COSTLY CONCESSIONS: AN EMPIRICAL FRAMEWORK FOR MATCHING WITH IMPERFECTLY TRANSFERABLE UTILITY

ALFRED GALICHON[†], SCOTT DUKE KOMINERS[♭], AND SIMON WEBER[§]

Abstract. We introduce an empirical framework for models of matching with imperfectly transferable utility and unobserved heterogeneity in tastes. Our framework allows us to characterize matching equilibrium in a flexible way that includes as special cases the classical fully- and non-transferable utility models, collective models, and settings with taxes on transfers. We allow for the introduction of a general class of additive unobserved heterogeneity on agents' preferences. We show existence and uniqueness of an equilibrium under minimal assumptions. We then provide two algorithms to compute the equilibrium in our model. The first algorithm operates under any structure of heterogeneity in preferences; the second is more efficient, but applies only in the case in which random utilities are logit. We show that the log-likelihood of the model has a simple expression and we compute its derivatives. An empirical illustration is provided in the appendix.

Date: March 12, 2018. Galichon gratefully acknowledges funding from NSF grant DMS-1716489, and from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013) / ERC grant agreement no. 312503. Kominers gratefully acknowledges the support of the NSF (grants CCF-1216095 and SES-1459912), the Harvard Milton Fund, and the Ng Fund of the Harvard Center of Mathematical Sciences and Applications. The authors thank the editor (James Heckman) and three anonymous reviewers for very helpful comments. Additionally, the authors appreciate the insightful comments of Pierre-André Chiappori, Edoardo Ciscato, Matthew Gentzkow, Yinghua He, Yu-Wei Hsieh, Murat Iyigun, Sonia Jaffe, Thierry Magnac, Ismael Mourifié, Bernard Salanié, Aloysius Siow, Heidi Williams, and seminar participants at the AEA Meetings, the University of California, Berkeley, Boston University, the University of Chicago, EIEF, the Fields Institute, Harvard, MIT, NYU, Penn State University, Princeton, Sciences Po, Stanford, the University of Toronto, UCLA, the University of Pennsylvania, Yale, and the University of Zürich. Yue Li provided great research assistance. Galichon is grateful for the hospitality of MIT, where part of this paper was written.

1. Introduction

The field of family economics has two principal approaches to modeling and understanding marriage patterns: *matching models* emphasize market-level forces and take heterogeneous tastes over partners as primitives; *collective models*, by contrast, focus on the impact of intra-household bargaining. However, these two approaches have not been combined yet, because collective models necessarily include nonlinearities of a form absent from classical matching frameworks. In this paper, we develop an Imperfectly Transferable Utility (ITU) matching model with random utility that allows us to unify marriage matching with unobserved heterogeneity in tastes and the collective framework. Our setting moreover allows for the introduction of a general class of additive unobserved heterogeneity on agents' preferences, under which existence and uniqueness of an equilibrium is obtained. These models can be computed efficiently and easily estimated.

Naturally, matching models have been extensively used to model the marriage market, in which men and women with heterogenous tastes may form pairs; this approach, pioneered in economics by Becker (1973) and Shapley and Shubik (1972), focuses mainly on matching patterns and the sharing of the surplus in a Transferable Utility (TU) setting; see a comprehensive survey of TU models of marriage in Chiappori (2017). While appealing from a theoretical point of view, TU matching models have a significant limitation: TU models rely on the assumption that there is a way to transfer utility between partners in an additive manner. Consequently, a man and a woman who match and generate a joint surplus Φ may decide on splitting this surplus between the utility of the man u and the utility of the woman v in any way such that $u + v \leq \Phi$. In this case, the bargaining frontier in the space of the partners' utilities is simply a straight line of slope -1. However, the assumption that the bargaining frontier has this particular shape may be inappropriate—one can think of many cases in which there are nonlinearities that partially impede the transfer of utility between matched partners. Such nonlinearities arise naturally in marriage markets, where the transfers between partners might take any form (e.g. cash, favor exchanges, and change in time use or consumption patterns), and the utility cost of a concession to one partner may not exactly equal the benefit to the other. An extreme case is the Nontransferable

Utility (NTU) framework (Gale and Shapley, 1962), in which there is no possibility of compensating transfer between partners. However, although NTU matching seems well-suited to settings like school choice, where transfers are often explicitly ruled out, it is also not, in general, the most realistic assumption.[1]

The collective model approach of Chiappori (1992), which focuses on intra-household bargaining over a potentially complex feasible utility set, cannot generally be expressed in terms of TU matching models, because imperfect substitutability in the bargaining process creates nonlinearities.[2] Indeed, in TU models, households are modelled as representative consumers, so that intra-household allocation of power does not affect the household's aggregate behavior, and public good consumption in particular. In spite of their complementarity, the matching and collective approaches to modeling marriage have not yet been combined into a single empirical framework. Choo and Siow (2006) observed this issue explicitly, stating that "[their] model of marriage matching should be integrated with models of intra-household allocations"—an integration which, in over ten years since Choo and Siow (2006) were writing, has not been achieved prior to our work.

In our Imperfectly Transferable Utility framework, partners participate in a one-to-one bilateral matching market, but utility transfers within matches are not necessarily additive. This allows us to embed TU, NTU, and collective approaches to the marriage market. Our framework also makes sense for modeling labor markets—because of taxation, an employer must pay more in wages than its employees actually receive (Dupuy et al., 2017). In contrast with prior ITU matching models, our setting allows for a compact characterization of equilibrium as a set of nonlinear equations, as well as efficient computational approaches. We prove existence and uniqueness of the equilibrium outcome in our ITU model with general heterogeneity in tastes. In the case that the heterogeneity is logit, we show how maximum likelihood estimation of our model can be performed in a very straightforward

[1]NTU matching appears as well in the "bargaining-in-marriage" approach of Pollak (2017).

[2]There are exceptions—see, e.g., the model described in Browning et al. (2014), pp. 83 and 118, in which one private good is assumed to provide the same marginal utility to both members of the household, and thus can be used to transfer utility additively.

manner, which we illustrate by estimating a simple collective model of matching in a market with marital preferences and private consumption.

Relation to the literature. The theory of ITU matching has (at least implicitly) been studied by a number of authors: Crawford and Knoer (1981), Kelso and Crawford (1982), and Hatfield and Milgrom (2005) found conditions and algorithms that can be used to find competitive equilibrium outcomes in ITU matching markets; they also analyzed the structure of the sets of equilibria. Kaneko (1982), Quinzii (1984), Alkan (1989), Alkan and Gale (1990), Gale (1984), Demange and Gale (1985), and Fleiner et al. (2017) provide results on the existence of equilibria and studied properties of the core. Pycia (2012) considers a general many-to-one matching setting with imperfectly transferable utility and characterizes the sharing rules that lead to pairwise alignment of preferences and existence of equilibria. Dupuy et al. (2017) study the problem of matching with linear taxes and provide comparative statics results. Legros and Newman (2007) find conditions under which positive assortativeness arises in ITU models; they apply these findings to problems of matching under uncertainty with risk aversion. Recently, Chiappori and Reny (2016) considered a similar model with risk sharing. Chade and Eeckhout (2014) extended the work of Legros and Newman (2007) to the case that agents have different risky endowments. Nöldeke and Samuelson (2015) connect ITU matching with abstract notions of convexity. Greinecker and Kah (2018) study the notion of pairwise stability in a ITU setting with a continuum of agents. Chiappori (2012) provides an illustrative example of how collective models naturally embed into ITU matching models.

However, the literature on the structural estimation of matching models has so far been restricted to the TU and NTU cases only. In the wake of the seminal work by Choo and Siow (2006), many papers have exploited heterogeneity in preferences for identification in the TU case (see Fox, 2010; Chiappori, Oreffice and Quintana-Domeque, 2012; Galichon and Salanié, 2015; Chiappori, Salanié, and Weiss, 2017; and Dupuy and Galichon, 2014). Choo and Seitz (2013) present one of the first attempts to reconcile the matching and the collective approaches, albeit still in the TU case. Other research in the collective model literature have endogenized the sharing rule, but mostly in a TU framework (see chapters 8 and 9 in the textbook by Browning et al. (2014) for a review, and references therein, e.g.

Chiappori et al. (2009) and Iyigun and Walsh (2007)). Cherchye et al. (2017) derive Afriat-style inequalities that result from ITU stability in collective models. Similar strategies have been successfully applied in the NTU case (see Dagsvik (2000), Menzel (2015), Hitsch, Hortaçsu, and Ariely (2010), and Agarwal (2015)). To the best of our knowledge, our work is the first to provide an empirical framework for general ITU models with random utility.

Organization of the paper. The remainder of the paper is organized as follows. Section 2 provides an introduction to the ITU framework building off the classic TU case. Section 3 formally describes the model we consider, introduces important technical machinery used throughout, and provides a number of examples. Section 4 introduces heterogeneity in tastes, defines the notion of aggregate equilibrium, and relates it with the classical notion of individual stability. Then, section 5 determines the equations characterizing the aggregate equilibrium, shows existence and uniqueness results, and provides an algorithm to find equilibria in our framework. Section 6 deals with the important special case of logit heterogeneity, providing a more efficient algorithm for find equilibria in that case, and discussing maximum likelihood estimation. Section 7 concludes. All proofs are presented in appendix A. The appendix also contains an illustrative example, and some additional results.

2. Prelude: From TU matching to ITU matching

We start with a brief overview of the structure of our model, which we hope will be particularly useful for readers who have already some degree of familiarity with TU matching models. To guide intuition, we start with the classical TU model, and show how it extends to the more general ITU model. A closely related discussion is presented in chapter 9 of Roth and Sotomayor (1990) and chapter 7 of Chiappori (2017); however, we innovate by introducing a model of ITU matching that is general enough to embed both the TU and the NTU models, while allowing the introduction of additive unobserved heterogeneity in preferences.

2.1. **The TU matching model.** We first recall the basics of the Transferable Utility model. In this model, it is assumed that there are finite sets $\mathcal{I}$ and $\mathcal{J}$ of men and women.

If a man $i \in \mathcal{I}$ and a woman $j \in \mathcal{J}$ decide to match, they respectively generate surplus terms α_{ij} and γ_{ij}; the vectors α and γ are primitives of the model.

If man i and woman j match, they also may agree on a transfer t_{ij} (determined at equilibrium) from the woman to the man (positive or negative), so that their utilities after transfer are respectively $\alpha_{ij} + t_{ij}$ and $\gamma_{ij} - t_{ij}$. If i and j choose to remain unmatched, they enjoy reservation utilities $\mathbf{U}_{i0}$ and $\mathbf{V}_{0j}$, which are exogenous.

Let μ_{ij} encode the "matching" (also determined at equilibrium), which is equal to 1 if i and j are matched, and 0 otherwise. Hence, a matching should satisfy the *feasibility conditions*

$$\textbf{(F)} \begin{cases} \mu_{ij} \in \{0,1\} \\ \sum_{j \in \mathcal{J}} \mu_{ij} \leq 1 \\ \sum_{i \in \mathcal{I}} \mu_{ij} \leq 1, \end{cases}$$

Let u_i and v_j be the indirect payoffs of man i and woman j, respectively. These quantities are determined at equilibrium, and we have $u_i = \max_{j \in \mathcal{J}} \{\alpha_{ij} + t_{ij}, \mathbf{U}_{i0}\}$ and $v_j = \max_{i \in \mathcal{I}} \{\gamma_{ij} - t_{ij}, \mathbf{V}_{0j}\}$, which implies in particular that for any i and j, the inequalities $u_i \geq \alpha_{ij} + t_{ij}$ and $v_j \geq \gamma_{ij} - t_{ij}$ jointly hold, implying that $u_i + v_j \geq \alpha_{ij} + \gamma_{ij}$ should hold for every $i \in \mathcal{I}$ and $j \in \mathcal{J}$. Likewise, $u_i \geq \mathbf{U}_{i0}$ and $v_j \geq \mathbf{V}_{0j}$ should hold for all i and j. Thus, the equilibrium payoffs should satisfy the *stability conditions*

$$\textbf{(S)} \begin{cases} u_i + v_j \geq \alpha_{ij} + \gamma_{ij} \\ u_i \geq \mathbf{U}_{i0} \\ v_j \geq \mathbf{V}_{0j}. \end{cases}$$

Finally, we relate the equilibrium matching μ and the equilibrium payoffs (u, v). If $\mu_{ij} > 0$, then $\mu_{ij} = 1$ and i and j are matched, so the first line of (**S**) should hold as an equality. On the contrary, if $\sum_j \mu_{ij} < 1$, then $\sum_j \mu_{ij} = 0$, so i is unmatched and $u_i = \mathbf{U}_{i0}$. Similar conditions hold for j. To summarize, the equilibrium quantities are related by the following set of *complementary slackness* conditions:

$$\textbf{(CS)} \begin{cases} \mu_{ij} > 0 \Longrightarrow u_i + v_j = \alpha_{ij} + \gamma_{ij} \\ \sum \mu_{ij} < 1 \Longrightarrow u_i = \mathbf{U}_{i0} \\ \sum \mu_{ij} < 1 \Longrightarrow v_j = \mathbf{V}_{0j} \end{cases}.$$

Following the classical definition, (μ, u, v) is an *equilibrium outcome* in the TU matching model if the feasibility conditions **(F)**, stability conditions **(S)**, and complementary slackness conditions **(CS)** are met. The characterization of the solutions to that problem in terms of linear programming is well known (see e.g. chapter 8 of Roth and Sotomayor 1990 or chapter 3 of Chiappori 2017). The equilibrium outcomes (μ, u, v) are such that μ maximizes the utilitarian social welfare $\sum_{ij} \mu_{ij} \left(\alpha_{ij} + \gamma_{ij} - \mathbf{U}_{i0} - \mathbf{V}_{0j}\right)$ with respect to $\mu \geq 0$ subject to $\sum_j \mu_{ij} \leq 1$ and $\sum_i \mu_{ij} \leq 1$, which is the primal problem; and the payoffs (u, v) are the solution of the corresponding dual problem, hence they minimize $\sum_i u_i + \sum_j v_j$ subject to $u_i + v_j \geq \alpha_{ij} + \gamma_{ij}$, and $u_i \geq \mathbf{U}_{i0}$, $v_j \geq \mathbf{V}_{0j}$. However, this interpretation in terms of optimality is very specific to the present TU case, as discussed in appendix C.

2.2. **The ITU matching model.** The ITU matching model is a natural generalization of the TU model. If man $i \in \mathcal{I}$ and woman $j \in \mathcal{J}$ agree to match with transfer t_{ij}, their utilities after transfer are respectively $\mathbf{U}_{ij}(t_{ij})$ and $\mathbf{V}_{ij}(t_{ij})$, where $\mathbf{U}_{ij}(.)$ is a continuous and nondecreasing function and $\mathbf{V}_{ij}(.)$ is a continuous and nonincreasing function. (Note that in the specialization to the TU case, $\mathbf{U}_{ij}(t_{ij}) = \alpha_{ij} + t_{ij}$ and $\mathbf{V}_{ij}(t_{ij}) = \gamma_{ij} - t_{ij}$). If i or j opt to remain unmatched, they enjoy respective payoffs $\mathbf{U}_{i0} \in \mathbb{R}$ and $\mathbf{V}_{0j} \in \mathbb{R}$, which are exogenous reservation utilities. As before, the matching μ has term μ_{ij} equal to 1 if i and j are matched, 0 otherwise; clearly, the set of conditions **(F)** defining feasible matchings is unchanged.

In equilibrium, the indirect payoffs are now given by $u_i = \max_{j \in \mathcal{J}} \{\mathbf{U}_{ij}(t_{ij}), \mathbf{U}_{i0}\}$ and $v_j = \max_{i \in \mathcal{I}} \{\mathbf{V}_{ij}(t_{ij}), \mathbf{V}_{0j}\}$, which implies in particular that for any i and j, the inequalities $u_i \geq \mathbf{U}_{ij}(t_{ij})$ and $v_j \geq \mathbf{V}_{ij}(t_{ij})$ jointly hold. However, in contrast to the TU case, adding up the utility inequalities does not cancel out the t_{ij} term. As a way out of this problem, we introduce in section 3 a function $D_{ij}(u, v)$, called *distance-to-frontier function*, which is non-decreasing in u and v and has $D_{ij}(\mathbf{U}_{ij}(t), \mathbf{V}_{ij}(t)) = 0$ for all t. Then $u_i \geq \mathbf{U}_{ij}(t_{ij})$ and $v_j \geq \mathbf{V}_{ij}(t_{ij})$ jointly imply that $D_{ij}(u_i, v_j) \geq D_{ij}(\mathbf{U}_{ij}(t), \mathbf{V}_{ij}(t)) = 0$.

Hence the equilibrium payoffs in an ITU model must satisfy the *nonlinear stability conditions*

$$\textbf{(S')} \begin{cases} D_{ij}\left(u_i, v_j\right) \geq 0 \\ u_i \geq \mathbf{U}_{i0} \\ v_j \geq \mathbf{V}_{0j}, \end{cases}$$

and the *nonlinear complementary slackness conditions*

$$\textbf{(CS')} \begin{cases} \mu_{ij} > 0 \Longrightarrow D_{ij}\left(u_i, v_j\right) = 0 \\ \sum \mu_{ij} < 1 \Longrightarrow u_i = \mathbf{U}_{i0} \\ \sum \mu_{ij} < 1 \Longrightarrow v_j = \mathbf{V}_{0j}. \end{cases}$$

A triple (μ, u, v) is an equilibrium outcome in the matching model with Imperfectly Transferable Utility whenever conditions **(F)**, **(S')** and **(CS')** are met.

3. Framework

We now give a complete description of the general framework sketched in the previous section. We consider a population of men and women who may decide either to remain single or to form heterosexual pairs. It will be assumed that if a man and a woman match, then they bargain over utility outcomes lying within a feasible set. In section 3.1, we shall describe the structure of these feasible sets. In section 3.2, we state our definition of an outcome, which specifies who is matched with whom and who gets what; we define pairwise stability, our equilibrium concept. In section 3.3 we give a number of examples of our model, including the classic TU and NTU models, matching with taxes, and collective models.

3.1. **The feasible bargaining sets.** If man $i \in \mathcal{I}$ and a woman $j \in \mathcal{J}$ are matched, then they bargain over a set of feasible utilities $(u_i, v_j) \in \mathcal{F}_{ij}$. We begin by describing the pairwise bargaining sets $\mathcal{F}_{ij}$; then, we provide two different—but equivalent—useful descriptions. First, we represent the feasible sets implicitly, by describing the bargaining frontier as the set of zeros of a function, and next, we shall represent the feasible sets explicitly, by finding a convenient parametrization of their frontiers.

3.1.1. *Assumptions on the feasible sets.* The following natural assumptions on the geometry of the sets $\mathcal{F}_{ij}$ is employed extensively throughout the paper.

Definition 1. The set $\mathcal{F}_{ij}$ is a *proper bargaining set* if the three following conditions are met:

(i) $\mathcal{F}_{ij}$ is closed and nonempty.

(ii) $\mathcal{F}_{ij}$ is *lower comprehensive*: if $u' \leq u$, $v' \leq v$, and $(u, v) \in \mathcal{F}_{ij}$, then $(u', v') \in \mathcal{F}_{ij}$.

(iii) $\mathcal{F}_{ij}$ is *bounded above*: Assume $u_n \to +\infty$ and v_n bounded below then for N large enough $(u_n, v_n) \notin \mathcal{F}$ for $n \geq N$; similarly for u_n bounded below and $v_n \to +\infty$.

Some comments on the preceding requirements are useful at this stage. The closedness of $\mathcal{F}_{ij}$ is classically needed for efficient allocations to exist. The fact that $\mathcal{F}_{ij}$ is lower comprehensive is equivalent to free disposal; in particular, it rules out the case in which $\mathcal{F}_{ij}$ has finite cardinality. The scarcity property rules out the possibility that both partners can obtain arbitrarily large payoffs. The fact that $\mathcal{F}_{ij}$ is nonempty, combined with condition (ii), implies that if both partners' demands are low enough, they can always be fulfilled. Finally, it is worth pointing out that these are the only restrictions that we shall impose on the bargaining sets; in particular, we do not require them to be convex sets.

3.1.2. *Implicit representation of the bargaining frontier.* We provide a first representation of the set $\mathcal{F}_{ij}$ as the lower level set of a function D_{ij}, which we have called "distance-to-frontier function" because $D_{\mathcal{F}_{ij}}(u, v)$ measures the signed distance (up to a factor $\sqrt{2}$) of (u, v) from the bargaining frontier of $\mathcal{F}_{ij}$, when running along the diagonal. (See figure 1a.) $D_{\mathcal{F}_{ij}}(u, v)$ is positive if (u, v) is outside of the feasible set, and negative if (u, v) is in the interior of the feasible set; its value is 0 at the frontier. Formally:

Definition 2. The distance-to-frontier function $D_{\mathcal{F}_{ij}} : \mathbb{R}^2 \to \mathbb{R}$ of a proper bargaining set $\mathcal{F}_{ij}$ is defined by

$$D_{\mathcal{F}_{ij}}(u, v) = \min \{z \in \mathbb{R} : (u - z, v - z) \in \mathcal{F}_{ij}\}. \tag{3.1}$$

The function $D_{\mathcal{F}_{ij}}$ defined by (3.1) exists: indeed, the set $\{z \in \mathbb{R} : (u - z, v - z) \in \mathcal{F}_{ij}\}$ is closed because $\mathcal{F}_{ij}$ is closed, bounded above because $\mathcal{F}_{ij}$ is bounded above, and nonempty

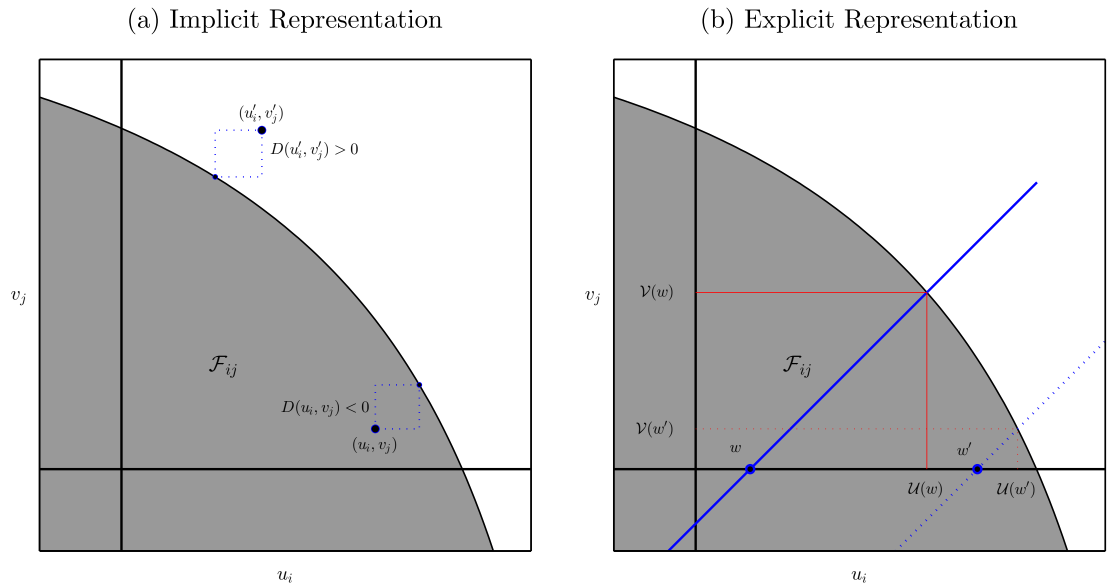

FIGURE 1. Implicit and explicit representations of the bargaining set $\mathcal{F}_{ij}$.

by condition (i) in definition 1; hence the minimum in (3.1) exists. By the definition of $D_{\mathcal{F}_{ij}}$, we have $\mathcal{F}_{ij}=\left\{(u,v) \in \mathbb{R}^2 : D_{\mathcal{F}_{ij}}(u,v) \leq 0\right\}$, and $D_{\mathcal{F}_{ij}}(u,v) = 0$ if and only if (u,v) lies on the frontier of $\mathcal{F}_{ij}$. The quantity $D_{\mathcal{F}_{ij}}(u,v)$ is interpreted as the distance (positive or negative) between (u,v) and the frontier of $\mathcal{F}_{ij}$ along the diagonal. In particular, $D_{\mathcal{F}_{ij}}(a+u, a+v) = a + D_{\mathcal{F}_{ij}}(u,v)$ for any real a, u and v. By the same token, if $D_{\mathcal{F}_{ij}}$ is differentiable at (u,v), then $\partial_u D_{\mathcal{F}_{ij}} + \partial_v D_{\mathcal{F}_{ij}} = 1$. The following lemma summarizes important properties of $D_{\mathcal{F}_{ij}}$.

Lemma 1. *Let $\mathcal{F}_{ij}$ be a proper bargaining set. Then:*

(i) $\mathcal{F}_{ij}=\left\{(u,v) \in \mathbb{R}^2 : D_{\mathcal{F}_{ij}}(u,v) \leq 0\right\}$.

(ii) For every $(u,v) \in \mathbb{R}^2$, $D_{\mathcal{F}_{ij}}(u,v) \in (-\infty, +\infty)$.

(iii) $D_{\mathcal{F}_{ij}}$ is $\gg$-isotone, meaning that $(u,v) \leq (u',v')$ implies $D_{\mathcal{F}_{ij}}(u,v) \leq D_{\mathcal{F}_{ij}}(u',v')$; and $u < u'$ and $v < v'$ jointly imply $D_{\mathcal{F}_{ij}}(u,v) < D_{\mathcal{F}_{ij}}(u',v')$.

(iv) $D_{\mathcal{F}_{ij}}$ is continuous.

(v) $D_{\mathcal{F}_{ij}}(a+u, a+v) = a + D_{\mathcal{F}_{ij}}(u,v)$.

The idea of describing a feasible set using a directional distance is not new; see e.g. Deaton and Muellbauer (1980), Chambers et al. (1998), and Parmeter and Kumbhakar (2014). Our contribution here is to show how this description is particularly useful for practical purposes in a matching context. Indeed, one important benefit of the representation of the bargaining sets in terms of distance functions is that basic geometric operations on these sets can be translated in terms of algebraic operations on the associated distance functions. In particular, the distance function associated with the union of sets is the minimum of the corresponding distance functions; and the distance function associated with the intersection of sets is the maximum of the distance functions. Formally:

Lemma 2. *Assume that $\mathcal{F}_1, ..., \mathcal{F}_K$ are K proper bargaining set, as introduced in definition 1. Then:*

(a) The sets $\mathcal{F}_\cup = \bigcup_{k\in\{1,...,K\}} \mathcal{F}_k$ and $\mathcal{F}_\cap = \bigcap_{k\in\{1,...,K\}} \mathcal{F}_k$ are proper bargaining sets, and

(b) The distance to frontier functions are given by

$$\begin{aligned} D_{\mathcal{F}_\cup}(u,v) &= \min\{D_{\mathcal{F}_1}(u,v), ..., D_{\mathcal{F}_K}(u,v)\} \\ D_{\mathcal{F}_\cap}(u,v) &= \max\{D_{\mathcal{F}_1}(u,v), ..., D_{\mathcal{F}_K}(u,v)\} \end{aligned}$$

This lemma is useful because it allows to create more complex models based on elementary ones. In particular, the bargaining set between a firm and a worker when there are progressive, piecewise linear taxes can be represented as an intersection of planar elementary bargaining sets, for which the distance function has a very simple expression. Likewise, the bargaining set between spouses in the case when there is a menu of public goods can be represented as a union of simpler bargaining sets. In examples 3.3.3 and 3.3.4, we will see applications of this idea.

3.1.3. *Explicit representation of the bargaining frontier.* We now give an explicit parametrization of the bargaining frontier, which will be useful in particular in section 5.

Given two utilities (u, v) such that $D_{\mathcal{F}_{ij}}(u, v) = 0$, let us introduce the *wedge* w to be the difference $w = u - v$.

Definition 3. Define $\mathcal{U}_{\mathcal{F}_{ij}}(w)$ and $\mathcal{V}_{\mathcal{F}_{ij}}(w)$ as the values of u and v such that

$$D_{\mathcal{F}_{ij}}(u,v) = 0 \text{ and } w = u - v. \tag{3.2}$$

See figure 1b. Definition 3 (and the existence of the functions $\mathcal{U}_{\mathcal{F}_{ij}}$ and $\mathcal{V}_{\mathcal{F}_{ij}}$) is motivated by the following result.

Lemma 3. *Let $\mathcal{F}_{ij}$ be a proper bargaining set. There are two 1-Lipschitz functions $\mathcal{U}_{\mathcal{F}_{ij}}$ and $\mathcal{V}_{\mathcal{F}_{ij}}$ defined on a nonempty open interval $(\underline{w}_{ij}, \bar{w}_{ij})$ such that $\mathcal{U}_{\mathcal{F}_{ij}}$ is nondecreasing and $\mathcal{V}_{\mathcal{F}_{ij}}$ is nonincreasing, and such that the set of (u,v) such that $D_{\mathcal{F}_{ij}}(u,v) = 0$ is given by $\{(\mathcal{U}_{\mathcal{F}_{ij}}(w), \mathcal{V}_{\mathcal{F}_{ij}}(w)) : w \in (\underline{w}_{ij}, \bar{w}_{ij})\}$. Furthermore, $\mathcal{U}_{\mathcal{F}_{ij}}(w)$ and $\mathcal{V}_{\mathcal{F}_{ij}}(w)$ are the unique values of u and v solving (3.2), and they are given by*

$$\mathcal{U}_{\mathcal{F}_{ij}}(w) = -D_{\mathcal{F}_{ij}}(0, -w), \text{ and } \mathcal{V}_{\mathcal{F}_{ij}}(w) = -D_{\mathcal{F}_{ij}}(w, 0). \tag{3.3}$$

Whenever $\mathcal{U}_{\mathcal{F}_{ij}}$ and $\mathcal{V}_{\mathcal{F}_{ij}}$ are differentiable, it is easy to see that $\mathcal{U}'_{\mathcal{F}_{ij}}(w) = \partial_v D_{\mathcal{F}_{ij}}(0, -w)$ and $\mathcal{V}'_{\mathcal{F}_{ij}}(w) = -\partial_u D_{\mathcal{F}_{ij}}(w, 0)$. Further, as $\mathcal{U}_{\mathcal{F}_{ij}}(w)$ is increasing and 1-Lipschitz, $\bar{w}_{ij}$ is finite if and only if the maximal utility u obtainable by the man for some feasible $(u,v) \in \mathcal{F}_{ij}$ is finite. Similarly, $\underline{w}_{ij}$ is finite if and only if the maximal utility v obtainable by the woman for some feasible $(u,v) \in \mathcal{F}_{ij}$ is finite.

3.2. **Basic model.** Having established the structure of the feasible bargains among matched couples, we describe the matching process. Men and women may form (heterosexual) pairs or decide to remain unmatched. If i (resp. j) decides to remain unmatched, he (resp. she) gets reservation utility $\mathcal{U}_{i0}$ (resp. $\mathcal{V}_{0j}$). If i and j decide to match with each other, they bargain over a set $\mathcal{F}_{ij}$ of feasible payoffs (u,v), where $\mathcal{F}_{ij}$ is a proper bargaining set, whose associated distance-to-frontier function is denoted $D_{ij} := D_{\mathcal{F}_{ij}}$ and whose functions $\mathcal{U}_{\mathcal{F}_{ij}}$ and $\mathcal{V}_{\mathcal{F}_{ij}}$ are respectively denoted $\mathcal{U}_{ij}$ and $\mathcal{V}_{ij}$. We denote by u_i (resp. v_j) the equilibrium outcome utility of man i (resp. woman j). At equilibrium, we must have $u_i \geq \mathcal{U}_{i0}$ and $v_j \geq \mathcal{V}_{0j}$ as it is always possible to leave an arrangement which yields less than the reservation utility. Similarly, at equilibrium, $D_{ij}(u_i, v_j) \geq 0$ must hold for every i and j; indeed, if this were not the case, there would be a pair (i,j) such that (u_i, v_j) is in the strict interior of the feasible set $\mathcal{F}_{ij}$, so that there would exist payoffs $u' \geq u_i$ and $v' \geq v_j$

(with at least one strict inequality) and $(u', v') \in \mathcal{F}_{ij}$, which would imply that i and j can be better off by matching together. Let μ_{ij} be an indicator variable which is equal to 1 if i and j are matched, and 0 otherwise. If $\mu_{ij} = 1$, we require that (u_i, v_j) be feasible, that is $D_{ij}(u_i, v_j) \leq 0$, hence equality should hold.

Combining the conditions just described, we are ready to define equilibrium in our ITU matching model. We call this equilibrium "individual" to distinguish it from the concept of "aggregate" equilibrium we introduce in section 4.

Definition 4 (Individual Equilibrium)**.** The triple $\left(\mu_{ij}, u_i, v_j\right)_{i \in \mathcal{I}, j \in \mathcal{J}}$ is an *individual equilibrium outcome* if the following three conditions are met:

(i) $\mu_{ij} \in \{0, 1\}$, $\sum_j \mu_{ij} \leq 1$ and $\sum_i \mu_{ij} \leq 1$;

(ii) for all i and j, $D_{ij}(u_i, v_j) \geq 0$, with equality if $\mu_{ij} = 1$;

(iii) $u_i \geq \mathcal{U}_{i0}$ and $v_j \geq \mathcal{V}_{0j}$, with equality respectively if $\sum_j \mu_{ij} = 0$, and if $\sum_i \mu_{ij} = 0$.

The vector $\left(\mu_{ij}\right)_{i \in \mathcal{I}, j \in \mathcal{J}}$ is an *individual equilibrium matching* if and only if there exists a pair of vectors $(u_i, v_j)_{i \in \mathcal{I}, j \in \mathcal{J}}$ such that (μ, u, v) is an individual equilibrium outcome.

As we detail in the next section, our setting embeds the TU and the NTU matching models, as well as many other matching frameworks, including collective models.

3.3. **Example Specifications.** Now, we provide examples of specifications of frontiers $\mathcal{F}$ (or equivalently, distance-to-frontier functions D) that illustrate a number of applications encompassed by our framework. See also further examples in appendix B.

3.3.1. *Matching with Transferable Utility (TU).* The classical TU matching model has been widely used in economics—it is the cornerstone of Becker's marriage model, which has found applications in labor markets, marriage markets, and housing markets (Shapley and Shubik, 1971; Becker, 1973). To recover the TU model in our framework, we take

$$\mathcal{F}_{ij} = \left\{(u, v) \in \mathbb{R}^2 : u + v \leq \Phi_{ij}\right\}, \tag{3.4}$$

that is, for some (potential) surplus matrix Φ, the partners can additively share the quantity $\Phi_{ij} = \alpha_{ij} + \gamma_{ij}$, which is interpreted as a *joint surplus* (see figure 2a). The Pareto efficient

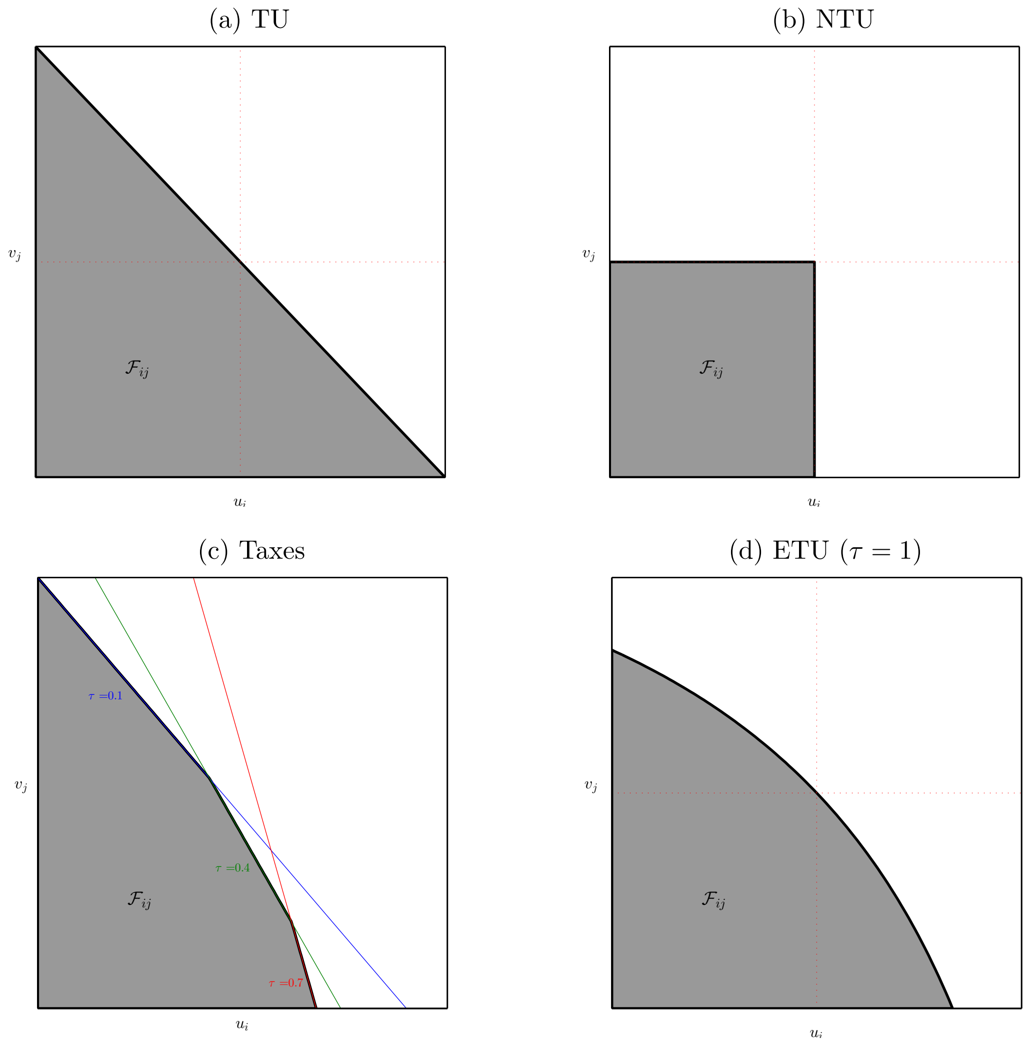

FIGURE 2. Examples of bargaining sets.

payoffs will be such that $u + v = \Phi_{ij}$. In this setting, utility is perfectly transferable: if one partner gives up one unit of utility, the other partner fully appropriates it.

It is easily verified that in the TU case,

$$D_{ij}(u, v) = \frac{u + v - \Phi_{ij}}{2}, \tag{3.5}$$

and as a result, $\mathcal{U}_{ij}(w) = (\Phi_{ij} + w)/2$, and $\mathcal{V}_{ij}(w) = (\Phi_{ij} - w)/2$.

3.3.2. *Matching with Non-Transferable Utility (NTU).* Equally important is the NTU matching model,[3] which has frequently been used to model school choice markets and centralized job assignment. In this case, utility is not transferable at all, and the maximum utility that each partner can obtain is fixed and does not depend on what the other partner gets. Like the TU model, we can embed the NTU model in our ITU framework: in this case,

$$\mathcal{F}_{ij} = \left\{ (u, v) \in \mathbb{R}^2 : u \leq \alpha_{ij}, v \leq \gamma_{ij} \right\}, \tag{3.6}$$

which means that the only efficient pair of payoffs has $u = \alpha_{ij}$ and $v = \gamma_{ij}$ (see figure 2b). It is easily checked that

$$D_{ij}(u, v) = \max\{u - \alpha_{ij}, v - \gamma_{ij}\}, \tag{3.7}$$

and we have $\mathcal{U}_{ij}(w) = \min\left\{\alpha_{ij}, w + \gamma_{ij}\right\}$ and $\mathcal{V}_{ij}(w) = \min\left\{\alpha_{ij} - w, \gamma_{ij}\right\}$. As argued elsewhere (see theorem 5 of Galichon and Hsieh (2017), recalled in appendix C.2 for completeness), our notion of equilibrium matching (implied by definition 4) when the feasible set is given by (3.6) can be shown to coincide with NTU stability in the sense of Gale and Shapley.

3.3.3. *Matching with a convex tax schedule.* Our framework embeds matching with nonlinear taxes, and our formulas take a convenient form when the tax schedule is convex (see Dupuy et al., 2017). Assume i is an employee, receiving gross wage w_{ij} from employer j. Assume that the utility of the employee is equal to α_{ij} plus net (after-tax) wage, while the profit of the firm equals γ_{ij} minus gross wage.

Assume the tax thresholds are given by $t^1, t^2, ..., t^K$, where no tax is due below t^1, so that the marginal rate before that threshold is $\tau^0 = 0$. Assume income between thresholds t^k and t^{k+1} is taxed at rate τ^k, and income above t^K is taxed at rate τ^K. It is assumed that the tax rates are increasing, which means that the tax schedule is convex.

Let α_{ij}^k be the utility of the worker with a gross wage t^k. On has $\alpha_{ij}^0 = \alpha_{ij}$, and $\alpha_{ij}^{k+1} = \alpha_{ij}^k + \left(1 - \tau^k\right)\left(t^{k+1} - t^k\right)$, and, more generally, the utility of a worker with gross

[3] This model is closely connected to the model of Gale and Shapley (1962); but we depart from that classical setting in allowing for free disposal. However, as we argue at the end of the paragraph, both models yield the same set of stable matchings.

wage w_{ij} is given by $u_i = \min\left\{\alpha_{ij}^k + \left(1-\tau^k\right) w_{ij}, k=0,...,K\right\}$, while the profit of the firm is given by $v_j = \gamma_{ij} - w_{ij}$. The feasible set is given by

$$\mathcal{F}_{ij} = \left\{(u,v) \in \mathbb{R}^2 : \forall k \in \{0,...,K\}, u_i \leq \alpha_{ij}^k + \left(1-\tau^k\right)\left(\gamma_{ij} - v_j\right)\right\},$$

and it follows from lemma 2 that the distance-to-frontier function is given by

$$D_{ij}(u,v) = \max_{k\in\{0,...,K\}} \left\{\frac{u - \alpha_{ij}^k + \left(1-\tau^k\right)\left(v-\gamma_{ij}\right)}{2-\tau^k}\right\}. \tag{3.8}$$

See figure 2c.

3.3.4. *Collective Models.* Following the "collective" approach initiated by Chiappori (1992),[4] assume that partners need to agree on a public good $g \in \mathcal{G}$ and private consumptions c_i and c_j such that utilities are given by

$$\mathfrak{u}_{ij}(c_i, g) = \alpha_{ij}(g) + \tau_{ij}\log c_i \text{ and } \mathfrak{v}_{ij}(c_j) = \gamma_{ij}(g) + \tau_{ij}\log c_j,$$

where the budget constraint for private consumption is $c_i + c_j = B_{ij}(g)$. The menu $\mathcal{G}$ of public consumptions is a closed set that may be finite or not, and therefore our setting applies also to the case when the set of available public goods is discrete, which is especially useful to model fertility decisions. Then it follows from lemma 2 that the distance-to-frontier function is given by

$$D_{ij}(u,v) = \min_{g\in\mathcal{G}} \tau_{ij}\log\left(\frac{\exp\left(\frac{u-\alpha_{ij}(g)}{\tau_{ij}}\right) + \exp\left(\frac{v-\gamma_{ij}(g)}{\tau_{ij}}\right)}{B_{ij}(g)}\right). \tag{3.9}$$

In particular, when there is no public good, $\mathcal{G}$ is a singleton and $\alpha_{ij}(g) = \alpha_{ij}$, $\gamma_{ij}(g) = \gamma_{ij}$, and $B_{ij}(g) = B_{ij}$, in which case the expression of D_{ij} reduces to:

$$D_{ij}(u,v) = \tau_{ij}\log\left(\frac{\exp\left(\frac{u-\alpha_{ij}}{\tau_{ij}}\right) + \exp\left(\frac{v-\gamma_{ij}}{\tau_{ij}}\right)}{B_{ij}}\right), \tag{3.10}$$

and we have in that case

$$\mathcal{U}_{ij}(w) = -\tau_{ij}\log\left(\frac{e^{\frac{-\alpha_{ij}}{\tau_{ij}}} + e^{\frac{-w-\gamma_{ij}}{\tau_{ij}}}}{B_{ij}}\right) \text{ and } \mathcal{V}_{ij}(w) = -\tau_{ij}\log\left(\frac{e^{\frac{w-\alpha_{ij}}{\tau_{ij}}} + e^{\frac{-\gamma_{ij}}{\tau_{ij}}}}{B_{ij}}\right). \tag{3.11}$$

[4]See a discussion on the efficiency assumption made in collective models in Del Boca and Flinn (2012).

We call the model in (3.10) an *Exponentially Transferable Utility* (ETU) model. A particular case of the ETU model can be found in Legros and Newman (2007, p. 1086). The terms α and γ play the role of "premuneration values," as defined in Liu et al. (2014) and Mailath et al. (2013). The corresponding feasible set is displayed in figure 2d for $\tau_{ij} = 1$.

Note that the ETU model imposes that the total household budget is 2, namely $B_{ij} = 2$. In this case, we recover the NTU model (3.7) as $\tau_{ij} \to 0$, and the TU model (3.5) as $\tau_{ij} \to +\infty$. Hence, the ETU model interpolates between the nontransferable and fully transferable utility models. Here, the parameter τ_{ij}, which captures the elasticity of substitution between marital well-being and consumption, equivalently parameterizes the *degree of transferability*.

4. Aggregate equilibrium: motivation and definition

In this section we add structure to our previous model by assuming that agents can be grouped into a finite number of types, which are observable to the econometrician and differ according to an unobserved taste parameter. Section 4.1 precisely describes this setting. The individual, or "microscopic" equilibrium defined in section 3 above has a "macroscopic" analog: the *aggregate equilibrium*, which describes the equilibrium matching patterns and systematic payoffs across observable types; we define this concept in section 4.3.

4.1. **Unobserved heterogeneity.** We assume that individuals may be gathered in groups of agents of similar observable characteristics, or types, but heterogeneous tastes. We let $\mathcal{X}$ and $\mathcal{Y}$ be the sets of *types* of men and women, respectively; we assume that $\mathcal{X}$ and $\mathcal{Y}$ are finite. Let $x_i \in \mathcal{X}$ (resp. $y_j \in \mathcal{Y}$) be the type of individual man i (resp. woman j). We let n_x be the mass of men of type x, and let m_y be the mass of women of type y. In the sequel, we denote by $\mathcal{X}_0 \equiv \mathcal{X} \cup \{0\}$ the set of marital options available to women (either type of male partner or singlehood, denoted 0); analogously, $\mathcal{Y}_0 \equiv \mathcal{Y} \cup \{0\}$ denotes the set of marital options available to men (either type of female partner or singlehood, again denoted 0). For a man $x \in \mathcal{X}$ and a woman $y \in \mathcal{Y}$, let $\mathcal{F}_{xy}$ be a proper bargaining set in the sense of definition 1. Let $D_{xy}(.,.)$ be the associated distance-to-frontier function, and recall from paragraph 3.1.3 that one can deduce an explicit representation of the feasible utilities by defining $\mathcal{U}_{xy}(w) = -D_{xy}(0, -w)$, and $\mathcal{V}_{xy}(w) = -D_{xy}(w, 0)$, so that $D_{xy}(u, v) \leq 0$ if and only if there exists a $w \in \mathbb{R}$ such that $u \leq \mathcal{U}_{xy}(w)$ and $v \leq \mathcal{V}_{xy}(w)$.

Consider a market in which men and women either decide to match or to remain single. Our first assumption requires the feasible set of utilities jointly obtainable by any pair of agents should be a random set whose stochasticity has the following structure.

Assumption 1. *There exist families of probability distributions* $(\mathbf{P}_x)_{x\in\mathcal{X}}$ *and* $(\mathbf{Q}_y)_{y\in\mathcal{Y}}$ *such that the feasible set of utilities* $\mathcal{F}_{ij}$ *jointly obtainable by* i *and* j *can be written*

$$\mathcal{F}_{ij} = \mathcal{F}_{x_i y_j} \oplus \left(\varepsilon_{iy_j}, \eta_{x_i j}\right) \tag{4.1}$$

where $\oplus$ *is the direct sum between sets, i.e.* $z \oplus C = \{z\} \oplus C = \{z + c, c \in C\}$, *while if* i *and* j *remain single, then they obtain utilities* ε_{i0} *and* η_{0j}, *where the random vectors* $\left(\varepsilon_{iy}\right)_{y\in\mathcal{Y}_0}$ *and* $\left(\eta_{xj}\right)_{x\in\mathcal{X}_0}$ *are i.i.d. draws from* $\mathbf{P}_x$ *and* $\mathbf{Q}_y$, *respectively.*

Assumption 1 immediately implies the following:

Lemma 4. *If* i *and* j *are matched, there exists* $(U_i, V_j) \in \mathcal{F}_{x_i y_j}$ *such that* $u_i = U_i + \varepsilon_{iy_j}$ *and* $v_j = V_j + \eta_{x_i j}$.

In the case of TU models (see example 3.3.1 above), the restriction implied by lemma 4 simply states that the joint surplus Φ_{ij} can be decomposed in the form $\Phi_{ij} = \Phi_{x_i y_j} + \varepsilon_{iy_j} + \eta_{x_i j}$. This is the "additive separability" assumption in Choo and Siow (2006), who were the first to realize its analytical convenience; it has played a central role in the subsequent literature,[5] see in particular Chiappori, Salanié, and Weiss (2017). Note that, while the transfers U_i and V_j are allowed to vary in an idiosyncratic manner within observable types, it will be a fundamental property of the equilibrium (stated in theorem 6 below in appendix E) that U_i is the same for all men i of type x matched with a woman of type y, while V_j is the same for all the women j of type y matched with a man of type x.

We now introduce a technical restriction on the bargaining sets $\mathcal{F}_{xy}$.

[5] In contrast, Dagsvik (2000) and Menzel (2015) assume that the heterogeneity in tastes is individual-specific; see appendix C.2.

Assumption 2. *The sets $\mathcal{F}_{xy}$ are such that for each man type $x \in \mathcal{X}$, either all the $\bar{w}_{xy}$, $y \in \mathcal{Y}$ are finite, or all the $\bar{w}_{xy}$, $y \in \mathcal{Y}$ coincide with $+\infty$ (where $\bar{w}_{xy}$ and $\underline{w}_{xy}$ are as defined in section 3.1.3). For each woman type $y \in \mathcal{Y}$, either all the $\underline{w}_{xy}$, $x \in \mathcal{X}$ are finite, or all the $\underline{w}_{xy}$, $x \in \mathcal{X}$ coincide with $-\infty$.*

This assumption expresses that given any agent (man or woman), the maximum utility that this agent can obtain with any partner is either always finite, or always infinite; this is needed to ensure existence of an equilibrium, and it is satisfied in all the examples we have.

We finally impose assumptions on $\mathbf{P}_x$ and $\mathbf{Q}_y$, the distributions of the idiosyncratic terms $(\varepsilon_{iy})_{y \in \mathcal{Y}_0}$ and $(\eta_{xj})_{x \in \mathcal{X}_0}$, which are i.i.d. random vectors respectively valued in $\mathbb{R}^{\mathcal{Y}_0}$ and $\mathbb{R}^{\mathcal{X}_0}$.

Assumption 3. $\mathbf{P}_x$ *and* $\mathbf{Q}_y$ *have non-vanishing densities on* $\mathbb{R}^{\mathcal{Y}_0}$ *and* $\mathbb{R}^{\mathcal{X}_0}$*.*

There are two components to assumption 3: the requirement that $\mathbf{P}_x$ and $\mathbf{Q}_y$ have full support, and the requirement that they are absolutely continuous. The full-support requirement implies that given any pair of types x and y, there are individuals of these types with arbitrarily large valuations for each other; this implies that at equilibrium, any matching between observable pairs of types will be observed. The absolute continuity requirement ensures that with probability 1 the men and the women's choice problems have a unique solution.

Transposing definition 4 to the framework with parameterized heterogeneity, we see that (μ_{ij}, u_i, v_j) is an individual equilibrium outcome when:

(i) $\mu_{ij} \in \{0, 1\}$, $\sum_j \mu_{ij} \leq 1$ and $\sum_i \mu_{ij} \leq 1$;

(ii) for all i and j, $D_{x_i y_j}(u_i - \varepsilon_{iy_j}, v_j - \eta_{x_i j}) \geq 0$, with equality if $\mu_{ij} = 1$;

(iii) $u_i \geq \varepsilon_{i0}$ and $v_j \geq \eta_{0j}$ with equality if respectively $\sum_j \mu_{ij} = 0$ and $\sum_i \mu_{ij} = 0$.

4.2. **Informal preview of the next steps.** To provide some intuition on the definition of aggregate equilibrium to follow, we summarize the next steps. We start with an equivalent condition to point (ii) in the definition of an individual equilibrium above (definition 4): for any pair of types $x \in \mathcal{X}$ and $y \in \mathcal{Y}$,

$$\min_{\substack{i:x_i=x \\ j:y_j=y}} D_{xy}\left(u_i - \varepsilon_{iy}, v_j - \eta_{xj}\right) \geq 0,$$

with equality if there is a matching between a man of type x and a woman of type y. Thus, defining $U_{xy} = \min_{i:x_i=x}\left\{u_i - \varepsilon_{iy}\right\}$ and $V_{xy} = \min_{j:y_j=y}\left\{v_j - \eta_{xj}\right\}$ yields $D_{xy}\left(U_{xy}, V_{xy}\right) \geq 0$. We show that under weak conditions, this is actually an equality, hence:

$$D_{xy}\left(U_{xy}, V_{xy}\right) = 0. \tag{4.2}$$

Further, one sees from the definition of U_{xy} and V_{xy} that $u_i \geq \max_{y\in\mathcal{Y}}\left\{U_{xy} + \varepsilon_{iy}, \varepsilon_{i0}\right\}$ and $v_j \geq \max_{x\in\mathcal{X}}\left\{V_{xy} + \eta_{xj}, \eta_{0j}\right\}$. Again under rather weak conditions (stated in appendix E), this actually holds as an equality, so that $u_i = \max_{y\in\mathcal{Y}}\left\{U_{xy} + \varepsilon_{iy}, \varepsilon_{i0}\right\}$ and $v_j = \max_{x\in\mathcal{X}}\left\{V_{xy} + \eta_{xj}, \eta_{0j}\right\}$. Hence, agents face discrete choice problems when choosing the type of their partner. At equilibrium, the mass of men of type x choosing type y women should coincide with the mass of women of type y choosing men of type x. Thus, we need to relate this common quantity μ_{xy} to the vector of systematic utilities (U_{xy}) and (V_{xy}). This is done in the next paragraph using results from the literature on Conditional Choice Probability (CCP) inversion (see Berry, 1994), which allows us to state a definition of aggregate equilibrium.

4.3. **Aggregate Equilibrium.** An *aggregate matching* (or just a *matching*, when no confusion is possible), is specified by a vector $(\mu_{xy})_{x\in\mathcal{X},y\in\mathcal{Y}}$ measuring the mass of matches between men of type x and women of type y. Let $\mathcal{M}$ be the set of matchings, that is, the set of $\mu_{xy} \geq 0$ such that $\sum_{y\in\mathcal{Y}} \mu_{xy} \leq n_x$ and $\sum_{x\in\mathcal{X}} \mu_{xy} \leq m_y$. For later purposes, we shall need to consider the strict interior of $\mathcal{M}$, denoted $\mathcal{M}^0$, i.e. the set of $\mu_{xy} > 0$ such that $\sum_{y\in\mathcal{Y}} \mu_{xy} < n_x$ and $\sum_{x\in\mathcal{X}} \mu_{xy} < m_y$. The elements of $\mathcal{M}^0$ are called *interior matchings.*

We look for an individual equilibrium $\left(\mu_{ij}, u_i, v_j\right)$ with the property that there exist two vectors (U_{xy}) and (V_{xy}) such that if i is matched with j, then $u_i = U_{x_i y_j} + \varepsilon_{i y_j}$, and $v_j = V_{x_i y_j} + \eta_{x_i j}$.[6]

[6]While this may look like a restriction, we show in appendix E that: (i) there always exists an individual equilibrium of this form, and (ii) under a very mild additional assumption on the feasible sets (namely, assumption 2' in appendix E), *any* individual equilibrium is of this form.

Under such an equilibrium, each agent is faced with a choice between the observable types of his or her potential partners, and man i and woman j solve respectively the following discrete choice problems

$$u_i = \max_{y \in \mathcal{Y}} \{U_{x_i y} + \varepsilon_{iy}, \varepsilon_{i0}\} \text{ and } v_j = \max_{x \in \mathcal{X}} \left\{V_{x y_j} + \eta_{xj}, \eta_{0j}\right\}.$$

This yields an important extension of Choo and Siow's (2006) original insight that the matching problem with heterogeneity in tastes is equivalent to a pair of discrete choice problems on both sides of the market. This allows us to relate the vector of utilities (U_{xy}) and (V_{xy}) to the equilibrium matching μ such that μ_{xy} is the mass of men of type x and women of type y mutually preferring each other. In order to establish this relation, we make use of the convex analytic apparatus of Galichon and Salanié (2015). We define the total indirect surplus of men and women by respectively

$$G(U) = \sum_{x \in \mathcal{X}} n_x \mathbb{E}\left[\max_{y \in \mathcal{Y}} \{U_{xy} + \varepsilon_{iy}, \varepsilon_{i0}\}\right] \quad \text{and} \quad H(V) = \sum_{y \in \mathcal{Y}} m_y \mathbb{E}\left[\max_{x \in \mathcal{X}} \left\{V_{xy} + \eta_{xj}, \eta_{0j}\right\}\right]. \tag{4.3}$$

By the Daly-Zachary-Williams theorem, the mass of men of type x demanding a partner of type y is a quantity $\mu_{xy} = \partial G(U) / \partial U_{xy}$, which we denote in vector notation by $\mu \equiv \nabla G(U)$. Similarly, the mass of women of type y demanding a partner of type x is given by ν_{xy}, where $\nu \equiv \nabla H(V)$. At equilibrium, the mass of men of type x demanding women of type y should coincide with the mass of women of type y demanding men of type x, thus $\mu_{xy} = \nu_{xy}$ should hold for any pair, so reexpresses as $\nabla G(U) = \nabla H(V)$. Of course, U and V are related by the feasibility equation $D_{xy}(U_{xy}, V_{xy}) = 0$ for each $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. This leads to the following definition.

Definition 5 (Aggregate Equilibrium)**.** The triple $\left(\mu_{xy}, U_{xy}, V_{xy}\right)_{x \in \mathcal{X}, y \in \mathcal{Y}}$ is an *aggregate equilibrium outcome* if the following three conditions are met:

(i) μ is an interior matching, i.e. $\mu \in \mathcal{M}^0$;

(ii) (U, V) is feasible, i.e.

$$D_{xy}(U_{xy}, V_{xy}) = 0, \ \forall x \in \mathcal{X}, y \in \mathcal{Y}; \tag{4.4}$$

(iii) μ, U, and V are related by the market clearing condition

$$\mu = \nabla G(U) = \nabla H(V). \tag{4.5}$$

The vector $\left(\mu_{xy}\right)_{x\in\mathcal{X},y\in\mathcal{Y}}$ is an *aggregate equilibrium matching* if and only if there exists a pair of vectors $(U_{xy}, V_{xy})_{x\in\mathcal{X},y\in\mathcal{Y}}$ such that (μ, U, V) is an aggregate equilibrium outcome.

We discuss the equivalence of individual and aggregate equilibrium in theorem 6 of appendix E.

4.4. **Aggregate matching equation.** Before ending this section, we rewrite the system of equations in definition 5 as a simpler system of equations which involves the matching vector μ only. To do this, we need to invert $\mu = \nabla G(U)$ and $\mu = \nabla H(V)$ in order to express U and V as a function of μ. For this purpose, we introduce the Legendre-Fenchel transform (a.k.a. convex conjugate) of G and H:

$$G^*(\mu) = \sup_U \left\{ \sum_{xy} \mu_{xy} U_{xy} - G(U) \right\} \quad \text{and} \quad H^*(\nu) = \sup_V \left\{ \sum_{xy} \nu_{xy} V_{xy} - H(V) \right\}. \tag{4.6}$$

It is a well-known fact from convex analysis (cf. Rockafellar 1970) that, under smoothness assumptions that hold here given assumption 3,

$$\mu = \nabla G(U) \iff U = \nabla G^*(\mu) \text{ and } \nu = \nabla H(V) \iff V = \nabla H^*(\nu),$$

so we may substitute out U and V as an expression of μ in the system of equations in definition 5, so that equilibrium is characterized by a set of $|\mathcal{X}| \times |\mathcal{Y}|$ equations expressed only in terms of μ.

Proposition 1. *Matching $\mu \in \mathcal{M}^0$ is an aggregate equilibrium matching if and only if*

$$D_{xy}\left(\frac{\partial G^*(\mu)}{\partial \mu_{xy}}, \frac{\partial H^*(\mu)}{\partial \mu_{xy}} \right) = 0 \text{ for all } x \in \mathcal{X},\ y \in \mathcal{Y}. \tag{4.7}$$

The reformulation in proposition 1 will be useful in section 6 when logit heterogeneity is considered; in that case, equation (4.7) can be inverted easily.

5. Aggregate Equilibrium: Existence, Uniqueness, Computation

In this section, we study aggregate equilibria by reformulating the ITU matching market in terms of a demand system. In section 5.1, we show that our demand system satisfies the gross substitutability property of Kelso and Crawford (1982); this observation is the basis of our existence and uniqueness proofs in section 5.2. The machinery developed in this section is also useful to obtain results on identification which is the focus of appendix D.

5.1. **Reformulation as a demand system.** Thanks to the explicit representation of the feasible sets, we obtain an alternative description of our matching model as a demand system, in the spirit of Azevedo and Leshno's (2016) approach to NTU models without unobserved heterogeneity. As we recall, $D_{xy}(U_{xy}, V_{xy}) = 0$ is equivalent to the existence of W_{xy} such that $U = \mathcal{U}(W)$ and $V = \mathcal{V}(W)$, where the xy-entries of $\mathcal{U}(W)$ and $\mathcal{V}(W)$ are $\mathcal{U}_{xy}(W_{xy})$ and $\mathcal{V}_{xy}(W_{xy})$, as introduced in definition 3. In our reformulation, the couple types xy will be treated as goods; men as producers, and women as consumers. Each man of type x chooses to produce one of the goods of type xy, where $y \in \mathcal{Y}_0$; similarly, each woman of type y chooses to consume one of the goods of type xy, where $x \in \mathcal{X}_0$. The wedges $W_{xy} = U_{xy} - V_{xy}$ are interpreted as prices, and $\partial G(\mathcal{U}(W)) / \partial U_{xy}$ is interpreted as the supply of the xy good, and $\partial H(\mathcal{V}(W)) / \partial V_{xy}$ is interpreted as the demand for that good if the price vector is W. An increase in W_{xy} raises the supply of the xy good and decreases the demand for it. We can define the excess demand function as

$$Z(W) := \nabla H(\mathcal{V}(W)) - \nabla G(\mathcal{U}(W)), \tag{5.1}$$

so that $Z_{xy}(W)$ is the mass of women of type y willing to match with men of type x minus the mass of men of type x willing to match with women of type y, if the vector of market wedges is W. We have therefore:

Proposition 2. *Outcome (μ, U, V) is an aggregate equilibrium outcome if and only if $\mu = \nabla G(U) = \nabla H(V)$, and there exists a vector (W_{xy}) such that $U = \mathcal{U}(W)$, $V = \mathcal{V}(W)$, and*

$$Z(W) = 0. \tag{5.2}$$

As a result, $Z(.)$ is to be interpreted as an excess demand function, and (W_{xy}) as a vector of market prices: if W_{xy} increases and all the other entries of W remain constant, the systematic utility V_{xy} of women in the xy category decreases and the utility U_{xy} of men in that category increases, hence Z_{xy}, the excess demand for category xy, decreases. It is possible to express that in this demand interpretation various categories of goods xy are gross substitutes, in the following sense:

Proposition 3 (Gross Substitutes)**.** *(a) If W_{xy} increases and all other entries of W remain constant, then:*

(a.1) $Z_{xy}(W)$ decreases,

(a.2) $Z_{x'y'}(W)$ increases if either $x = x'$ or $y = y'$ (but both equalities do not hold),

(a.3) $Z_{x'y'}(W)$ remains constant if $x \neq x'$ and $y \neq y'$.

(b) for any $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, the sum $\sum_{x' \in \mathcal{X}, y' \in \mathcal{Y}} Z_{x'y'}(W)$ is a decreasing function of W_{xy}.

The result implies that the excess demand function Z satisfies the *gross substitutability* condition. Point (a.1) means that when W_{xy} increases, one moves along the Pareto frontier of the feasible set $\mathcal{F}_{xy}$ towards a direction which is more favorable to the men (U_{xy} increases, V_{xy} decreases), and thus there is *ceteris paribus* less demand from women and more from men for the category xy, and excess demand Z_{xy} decreases. Point (a.2) expresses that when the price of some category, say W_{xy} increases, and all the other entries of W remain constant, then the prospects of women in the category xy deteriorates, thus some of these women will switch to category $x'y$, and hence the excess demand $Z_{x'y}$ for category $x'y$ increases. Point (a.3) simply means that an agent (man or woman) does not respond to the price change of a category which does not involve his or her type. Finally, point (b) expresses that when the price of category xy increases, then singlehood becomes weakly less attractive for all men, and strongly less so for men of category x; while singlehood becomes more attractive for women, which explains that the sum of $Z_{x'y'}$ over all categories, decreases.

5.2. **Existence, uniqueness, and computation.** We now state and prove a theorem that ensures the existence and uniqueness of an equilibrium using the characterization of aggregate equilibrium as a demand system introduced in proposition 2. We show that there is a unique vector of prices (W_{xy}) at which the value of excess demand is 0. This is stated in the following result:

Theorem 1 (Existence and uniqueness of a price equilibrium)**.** *Under assumptions 1, 2, and 3, there exists a unique vector W such that*

$$Z(W) = 0. \tag{5.3}$$

5.2.1. *Existence and computation.* The proof of equilibrium existence is constructive, and W is obtained as the outcome of the following algorithm. It is shown in the proof of theorem 1 that one can find an initial vector of prices $\left(W^0_{xy}\right)$ for which excess demand is negative, that is $Z\left(W^0\right) \leq 0$. This suggests that prices $\left(W^0_{xy}\right)$ are too high. The iteration consists of lowering these prices such that at each step, the excess demand at current price $Z\left(W^t\right)$ remains negative. More precisely, we set W^t_{xy}, the price of category xy at time t, to be such that $Z\left(W^t_{xy}, W^{t-1}_{-xy}\right) = 0$, where $\left(W^t_{xy}, W^{t-1}_{-xy}\right)$ denotes the price vector which coincides with W^{t-1} on all entries except on the xy entry and which sets price W^t_{xy} to the xy entry. In other words, the prices of each category are updated in order to cancel the corresponding excess demand, holding the prices of other categories constant. Formally, it is possible to define a map $\mathcal{W} : \mathbb{R}^{\mathcal{X}\times\mathcal{Y}} \to \mathbb{R}^{\mathcal{X}\times\mathcal{Y}}$ such that $W' = \mathcal{W}(W)$ if and only if for all $xy \in \mathcal{X}\times\mathcal{Y}$

$$Z_{xy}\left(W'_{xy}, W_{-xy}\right) = 0,$$

and the procedure simply consists in setting $W^t = \mathcal{W}\left(W^{t-1}\right)$. By proposition 3, point (a.1), it follows that $W^t_{xy} \leq W^{t-1}_{xy}$ for each xy. Because of the gross substitutability property (proposition 3, point (a.2)), $Z\left(W^t_{xy}, W^t_{-xy}\right) \leq Z\left(W^t_{xy}, W^{t-1}_{-xy}\right) = 0$, so that excess demand is still negative at step t. Finally, it is possible to show that W^t_{xy} remains bounded by below; thus, it converges monotonically. The limit is therefore a fixed point of $\mathcal{W}$, hence a zero of Z. This leads to the following algorithm:

Algorithm 1.

Step 0	Start with $W^0 = \bar{W}$.
Step t	For each $x \in \mathcal{X}, y \in \mathcal{Y}$, define $W^{t+1} = \mathcal{W}\left(W^t\right)$.

The algorithm terminates when $\sup_{xy\in\mathcal{X}\times\mathcal{Y}} |W^{t+1}_{xy} - W^t_{xy}| < \epsilon$.

5.2.2. *Uniqueness.* The proof of uniqueness is based on a result of Berry, Gandhi, and Haile (2013), that implies that Z is inverse isotone. Hence, if there are two vectors W and $\tilde{W}$ such that $Z(W) = Z\left(\tilde{W}\right) = 0$, it would follow that $W \leq \tilde{W}$ and $\tilde{W} \leq W$ altogether, hence $W = \tilde{W}$. Note that the uniqueness of an equilibrium follows crucially from the full support assumption (assumption 3).

Combining theorem 1 and proposition 2, it follows that there exists a unique equilibrium outcome (μ, U, V), where μ, U, and V are related to W by $U_{xy} = \mathcal{U}_{xy}(W_{xy})$, $V_{xy} = \mathcal{V}_{xy}(W_{xy})$, and $\mu = \nabla G(U) = \nabla H(V)$.

Corollary 1 (Existence and uniqueness of an equilibrium outcome)**.** *Under assumptions 1, 2, and 3, there exists a unique equilibrium outcome* (μ, U, V)*, and* μ*,* U*, and* V *are related to the solution* W *to system (5.3) by* $U_{xy} = \mathcal{U}_{xy}(W_{xy})$*,* $V_{xy} = \mathcal{V}_{xy}(W_{xy})$*, and* $\mu = \nabla G(U) = \nabla H(V)$*.*

6. The ITU-logit model

In this section, we consider the model of matching with Imperfectly Transferable Utility and *logit* heterogeneity. We therefore replace assumption 3 by the stronger:

Assumption 3'. $\mathbf{P}_x$ *and* $\mathbf{Q}_y$ *are the distributions of i.i.d. Gumbel (standard type I extreme value) random variables.*

Of course, assumption 3' is a specialization of assumption 3, as the Gumbel distribution has a positive density on the real line. As we show in this section, the logit assumption carries strong implications. We show in section 6.1 that under assumption 3', the equilibrium matching equations (4.7) can be drastically simplified, and an algorithm more efficient than algorithm 1 can be used to solve them. Next, in section 6.2, we provide a number of illustrative applications of the logit assumption in the various example instances introduced in section 3.3. Finally, we will show in section 6.3 that maximum likelihood estimation is particularly straightforward in the logit context.

6.1. Equilibrium and computation, logit case.

With logit random utilities, it is well-known (McFadden, 1974) that the systematic part of the utility U_{xy} can be identified by the log of the ratio of the odds of choosing alternative y, relative to choosing the default option, and a similar formula applies to V_{xy}, hence $U_{xy} = \log(\mu_{xy}/\mu_{x0})$ and $V_{xy} = \log(\mu_{xy}/\mu_{0y})$, where $\mu_{x0} = n_x - \sum_{y\in\mathcal{Y}} \mu_{xy}$, and $\mu_{0y} = m_y - \sum_{x\in\mathcal{X}} \mu_{xy}$. Hence, the feasibility equation $D_{xy}(U_{xy}, V_{xy}) = 0$ in expression (4.7) becomes $D_{xy}\left(\log \mu_{xy} - \log \mu_{x0}, \log \mu_{xy} - \log \mu_{0y}\right) = 0$, which, given the translation invariance property (v) of lemma 1, yields

$$\log \mu_{xy} = -D_{xy}\left(-\log \mu_{x0}, -\log \mu_{0y}\right),$$

which explicitly defines μ_{xy} as a function of μ_{x0} and μ_{0y}:

$$\mu_{xy} = M_{xy}\left(\mu_{x0}, \mu_{0y}\right), \text{ where } M_{xy}\left(\mu_{x0}, \mu_{0y}\right) = \exp\left(-D_{xy}\left(-\log \mu_{x0}, -\log \mu_{0y}\right)\right). \quad (6.1)$$

Remark 6.1. By construction, the aggregate matching functions derived from our framework are homogeneous of degree 1 in the number of singles. That is, if μ^*_{xy}, μ^*_{x0} and μ^*_{0y} is the equilibrium matching for given population supplies, $\lambda\mu^*_{xy}$, $\lambda\mu^*_{x0}$ and $\lambda\mu^*_{0y}$ is the equilibrium matching when the population supplies are multiplied by λ. It is well known that the Choo and Siow model satisfies this constant return to scale property, while other models, such as those explored by Mourifié and Siow (2014) and Menzel (2015), do not.

The expression of μ_{xy} as a function of μ_{x0} and μ_{0y}, combined with the requirement that $\mu \in \mathcal{M}^0$, provides a set of equations that fully characterize the aggregate matching equilibrium, as argued in the following result:

Theorem 2. *Under assumptions 1, 2, and 3', the equilibrium outcome (μ, U, V) in the ITU-logit model is given by*

$$\mu_{xy} = M_{xy}\left(\mu_{x0}, \mu_{0y}\right), \; U_{xy} = \log \frac{\mu_{xy}}{\mu_{x0}}, \; V_{xy} = \log \frac{\mu_{xy}}{\mu_{0y}},$$

where the pair of vectors $(\mu_{x0})_{x\in\mathcal{X}}$ and $\left(\mu_{0y}\right)_{y\in\mathcal{Y}}$ is the unique solution to the system of equations

$$\begin{cases} \sum_y M_{xy}\left(\mu_{x0}, \mu_{0y}\right) + \mu_{x0} = n_x \\ \sum_x M_{xy}\left(\mu_{x0}, \mu_{0y}\right) + \mu_{0y} = m_y. \end{cases} \quad (6.2)$$

Theorem 2 implies that computing aggregate equilibria in the logit case is equivalent to solving the system of nonlinear equations (6.2)—a system of $|\mathcal{X}| + |\mathcal{Y}|$ equations in the same number of unknowns. It turns out that a simple iterative procedure provides a efficient means of solving (6.2). The basic idea is each equation in the first set of equations is an equation in the full set of (μ_{0y}), but in the single unknown μ_{x0}. Hence, these can be inverted to obtain the values of (μ_{x0}) from the values of (μ_{0y}). A similar logic applies to the second set of equations, where the values of (μ_{0y}) can be obtained from the values of (μ_{x0}). The proposed algorithm operates by iterating the expression of (μ_{x0}) from (μ_{0y}) and vice-versa. Provided the initial choice of (μ_{0y}) is high enough, the procedure converges isotonically, as argued in the theorem below.

Algorithm 2.

Step 0	Fix the initial value of μ_{0y} at $\mu_{0y}^{0} = m_y$.
Step $2t+1$	Keep the values μ_{0y}^{2t} fixed. For each $x \in \mathcal{X}$, solve for the value, μ_{x0}^{2t+1}, of μ_{x0} such that equality $\sum_{y \in \mathcal{Y}} M_{xy}(\mu_{x0}, \mu_{0y}^{2t}) + \mu_{x0} = n_x$ holds.
Step $2t+2$	Keep the values μ_{x0}^{2t+1} fixed. For each $y \in \mathcal{Y}$, solve for which is the value, μ_{0y}^{2t+2}, of μ_{0y} such that equality $\sum_{x \in \mathcal{X}} M_{xy}(\mu_{x0}^{2t+1}, \mu_{0y}) + \mu_{0y} = m_y$ holds.

The algorithm terminates when $\sup_y |\mu_{0y}^{2t+2} - \mu_{0y}^{2t}| < \epsilon$.

Theorem 3. *Under assumptions 1, 2, and 3', algorithm 2 converges toward the solution (6.2), in such a way that $\left(\mu_{0y}^{t}\right)$ is nonincreasing with t, and $\left(\mu_{x0}^{t}\right)$ is nondecreasing with t.*

6.2. **Example specifications, logit case.**

6.2.1. *TU-logit specification.* In the logit case of the TU specification introduced in paragraph 3.3.1, the matching function becomes

$$M_{xy}\left(\mu_{x0}, \mu_{0y}\right) = \mu_{x0}^{1/2} \mu_{0y}^{1/2} \exp \frac{\Phi_{xy}}{2}, \tag{6.3}$$

which is Choo and Siow's (2006) formula.

6.2.2. *NTU-logit specification.* In the logit case of the NTU specification introduced in paragraph 3.3.2, the matching function becomes

$$M_{xy}\left(\mu_{x0}, \mu_{0y}\right) = \min\left(\mu_{x0} e^{\alpha_{xy}}, \mu_{0y} e^{\gamma_{xy}}\right). \tag{6.4}$$

When $\mu_{x0}e^{\alpha_{xy}} \leq \mu_{0y}e^{\gamma_{xy}}$, $\mu_{xy} = \mu_{x0}e^{\alpha_{xy}}$ is constrained by the choice problem of men; we say that, relative to pair xy, men are on the *short side (of the market)* and women are on the *long side (of the market)*, and visa versa. Galichon and Hsieh (2017) study this model in detail. In particular, they show that existence and computation of equilibria in a more general version of this model can alternatively be provided via an aggregate version of the Gale–Shapley (1962) algorithm; we refer to appendix C.2 for more.

6.2.3. *Convex tax schedule and logit specification.* Recall from paragraph 3.3.3 and expression (3.8) that when there is a convex tax schedule, the distance-to-frontier function is expressed as a maximum of linear terms, more specifically

$$D_{xy}\left(u_x, v_y\right) = \max_{k \in \{0,\dots,K\}} \left\{ \lambda^k_{xy}(u - \alpha^k_{xy}) + \zeta^k_{xy}(v - \gamma_{xy}) \right\}$$

where $\lambda^k_{xy} = 1/\left(2 - \tau^k_{xy}\right)$, and $\zeta^k_{xy} = \left(1 - \tau^k_{xy}\right)/\left(2 - \tau^k_{xy}\right)$, and α^k_{xy} is obtained recursively by $\alpha^0_{xy} = \alpha_{xy}$, and $\alpha^{k+1}_{xy} = \alpha^k_{xy} + \left(1 - \tau^k_{xy}\right)\left(t^{k+1}_{xy} - t^k_{xy}\right)$. In this case, the corresponding matching function obtains as

$$M_{xy}\left(\mu_{x0}, \mu_{0y}\right) = \min_{k \in \{0,\dots,K\}} \mu_{x0}^{\lambda^k_{xy}} \mu_{0y}^{\zeta^k_{xy}} e^{\lambda^k_{xy}\alpha^k_{xy} + \zeta^k_{xy}\gamma_{xy}}. \tag{6.5}$$

Of course, the NTU-logit case of the previous paragraph is a particular case of this expression of $K = 1$, $\lambda^0_{xy} = 0$, $\zeta^0_{xy} = 1$ and $\lambda^1_{xy} = 1$, $\zeta^1_{xy} = 0$.

6.2.4. *Collective model and logit specification.* In the case of the collective model introduced in paragraph 3.3.4 with logit heterogeneity, one obtains for the matching function:

$$M_{xy}\left(\mu_{x0}, \mu_{0y}\right) = \max_{g \in \mathcal{G}} \left(\frac{e^{-\alpha_{xy}(g)/\tau_{xy}} \mu_{x0}^{-1/\tau_{xy}} + e^{-\gamma_{xy}(g)/\tau_{xy}} \mu_{0y}^{-1/\tau_{xy}}}{B_{xy}\left(g\right)} \right)^{-\tau_{xy}} \tag{6.6}$$

In particular, if there is no public good (hence in the ETU-logit case), the maximum is taken over a single term, and the matching function becomes

$$M_{xy}\left(\mu_{x0}, \mu_{0y}\right) = \left(\frac{e^{-\alpha_{xy}/\tau_{xy}} \mu_{x0}^{-1/\tau_{xy}} + e^{-\gamma_{xy}/\tau_{xy}} \mu_{0y}^{-1/\tau_{xy}}}{B_{xy}} \right)^{-\tau_{xy}}. \tag{6.7}$$

As expected, when $B_{xy} = 2$ and $\tau_{xy} \to 0$, formula (6.7) converges to the NTU-logit formula, (6.4). Likewise, when $B_{xy} = 2$ and $\tau_{xy} \to +\infty$, (6.7) converges to the TU-logit formula, (6.3). But when $\tau_{xy} = 1$, then (up to multiplicative constants) μ_{xy} becomes the harmonic mean between μ_{x0} and μ_{0y}. We thus recover a classical matching function form—the "Harmonic Marriage Matching Function" that has been used by demographers for decades, see, e.g., Schoen (1981). To our knowledge, our framework gives the first behavioral/microfounded justification of the harmonic marriage matching function—see Siow (2008, p. 5).

6.3. **Maximum likelihood estimation, logit case.** In this paragraph, we assume that (D_{xy}) belongs to a parametric family $\left(D^\theta_{xy}\right)$ and we estimate $\theta \in \mathbb{R}^d$ by maximum likelihood. We refer the reader to appendix D for a discussion of identification.[7] In this case, the matching function is then $M^\theta_{xy}\left(\mu_{x0}, \mu_{0y}\right) = \exp\left(-D^\theta_{xy}\left(-\log\mu_{x0}, -\log\mu_{0y}\right)\right)$. We shall assume sufficient smoothness on the parametrization.

Assumption 4. *For each $xy \in \mathcal{XY}$, the map $(\theta, u, v) \mapsto \left(D^\theta_{xy}\left(u_x, v_y\right)\right)_{xy}$ is twice continuously differentiable from $\mathbb{R}^d \times \mathbb{R}^{|\mathcal{X}|} \times \mathbb{R}^{|\mathcal{Y}|}$ to $\mathbb{R}^{|\mathcal{X}|\times|\mathcal{Y}|}$.*

The sample is made of N i.i.d. draws of household types $xy \in \mathcal{XY} \cup \mathcal{X}_0 \cup \mathcal{Y}_0$. Let $\zeta = (n, m)$ be the distribution vector of men and women's types, respectively. Given the model's primitives θ and ζ, let $(\mu^{\theta,\zeta}_{x0}, \mu^{\theta,\zeta}_{0y})$ be the solution to the system of equations (6.2), and let $\mu^{\theta,\zeta}_{xy} = M^\theta_{xy}\left(\mu_{x0}, \mu_{0y}\right)$ be the equilibrium matching patterns. We define the predicted frequency $\Pi_{xy}(\theta, \zeta)$ of a household of type xy by

$$\Pi_{xy}(\theta, \zeta) = \frac{\mu^{\theta,\zeta}_{xy}}{N^h(\theta, \zeta)}, \text{ where } N^h(\theta, \zeta) := \sum_{xy \in \mathcal{XY}\cup\mathcal{X}_0\cup\mathcal{Y}_0} \mu^{\theta,\zeta}_{xy}, \tag{6.8}$$

and $N^h(\theta, \zeta)$ is the total mass of households predicted if the masses of individual men and women are given by ζ.

We let $\hat{\mu}_{xy}$ be the number of households of type xy in the sample, $\hat{N}^h = \sum_{xy\in\mathcal{XY}\cup\mathcal{X}_0\cup\mathcal{Y}_0} \hat{\mu}_{xy}$ the number of households in the sample, and $\hat{\pi}_{xy} = \hat{\mu}_{xy}/\hat{N}^h$ be the empirical frequency of household xy in the sample. The log-likelihood of observation $\hat{\pi}$ is

$$l\left(\hat{\pi}|\theta, \zeta\right) = \sum_{xy\in\mathcal{XY}\cup\mathcal{X}_0\cup\mathcal{Y}_0} \hat{\pi}_{xy} \log \Pi_{xy}(\theta, \zeta).$$

[7] As we shall see in appendix D, in general, we cannot identify the preferences on both sides of the market in ITU models unless we observe the transfers. However, in contrast with the TU case, we may identify the preferences on both sides of the market if one observes multiple markets.

Consider $\mathcal{I}$ the Fisher information matrix of $l\left(\pi|\theta,\zeta\right)$ with respect to its parameter vector (θ,ζ), which is written blockwise as

$$\mathcal{I}=\begin{pmatrix}\mathcal{I}_{11} & \mathcal{I}_{12}\\ \mathcal{I}_{12}' & \mathcal{I}_{22}\end{pmatrix},$$

where $\mathcal{I}_{11}$ is the Hessian of $-l\left(\hat{\pi}|\theta,\zeta\right)$ with respect to θ, $\mathcal{I}_{12}$ is the matrix of cross-derivatives of $-l\left(\hat{\pi}|\theta,\zeta\right)$ with respect to the entries of θ and ζ, and $\mathcal{I}_{22}$ is the Hessian of $-l\left(\hat{\pi}|\theta,\zeta\right)$ with respect to ζ. Throughout this section, we shall assume that $\mathcal{I}_{11}$ is invertible; the partially identified case when this matrix is no longer invertible should be handled using specific techniques, which are not the focus here.

In principle, (θ,ζ) could be estimated jointly by maximum likelihood; however, the dimensionality of ζ, the vector of the types distribution in the population is potentially large, so this could pose computational difficulties. Further, ζ is a nuisance parameter as the focus of the estimation procedure is on the estimation of θ; thus we choose to use the consistent estimator of ζ provided by the distribution of types in the sample. Letting A be the matrix acting on $\mu=\left(\mu_{xy},\mu_{x0},\mu_{0y}\right)$ such that $(A\mu)=\binom{n}{m}$ where $n_x=\sum_{y\in\mathcal{Y}_0}\mu_{xy}$ and $m_y=\sum_{x\in\mathcal{X}_0}\mu_{xy}$, we get that a consistent estimator of ζ is $A\hat{\pi}$. Thus, we shall define $\hat{\theta}$ to be the maximum likelihood estimator of θ conditional on the distribution of types being estimated by $A\hat{\pi}$, that is $\hat{\theta}$ is the value of θ that maximizes $l\left(\hat{\pi}|\theta,A\hat{\pi}\right)$.

Theorem 4. *(i) The log-likelihood is expressed using*

$$-l\left(\hat{\pi}|\theta,A\hat{\pi}\right)=\sum_{xy\in\mathcal{X}\times\mathcal{Y}}\hat{\pi}_{xy}D_{xy}^{\theta}\left(u_x^{\theta},v_y^{\theta}\right)+\sum_{x\in\mathcal{X}}\hat{\pi}_{x0}u_x^{\theta}+\sum_{y\in\mathcal{Y}}\hat{\pi}_{0y}v_y^{\theta}+\log N^h\left(\theta,\zeta\right), \qquad (6.9)$$

where the quantities $u_x^{\theta}=-\log\mu_{x0}^{\theta}$ and $v_y^{\theta}=-\log\mu_{0y}^{\theta}$ form the unique pair vectors (u,v) solution to

$$\begin{cases} e^{-u_x}+\sum_{y\in\mathcal{Y}}e^{-D_{xy}^{\theta}(u_x,v_y)}=\hat{n}_x\\ e^{-v_y}+\sum_{x\in\mathcal{X}}e^{-D_{xy}^{\theta}(u_x,v_y)}=\hat{m}_y.\end{cases} \qquad (6.10)$$

(ii) Letting $\hat{\theta}$ be the Maximum Likelihood Estimator, $N^{1/2}\left(\hat{\theta}-\theta\right)\Rightarrow\mathcal{N}\left(0,V_{\theta}\right)$ as the sample size $\hat{N}^h\to+\infty$, where the variance-covariance matrix V_{θ} can be consistently estimated by

$$\hat{V}_{\theta}=\left(\mathcal{I}_{11}\right)^{-1}\left(\mathcal{I}_{12}\right)AV_{\pi}A'\mathcal{I}_{12}'\left(\mathcal{I}_{11}\right)^{-1},\ \textit{with } V_{\pi}=\operatorname{diag}\left(\pi\right)-\pi\pi'. \qquad (6.11)$$

Expression (6.9) in part (i) of the result has an interesting interpretation. For a matched pair (x,y), $D_{xy}(u_x,v_y)$ is the signed distance to the bargaining frontier, which will be typically negative (i.e. (u_x,v_y) is an interior point in general). For a single individual man or woman of type x or y, u_x or v_y is also the signed distance to the bargaining frontier, which is 0. Hence, the value of minus the likelihood is the sum of two terms: (i) a first term

$$\sum_{xy\in\mathcal{X}\times\mathcal{Y}}\hat{\pi}_{xy}D^{\theta}_{xy}\left(u^{\theta}_x,v^{\theta}_y\right)+\sum_{x\in\mathcal{X}}\hat{\pi}_{x0}u^{\theta}_x+\sum_{y\in\mathcal{Y}}\hat{\pi}_{0y}v^{\theta}_y$$

which is the average distance of $u^{\theta}_x = -\log\mu^{\theta}_{x0}$ *and* $v^{\theta}_y = -\log\mu^{\theta}_{0y}$ to the bargaining frontier, and (ii) a second term which is $\log N^h(\theta,\zeta)$, the logarithm of the predicted number of households.

Let us comment on the intuitive explanation for the second term, i.e. the logarithm of the number of predicted households in the expression of the opposite of the likelihood. The maximization of the second term pulls the estimation towards overestimating the number of matched household. On the contrary, in the first term, single households are given the same weight as matched households in the objective function, thus the maximization of the first term tends to underestimate the number of matched households. The trade-off between the two effects is expressed by the $\log N^h(\theta,\zeta)$ term.

In appendix F, we provide an illustrative empirical example of this strategy.

7. Conclusion

The present contribution brings together a number of approaches. In terms of the techniques used, it builds on concepts from game theory, general equilibrium, and econometrics. In terms of models allowed, it embeds models with and without transferable utility. It also provides an integrated approach for both matching models and collective models. Lastly, we note that our work can be used in conjunction with reduced-form methodologies, as it allows us to compute the equilibrium outcome's response to a shock in the matching primitives (e.g., a demographic shock) and to regress the former on the latter.

Beyond the class of problems investigated in the present paper, the methods developed here, based on fixed point theorems for isotone functions, may be more broadly applicable.

In particular, they may be a useful tool for the investigation of matching problems with peer effects put forward by Mourifié and Siow (2014). They may also prove useful for studying certain commodity flow problems in trading networks, and may also extend to one-to-many matching problems. We leave this last extension for further work.

† *Economics Department, FAS, and Mathematics Department, Courant Institute, New York University; and Toulouse School of Economics. Email: ag133@nyu.edu.*

♭ *Society of Fellows and Department of Economics, Harvard University, and Harvard Business School. Email: kominers@fas.harvard.edu.*

§ *Becker Friedman Institute, University of Chicago. Email: simonweber@uchicago.edu.*

References

[1] Agarwal, N. (2015): "An Empirical Model of the Medical Match," *American Economic Review* 105 (7), pp. 1939–1978.

[2] Alkan, A. (1989): "Existence and computation of matching equilibria," *European Journal of Political Economy*, 5(2-3), pp. 285–296.

[3] Alkan, A., and Gale, D. (1990): "The core of the matching game," *Games and Economic Behavior*, 2(3), pp. 203–212.

[4] Azevedo, E., and Leshno, J. (2016): "A Supply and Demand Framework for Two-Sided Matching Markets," *Journal of Political Economy*, 124: (5), pp. 1235–1268.

[5] Becker, G. S. (1973): "A Theory of Marriage: Part I," *Journal of Political Economy*, 81(4), pp. 813–46.

[6] Berry, S. T. (1994): "Estimating discrete-choice models of product differentiation". *The RAND Journal of Economics* 25(2), pp. 242-262.

[7] Berry, S., Gandhi, A., and Haile, P. (2013). "Connected substitutes and invertibility of demand". *Econometrica* 81 (5), pp. 2087–2111.

[8] Browning, M., P.-A. Chiappori, and Y. Weiss (2014): *Economics of the Family*. Cambridge University Press.

[9] Chade, H., and J. Eeckhout (2014): "Stochastic Sorting," working paper.

[10] Chambers, R. G., Y. Chung, and R. Färe (1998), "Profit, directional distance functions, and nerlovian efficiency," Journal of Optimization Theory and Applications, 98, 351–364.

[11] Cherchye, L., T. Demuynck, B. De Rock and F. Vermeulen (2017): "Household consumption when the marriage is stable," *American Economic Review*, 107(6), pp. 1507-1534.

[12] Chiappori, P.-A. (1992): "Rational Household Labour Supply," *Econometrica* 56 (1), pp. 63–89.

[13] Chiappori, P.-A. (2012): "Modèles d'appariement en économie: quelques avancées récentes," *Revue économique* 63 , pp. 433–448.

[14] Chiappori, P.-A. (2017): "Matching with Transfers: The Economics of Love and Marriage," Princeton University Press.

[15] Chiappori, P.-A., M. Iyigun, and Y. Weiss (2009): "Investment in Schooling and the Marriage Market," *American Economic Review*, 99(5), pp. 1689–1713.

[16] Chiappori, P.-A., Oreffice, S., and C. Quintana-Domeque (2012): "Fatter Attraction: Anthropometric and Socioeconomic Matching on the Marriage Market." *Journal of Political Economy*, 120(4), pp. 659–695.

[17] Chiappori, P.-A. and Reny, P. J (2016): "Matching to share risk". *Theoretical Economics*, 11, pp. 227-251.

[18] Chiappori, P.-A., B. Salanié, and Weiss, Y. (2017): "Partner Choice, Invetsment in Children, and the Marital College Premium," *American Economic Review*, forthcoming.

[19] Choo, E. and Seitz, S. (2013): "The Collective Matching Model: Identification, Estimation and Testing," chapter in *Advances in Econometrics* 32, Emerald Publishing Group.

[20] Choo, E., and A. Siow (2006): "Who Marries Whom and Why," *Journal of Political Economy*, 114(1), pp. 175–201.

[21] Crawford, V. P., & Knoer, E. M. (1981): "Job Matching with Heterogeneous Firms and Workers," *Econometrica*, 49(2), pp. 437–50.

[22] Dagsvik, J. K. (2000): "Aggregation in Matching Markets," *International Economic Review*, 41(1), pp. 27–57.

[23] Deaton, A., and Muellbauer, J. (1980). *Economics and consumer behavior*. Cambridge University Press.

[24] Demange, G., & Gale, D. (1985): "The Strategy Structure of Two-Sided Matching Markets," *Econometrica*, 53(4), pp. 873–888.

[25] Dupuy, A., and A. Galichon (2014): "Personality traits and the marriage market," *Journal of Political Economy*, 122 (6), pp. 1271-1319.

[26] Dupuy, A., Galichon, A., Jaffe, S., and Scott D. Kominers (2017): "On the Incidence of Taxation in Matching Markets," working paper.

[27] Flinn, C., Del Boca, D. (2012): "Endogenous Household Interaction," *Journal of Econometrics* 166, pp. 49-65.

[28] Fleiner, T., Jagadeesan, R., Jankó, Z., and Teytelboym, A. (2017): "ETrading Networks with Frictions, working paper.

[29] Fox, J. (2010): "Identification in Matching Games," *Quantitative Economics* 1, pp. 203–254.

[30] Ekeland, I., J. Heckman, and L. Nesheim (2004): "Identification and estimation of hedonic models," *Journal of Political Economy*, 112, 60–109.

[31] Gale, D. (1984): "Equilibrium in a discrete exchange economy with money," *International Journal of Game Theory*, 13(1), pp. 61–64.

[32] Gale, D., & Shapley, L. S. (1962): "College Admissions and the Stability of Marriage," *The American Mathematical Monthly*, 69(1), pp. 9–15.

[33] Galichon, A. and Hsieh, Y.-W. (2017): "A Theory of Decentralized Matching Markets Without Transfers, with an Application to Surge Pricing," working paper, SSRN #2887732.

[34] Galichon, A., and B. Salanié (2015): "Cupid's Invisible Hand: Social Surplus and Identification in Matching Models," working paper, available at SSRN: http://dx.doi.org/10.2139/ssrn.1804623.

[35] Greinecker, M., and Kah, C. (2018): "Pairwise stable matching in large economies," working paper.

[36] Hatfield, J. W., and Milgrom, P. R. (2005): "Matching with Contracts," *American Economic Review*, 95(4), pp. 913–935.

[37] Heckman, J., R. Matzkin, and L. Nesheim (2010): "Nonparametric identification and estimation of nonadditive hedonic models," *Econometrica*, 78(5), 1569–1591.

[38] Hitsch, G. J., Hortasu, A., & Ariely, D. (2010): "Matching and Sorting in Online Dating," *American Economic Review*, 100(1), pp. 130–63.

[39] Iyigun, M. and R. P. Walsh (2007): "Building the family nest: Premarital investments, marriage markets, and spousal allocations," *Review of Economic Studies*, 74, pp. 507–535.

[40] Kaneko, M. (1982): "The central assignment game and the assignment markets," *Journal of Mathematical Economics*, 10(2-3), pp. 205–232.

[41] Legros, P., and Newman, A. F. (2007): "Beauty Is a Beast, Frog Is a Prince: Assortative Matching with Nontransferabilities," *Econometrica*, 75(4), pp. 1073–1102.

[42] Liu, Q., Mailath, G., Postlewaite, A., and Samuelson, L. (2014): "Stable Matching with Incomplete Information." *Econometrica* 82 (2), pp. 541–587.

[43] Mailath, G., Postlewaite, A., and Samuelson, L. (2013): "Pricing and Investments in Matching Markets." *Theoretical Economics* 8 (2), pp. 535–590.

[44] McFadden, D. (1974): "Conditional Logit Analysis of Qualitative Choice Behavior". In *Frontiers in Econometrics*, edited by Zarembka, P. New York: Academic Press.

[45] Menzel, K. (2015): "Large Matching Markets as Two-Sided Demand Systems," *Econometrica*, 83(3), pp. 897-941.

[46] Moré, J. (1972), "Nonlinear Generalizations of Matrix Diagonal Dominance with Applications to Gauss-Seidel Iterations." *SIAM Journal on Numerical Analysis* 9 (2), 1972.

[47] Mourifié, I., and Siow, A. (2014): "Cohabitation vs Marriage: Marriage matching with peer effects," working paper, University of Toronto.

[48] Nöldeke, G., & Samuelson, L. (2015): "The Implementation Duality," *Cowles Foundation Discussion Paper*.

[49] Parmeter, C.F. and Kumbhakar, S.C. (2014), "Efficiency Analysis: A Primer on Recent Advances," *Foundations and Trends in Econometrics* 7 (3–4), pp 191–385.

[50] Pollak, R. (2017): "How Bargaining in Marriage drives Marriage Market Equilibrium," working paper.

[51] Quinzii, M. (1984): "Core and competitive equilibria with indivisibilities," *International Journal of Game Theory*, 13(1), pp. 41–60.

[52] Rheinboldt, W. (1970). "On M-Functions and their Application to Nonlinear Gauss-Seidel Iterations and to Network Flows." *Journal of Mathematical Analysis and Applications* 32, pp. 274–307.

[53] Rockafellar, R.T. (1970): *Convex Analysis*. Princeton University Press.

[54] Roth, A. and M. Sotomayor (1990): *Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis*. Cambridge University Press.

[55] Schoen, R. (1981). "The Harmonic Mean as the Basis of a Realistic Two-Sex Marriage Model." *Demography* 18 (2), pp. 201–216.

[56] Shapley, L., and M. Shubik (1972): "The Assignment Game I: The Core," *International Journal of Game Theory*, 1, pp. 111–130.

[57] Siow, A. (2008). "How does the marriage market clear? An empirical framework." *Canadian Journal of Economics* 41 (4), pp. 1121–1155.

[58] van der Vaart, A. (2000). *Asymptotic Statistics*. Cambridge University Press.

Appendix to "Costly Concessions: An Empirical Framework for Matching with Imperfectly Transferable Utility"

by Alfred Galichon, Scott Duke Kominers and Simon Weber

This appendix contains supplementary material that could not be included in the main text. Appendix A contains the proofs of the mathematical results in the main text. Appendix B provides examples of other models of interest. Appendix C offers remarks on equilibrium versus optimality in the matching models considered in the main paper. Appendix D provides formal results on identification. Appendix E provides results relating aggregate and individual equilibria. Finally, appendix F contains an illustrative example based on the Living Costs and Food Survey (ONS, 2015).

Appendix A. Proofs of the results in the main text

A.1. **Proof of lemma 1.**

Proof. (i) directly follows from the definition of D. (ii) is straightforward given requirements (i)–(iii) of definition 1. Let us show (iii). Assume $(u,v) \leq (u',v')$. Then by requirement (ii) of definition 1, for any $z \in \mathbb{R}$, $(u'-z, v'-z) \in \mathcal{F}_{ij}$ implies $(u-z, v-z) \in \mathcal{F}_{ij}$. Thus $D_{\mathcal{F}}(u,v) \leq D_{\mathcal{F}}(u',v')$, which is the first part of the claim.

Now assume $u < u'$ and $v < v'$ and $D_{\mathcal{F}}(u,v) = D_{\mathcal{F}}(u',v')$. Then $u - D_{\mathcal{F}}(u,v) < u' - D_{\mathcal{F}}(u',v')$ and $v - D_{\mathcal{F}}(u,v) < v' - D_{\mathcal{F}}(u',v')$. But this implies that there exists $\epsilon > 0$ such that $u - D_{\mathcal{F}}(u,v) + \epsilon \leq u' - D_{\mathcal{F}}(u',v')$ and $v - D_{\mathcal{F}}(u,v) + \epsilon \leq v' - D_{\mathcal{F}}(u',v')$; however, as $(u' - D_{\mathcal{F}}(u',v'), v' - D_{\mathcal{F}}(u',v')) \in \mathcal{F}_{ij}$, this implies, still by requirement (ii) of definition 1, that $(u - D_{\mathcal{F}}(u,v) + \epsilon, v - D_{\mathcal{F}}(u,v) + \epsilon) \in \mathcal{F}_{ij}$, a contradiction. Thus $D_{\mathcal{F}}(u,v) < D_{\mathcal{F}}(u',v')$, which completes the proof that $D_{\mathcal{F}}$ is $\gg$-isotone.

To show point (iv) ($D_{\mathcal{F}}$ is continuous), consider (u,v) and (u',v'), and assume that $v - u \geq v' - u'$. Then $u - D(u,v) \leq u' - D_{\mathcal{F}}(u',v')$; indeed, assume by contradiction that $u - D(u,v) > u' - D_{\mathcal{F}}(u',v')$, then by summation $v - D(u,v) > v' - D_{\mathcal{F}}(u',v')$. By the same argument as above, this leads to a contradiction; hence, $u - D(u,v) \leq u' - D_{\mathcal{F}}(u',v')$. It is easy to check that $v - u \leq v' - u'$ implies $u - D(u,v) \geq u' - D_{\mathcal{F}}(u',v')$. Hence, in general

$$\min\left(v'-v, u'-u\right) \leq D_{\mathcal{F}}\left(u',v'\right) - D_{\mathcal{F}}(u,v) \leq \max\left(u'-u, v'-v\right)$$

which shows continuity of $D_{\mathcal{F}}$.

(v) One has $D_{\mathcal{F}}(u+a, v+a) = \min\{z \in \mathbb{R} : (u+a-z, v+a-z) \in \mathcal{F}\}$ by the very definition of $D_{\mathcal{F}}$, which immediately shows that $D_{\mathcal{F}}(u+a, v+a) = a + D_{\mathcal{F}}(u,v)$. ∎

A.2. **Proof of lemma 2.**

Proof. Assume that $K = 2$, as the case $K \geq 2$ follows by induction. To prove (a), we must show that $\mathcal{F}_{\cup}$ and $\mathcal{F}_{\cap}$ are (i) nonempty and closed, (ii) lower comprehensive and (iii) bounded above, as required by definition 1.

Proof of (a) for $\mathcal{F}_\cup$: The union of nonempty closed sets is nonempty and closed, showing (i). Take $(u,v) \in \mathcal{F}_\cup$, say $(u,v) \in \mathcal{F}$. If $u' \leq u$ and $v' \leq v$, then $(u',v') \in \mathcal{F}$ since $\mathcal{F}$ is lower comprehensive. Hence, $(u',v') \in \mathcal{F}_\cup$, showing (ii). Finally, assume that $u_n \to +\infty$ and v_n bounded below. There exist N and N' such that $(u_n, v_n) \notin \mathcal{F}$ for $n \geq N$ and $(u_n, v_n) \notin \mathcal{F}'$ for $n \geq N'$. Hence, $(u_n, v_n) \notin \mathcal{F}_\cup$ for $n \geq \max(N, N')$, showing (iii).

Proof of (a) for $\mathcal{F}_\cap$: The intersection of closed sets is closed. Take $(u,v) \in \mathcal{F}$ and $(u',v') \in \mathcal{F}'$. Then $(\min(u,u'), \min(v,v')) \in \mathcal{F} \cap \mathcal{F}'$ by lower comprehensiveness of $\mathcal{F}$ and $\mathcal{F}'$, hence $\mathcal{F}_\cap$ is nonempty, showing (i). Take $(u,v) \in \mathcal{F}_\cap$. If $u' \leq u$ and $v' \leq v$, then $(u',v') \in \mathcal{F}$ and $(u',v') \in \mathcal{F}'$, since all the $\mathcal{F}$ and $\mathcal{F}'$ are lower comprehensive. Hence, $(u',v') \in \mathcal{F}_\cap$. Finally, assume that $u_n \to +\infty$ and v_n bounded below. There exist N and N' such that $(u_n, v_n) \notin \mathcal{F}$ for $n \geq N$ and $(u_n, v_n) \notin \mathcal{F}'$ for $n \geq N'$. Hence, $(u_n, v_n) \notin \mathcal{F}_\cup$ for $n \geq \min(N, N')$, showing (iii).

We have shown that the sets $\mathcal{F}_\cup$ and $\mathcal{F}_\cap$ are proper bargaining sets. In the following, and without loss of generality, we consider a point (u,v) and denote $\underline{D} = \min\{D_{\mathcal{F}}(u,v), D_{\mathcal{F}'}(u,v)\}$ and $\bar{D} = \max\{D_{\mathcal{F}}(u,v), D_{\mathcal{F}'}(u,v)\}$ for a given (u,v).Let us prove (b).

Proof of (b) for $\mathcal{F}_\cup$: Assume that $D_{\mathcal{F}_\cup}(u,v) > \underline{D}$. Without loss of generality, say $\underline{D} = D_{\mathcal{F}}(u,v)$, then $(u - \underline{D}, v - \underline{D})$ belongs to $\mathcal{F}$ and therefore to $\mathcal{F}_\cup$, hence $D_{\mathcal{F}_\cup}(u,v) \neq \min\{z \in \mathbb{R} : (u-z, v-z) \in \mathcal{F})$, a contradiction. Assume that $D_{\mathcal{F}_\cup}(u,v) < \underline{D}$. Then $u - D_{\mathcal{F}_\cup}(u,v) > u - D_{\mathcal{F}}(u,v)$ and $v - D_{\mathcal{F}_\cup}(u,v) > v - D_{\mathcal{F}}(u,v)$, so that $(u - D_{\mathcal{F}_\cup}(u,v), v - D_{\mathcal{F}_\cup}(u,v)) \notin \mathcal{F}$. Similarly, we can show that $(u - D_{\mathcal{F}_\cup}(u,v), v - D_{\mathcal{F}_\cup}(u,v)) \notin \mathcal{F}'$. Hence, $(u - D_{\mathcal{F}_\cup}(u,v), v - D_{\mathcal{F}_\cup}(u,v)) \notin \mathcal{F}_\cup$, a contradiction.

Proof of (b) for $\mathcal{F}_\cap$: Assume that $D_{\mathcal{F}_\cap}(u,v) > \bar{D}$. Note that $u - \bar{D} \leq u - D_{\mathcal{F}}(u,v)$ and $v - \bar{D} \leq v - D_{\mathcal{F}}(u,v)$, so that $(u - \bar{D}, v - \bar{D}) \in \mathcal{F}$ by lower comprehensiveness. Similarly, we can show that $(u - \bar{D}, v - \bar{D}) \in \mathcal{F}'$. Hence, $(u - \bar{D}, v - \bar{D}) \in \mathcal{F}_\cap$, so $D_{\mathcal{F}_\cap}(u,v) \neq \min\{z \in \mathbb{R} : (u-z, v-z) \in \mathcal{F})$, a contradiction. Assume that $D_{\mathcal{F}_\cap}(u,v) < \bar{D}$. Assume wlog that $D_{\mathcal{F}}(u,v) = \bar{D}$ and notice that $u - D_{\mathcal{F}_\cap}(u,v) > u - D_{\mathcal{F}}(u,v)$ and $v - D_{\mathcal{F}_\cap}(u,v) > v - D_{\mathcal{F}}(u,v)$. Thus, $(u - D_{\mathcal{F}_\cap}(u,v), v - D_{\mathcal{F}_\cap}(u,v)) \notin \mathcal{F}$, hence $(u - D_{\mathcal{F}_\cap}(u,v), v - D_{\mathcal{F}_\cap}(u,v)) \notin \mathcal{F}_\cap$, a contradiction. ∎

A.3. **Proof of lemma 3.**

Proof. The proof is divided in several parts.

First part: let us show that the set of wedges w that can be expressed as $w = u - v$ for u and v such that $D_{\mathcal{F}}(u, v) = 0$ is an open interval. Consider u, u', v and v' such that $D(u, v) = 0$ and $D(u', v') = 0$, and $(u, v) \neq (u', v')$. Let $w = u - v$ and $w' = u' - v'$. Assume w.l.o.g $u' > u$, then one has necessarily $v \geq v'$, hence $u' - v' > u - v$. In this case, let $u_t = t(u' - u) + u$ and $v_t = t(v' - v) + v$. Let $\tilde{u}_t = u_t - D(u_t, v_t)$ and $\tilde{v}_t = v_t - D(u_t, v_t)$, so that $D(\tilde{u}_t, \tilde{v}_t) = 0$. One has $\tilde{u}_t - \tilde{v}_t = u_t - v_t = t(u' - v') + (1 - t)(u - v) = tw' + (1 - t)w$, which shows that the set of wedges is an interval, denoted I. Let us now show that this interval is open. Call $\underline{w}$ the infimum of the interval, and assume it is finite. Then there is a sequence (u_n, v_n) such that u_n is decreasing, v_n is increasing, $D(u_n, v_n) = 0$ and $u_n - v_n \to \underline{w}$. Then by the scarcity of $\mathcal{F}$, u_n and v_n need to remain bounded, hence they converge in $\mathcal{F}$. Let (u^*, v^*) be their limit; one has $D(u^*, v^*) = 0$ and $u^* - v^* = \underline{w}$. For any $u' < u^*$, one has $D(u', v^*) \leq 0$; hence, by scarcity of $\mathcal{F}$, there is some $v' \geq v^*$ such that $D(u', v') = 0$. $u' < u^*$ and $v' \geq v^*$, thus $u' - v' < u^* - v^* = \underline{w}$, a contradiction. Thus $\underline{w} \in I$. A symmetric argument shows that if the supremum of I is finite, then it belongs in I. Thus, I is an open interval.

Second part: let us show that $\mathcal{U}$ and $\mathcal{V}$ are well defined on I. For $w \in I$, there exists by definition (u, v) such that $D(u, v) = 0$ and $u - v = w$. The argument at the beginning of part (i) implies that (u, v) is unique. Hence $\mathcal{U}$ and $\mathcal{V}$ are well defined.

Third part: let us show that $\mathcal{U}$ is increasing and $\mathcal{V}$ is decreasing. Suppose $w < w'$ and $\mathcal{U}(w) \geq \mathcal{U}(w')$. Then $w - \mathcal{U}(w) < w' - \mathcal{U}(w')$, hence $\mathcal{V}(w) > \mathcal{V}(w')$, a contradiction. Thus $\mathcal{U}(w) < \mathcal{U}(w')$, which shows that $\mathcal{U}$ is increasing. By a similar logic, $\mathcal{V}(w) > \mathcal{V}(w')$, and $\mathcal{V}$ is decreasing.

Fourth part: let us show that $\mathcal{U}$ is 1-Lipschitz. Take $\epsilon > 0$ and assume by contradiction $u' > u + \epsilon$ where $u = \mathcal{U}(w)$ and $u' = \mathcal{U}(w + \epsilon)$. Then $D(u, u - w) = 0$ with and $D(u', u' - w - \epsilon) = 0$. Then because $u' > u$ and $u' - \epsilon > u$, it follows that $D(u', u' - w - \epsilon) > D(u, u - w) = 0$, a contradiction. Hence, $0 \leq \mathcal{U}(w + \epsilon) - \mathcal{U}(w) \leq \epsilon$, and thus $\mathcal{U}$ is 1-Lipschitz. A similar argument for $\mathcal{V}$ completes the proof.

Fifth part: let us show that expression (3.3) holds. By applying point (v) of lemma 1 twice, once with $a = -u$ and once $a = -v$, it follows respectively that $D_{\mathcal{F}}(0, v-u) = D_{\mathcal{F}}(u, v) - u$ and that $D_{\mathcal{F}}(u-v, 0) = D_{\mathcal{F}}(u, v) - v$. Hence, if (u, v, w) are solutions to (3.2), it follows that $u = -D_{\mathcal{F}}(0, -w)$, and thus $v = -D_{\mathcal{F}}(w, 0)$. Hence (3.3) holds. ∎

A.4. **Proof of lemma 4.**

Proof. Set $U_i = u_i - \varepsilon_{iy_j}$ and $V_j = v_j - \eta_{x_i j}$. Then by assumption 1, there exists $w_{ij} \in \mathbb{R}$ such that $U_i \leq \mathcal{U}_{x_i y_j}(w_{ij})$ and $V_j \leq \mathcal{V}_{x_i y_j}(w_{ij})$; by definition, $\left(\mathcal{U}_{x_i y_j}(w_{ij}), \mathcal{V}_{x_i y_j}(w_{ij})\right) \in \mathcal{F}_{x_i y_j}$, and because the latter set is a proper bargaining set, it follows from definition 1, part (ii) that $(U_i, V_j) \in \mathcal{F}_{x_i y_j}$. ∎

A.5. **Proof of proposition 1.**

Proof. Assume μ is an aggregate equilibrium matching. Then, by definition, there exists a pair of vectors U and V such that (μ, U, V) is an aggregate equilibrium outcome. Thus $D_{xy}(U_{xy}, V_{xy}) = 0$ for every $x \in \mathcal{X}$, $y \in \mathcal{Y}$, and $\mu_{xy} = \partial G(U)/\partial U_{xy}$ and $\mu_{xy} = \partial H(V)/\partial V_{xy}$, which inverts into

$$U_{xy} = \partial G^*(\mu)/\partial \mu_{xy} \text{ and } V_{xy} = \partial H^*(\mu)/\partial \mu_{xy}, \tag{A.1}$$

and thus by substitution,

$$D_{xy}\left(\partial G^*(\mu)/\partial \mu_{xy}, \partial H^*(\mu)/\partial \mu_{xy}\right) = 0 \tag{A.2}$$

holds for every $x \in \mathcal{X}$, $y \in \mathcal{Y}$. Conversely, assume (A.2) holds. Then, defining U and V by (A.1), one sees that (μ, U, V) is an aggregate equilibrium outcome. ∎

A.6. **Proof of proposition 2.**

Proof. Assume (μ, U, V) is an aggregate equilibrium outcome. Then $D_{xy}(U_{xy}, V_{xy}) = 0$ for every $x \in \mathcal{X}$, $y \in \mathcal{Y}$, and

$$\mu_{xy} = \partial G(U)/\partial U_{xy} = \partial H(V)/\partial V_{xy} \tag{A.3}$$

thus, by lemma 3, there exists a vector (W_{xy}) such that for every $x \in \mathcal{X}$, $y \in \mathcal{Y}$,

$$U_{xy} = \mathcal{U}_{xy}(W_{xy}) \text{ and } V_{xy} = \mathcal{V}_{xy}(W_{xy}), \tag{A.4}$$

where $\mathcal{U}_{xy}$ and $\mathcal{V}_{xy}$ are defined in (3.3). Thus it follows that $Z(W) = 0$. Conversely, assume that $Z(W) = 0$. Then letting U and V as in (A.4), and μ such that $\mu_{xy} = \partial G(U)/\partial U_{xy}$, it follows that (μ, U, V) is an aggregate equilibrium outcome. ∎

A.7. **Proof of proposition 3.**

Proof. Recall that

$$Z_{x'y'}(W) = \frac{\partial H}{\partial V_{x'y'}}(\mathcal{V}(W)) - \frac{\partial G}{\partial U_{x'y'}}(\mathcal{U}(W))$$

and, because of assumption 3, $\partial G/\partial U_{x'y'}(U)$ is increasing in $U_{x'y'}$, decreasing in U_{xy} if either of the conditions $x = x'$ or $y = y'$ holds (but not both), a similar condition holds for H, and $W_{xy} \to \mathcal{V}_{xy}(W_{xy})$ is nonincreasing, while $W_{xy} \to \mathcal{U}_{xy}(W_{xy})$ is nondecreasing. At the same time, $\mathcal{U}_{xy}(W_{xy}) - \mathcal{V}_{xy}(W_{xy}) = W_{xy}$, so $\mathcal{U}_{xy}$ and $\mathcal{V}_{xy}$ cannot be stationary at the same point W_{xy}.

Proof of (a.1): One has $Z_{x'y'}(W) = \partial H/\partial V_{x'y'}(\mathcal{V}(W)) - \partial G/\partial U_{x'y'}(\mathcal{U}(W))$, thus the map $W_{x'y'} \to Z_{x'y'}(W)$ is nonincreasing. At the same time, as $\partial G/\partial U_{x'y'}(U)$ is increasing in $U_{x'y'}$ and $\partial H/\partial V_{x'y'}(V)$ is increasing in $V_{x'y'}$ and as $\mathcal{U}_{xy}$ and $\mathcal{V}_{xy}$ cannot be stationary at the same point W_{xy}, it follows that $W_{x'y'} \to Z_{x'y'}(W)$ is decreasing.

Proof of (a.2): The proof is based on the same logic as above.

Proof of (a.3): When $x \neq x'$ and $y \neq y'$, then the quantity $\partial H/\partial V_{x'y'}(\mathcal{V}(W))$ does not depend on W_{xy} and nor does $\partial G/\partial U_{x'y'}(\mathcal{U}(W))$. Thus $Z_{x'y'}(W)$ does not depend on W_{xy}.

Proof of (b): One has

$$\sum_{x'\in\mathcal{X}, y'\in\mathcal{Y}} Z_{x'y'}(W) = \sum_{y'} m_{y'} - \sum_{x'} n_{x'} + \sum_{x'} \mu_{x'0}(\mathcal{U}(W)) - \sum_{y'} \mu_{0y'}(\mathcal{V}(W))$$

where $\mu_{x'0}(U)$ is defined as $n_{x'} - \sum_{y'\in\mathcal{Y}} \partial G(U)/\partial U_{x'y'}$, and $\mu_{0y'}(V)$ is defined as $m_{y'} - \sum_{x'\in\mathcal{X}} \partial H(V)/\partial V_{x'y'}$. But it is easy to check that $\mu_{x'0}(U) = n_{x'} \Pr(\varepsilon_{i0} > \max_{y'\in\mathcal{Y}}\{U_{x'y'} + \varepsilon_{iy'}\})$, thus $\mu_{x'0}(U)$ is decreasing with respect to all the entries of vector $U_{x'y'}$, $y' \in \mathcal{Y}$. A similar logic applies to show that $\mu_{0y'}(V)$ is decreasing with respect to all the entries of vector $V_{x'y'}$, $x' \in \mathcal{X}$. Hence, $\sum_{x'\in\mathcal{X}, y'\in\mathcal{Y}} Z_{x'y'}(W)$ is decreasing with respect to any entry of the vector W.

Remark: Conditions (a.1)–(a.3) express that $-Z$ is a Z-function, while conditions (a) and (b) together imply that $-Z$ is a M-function. See Rheinboldt (1970). ∎

A.8. **Proof of theorem 1.** The existence part of theorem 1 is constructive, and consists in showing that algorithm 1 converges to a solution of equations (4.7); this convergence in turns follows from two claims, which are rather classical but included here for completeness. The uniqueness part relies on the fact that, by a result of Berry et al. (2013), the Gross Substitute property established in proposition 3 implies that the excess demand function Z is inverse antitone, thus injective.

We show that:

Claim 1. *There exist two vectors w^l and w^u such that $w^l \leq w^u$ and*

$$Z\left(w^u\right) \leq 0 \leq Z\left(w^l\right).$$

Proof. By assumption 2 (iii), for each $x \in \mathcal{X}$, either all the men's payoffs U_{xy} are bounded above or they all converge to $+\infty$. Let $\mathcal{X}_1$ be the set of $x \in \mathcal{X}$ such that for each $y \in \mathcal{Y}$, $\mathcal{U}_{xy}\left(w_{xy}\right)$ all converge to $+\infty$ as $w_{xy} \to \bar{w}_{xy}$. For $x \in \mathcal{X}_1$, let $p_y = n_x \left(1 - 1/k\right) / \left|\mathcal{Y}\right|$, and let $U^k_{xy} = \partial G^* / \partial \mu_{xy}\left(p\right)$. It is easy to see that $U^k_{xy} \to +\infty$, thus $V^k_{xy} \to -\infty$. Hence there exists w_{xy} such that $\mathcal{U}_{xy}\left(w_{xy}\right) = U^k_{xy}$ and $V^k_{xy} = \mathcal{V}_{xy}\left(w^k\right)$. Now for $x \notin \mathcal{X}_1$, then for each $y \in \mathcal{Y}$, $\bar{w}_{xy}$ is finite, and $\mathcal{U}_{xy}\left(w_{xy}\right)$ all converge to a finite value $\bar{U}_{xy} \in \mathbb{R}$. Then, let $U^k_{xy} = \mathcal{U}_{xy}\left(\bar{w}_{xy} - 1/k\right)$ and $V^k_{xy} = \mathcal{V}_{xy}\left(\bar{w}_{xy} - 1/k\right)$, so that $V^k_{xy} \to -\infty$ and $U^k_{xy} \to \bar{U}_{xy} \in \mathbb{R}$. We have thus constructed vectors w^k such that $w^k_{xy} \to \bar{w}_{xy}$ for all x and y, and $\mathcal{V}_{xy}\left(w^k\right) \to -\infty$, while $\mathcal{U}_{xy}\left(w^k\right)$ converges to a vector of positive numbers. Thus, for k large enough, setting $w^u = w^k$ implies $Z\left(w^u\right) \leq 0$. A similar logic implies that there exists w^l such that $Z\left(w^l\right) \geq 0$. ∎

Claim 2. *Z is inverse antitone: if $Z\left(w\right) \leq Z\left(w'\right)$ for some two vectors w and w', then $w \geq w'$.*

Proof. We show that $-Z$ satisfies the assumptions in Berry et al. (2013), theorem 1, see also related results in Moré (1972), theorem 3.3. We verify the three assumptions in Berry et al. (2013). Assumption 1 in that paper is met because Z is defined on the Cartesian product of the intervals $(\underline{w}_{xy}, \bar{w}_{xy})$. Next, by part (a.2) of proposition 3 above, $-Z_{xy}(w)$ is weakly decreasing in $w_{x'y'}$ for $x'y' \neq xy$, and letting

$$Z_0(w) = \sum_{y'} m_{y'} - \sum_{x'} n_{x'} - \sum_{\substack{x \in \mathcal{X} \\ y \in \mathcal{Y}}} Z_{xy}(w),$$

it follows from part (b) of proposition 3 above that $-Z_0$ is strictly decreasing in all the w_{xy}. Thus assumption 2 and 3 in Berry et al. (2013) are also satisfied, hence $-Z$ is inverse isotone, Z is inverse antitone. ∎

With these preparations, a proof of theorem 1 can be provided.

Proof of theorem 1. We prove existence first, then uniqueness.

Proof of existence: It is easy to see that Z is continuous, and by the results of proposition 3, it is strictly diagonally antitone, and off-diagonally isotone. Existence follows from theorem 3.1 in Rheinboldt (1970) jointly with proposition 3 and Claim 1. The proof there is based on a constructive argument based on nonlinear Gauss-Seidel iterations, as discussed in section 5.2.1.

Proof of uniqueness: As noted in Berry et al. (2013), uniqueness follows from Claim 2 as in corollary 1. Indeed, assume $Z(w) = Z(w')$. Then, by Claim 2, both inequalities $w \geq w'$ and $w' \geq w$ hold, and thus $w = w'$. ∎

A.9. **Proof of corollary 1.**

Proof. This corollary directly follows from a combination of proposition 2 and theorem 1. ∎

A.10. **Proof of theorem 2.**

Proof. By combining theorem 1 with proposition 1, it follows that equation (4.7), namely

$$D_{xy}\left(\frac{\partial G^*}{\partial \mu_{xy}}(\mu), \frac{\partial H^*}{\partial \mu_{xy}}(\mu)\right) = 0$$

has a unique solution. But when assumption 3 is strengthened into assumption 3', then

$$\partial G^*/\partial \mu_{xy}(\mu) = \log\left(\mu_{xy}/\mu_{x0}\right) \text{ and } \partial H^*/\partial \mu_{xy}(\mu) = \log\left(\mu_{xy}/\mu_{0y}\right)$$

where $\mu_{x0} = n_x - \sum_{y\in\mathcal{Y}} \mu_{xy}$ and $\mu_{0y} = m_y - \sum_{x\in\mathcal{X}} \mu_{xy}$. Hence equation (4.7) rewrites as

$$\begin{cases} D_{xy}\left(\log\mu_{xy} - \log\mu_{x0}, \log\mu_{xy} - \log\mu_{0y}\right) = 0 \\ \mu_{x0} + \sum_{y\in\mathcal{Y}} \mu_{xy} = n_x \\ \mu_{0y} + \sum_{x\in\mathcal{X}} \mu_{xy} = m_y \end{cases} \tag{A.5}$$

but $D_{xy}\left(\log\mu_{xy} - \log\mu_{x0}, \log\mu_{xy} - \log\mu_{0y}\right) = \log\mu_{xy} + D_{xy}\left(-\log\mu_{x0}, -\log\mu_{0y}\right)$, thus system (A.5) rewrites as system (6.2). Conversely, assume $\left(\mu_{x0}, \mu_{0y}\right)$ satisfy the system (6.2). Then, letting $\mu_{xy} = M_{xy}\left(\mu_{x0}, \mu_{0y}\right)$, $U_{xy} = \log\left(\mu_{xy}/\mu_{x0}\right)$ and $V_{xy} = \log\left(\mu_{xy}/\mu_{0y}\right)$, one has $\mathcal{M}^{int}$, $D_{xy}\left(U_{xy}, V_{xy}\right) = 0$ and $U_{xy} = \log\left(\mu_{xy}/\mu_{x0}\right)$ and $V_{xy} = \log\left(\mu_{xy}/\mu_{0y}\right)$, thus (μ, U, V) is an aggregate equilibrium outcome. ∎

A.11. **Proof of theorem 3.**

Proof. The proof of theorem 2 is based on the following set of properties of $M_{xy}\left(\mu_{x0}, \mu_{0y}\right) = \exp\left(-D_{xy}\left(-\log\mu_{x0}, -\log\mu_{0y}\right)\right)$, which are direct consequences of definition 1 and of lemma 1. For every pair $x \in \mathcal{X}$, $y \in \mathcal{Y}$:

(i) Map $M_{xy} : (a, b) \mapsto M_{xy}(a, b)$ is continuous.

(ii) Map $M_{xy} : (a, b) \mapsto M_{xy}(a, b)$ is weakly isotone, i.e. if $a \leq a'$ and $b \leq b'$, then $M_{xy}(a, b) \leq M_{xy}(a', b')$.

(iii) For each $a > 0$, $\lim_{b\to 0^+} M_{xy}(a, b) = 0$, and for each $b > 0$, $\lim_{a\to 0^+} M_{xy}(a, b) = 0$.

Given these properties, the existence of a solution (μ_{x0}, μ_{0y}) is essentially an application of Tarski's fixed point theorem; we provide an explicit proof for concreteness. We show that the construction of μ_{x0}^{2t+1} and μ_{0y}^{2t+2} at each step is well defined. Consider step $2t+1$. For each $x \in \mathcal{X}$, the equation to solve is

$$\sum_{y\in\mathcal{Y}} M_{xy}(\mu_{x0}, \mu_{0y}) + \mu_{x0} = n_x$$

but the right-hand side is a continuous and increasing function of μ_{x0}, tends to 0 when $\mu_{x0} \to 0$ and tends to $+\infty$ when $\mu_{x0} \to +\infty$. Hence μ_{x0}^{2t+1} is well defined and belongs in $(0, +\infty)$. Denoting

$$\mu_{x0}^{2t+1} = \mathfrak{F}_x(\mu_{0.}^{2t}),$$

we see that $\mathfrak{F}$ is antitone, meaning that $\mu_{0y}^{2t} \leq \tilde{\mu}_{0y}^{2t}$ for all $y \in \mathcal{Y}$ implies $\mathfrak{F}_x(\tilde{\mu}_{0.}^{2t}) \leq \mathfrak{F}_x(\mu_{0.}^{2t})$ for all $x \in \mathcal{X}$. By the same token, at step $2t+2$, μ_{0y}^{2t+2} is well defined in $(0, +\infty)$, and we can denote

$$\mu_{0y}^{2t+2} = \mathfrak{G}_y(\mu_{.0}^{2t+1})$$

where, similarly, $\mathfrak{G}$ is antitone. Thus, $\mu_{0.}^{2t+2} = \mathfrak{G} \circ \mathfrak{F}\left(\mu_{0.}^{2t}\right)$, where $\mathfrak{G} \circ \mathfrak{F}$ is isotone. But $\mu_{0y}^{2} \leq m_y = \mu_{0y}^{0}$ implies that $\mu_{0.}^{2t+2} \leq \mathfrak{G} \circ \mathfrak{F}\left(\mu_{0.}^{2t}\right)$. Hence $\left(\mu_{0.}^{2t+2}\right)_{t \in \mathbb{N}}$ is a decreasing sequence, bounded from below by 0. As a result $\mu_{0.}^{2t+2}$ converges. Letting $\bar{\mu}_{0.}$ be its limit, and letting $\bar{\mu}_{.0} = \mathfrak{F}(\bar{\mu}_{0.})$, it is not hard to see that $\left(\bar{\mu}_{0x}, \bar{\mu}_{0y}\right)$ is a solution to (6.2). ∎

A.12. **Proof of theorem 4.**

Proof. Proof of part (i): One has $l\left(\hat{\pi}|\theta,\zeta\right) = \sum_{xy \in \mathcal{X}\mathcal{Y} \cup \mathcal{X}_0 \cup \mathcal{Y}_0} \hat{\pi}_{xy} \log \Pi_{xy}\left(\theta,\zeta\right)$, hence

$$l\left(\hat{\pi}|\theta,\zeta\right) = \sum_{xy \in \mathcal{X}\mathcal{Y} \cup \mathcal{X}_0 \cup \mathcal{Y}_0} \hat{\pi}_{xy} \log \mu_{xy}^{\theta,\zeta} - \log N^h\left(\theta,\zeta\right).$$

But as $\log \mu_{xy}^{\theta,\zeta} = -D_{xy}^{\theta}\left(u_x^{\theta}, v_y^{\theta}\right)$, we get

$$-l\left(\hat{\pi}|\theta, A\hat{\pi}\right) = \sum_{xy \in \mathcal{X}\mathcal{Y}} \hat{\pi}_{xy} D_{xy}^{\theta}\left(u_x^{\theta}, v_y^{\theta}\right) + \sum_{x \in \mathcal{X}} \hat{\pi}_{x0} u_x^{\theta} + \sum_{y \in \mathcal{Y}} \hat{\pi}_{0y} v_y^{\theta} + \log N^h\left(\theta,\zeta\right).$$

Proof of part (ii): This asymptotic result is classical in nonparametric estimation in the presence of a nuisance parameter; see e.g. van der Vaart (2000), Chapter 25. θ is such that $\nabla_\theta l\left(\pi|\theta, A\pi\right) = 0$, and $\hat{\theta}$ is such that $\nabla_\theta l\left(\hat{\pi}|\hat{\theta}, A\hat{\pi}\right) = 0$. As a result, $\nabla_\theta l\left(\hat{\pi}|\hat{\theta}, A\hat{\pi}\right) - \nabla_\theta l\left(\pi|\theta, A\pi\right) = 0$, and

$$\left(D_{\theta\pi} l\right)\left(\hat{\pi} - \pi\right) + \left(D_{\theta\zeta} l\right) A\left(\hat{\pi} - \pi\right) + \left(D_{\theta\theta} l\right)\left(\hat{\theta} - \theta\right) = o\left(n^{-1/2}\right)$$

but by first order conditions, the first term is equal to zero, and the equation becomes

$$\hat{\theta} - \theta = \mathcal{I}_{11}^{-1}\mathcal{I}_{12} A\left(\hat{\pi} - \pi\right) + o\left(n^{-1/2}\right)$$

and as the convergence in distribution $n^{1/2}\left(\hat{\pi} - \pi\right) \Rightarrow V_\pi$ holds as $n \to \infty$, expression (6.11) follows. ∎

A.13. **Proof of theorem 5.**

Proof. (i) and (ii) follow as a direct consequence of equations (D.1), while (iii) follows from taking the pairwise difference of equations (D.2) and using the fact that $\mathcal{U}_{xy}(w_{xy}) - \mathcal{V}_{xy}(w_{xy}) = w_{xy}$. ∎

A.14. **Proof of theorem 6.**

Proof. Proof of part (i): Let (μ, U, V) be an aggregate equilibrium matching, and let u_i and v_j as in (E.1). By definition of these quantities, one has $u_i - \varepsilon_{iy} \geq U_{xy}$ and $v_j - \eta_{xj} \geq V_{xy}$, thus $D_{xy}(u_i - \varepsilon_{iy}, v_j - \eta_{yj}) \geq D_{xy}(U_{xy}, V_{xy}) = 0$. Further, $u_i \geq \varepsilon_{i0}$ and $v_j \geq \eta_{0j}$, hence the stability condition holds. Let us show that one can construct μ_{ij} so that (μ_{ij}, u_i, v_j) is feasible. For $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, let $\mathcal{I}_{xy}$ be the set of $i \in \mathcal{I}$ such that $x_i = x$ and $y = \arg\max_{y \in \mathcal{Y}_0} \{U_{xy} + \varepsilon_{iy}, \varepsilon_{i0}\}$. Similarly, let $\mathcal{J}_{xy}$ be the set of $j \in \mathcal{J}$ such that $y_j = y$ and $x = \arg\max_{x \in \mathcal{X}_0} \{V_{xy} + \eta_{xj}, \eta_{0j}\}$. The mass of $\mathcal{I}_{xy}$ is $\partial G(U) / \partial U_{xy}$ and the mass of $\mathcal{J}_{xy}$ is $\partial H(V) / \partial V_{xy}$. The equilibrium condition $\mu = \nabla G(U) = \nabla H(V)$ implies therefore that the mass of $\mathcal{I}_{xy}$ and the mass of $\mathcal{J}_{xy}$ coincide. One can therefore take any assignment of men in $\mathcal{I}_{xy}$ to women in $\mathcal{J}_{xy}$. Let μ_{ij} be the resulting individual assignment. If $\mu_{ij} > 0$, then $i \in \mathcal{I}_{x_i y_j}$ and $j \in \mathcal{J}_{x_i y_j}$, therefore $u_i = U_{xy} + \varepsilon_{iy}$ and $v_j = V_{xy} + \eta_{xj}$, thus $D_{xy}(u_i - \varepsilon_{iy}, v_j - \eta_{yj}) = D_{xy}(U_{xy}, V_{xy}) = 0$. Assume i is unassigned under (μ_{ij}); then for all $y \in \mathcal{Y}$, $u_i > U_{xy} + \varepsilon_{iy}$, and thus $u_i = \varepsilon_{i0}$. Similarly, if j is unassigned under (μ_{ij}), then $v_j = \eta_{0j}$. Hence, (μ_{ij}, u_i, v_j) is an individual equilibrium.

Proof of part (ii): Now assume (μ_{ij}, u_i, v_j) is an individual equilibrium. Then for all i and j, the stability condition

$$D_{x_i y_j}(u_i - \varepsilon_{iy}, v_j - \eta_{jx}) \geq 0,$$

holds, and holds with equality if $\mu_{ij} > 0$. Hence, for all pairs x and y, we have the inequality

$$\min_{i: x_i = x} \min_{j: y_j = y} \left\{ D_{x_i y_j}(u_i - \varepsilon_{iy}, v_j - \eta_{jx}) \right\} \geq 0,$$

with equality if $\mu_{xy} > 0$, that is, if there is at least one marriage between a man of type x and a woman of type y. Taking U and V as (E.2), and making use of the strict monotonicity of D_{xy} in both its arguments, matching $\mu \in \mathcal{M}$ is an equilibrium matching if inequality

$D_{xy}(U_{xy}, V_{xy}) \geq 0$ holds for any x and y, with equality if $\mu_{xy} > 0$. By definition of U and V, one has

$$u_i \geq \max_y \{U_{xy} + \varepsilon_{iy}, \varepsilon_{i0}\} \text{ and } v_j \geq \max_{x \in \mathcal{X}} \left\{V_{xy_j} + \eta_{xj}, \eta_{0j}\right\}.$$

Assume one of these inequalities holds strict, for instance $u_i > \max_y \{U_{xy} + \varepsilon_{iy}, \varepsilon_{i0}\}$. Then $u_i - \varepsilon_{iy} > U_{xy}$. Because D was assumed strictly increasing, this implies that for all j

$$D_{xy}\left(u_i - \varepsilon_{iy}, v_j - \eta_{yj}\right) > D_{xy}\left(U_{xy}, v_j - \eta_{yj}\right) \geq D_{xy}(U_{xy}, V_{xy}) \geq 0$$

thus for all j, $\mu_{ij} = 0$. Therefore i is single, but $u_i > \varepsilon_{i0}$ yields a contradiction. Now assumption 3 implies $\mu_{xy} > 0$ for all x and y, thus $D_{xy}(U_{xy}, V_{xy}) = 0$. ∎

A.15. **Proof of corollary 2.**

Proof. Let $\left(\mu_{ij}, u_i, v_j\right)$ be an individual outcome. By part (ii) of theorem 6, the aggregate outcome (μ, U, V) is such that

$$U_{xy} = \min_{i:x_i=x} \{u_i - \varepsilon_{iy}\} \text{ and } V_{xy} = \min_{j:y_j=y} \left\{v_j - \eta_{jx}\right\},$$

hence $U_{xy} \geq u_i - \varepsilon_{iy}$ and $V_{xy} \geq v_j - \eta_{jx}$, but $D_{xy}(U_{xy}, V_{xy}) = 0$ and $D_{xy}\left(u_i - \varepsilon_{iy}, v_j - \eta_{jx}\right) = 0$, thus, by assumption 2', $U_{xy} = u_i - \varepsilon_{iy}$ and $V_{xy} = v_j - \eta_{jx}$. Hence $u_i = U_{xy} + \varepsilon_{iy}$ and $v_j = V_{xy} + \eta_{jx}$. ∎

Appendix B. Other models of interest

B.1. **Matching with a Linear Tax Schedule.** Our framework can also model a labor market with linear tax: Assume the nominal wage W_{ij} is taxed at rate $1 - R_{ij}$ on the employee's side (income tax) and at rate $1 + C_{ij}$ on the firm's side (social contributions). The tax rates are allowed to depend on both employer and employee characteristics. Assume that if employee i and employer j match and decide on a wage W_{ij}, they respectively have (post-transfer) utilities $u_i = \alpha_{ij} + R_{ij} W_{ij}$ and $v_j = \gamma_{ij} - C_{ij} W_{ij}$, where α_{ij} is job j's amenity to worker i, and γ_{ij} is the productivity of worker i on job j. This specification is called the *Linearly Transferable Utility* (LTU) model, and the feasible set is given by

$$\mathcal{F}_{ij} = \left\{(u, v) \in \mathbb{R}^2 : \lambda_{ij} u + \zeta_{ij} v \leq \Phi_{ij}\right\},$$

where $\lambda_{ij} = 1/R_{ij} > 0$, and $\zeta_{ij} = 1/C_{ij} > 0$, and $\Phi_{ij} = \lambda_{ij}\alpha_{ij} + \zeta_{ij}\gamma_{ij}$. Note that the TU case is recovered when $\lambda_{ij} = 1$ and $\zeta_{ij} = 1$. A simple calculation yields

$$D_{ij}(u,v) = \frac{\lambda_{ij}u + \zeta_{ij}v - \Phi_{ij}}{\lambda_{ij} + \zeta_{ij}}, \tag{B.1}$$

and we have

$$\mathcal{U}_{ij}(w) = \frac{\Phi_{ij} + \zeta_{ij}w}{\lambda_{ij} + \zeta_{ij}} \text{ and } \mathcal{V}(w) = \frac{\Phi_{ij} - \lambda_{ij}w}{\lambda_{ij} + \zeta_{ij}}.$$

The LTU model (B.1) is studied in depth in Dupuy et al. (2017), who carry welfare analysis. In the case when the heterogeneity is logit, the matching function becomes

$$M_{xy}\left(\mu_{x0}, \mu_{0y}\right) = e^{\left(\lambda_{xy}\alpha_{xy} + \zeta_{xy}\gamma_{xy}\right)/\left(\lambda_{xy} + \zeta_{xy}\right)} \mu_{x0}^{\lambda_{xy}/\left(\lambda_{xy} + \zeta_{xy}\right)} \mu_{0y}^{\zeta_{xy}/\left(\lambda_{xy} + \zeta_{xy}\right)}. \tag{B.2}$$

In particular, when $\lambda_{xy} = 1$ and $\zeta_{xy} = 1$, one recovers the Choo and Siow (2006) matching function.

B.2. **Matching with Uncertainty.** Now, we consider a model of matching with risksharing; such a model is considered by Legros and Newman (2007), Chade and Eeckhout (2014), and Chiappori and Reny (2016) who all focus on characterizing (positive or negative) assortativeness. Assume $i \in \mathcal{I}$ are the men and $j \in \mathcal{J}$ are the women. Assume that the joint endowment of the household ij is $\tilde{b}_{ij} \in \mathbb{R}^d$, which is stochastic. Let $\tilde{c}_i \in \mathbb{R}^d$ be the contingent consumption of the man, and $\tilde{c}_j \in \mathbb{R}^d$ be the contingent consumption of the woman. It is assumed that i and j are expected utility maximizers with respective (concave) utilities U_i and V_j. Hence, they enjoy respective utilities $\mathbb{E}\left[U_i\left(\tilde{c}_i\right)\right]$ and $\mathbb{E}\left[V_j\left(\tilde{c}_j\right)\right]$. Letting λ and $(1-\lambda)$ be the respective Pareto weights associated to i and j's consumption, the contingent consumptions conditional on household budget b are given by the program

$$S_{ij}(\lambda; b) = \max_{c_i, c_j \geq 0} \left\{\lambda U_i(c_i) + (1-\lambda) V_j(c_j) : c_i + c_j \leq b\right\}.$$

The set of feasible utilities for household ij is

$$\mathcal{F}_{ij} = \left\{(u,v) \in \mathbb{R}^2 : \max_{\lambda \in [0,1]} \left\{\lambda u + (1-\lambda) v - \mathbb{E}\left[S_{ij}\left(\lambda; \tilde{b}_{ij}\right)\right]\right\} \leq 0\right\},$$

from which it is immediate that the corresponding distance-to-frontier is

$$D_{ij}(u,v) = \max_{\lambda \in [0,1]} \left\{\lambda u + (1-\lambda) v - \mathbb{E}\left[S_{ij}\left(\lambda; \tilde{b}_{ij}\right)\right]\right\}.$$

Appendix C. Further remarks

C.1. **Equilibrium vs optimality.** In this appendix, we comment on the contrast between equilibrium and optimality, which manifests itself in our model. It is a well known fact (which is studied in Monge-Kantorovich theory) that in the TU case, equilibrium and optimality coincide. This coincidence is not preserved under the more general ITU framework. Indeed, as argued in example 3.3.1 above, the TU matching model (also sometimes called the *optimal assignment model*), is recovered in the case $D_{ij}(u,v) = u + v - \Phi_{ij}$ for some joint surplus matrix Φ_{ij}, shared additively between partners. It is well known in this case that the equilibrium conditions are the complementary slackness conditions for optimality in a linear programming problem, so in this case, equilibrium and optimality coincide. However, outside of this case, our conditions are *not* the first-order conditions associated to an optimization problem, and equilibrium does not have an interpretation as the maximizer of some welfare function.

In the TU setting, $D_{xy}(u,v) = (u + v - \Phi_{xy})/2$; thus, the matching equation (4.7) can be rewritten as $\nabla G^*(\mu) + \nabla H^*(\mu) = \Phi_{xy}$. In this case, Galichon and Salanié (2015) have shown the existence and uniqueness of a solution to (4.7) by showing that this equation coincides with the first-order conditions associated to the utilitarian welfare maximization problem, namely

$$\max_{\mu} \left\{ \sum_{xy} \mu_{xy} \Phi_{xy} - \mathcal{E}(\mu) \right\},$$

where $\Phi = \alpha + \gamma$ is the systematic part of the joint affinity, and $\mathcal{E} := G^* + H^*$ is an entropy penalization that trades-off against the maximization of the observable part of the joint affinity. However, besides the particular case of Transferable Utility, (4.7) cannot be interpreted in general as the first-order conditions of an optimization problem because the function defined by the left hand-side of (4.7) does not have a symmetric Jacobian. Hence, the methods developed in the present paper, which are based on gross substitutability, are very different than those of Galichon and Salanié (2015), which rely on convex optimization.

C.2. **The case of non-transferable utility.** Recall that in the case of non-transferable utility, the frontier of the feasible set $\mathcal{F}_{ij}$ are the pairs of payoffs (u_i, v_j) such that

$$\max\left(u_i - \alpha_{ij}, v_j - \gamma_{ij}\right) = 0.$$

Combined with our definition of stable matching, this means that if i and j are matched, then $u_i \leq \alpha_{ij}$ and $v_j \leq \gamma_{ij}$ with at least one of these inequalities holding as an equality. This, however, does not mean that both these inequalities should be equalities, which means that the stable payoffs are not necessarily the (only) efficient point $\left(\alpha_{ij}, \gamma_{ij}\right)$. Intuitively, this means that if the market conditions are such that there is a shortage of women in the population, the men will engage in a rat race to compete for the women's attention, and will therefore burn utility as a result of this wasteful competition.

While this notion of NTU stability involves utility burning, and therefore slightly differs from the usual notion of NTU stability à-la Gale and Shapley, there is in fact a strong connection between the two, which is exposed in section 5 of Galichon and Hsieh (2017), and which we recall here for convenience. In particular, when there is one agent of each type, theorem 5 in that paper shows that our notion of competitive NTU equilibrium exactly coincides with Gale and Shapley's stability. For convenience, we provide here this result and its proof:

Theorem (Galichon and Hsieh, 2017)**.** *Let $n_i = 1$ for all $i \in \mathcal{I}$ and $m_j = 1$ for all $j \in \mathcal{J}$. Assume $\mu_{ij} \in \{0,1\}^{\mathcal{I}\times\mathcal{J}}$ is a stable matching in the Gale-Shapley sense. Then, defining $\overline{u}_i^\mu := \sum_{j'\in\mathcal{J}} \mu_{ij'}\alpha_{ij'}$ and $\overline{v}_j^\mu := \sum_{i'\in\mathcal{I}} \mu_{i'j}\alpha_{i'j}$, the agents' payoffs in that stable outcome, we have that $(\mu, \overline{u}, \overline{v})$ is a competitive NTU equilibrium in our sense.*

Conversely, assume (μ, u, v) is a competitive NTU equilibrium in our sense. Then μ is a stable matching in the Gale-Shapley sense, and $u_i \leq \overline{u}_i^\mu$ for all $i \in \mathcal{I}$ and $v_j \leq \overline{v}_j^\mu$ for all $j \in \mathcal{J}$.

Proof. Assume μ is a stable matching in the Gale-Shapley sense. Then the stability inequality $\max\left(\overline{u}_i^\mu - \alpha_{ij}, \overline{v}_j^\mu - \gamma_{ij}\right) \geq 0$ holds, with equality if $\mu_{ij} = 1$; while $\overline{u}_i^\mu \geq 0$ with equality if $\mu_{i0} = 1$ and $\overline{v}_j^\mu \geq 0$ with equality if $\mu_{0j} = 0$. Thus, $\left(\mu, \overline{u}_i^\mu, \overline{v}_j^\mu\right)$ is a competitive NTU equilibrium (in our definition). Conversely, consider (μ, u, v) a competitive NTU equilibrium (in our definition), and let us show that μ is a stable matching in the Gale-Shapley sense. Assume ij is a blocking pair such that $\alpha_{ij} > \overline{u}_i^{\mu}$ and $\gamma_{ij} > \overline{v}_j^{\mu}$, and let i' be the partner of j under μ and j' be the partner of i under μ, hence $\alpha_{ij} > \alpha_{ij'}$ and $\gamma_{ij} > \gamma_{i'j}$. One has $u_i - \alpha_{ij'} > u_i - \alpha_{ij}$ and $v_j - \gamma_{i'j} > v_j - \gamma_{ij}$, hence $\max\left(u_i - \alpha_{ij'}, v_j - \gamma_{i'j}\right) > \max\left(u_i - \alpha_{ij}, v_j - \gamma_{ij}\right) \geq 0$. Thus either $u_i > \alpha_{ij'}$ or $v_j > \gamma_{i'j}$; without loss of generality assume that $u_i > \alpha_{ij'}$. But because i and j' are matched under μ, it follows from the fact that (μ, u, v) is a competitive NTU equilibrium (in our definition) that $\max\left(u_i - \alpha_{ij'}, v_{j'} - \gamma_{ij'}\right) = 0$, hence $u_i \leq \alpha_{ij'}$, a contradiction. ∎

Note however, that as soon as there are more than a unit mass of agent per observable types, or unobservable heterogeneities, the two notions yield different equilibria. In particular, it is worth contrasting our matching function in the NTU-logit case (based on our notion of competitive NTU equilibrium)

$$\mu_{xy} = \min\left(\mu_{x0} e^{\alpha_{xy}}, \mu_{0y} e^{\gamma_{xy}}\right) \tag{C.1}$$

with the matching function derived by Dagsvik (2000) and Menzel (2015) (based on the classical Gale-Shapley definition of stability), who obtain

$$\mu_{xy} = \mu_{x0}\mu_{0y} e^{\alpha_{xy}+\gamma_{xy}}. \tag{C.2}$$

One main difference between these settings is that we assume that if i and j match, they get utilities

$$\alpha_{xy} - w_{xy}^{-} + \varepsilon_{iy} \text{ and } \gamma_{xy} - w_{xy}^{+} + \eta_{xj}$$

where both w_{xy}^{-} and w_{xy}^{+} are nonnegative and at least one of these terms is equal to zero, and the ε_{iy} and η_{xj} terms are i.i.d. type I extreme value distributions. In contrast, the Dagsvik-Menzel model assumes that these utilities are given by

$$\alpha_{xy} + \varepsilon_{ij} \text{ and } \gamma_{xy} + \eta_{ij},$$

where the ε_{ij} and η_{ij} terms are i.i.d. type I extreme value distributions. Hence, the Dagsvik-Menzel model assumes i.i.d. preference shocks for individuals, while our model models preference shocks as contingent upon observable characteristics. Thus, the resulting matching function is not positive homogenous of degree one as it can be seen in equation (C.2); as a result, when the mass of individuals per type increases, the fraction of married individuals increases in the Dagsvik-Menzel model (exemplifying increasing returns to scale), while it remains constant in our model, which has constant returns to scale.

C.3. **Link between Pareto weights and distance function.** When the feasible set $\mathcal{F}$ is smooth and convex, the associated distance function D and the explicit representation of the frontier by the means of $\mathcal{U}(w)$ and $\mathcal{V}(w)$ can be connected to the Pareto weight, which is the classical approach for parameterizing the frontier. This paragraph documents the precise connection.

Calling λ the Pareto weight associated with the man and $(1-\lambda)$ the Pareto weights associated with the woman, (u,v) maximize $\lambda u + (1-\lambda)v$ subject to feasibility condition $(u,v) \in \mathcal{F}$. Hence, (u,v) solves $\max\{\lambda u + (1-\lambda)v : D(u,v) \leq 0\}$. Thus, letting z be the Lagrange multiplier of the constraint, $\lambda = z\partial_u D$ and $(1-\lambda) = z\partial_v D$, hence by summation $z = 1$, so

$$\lambda = \partial_u D(u,v), \tag{C.3}$$

so the Pareto weight is readily obtained as the derivative of D with respect to its first variable at point (u,v) on the frontier.

Coming to the connection with the wedge parameterization $w \to (\mathcal{U}(w), \mathcal{V}(w))$, observe that if (u,v) are associated with the Pareto weights $(\lambda, 1-\lambda)$, then the wedge $w = u - v$ is given by the maximization of $\{\lambda\mathcal{U}(w) + (1-\lambda)\mathcal{V}(w)\}$ over $w \in \mathbb{R}$, yielding first order conditions $\lambda\mathcal{U}'(w) + (1-\lambda)\mathcal{V}'(w) = 0$. Hence, using the property $\mathcal{U}'(w) - \mathcal{V}'(w) = 1$, we obtain that the correspondence between the wedge and the Pareto weights is given by

$$\lambda = 1 - \mathcal{U}'(w) = -\mathcal{V}'(w). \tag{C.4}$$

To summarize, there is a very close link between our approach (using the distance-to-frontier function and the wedge parameterization) and the classical approach (using Pareto weights), when the latter applies. The real interest of our approach, however, lies in cases that cannot be handled by the traditional approach, in particular when the feasible set is not smooth or convex.

C.4. **Abstract convexity.** When D_{ij} is strictly increasing in each of its arguments (or equivalently, when the upper frontier of $\mathcal{F}_{ij}$ is strictly downward sloping), one may define

$$\mathbb{U}_{ij}(v) = \max\{u : D_{ij}(u,v) \leq 0\} \text{ and } \mathbb{V}_{ij}(u) = \max\{u : D_{ij}(u,v) \leq 0\}$$

and it can be verified that $\mathbb{U}_{ij}$ and $\mathbb{V}_{ij}$ are continuous, strictly decreasing, and inverses of each other. In this case, if u and v are equilibrium payoff vectors, then

$$v_j = \max_{i \in \mathcal{I}} \{\mathbb{V}_{ij}(u_i), \mathcal{V}_{0j}\} \text{ and } u_i = \max_{j \in \mathcal{J}} \{\mathbb{U}_{ij}(v_j), \mathcal{U}_{i0}\}.$$

In particular, in the TU case studied in example 3.3.1 above, $\mathbb{U}_{ij}(v) = \Phi_{ij} - v$ and $\mathbb{V}_{ij}(u) = \Phi_{ij} - u$. The maps $\mathbb{U}_{ij}$ and $\mathbb{V}_{ij}$ are called *Galois connections*, and are investigated directly by Nöldeke and Samuelson (2015); they also appear implicitly in Legros and Newman (2007). Our setting is more general than Galois connections, as the NTU case studied in example 3.3.2 above cannot be described using Galois connections.

Appendix D. Identification

In this appendix, we discuss identification issues. Assume that the parameter to be identified is $\theta = (\alpha_{xy}, \gamma_{xy}, \tau_{xy})$, and the feasible sets are parameterized by θ so that the distance-to-frontier function is given by

$$D^{\theta}(u,v) = \tau_{xy} d_{xy}\left(\frac{u - \alpha_{xy}}{\tau_{xy}}, \frac{v - \gamma_{xy}}{\tau_{xy}}\right),$$

where $d_{xy}(a,b)$ is a known function, for instance, in the ETU model of paragraph 3.3.4, $d_{xy}(a,b) = \log\left(\frac{\exp(a)+\exp(b)}{2}\right)$. The interpretation of the parameters is straightforward; α_{xy} represents the men's preferences; γ_{xy} represents the women's preferences, and τ_{xy} represents the curvature of the efficient bargaining frontier. Letting $\mathcal{U}_{xy}(w) = -d_{xy}(0,-w)$ and $\mathcal{V}_{xy}(w) = -d_{xy}(w,0)$, one has that the feasibility condition $D^{\theta}(U_{xy}, V_{xy}) = 0$ holds if and only if $U_{xy} = \alpha_{xy} + \tau_{xy}\mathcal{U}_{xy}(w_{xy})$ and $V_{xy} = \gamma_{xy} + \tau_{xy}\mathcal{V}_{xy}(w_{xy})$, which, coupled with the optimality conditions $\nabla G^*(\mu) = U$ and $\nabla H^*(\mu) = V$, yields

$$\begin{cases} \nabla G^*(\mu) = \alpha_{xy} + \tau_{xy}\mathcal{U}_{xy}(w_{xy}) \\ \nabla H^*(\mu) = \gamma_{xy} + \tau_{xy}\mathcal{V}_{xy}(w_{xy}) \end{cases}, \tag{D.1}$$

for some $w_{xy} = \frac{(U_{xy}-\alpha_{xy})-(V_{xy}-\gamma_{xy})}{\tau_{xy}}$, which is equal to the (algebraic) quantity received by x minus the quantity received by y. Note that system (D.1) exhausts the equilibrium conditions of the model.

Theorem 5. *(i) Assume the matching patterns (μ_{xy}) and the transfers (w_{xy}) are observed, and $\tau = (\tau_{xy})$ is known. Then α and γ are point-identified by*

$$\alpha_{xy} = \frac{\partial G^*(\mu)}{\partial \mu_{xy}} - \tau_{xy}\mathcal{U}_{xy}(w_{xy}) \text{ and } \gamma_{xy} = \frac{\partial H^*(\mu)}{\partial \mu_{xy}} - \tau_{xy}\mathcal{V}_{xy}(w_{xy}). \tag{D.2}$$

(ii) Assume only the matching patterns (μ_{xy}) are observed, and $\tau = (\tau_{xy})$ is known. Then α and γ are set-identified by

$$(\alpha_{xy}, \gamma_{xy}) \in \left\{ \left(\frac{\partial G^*(\mu)}{\partial \mu_{xy}} - \tau_{xy}\mathcal{U}_{xy}(w_{xy}), \frac{\partial H^*(\mu)}{\partial \mu_{xy}} - \tau_{xy}\mathcal{V}_{xy}(w_{xy}) \right) : w_{xy} \in \mathbb{R} \right\}. \tag{D.3}$$

(iii) Assume there are K markets where the matching patterns (μ^k_{xy}) and the transfers (w^k_{xy}) are observed. Then τ is identified by the fixed-effect regression

$$\frac{\partial G^*(\mu^k)}{\partial \mu_{xy}} - \frac{\partial H^*(\mu^k)}{\partial \mu_{xy}} = (\alpha_{xy} - \gamma_{xy}) + \tau_{xy} w^k_{xy}, \tag{D.4}$$

and α and γ are in turn identified by (D.2).

Let $U^k_{xy} = \partial G^*(\mu^k)/\partial \mu_{xy}$ and $V^k_{xy} = \partial H^*(\mu^k)/\partial \mu_{xy}$. We have $\Delta U^k_{xy} - \Delta V^k_{xy} = \tau_{xy}\Delta w^k_{xy}$, where $\Delta z^k = z^k - \bar{z}$ and $\bar{z} = K^{-1}\sum_{k'} z_{k'}$. Hence

$$\hat{\tau}_{xy} = \frac{\sum_{k=1}^K (\Delta U^k_{xy} - \Delta V^k_{xy})\Delta w^k_{xy}}{\sum_{k=1}^K (\Delta w^k_{xy})^2},$$

and

$$\begin{cases} \alpha_{xy} = \bar{U}_{xy} - \hat{\tau}_{xy} K^{-1} \sum_{k=1}^K \mathcal{U}_{xy}(w^k_{xy}) \\ \gamma_{xy} = \bar{V}_{xy} - \hat{\tau}_{xy} K^{-1} \sum_{k=1}^K \mathcal{V}_{xy}(w^k_{xy}) \end{cases}.$$

Theorem 5 clarifies what can be identified depending on how much is observed. If only the matching patterns μ are observed, then part (i) of the theorem expresses that if in addition the transfers are observed too, then the matching preferences α and γ on both sides of the market can be identified conditional on the knowledge of the curvature of the bargaining frontier. Part (ii) expresses that it is not possible to identify both α and γ simultaneously without observing the transfers. In order to identify simultaneously α, γ, and τ, then observations on multiple markets are needed, as shown in part (iii) of the theorem.

Appendix E. Relating individual and aggregate equilibria

In this appendix, we establish a precise connection between individual equilibria (defined in section 3), and aggregate equilibria (defined in section 4). It will be useful at some point to introduce a slightly stronger assumption than assumption 2, to handle the case when the frontiers of the bargaining sets are strictly downward sloping. This leads us to formulate:

Assumption 2'. *The sets $\mathcal{F}_{ij}$ satisfy assumption 2, and in addition, D_{ij} is strictly increasing in both its arguments for all i and j.*

Note that in the NTU case, the frontier of the feasible set is not strictly downward-sloping, and therefore assumption 2' is not satisfied, while it is satisfied for all the other examples in section 3.3.

The following result relates the individual and aggregate equilibria.

Theorem 6. *(i) Under assumptions 1, 2, and 3, let (μ, U, V) be an aggregate equilibrium outcome. Then, defining*

$$u_i = \max_{y \in \mathcal{Y}} \{U_{x_i y} + \varepsilon_{iy}, \varepsilon_{i0}\} \text{ and } v_j = \max_{x \in \mathcal{X}} \left\{V_{x y_j} + \eta_{xj}, \eta_{0j}\right\}, \tag{E.1}$$

there is an individual matching μ_{ij} such that $\left(\mu_{ij}, u_i, v_j\right)$ is an individual equilibrium outcome, which is such that $\mu_{ij} > 0$ implies $u_i = U_{x_i y} + \varepsilon_{iy}$ and $v_j = V_{x y_j} + \eta_{xj}$.

(ii) Under assumptions 1, 2', and 3, let $\left(\mu_{ij}, u_i, v_j\right)$ *be an individual equilibrium outcome. Then, defining*

$$U_{xy} = \min_{i:x_i=x} \{u_i - \varepsilon_{iy}\} \text{ and } V_{xy} = \min_{j:y_j=y} \left\{v_j - \eta_{jx}\right\}, \tag{E.2}$$

and $\mu_{xy} = \sum_{ij\in\mathcal{I}\times\mathcal{J}} \mu_{ij} \mathbb{1}\{x_i = x\} \mathbb{1}\{y_j = y\}$*, it follows that* (μ, U, V) *is an aggregate equilibrium outcome.*

Note that deducing an aggregate equilibrium based on an individual equilibrium (part ii) requires a slightly stronger assumption than deducing an individual equilibrium based on an aggregate equilibrium (part i). The NTU case (not covered under assumption 2') thus deserves further investigations, which are carried in Galichon and Hsieh (2017).

Part (ii) of theorem 6 implies that agents keep their entire utility shocks at equilibrium, even when they could transfer them fully or partially.

Corollary 2. *Under assumptions 1, 2', and 3, consider a pair of matched individuals* i *and* j *of types* x *and* y *respectively. Then the equilibrium payoffs of* i *and* j *are respectively given by* $u_i = U_{xy} + \varepsilon_{iy}$ *and* $v_j = V_{xy} + \eta_{xj}$*, where* U *and* V *are aggregate equilibrium payoffs. Therefore, individuals keep their idiosyncratic utility shocks at equilibrium.*

This finding, which carries strong testable implications, was known in the TU case (see Chiappori, Salanié, and Weiss (2017)). Our theorem clarifies the deep mechanism that drives this result: the crucial assumption is that the distance-to-frontier function D_{ij} should only depend on i and j through the observable types x_i and y_j, and that some transfers are possible.

Appendix F. Illustrative example

F.1. **A simple marriage model.** We estimate the model introduced in paragraph 3.3.4—a model with marital complementarities and private consumption—using household consumption data. Following the setup of section 6, we assume logit heterogeneity. In the spirit of paragraph 3.3.4, the systematic utilities of a man of type x and a woman of type y paired together are specified as $\alpha_{xy} + \tau \log c^m_{xy}$ and $\gamma_{xy} + \tau \log c^w_{xy}$, respectively, where c^m_{xy} and c^w_{xy}

are respectively the private consumption levels of the man and the woman. Private consumption should satisfy the budget constraint $c_{xy}^m + c_{xy}^w = I_x + I_y$, where I_x and I_y are the income of men of type x and women of type y, respectively. It follows from theorem 6 that at equilibrium, c_{xy}^m and c_{xy}^w only depend on the man and the woman's observable types. In addition, we specify the utility of men and women of remaining single as $\alpha_{x0} + \tau \log I_x$ and $\gamma_{0y} + \tau \log I_y$.

The systematic parts of the matching surpluses, relative to singlehood, for a married pair x, y are given by

$$U_{xy} = \alpha_{xy} - \alpha_{x0} + \tau \log\left(\frac{c_{xy}^m}{I_x}\right) \text{ and } V_{xy} = \gamma_{xy} - \gamma_{0y} + \tau \log\left(\frac{c_{xy}^w}{I_y}\right).$$

Without loss of generality, we assume in the sequel that $\alpha_{x0} = 0$ and $\gamma_{0y} = 0$. The budget constraint $c_{xy}^m + c_{xy}^w = I_x + I_y$ implies an expression for the feasible set $\mathcal{F}$ and the distance-to-frontier function D_{xy}. A calculation similar to the one in section 3 shows that

$$D_{xy}(U_{xy}, V_{xy}) = \tau \log\left(\rho_{xy} \exp\left(\frac{U_{xy} - \alpha_{xy}}{\tau}\right) + (1 - \rho_{xy}) \exp\left(\frac{V_{xy} - \gamma_{xy}}{\tau}\right)\right),$$

where $\rho_{xy} = \frac{I_x}{I_x + I_y}$ denotes the man's share of contribution to total income of the household.

We estimate the model on a marriage market based on data from the British Living Costs and Food survey (ONS, 2015), which contains the relevant information on incomes, demographics (such as age and education), and a proxy for private consumption. The moderate size of the sample we retain is well suited for the purpose of illustration; however, estimation scales up well in the ITU-logit framework, as we discuss in section F.4 below.

F.2. **The data.** To estimate our model, we use the British Living Costs and Food Survey data set (which replaced the Family Expenditure Survey in 2008) for the year 2013 (see ONS, 2015). The data allows us to construct a toy marriage market that includes raw estimates of personal expenditures. We focus on married heterosexual pairs, in which case we gather information on both partners, as well as singles (never married, divorced, separated or widowed)[8] who are heads of their households. We only keep couples in which both members have positive income, and singles with positive income. Additionally, we restrict

[8]Ideally, it would be preferable to focus on first-time married couples and never-married singles, but such detailed information on marital history is usually missing in expenditures data sets.

our attention to households of size 1 for singles and size 2 for couples (hence excluding households with children or relatives and non-relatives), as we focus our attention on the sharing of resources between the married partners. Another advantage of such restriction is that we exclude from the analysis a major public good, namely, investment in children and their education. Finally, we select households in which the head is between 25 and 40 years old, and drop singles or couples with missing information.

The total income of a matched pair is the sum of the partners' personal incomes. Ideally, our application would combine income data with data on private consumption. Of course, private consumption variables are rarely available, and researchers must instead use a proxy of personal expenditures. The data offers a variable called "Total Personal Expenditures"; this is an imperfect measure of consumption, however, it excludes major public goods such as rent, heating or car purchases, while aggregating individual-level expenditures on food, household equipment, leisure goods and services, and clothing. For singles, the variable is set equal to total personal income. For couples, personal expenditure is taken by breaking down the total income proportionally according to each partner's share of personal expenditures. This ensures that the sum of personal expenditures across partners coincides with the couple's total income.

Table 1. Summary statistics, full sample

	Married				Single			
	Male		Female		Male		Female	
	mean	sd	mean	sd	mean	sd	mean	sd
Age	32.37	4.60	30.33	4.94	33.43	4.61	32.58	4.35
White	0.92	0.27	0.90	0.30	0.87	0.34	0.89	0.31
Black	0.01	0.11	0.02	0.14	0.04	0.20	0.05	0.21
Education	19.84	2.99	20.14	3.08	18.80	2.97	19.17	2.65
Personal Income	638.05	325.88	485.36	264.95	544.27	330.86	478.32	275.12
Share Expenditures	0.47	0.23	0.53	0.23	.	.	.	.
Observations	161		161		76		66	

Summary statistics are displayed in table 1. Our sample is mostly composed of White individuals. Married men appear to be older than married women (with an average age difference of two years, a fairly standard fact in marriage markets), but somewhat less educated. The data displays large variations in personal income, and shows that women account for a slightly larger share of personal expenditures than men. However, this may be a consequence of measurement error on private consumption, as the latter is only imperfectly observed. Finally, due to our restrictions on couples and singles selection (children and age requirements), our sample is rather small but is well suited for our illustration exercise.

F.3. **Estimation.** Estimation follows the steps described in section 6. We assume that the weight of each man and woman in our sample is uniform, so that $n_x = m_y = 1$ for all $x, y \in \mathcal{X} \times \mathcal{Y}$. We do not worry here about the fact that the types are sampled from a continuous distribution; if we did, our model would have to be amended to the continuous logit framework used by Dupuy and Galichon (2014) and Menzel (2015), but the estimation would be identical. The likelihood function is similar to expression (6.9), but we augment it by making use of the fact that our model predicts private consumption, as described in section F.4 below.

We use a simple parametrization of couples' pre-transfer utilities:

$$\begin{aligned} \alpha_{xy} &= \alpha_1 |\text{educ}_x - \text{educ}_y| + \alpha_2 |\text{age}_x - \text{age}_y| \\ \gamma_{xy} &= \gamma_1 |\text{educ}_x - \text{educ}_y| + \gamma_2 |\text{age}_x - \text{age}_y|, \end{aligned}$$

where educ_x and educ_y are the (standardized) ages at which the members of the couple left the schooling system—a proxy for years of education (we also standardized the age variables, age_x and age_y).

The results of our maximum likelihood estimation are presented in table 2. They are robust to random selection of starting points (multistart). As an additional robustness check, we estimated the model for a range of fixed values of τ (from low, NTU-limit values to high, TU-limit values). The value of the log-likelihood is decreasing for small and large values of τ. Finally, we computed the hessian of the log-likelihood function at the optimal value of the parameters and checked its invertibility.

Table 2. Estimates

Parameters	α_1	α_2	γ_1	γ_2	s_ϵ	τ
Estimates	-1.26	-1.90	-1.80	-2.04	301.75	3.26
CI	$[-1.84, -0.76]$	$[-2.42, -1.39]$	$[-2.35, -1.23]$	$[-2.78, -1.50]$	$[256.22, 353.61]$	$[1.79, 7.57]$

Note: These estimates are obtained using the TraME package (Galichon and O'hara, 2017) and the NLOPT optimization routine. Parameters α_1 and γ_1 measure education assortativeness, α_2 and γ_2 measure age assortativeness, and τ captures the curvature of the bargaining frontier. The standard deviation of our measurement error is estimated as s_ϵ. Confidence intervals at the 5% level are computed using 200 bootstrap estimates.

Table 2 calls for several comments. We provide bootstrapped confidence intervals for our parameter estimates. We used 200 replications and report percentiles intervals at the 5% level. The coefficients corresponding to education and age assortativeness are in line with the prior literature on marriage—they indicate that utility decreases as distance between the education level or age of the partners increases. Hence, our results unsurprisingly suggest positive assortative mating in education and age. We also obtain an estimate of the curvature of the efficient bargaining frontier that is suggestive an intermediate case between NTU and TU. Although our estimates are too imprecise to reject either TU or NTU, our illustration—focusing on a relatively simple model of marriage with education assortativeness and consumption, and making use of crude expenditure data—suggests the potential of the approach.

F.4. **Computational Details.** We modify the likelihood in expression (6.9) to account for the fact that the model predicts private consumption, which is imperfectly observed in the data. For man i, we have

$$c^m_{x_i y_j} = I_{x_i} \exp\left(u_{x_i} - D_{x_i y_j}(u_{x_i}, v_{y_j}) - \alpha_{x_i y_j}\right)^{1/\tau}$$

with the notation $u_x = -\log \mu_{x0}$ and $v_y = -\log \mu_{0y}$. We assume further that we measure private expenditures with some measurement error, that is we observe men's private consumptions as $\hat{c}^m_i = c^m_{x_i, y_j} + \epsilon_{ij}$, where ϵ is a Gaussian measurement error with variance s^2_ϵ, and independently distributed across the (x, y) pairs. Letting θ be a parameterizations of

(α, γ, τ), the log-likelihood (up to constants) is:

$$\log \mathcal{L}(\theta, s_\epsilon) = -\sum_{(i,j)\in\mathcal{C}} D^\theta(u^\theta_{x_i}, v^\theta_{y_j}) - \sum_{i\in\mathcal{S}_M} u^\theta_{x_i} - \sum_{j\in\mathcal{S}_F} v^\theta_{y_j} - \sum_{(i,j)\in\mathcal{C}} \frac{\left(\hat{c}^m_i - c^m_{x_i,y_j}\right)^2}{2s^2_\epsilon} - |\mathcal{C}| \log s_\epsilon,$$

where $\mathcal{C}$ denotes the set of matched pairs (i, j) observed in the data, $\mathcal{S}_M$ and $\mathcal{S}_F$ respectively denote the set of single men and the set of single women observed in the data, and where u^θ_x and v^θ_y satisfy equilibrium equations (6.2).

We maximize the likelihood using the NLOPT package and the BFGS algorithm, with bound constraints on τ and on s_ϵ (these parameters are restricted to be positive). We test the robustness of the results against different starting points for the parameter values. We compute analytically the gradient to improve performance. At each step of the estimation process, the following computations are performed:

(i) The parameters (θ, s_ϵ) are updated using the gradient computed in the previous step.
(ii) The updated values of (θ, s_ϵ) are deduced.
(iii) The equilibrium quantities u and v are computed using algorithm 2, and the predicted consumption levels are constructed.
(iv) The log-likelihood is updated.

This procedure is part of a R package named TraME (Transportation Methods for Econometrics; Galichon and O'hara, 2017). It simplifies the computation and the estimation of a wide range of discrete choice and matching problems, as it relies on a flexible formulation of these models in terms of transferability or heterogeneity structure. Under TraME, user-defined models can be solved using core equilibrium algorithms (mainly via Linear Programming, Convex Optimization, Jacobi iterations, Deferred Acceptance, or Iterative Fitting of which algorithm 2 is an instance of) and estimated by maximum likelihood.

From the estimation steps mentioned above, step (iii) is the most time-consuming. However, the IPFP algorithm is quite fast, especially given that we have a small sample. It takes about 3 seconds to obtain the equilibrium quantities u and v in a market of this size. With 300 men and women, computation time raises to 3.5 seconds, and to 9.5 seconds with

500 men and women. Overall, the max-likelihood estimation procedure converges in 45 minutes.

References

[1] Office for National Statistics, Department for Environment, Food and Rural Affairs (2015): "Living Costs and Food Survey, 2013". UK Data Service. Available from: http://doi.org/10.5255/UKDA-SN-7702-1

[2] Galichon, A. and K. O'hara (2015–2018+): "TraME: Transportation Methods for Econometrics". R package available at https://github.com/TraME-Project/.