mathbb{L}_{i-1}}, \tilde{\chi}_{-i}\right)\right| \leq (1 + 2\hat{\varepsilon})\, F, \qquad (100)$$

$$\mathbb{E}\left[\tilde{\pi}_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, \tilde{h}_{-i}^{\mathbb{L}_{i-1}}, \tilde{\chi}_{-i}\right) | x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}} \chi_{-i}\right] = \pi_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}} \chi_{-i}\right), \qquad (101)$$

$$\text{sign}\,(x_{i-1})\, \tilde{\pi}_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, \tilde{h}_{-i}^{\mathbb{L}_{i-1}}, \tilde{\chi}_{-i}\right) \geq -(1 + \hat{\varepsilon})c + \hat{\varepsilon}F \;\forall x_{-i}, h_{-i}^{\mathbb{T}'}, \tilde{h}_{-i}^{\mathbb{L}_{i-1}}, \tilde{\chi}_{-i}, \text{ and} \qquad (102)$$

$$\tilde{\pi}_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, \tilde{h}_{-i}^{\mathbb{L}_{i-1}}, \tilde{\chi}_{-i}\right) \text{ is minimized when } s_{i-1} = 0. \qquad (103)$$

We define the reward function

$$\pi_i^{**}\left(x_{-i}, h_{-i}^{\mathbb{T}''}\right) = \tilde{\pi}_i^{*}\left(x_{-i}, h_{-i}^{\mathbb{T}'}, \tilde{h}_{-i}^{\mathbb{L}_{i-1}}, \tilde{\chi}_{-i}\right) + \sum_{\substack{t=1,\dots,T_1 \\ t\notin\bigcup_{l=1}^{L}\mathbb{T}(\text{main}(l))}} \tilde{\pi}_i^{\text{cancel}}\left(x_{i-1}, a_{-i,t}, \omega_{-i,t}\right)$$
$$+ \sum_{\substack{t\in\mathbb{T}(\text{final},1,i): \\ \text{verified protocol}}} \pi_i^{\text{verify}}(h_{-i}^{\mathbb{T}''}) + \sum_{\substack{t\in\mathbb{T}(\text{final},1,i): \\ \text{secure protocol}}} \pi_i^{\text{secure}}(h_{-i}^{\mathbb{T}''}).$$

Here, the rewards $\pi_i^{\text{verify}}(h_{-i}^{\mathbb{T}''})$ and $\pi_i^{\text{secure}}(h_{-i}^{\mathbb{T}''})$ are defined analogously to (42) and (30) for the periods where players $-i$ communicate by the verified and secure communication modules in phase $(\text{final}, 1, i)$. Note that these rewards depend only on the history in phase $(\text{final}, 1, i)$, and the per-period reward is bounded by 1. Also, $\tilde{\pi}_i^{\text{cancel}}$ is bounded by $[-\bar{u}, \bar{u}]$. So, we have

$$\left|\pi_i^{**}\left(x_{-i}, h_{-i}^{\mathbb{T}''}\right)\right| \leq |\mathbb{T}''|\,(1+\bar{u}) + (1+2\hat{\varepsilon})\,7\bar{u}T^* \leq_{\text{by (17)}} 8\bar{u}T^*. \tag{104}$$

Since $\pi_i^{**}\left(x_{-i}, h_{-i}^{\mathbb{T}''}\right)$ satisfies (37) (given (104)), it remains to verify the other three conditions of Lemma 9.

[Sequential Rationality:] We verify (35) for $t = 1, \dots, T_1$ by backward induction. Given $\tilde{\pi}_i^{\text{cancel}}$, for periods $t' = T^* + 1, \dots, T_1$, player i maximizes the conditional expectation of

$$\tilde{\pi}_i^{*}\left(x_{-i}, h_{-i}^{\mathbb{T}'}, \tilde{h}_{-i}^{\mathbb{L}_{i-1}}, \tilde{\chi}_{-i}\right) + \sum_{\substack{t\in\mathbb{T}(\text{final},1,i): \\ \text{verified}}} \pi_i^{\text{verify}}(h_{-i}^{\mathbb{T}''}) + \sum_{\substack{t\in\mathbb{T}(\text{final},1,i): \\ \text{secure}}} \pi_i^{\text{secure}}(h_{-i}^{\mathbb{T}''}).$$

Since the reward $\tilde{\pi}_i^{*}\left(x_{-i}, h_{-i}^{\mathbb{T}'}, \tilde{h}_{-i}^{\mathbb{L}_{i-1}}, \tilde{\chi}_{-i}\right)$ depends only on the histories in $\mathbb{T}(\text{final}, 1, i)$, player i follows the equilibrium strategy in phases $(\text{final}, 1, j)_{j\neq i}$.

For phase $(\text{final}, 1, i)$, given (100) and (103), the premise for secure communication with magnitude $(1+2\hat{\varepsilon})\,F$ for player i is satisfied for all $x \in \{G, B\}^N$. In addition, as $v_i^E = v_i^0 =$ [value of $\tilde{\pi}_i^*$ given $s_{i-1} = 0$], the premise for verified communication with magnitude $(1+2\hat{\varepsilon})\,F$ for player i is satisfied for all $x \in \{G, B\}^N$. Since $\mathcal{I}_{\text{jam}} = -i$ for verified communication, Condition (17) implies Conditions (38), (43), and (44) (for verified communication), as well as Condition (32) (for secure communication). In total, Lemmas 8 and 13 imply sequential rationality for $t' \in \mathbb{T}(\text{final}, 1, i)$. Finally, since π_i^* and $\tilde{\pi}_i^*$ are equal in expectation given $x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}, \chi_{-i}$, (47) implies (35) for $t = 1, \ldots, T^*$.

[Promise Keeping:] Since π_i^* and $\tilde{\pi}_i^*$ are equal in expectation given $x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}$, (48) holds.

[Self Generation:] By (17) and (102), $\text{sign}\,(x_{i-1})\,\tilde{\pi}_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, \tilde{h}_{-i}^{\mathbb{L}_{i-1}}, \tilde{\chi}_{-i}\right) \geq -3\bar{u}T^*$. Other terms in $\pi_i^{**}\left(x_{-i}, h_{-i}^{\mathbb{T}''}\right)$ are bounded by $(1+\bar{u})\,|\mathbb{T}''| + 2\varepsilon^* L(T_0)^3 \leq_{\text{by (17)}} -2\bar{u}T^*$. So, (37) holds.

N.8 Proof of Lemma 15

Compared to Lemma 14, we introduce (50) and replace (46) with (51) (a more restrictive condition), (47) with (52) (less restrictive) and (48) with (53) (less restrictive). We show that the third replacement is without loss, and then show the same for the second. Given (52), let

$\hat{v}_i\,(x_{-i}) := \frac{1}{L(T_0)^3}\mathbb{E}^{\sigma^*(x)}\left[\sum_{t\in\bigcup_{l=1}^{L}\mathbb{T}(\text{main}(l))} \hat{u}_i\,(\mathbf{a}_t) + \pi_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}, \chi_{-i}\right)\right]$.

Since $v_i(x_{i-1}) \in [-\bar{u}, \bar{u}]$, Conditions (49) and (53) imply

$$\hat{v}_i\,(x_{-i}) - (v_i(x_{i-1}) + 2\text{sign}(x_{i-1})\varepsilon^*) \in [-2\bar{u}, 2\bar{u}]\,. \tag{105}$$

Define $\tilde{\pi}_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}, \chi_{-i}\right) = \pi_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}, \chi_{-i}\right) - \left(\hat{v}_i\,(x_{-i}) - (v_i(x_{i-1}) + 2\text{sign}(x_{i-1})\varepsilon^*)\right)T^*$. Note that changing the reward function from π_i^* to $\tilde{\pi}_i^*$ only subtracts a constant and thus does not affect sequential rationality. In addition, since $\text{sign}(x_{i-1})\left(\hat{v}_i\,(x_{-i}) - (v_i(x_{i-1}) + \text{sign}(x_{i-1})2\varepsilon^*)\right) \geq 0$ by (53), (49) implies $\text{sign}(x_{i-1})\tilde{\pi}_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}, \chi_{-i}\right) \geq -2\varepsilon^* T^*$. Hence, self-generation also holds with reward function $\tilde{\pi}_i^*$. Finally, since $(\hat{v}_i\,(x_{-i}) - (v_i(x_{i-1}) + 2\text{sign}(x_{i-1})\varepsilon^*))T^*$ is bounded by $2\bar{u}T^*$ by (105), (51) implies

$$\sup_{x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}, \chi_{-i}} \left|\tilde{\pi}_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}, \chi_{-i}\right)\right| \leq 7\varepsilon^* T^*.$$

Hence, (46) also holds with reward function $\tilde{\pi}_i^*$. Therefore, the premise of Lemma 14 holds. We now show that it is also without loss to replace (47) with (52). We assume that, before the end of main phase l, player i believes that $t_{i-1}(l)$ is uniformly distributed over $\mathbb{T}(\text{main}(l))$.[27]

[27] This belief results if trembles in periods $t = 1, ..., T^*$ are independent of $(\mathbb{L}_i, h_i^{t-1})$, and thus is consistent.

Define

$$\tilde{\pi}_i^{\text{cancel}}(x_{i-1}, a_{-i}, \omega_{-i}) := \pi_i^{\text{cancel}}(x_{i-1}, a_{-i}, \omega_{-i}) + \text{sign}(x_{i-1}) \max_{\tilde{x}_{i-1}, \tilde{a}_{-i}, \tilde{\omega}_{-i}} \pi_i^{\text{cancel}}(\tilde{x}_{i-1}, \tilde{a}_{-i}, \tilde{\omega}_{-i}).$$

We have $\tilde{\pi}_i^{\text{cancel}}(x_{i-1}, a_{-i}, \omega_{-i}) \in [-2\bar{u}, 2\bar{u}]$, by (7). Note that

$$\mathbb{E}\left[\hat{u}_i(\mathbf{a}) + \pi_i^{\text{cancel}}(x_{i-1}, a_{-i}, \omega_{-i}) | a\right] = v_i(x_{i-1}) + \text{sign}(x_{i-1}) \max_{\tilde{x}_{i-1}, \tilde{a}_{-i}, \tilde{\omega}_{-i}} \pi_i^{\text{cancel}}(\tilde{x}_{i-1}, \tilde{a}_{-i}, \tilde{\omega}_{-i}) \tag{106}$$

and $\text{sign}(x_{i-1})\tilde{\pi}_i^{\text{cancel}}(x_{i-1}, a_{-i}, \omega_{-i}) \geq 0$. Since $T^* \in \mathbb{T}'$, we can define

$$\tilde{\pi}_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}, \chi_{-i}\right) := \begin{cases} \pi_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}\right) & \text{if } \chi_n = 0 \text{ for all } n \neq i, \\ \sum_{t \in \mathbb{T}'} \tilde{\pi}_i^{\text{cancel}}(x_{i-1}, a_{-i}, \omega_{-i}) \\ + (T_0)^3 \sum_{l=1}^{L} \tilde{\pi}_i^{\text{cancel}}\left(x_{i-1}, a_{-i,t_{i-1}(l)}, \omega_{-i,t_{i-1}(l)}\right) & \text{if } \chi_n = 1 \text{ for some } n \neq i. \end{cases}$$

The $(T_0)^3$ term cancels the probability that $t_{i-1}(l) = t$ for each $t \in \mathbb{T}(\text{main}(l))$, so with this reward function player i is indifferent over all action profiles when $\chi_n = 1$ for some $n \neq i$.

Given reward function $\tilde{\pi}_i^*$, (47) and (49) hold. Moreover, given (51) for $\pi_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}\right)$,

$$\sup_{x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}, \chi_{-i}} \left|\tilde{\pi}_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}, \chi_{-i}\right)\right| \leq \max\{7\bar{u}T^*, 2\bar{u}T^*\} \leq 7\bar{u}T^*.$$

Therefore, the premise of Lemma 14 holds.

N.9 Proof of Lemma 17

Definition of the Reward Function

We must define $\pi_{i,t}^{\text{indiff}}(h_{-i})$. Given h_{-i}, fix h_i uniquely identified from h_{-i} by Lemma 2. Let H_i^0 be the set of histories for player i with $\omega_{i,1} \neq a^1$ and $\omega_{i,2} \neq a^1$. Given the resulting profile $h = (h_i, h_{-i})$, for $t = 2$, we define $\Delta v_{i,t}(h_{-i})$ as follows:

1. If $\omega_{i,t-1} = a^1$, then $\Delta v_{i,t}(h_{-i}) := 0$.

2. Otherwise, define $\Pr(\mathcal{I}_{\text{jam}} \setminus \{i\} | h^{t-1}, H_i^0, a_i)$ as the conditional probability that the realized set of jamming players other than i at the end of the protocol equals $\mathcal{I}_{\text{jam}} \setminus \{i\}$, given

that players $-i$ follow the protocol, $h_i \in H_i^0$, and player i plays a_i in period t. Let

$$\Delta v_{i,t}(h_{-i}) = \sum_{\mathcal{I}_{\text{jam}} \setminus \{i\}} \begin{pmatrix} \Pr\left(\mathcal{I}_{\text{jam}} \setminus \{i\} | h^{t-1}, H_i^0, a^1\right) \\ -\Pr\left(\mathcal{I}_{\text{jam}} \setminus \{i\} | h^{t-1}, H_i^0, a^0\right) \end{pmatrix} v_i\left(\mathcal{I}_{\text{jam}} \setminus \{i\}\right).$$

Note that $|\Delta v_{i,t}(h_{-i})| \leq K$, by the bound (63).

Finally, for $t = 2$, we define

$$\pi_{i,t}^{\text{indiff}}(h_{-i}) = -\mathbf{1}_{\{a_{i,t}=a^1\}} \Delta v_{i,t}(h_{-i}). \tag{107}$$

For $t = 1$, define $\Pr\left(\mathcal{I}_{\text{jam}} \setminus \{i\} | h^{t-1}, H_i^0, a_i\right)$ as the conditional probability that the realized set of jamming players other than i at the end of the protocol equals $\mathcal{I}_{\text{jam}} \setminus \{i\}$, given that players $-i$ follow the protocol, $h_i \in H_i^0$, and player i plays a_i in period t and a^0 in period $t+1$. The resulting definitions of $\Delta v_{i,t}(h_{-i})$ and $\pi_{i,t}^{\text{indiff}}(h_{-i})$ are the same as for $t = 2$.

Note that $\left|\pi_{i,t}^{\text{indiff}}(h_{-i})\right| \leq K$ for $t = 1, 2$. Hence, condition (i) holds.

Incentive Compatibility

We show that, for every player i and period $t = 1, 2$, it is optimal for player i to follow the protocol in period t given that she will follow the protocol in every later period.

Recall that $\Pr\left(h_i \in H_i^0\right)$ is independent of player i's strategy, and Condition 2 of the premise implies that $w_i(h) = w_i(\tilde{h})$ for all h and $\tilde{h}$ satisfying $h_i \notin H_i^0$ and $\tilde{h}_i \notin H_i^0$. Moreover, $w_i(h) = v_i\left(\mathcal{I}_{\text{jam}} \setminus \{i\}\right)$ if $h_i \in H_i^0$. Hence, player i maximizes her payoff by maximizing $\sum_{t=1}^{2} \pi_{i,t}^{\text{indiff}}(h_{-i}) + v_i\left(\mathcal{I}_{\text{jam}} \setminus \{i\}\right)$ conditional on $h_i \in H_i^0$.

For $t = 2$, ignoring sunk payoffs, player i maximizes $\pi_{i,t}^{\text{indiff}}(h_{-i}) + v_i\left(\mathcal{I}_{\text{jam}} \setminus \{i\}\right)$ conditional on $h_i \in H_i^0$. By (107), player i is indifferent between a^0 and a^1. Moreover, she is also indifferent between a^0 and any $a \notin \{a^0, a^1\}$, since (i) the distribution of $\mathcal{I}_{\text{jam}} \setminus \{i\}$ is the same whether she takes a^0 or $a \notin \{a^0, a^1\}$, and (ii) (107), $\pi_i^{\text{indiff}}(h_{-i})$ is the same as well.

For $t = 1$, noting that her period 1 action does not affect the distribution of anyone's action in period 2, player i again maximizes payoff $\pi_{i,t}^{\text{indiff}}(h_{-i}) + v_i\left(\mathcal{I}_{\text{jam}} \setminus \{i\}\right)$ conditional on $h_i \in H_i^0$. Again, (107) implies she is indifferent among all actions.