Technology, Skill and Long Run Growth

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Abstract

This paper develops a model in which two factors contribute to growth: investments in technology and investments in human capital. Growth in turn takes two forms: TFP growth and growth in variety. Although both factors contribute to both forms of growth, the rate of TFP growth depends mainly on parameters governing skill accumulation, while growth in variety depends, roughly, on the difference between the parameters governing technology and skill accumulation. Conditions for a BGP are established and the effects of parameters on growth are characterized. In addition, "no growth" results are established for economies where only one factor can invest.

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1. INTRODUCTION

This paper develops a model in which two factors contribute to growth: investments in technology by heterogeneous firms and investments in human capital by heterogeneous workers. Income growth in turn takes two forms: growth in the quantity produced of each differentiated good and growth in the number of goods available. Call these two forms total factor productivity (TFP) growth and variety growth.

Both types of investment affect both forms of growth, but the contributions are not symmetric. Improvements in the parameters governing investment in skill raise the rate of TFP growth and reduce the rate of variety growth. Improvements in the parameters governing investment in technology raise the rate of variety growth, while the effect on TFP growth is positive, zero, or negative as the elasticity of intertemporal substitution (EIS) is greater than, equal to or less than unity.

This asymmetry appears despite the fact that skill and technology are modeled as symmetric in many respects, and on balanced growth paths (BGPs) the rate of TFP growth is also the (common) growth rate of technology and human capital. But the factors are fundamentally different in two respects. First, human capital is a rival input while technology is nonrival. That is, an increase in a worker's human capital affects only his own productivity, while an improvement in a firm's technology can be exploited by all its workers.

In addition, the two factors differ in the way entry occurs. Growth in the size of the workforce is exogenous. Entry by new firms is endogenous, and entering firms must invest to obtain technologies for new goods. Thus, the expected profitability of a new product affects the incentives of entrants, and the entry rate is governed by a zero-profit condition.

Analyzing both types of investment together is important because there is strategic complementarity in the incentives to invest. Incumbent workers invest in skill to increase their wages. But without continued improvement in the set of technologies used by firms, the returns to workers' investments would decline and, eventually, be too small to justify further investment. Similarly, incumbent firms invest in better technologies to increase their profits, but without continued improvement in the skill distribution of the workforce, their returns would eventually be too small to justify further investments. Sustained growth requires continued investment in both factors, and the contribution of this paper is to characterize the interplay between the two types of investment.

The rest of the paper is organized as follows. Related literature is discussed in section 2. Section 3 sets out the production technologies and characterizes the (static) production equilibrium. The production function for differentiated intermediates has two inputs, technology and human capital, and it is log-supermodular. Hence the competitive equilibrium features positively assortative matching between technology and skill. Proposition 1 establishes the existence, uniqueness and efficiency of a production equilibrium, describing the allocation of labor across technologies and the resulting prices, wages, output levels, and profits. Lemmas 2 and 3 establish some homogeneity properties. The first main result, Proposition 4, shows that if the technology and skill distributions are Pareto, with locations that are appropriately aligned, then the equilibrium allocation of skill to technology is linear, and the wage, price, output, and profit functions are isoelastic.

Section 4 treats dynamics: the investment decisions of incumbent firms, new entrants, and workers; the evolution of the technology and skill distributions; and the interest rate and consumption growth. Section 5 provides formal definitions of a competitive equilibrium and a balanced growth path. A balanced growth path features stationary, nondegenerate distributions of relative technology and relative human capital, with both growing at a common, constant rate.

Section 6 specializes to the case where technology and skill have Pareto distributions, showing that the isoelastic forms for the profit and wage functions are inherited by the value functions for producers and workers. This fact leads to a tractable set of conditions describing investment and the evolution of the technology and skill distributions on a BGP. The second main result, Proposition 5, provides conditions that ensure the existence of a BGP.

Section 7 looks at the effects of various parameters and policies on TFP and variety growth. Proposition 6, the third main result, describes the effects of parameter changes. Because the model has an important positive external effect, the competitive equilibrium investment rates are inefficient: they are too low. The effects of subsidies to investments by workers and firms are described in Proposition 7.

Section 8 shows that growth in one factor alone cannot be sustained in the long run. If there is no investment in technology, and the distribution of skill is sufficiently far to the right, then there is no incentive for further investment in skill. The same is true of the roles of technology and skill are reversed.

Section 9 looks at some positive implications of the model: the wage dynamics for entering cohorts of workers and the revenue and employment dynamics for cohorts of entering firms. Section 10 concludes. Proofs and technical derivations and arguments are gathered in the Appendix.

2. RELATED LITERATURE

In most of the endogenous growth literature, growth has only one source: either human capital accumulation or innovations in technology.

Among the human capital models, growth can arise from on-the-job learning, as in the learning-by-doing models of Arrow (1962), Stokey (1988), Young (1991, 1993), Matsuyama (1992). In others, human capital accumulation competes with production as a use of time, as in Uzawa (1965), Romer (1986), Lucas (1988, 2009), Lucas and Moll (2014), Perla and Tonetti (2014), and others.

In the literature on technology-driven growth, some models emphasize creative destruction, as in Romer (1990), Grossman and Helpman (1991), Aghion and Howitt (1992), Jones (1995), Stokey (1995), Acemoglu (2002), and Klette and Kortum (2004), while in others quality improvements are critical, as in Atkeson and Burstein (2010) and Luttmer (2007).

The model here is also related to the model of technology and wage inequality in Jovanovic (1998) and the model of skill and technology growth in Lloyed-Ellis and Roberts (2002).

The framework here builds on the model of technology growth across firms in Perla and Tonetti (JPE, 2014), adding a similar investment model on the human capital side.

[To be completed.]

3. PRODUCTION AND PRICES

The single final good is produced by competitive firms using intermediate goods as inputs. Intermediate goods are produced by heterogeneous, monopolistically competitive firms. Each intermediate firm produces a unique variety, and all intermediates enter symmetrically into final good production. But intermediate firms differ in their technology level x, which affects their productivity. Let N_p be the number (mass) of intermediate good producers, and let $F(x)$, with with continuous density f, denote the distribution function for technology.

Intermediate good producers use heterogeneous labor, differentiated by its human capital level h, as the only input. Let L_w, be the size of the workforce, and let $\Psi(h)$, with continuous density ψ, denote the distribution function for human capital. This section looks at the the allocation of labor across producers, and wages, prices, output levels, and profits, given N_p, F, L_w, Ψ.

A. Technologies

Each final good producer has the CRS technology

$$y_F = \left[N_p^{1-\chi} \int y(x)^{(\rho-1)/\rho} f(x) dx \right]^{\rho/(\rho-1)}, \tag{1}$$

where $\rho > 1$ is the substitution elasticity and $\chi \in (0, 1/\rho]$ measures diminishing returns to increased variety. Let $p(x)$ denote the price charged by a producer with technology x. Then input demands are

$$y^d(x) = N_p^{-\rho\chi} p(x)^{-\rho} y_F, \qquad \text{all } x,$$

where the price of the final good is normalized to unity,

$$1 = p_F = \left[N_p^{1-\rho\chi} \int p(x)^{1-\rho} f(x) dx \right]^{1/(1-\rho)}. \tag{2}$$

The output of a firm depends on the size and quality of its workforce, as well as its technology. In particular, if a producer with technology x employs ℓ workers with human capital h, then its output is

$$y = \ell \phi(h, x),$$

where $\phi(h, x)$ is the CES function

$$\phi(h, x) \equiv \left[\omega h^{(\eta-1)/\eta} + (1-\omega)\, x^{(\eta-1)/\eta}\right]^{\eta/(\eta-1)}, \qquad \eta, \omega \in (0,1)\,. \tag{3}$$

The elasticity of substitution between technology and human capital is assumed to be less than unity, $\eta < 1$. Firms could employ workers with different human capital levels, and in this case their outputs would simply be summed. In equilibrium firms never choose to do so, however, and for simplicity the notation is not introduced.

B. Intermediate goods: price, output, labor

Let $w(h)$ denote the wage function. For a firm with technology x, the cost of producing a unit of output with labor of quality h is $w(h)/\phi(h, x)$. Optimal labor quality $h^*(x)$ minimizes this expression, so h^* satisfies

$$\frac{w'(h^*)}{w(h^*)} = \frac{\phi_h(h^*, x)}{\phi(h^*, x)}. \tag{4}$$

It is straightforward to show that if the (local, necessary) second order condition for cost minimization holds, then $\eta < 1$ implies h^* is strictly increasing in x. The labor market is competitive, and since the production function in (3) is log-supermodular, efficiency requires positively assortative matching (Costinot, 2009).

Unit cost

$$c(x) = \frac{w(h^*(x))}{\phi(h^*(x), x)},$$

is strictly decreasing in x,

$$\frac{c'(x)}{c(x)} = -\frac{\phi_x(h^*(x), x)}{\phi(h^*(x), x)} < 0.$$

As usual, profit maximization by intermediate good producers entails setting a price that is a markup of $\rho/(\rho-1)$ over unit cost. Output is then determined by demand, and labor input by the production function. Hence price, quantity, labor input, and operating profits for the intermediate firm are

$$\begin{aligned} p(x) &= \frac{\rho}{\rho-1}\frac{w(h^*(x))}{\phi(h^*(x),x)}, \\ y(x) &= y_F N_p^{-\rho\chi} p(x)^{-\rho}, \\ \ell(x) &= \frac{y(x)}{\phi(h^*(x),x)}, \\ \pi(x) &= \frac{1}{\rho}p(x)y(x), \qquad \text{all } x, \end{aligned} \tag{5}$$

where the price normalization requires (2). Firms with higher technology levels x have lower prices, higher sales, and higher profits. They may or may not employ more labor.

Each worker inelastically supplies one unit of labor. Let x_m and h_m denote the lower bounds for the supports of F and Ψ. Then markets clear for all types of labor if

$$h_m = h^*(x_m), \tag{6}$$

$$L_w\left[1-\Psi(h^*(x))\right] = N_p\int_x^\infty \ell(\xi)f(\xi)d\xi, \qquad \text{all } x \geq x_m. \tag{7}$$

C. Production equilibrium

At any instant, the economy is described by its production parameters, the number of firms and workers, and the distributions of technology and skill.

Definition: A *production environment* $\mathcal{E}_p$ is described by

i. parameters (ρ,χ,ω,η), with $\rho>1$, $\chi\in(0,1/\rho]$, $\omega\in(0,1)$, $\eta\in(0,1)$;

ii. numbers of producers and workers $N_p>0$ and $L_w>0$;

iii. distribution functions $F(x)$ with continuous density $f(x)$ and lower bound

$x_m \geq 0$ on its support, and $\Psi(h)$ with continuous density $\psi(h)$ and lower bound $h_m \geq 0$ on its support.

A production equilibrium consists of price functions and an allocation that satisfy profit maximization and labor market clearing.

Definition: Given a production environment $\mathcal{E}_p$, the prices $w(h), p(x)$, and allocation $h^*(x)$, $y(x)$, $\ell(x)$, $\pi(x)$, y_F, are a *production equilibrium* if (2) and (4)-(7) hold.

The following result is then straightforward.

Proposition 1: For any production environment $\mathcal{E}_p$, an equilibrium exists, and it is unique and efficient.

D. Homogeneity properties

The analysis of BGPs will exploit the fact that production equilibria have certain homogeneity properties. Lemma 2 deals with proportionate shifts in the two distribution functions.

Lemma 2: Fix $\mathcal{E}_p$, and let $\mathcal{E}_{pA}$ be a production environment with the same parameters $(\rho, \chi, \omega, \eta)$ and numbers N_p, L_w, but with distribution functions F_A, Ψ_A satisfying

$$\begin{aligned} F_A(X) &= F(X/Q), \qquad \text{all } X, \\ \Psi_A(H) &= \Psi(H/Q), \qquad \text{all } H. \end{aligned}$$

If $[w, p, h^*, y, \ell, \pi, y_F]$ is the production equilibrium for $\mathcal{E}_p$, then the equilibrium for $\mathcal{E}_{pA}$ is

$$\begin{aligned} &w_A(H) = Qw(H/Q), && p_A(X) = p(X/Q) \\ &h_A^*(X) = Qh^*(X/Q), && y_A(X) = Qy(X/Q), \\ &\ell_A(X) = \ell(X/Q), && \pi_A(X) = Q\pi(X/Q), \\ &y_{FA} = Qy_F, && \text{all } X, H. \end{aligned}$$

Price and employment for any firm depend only on its relative technology $x = X/Q$, while its labor quality, output, and profits are scaled by Q. Wages and final output are also scaled by Q.

Lemma 3 deals with the effects of changes in the numbers of producers and workers. The impact of variety growth depends on

$$\Omega \equiv \frac{1-\rho\chi}{\rho-1}, \tag{8}$$

where $\Omega \in [0, 1/(\rho-1))$. In the limiting case $\chi = 1/\rho$, growth in variety is not valued amd $\Omega = 0$.

Lemma 3: Fix $\mathcal{E}_p$, and let $\mathcal{E}_{pB}$ be a production environment with the same parameters and distribution functions, but with $L_{wB} = e^{\upsilon} L_w$ and $N_{pB} = e^n N_p$. If $[w, p, h^*, y, \ell, \pi, y_F]$ is the production equilibrium for $\mathcal{E}_p$, then the equilibrium for $\mathcal{E}_{pB}$ is

$$\begin{aligned}
w_B &= e^{\Omega n} w, & p_B &= e^{\Omega n} p, \\
h_B^* &= h^*, & y_B &= e^{\upsilon - n} y, \\
\ell_B &= e^{\upsilon - n} \ell, & \pi_B &= e^{\upsilon + (\Omega - 1)n} \pi, \\
y_{FB} &= e^{\upsilon + \Omega n} y_F, & &\text{all } X, H.
\end{aligned}$$

A change in L_w leads to proportionate changes in employment, output and profits at each firm and in final output, with wages, prices and the allocation of skill to technology unaffected.

An increase in N_p leads to proportionate decreases in employment and output at each firm. Final output, the price of each intermediate, and all wage rates change with an elasticity of $\Omega \geq 0$. Thus, all increase if variety is valued, if $\Omega > 0$, and all are unchanged if it is not, if $\Omega = 0$.

Profits per firm—which reflect both the increase in price and decrease in scale—can change in either direction. If $\Omega > 1$, then the love of variety is strong enough so that an increase in the number of producers actually increases the profit of each incumbent. This case occurs only if $\rho < 2$ and, in addition, the parameter χ is not too large. In the analysis of BGPs we will impose the restriction $\rho \geq 2$, to rule out this case.

E. Pareto distributions

In this section we will show that if the distribution functions F and Ψ are Pareto, with shape parameters that are not too different and location parameters that are appropriately aligned, the production equilibrium has a linear assignment of skill to technology, and wage, price, and profit functions that are isoelastic.

PROPOSITION 4: Let $\mathcal{E}_p$ be a production environment for which F and Ψ are Pareto distributions with parameters (α_x, x_m) and (α_h, h_m). Assume that $\alpha_x > 1$, $\alpha_h > 1$, and

$$-1 < \alpha_x - \alpha_h < \rho - 1. \tag{9}$$

Define

$$\begin{aligned} \varepsilon &\equiv \frac{1}{\rho}\left(1 + \alpha_x - \alpha_h\right), & (10)\\ \zeta &\equiv \varepsilon + \alpha_h - \alpha_x, & (11)\\ a_h &\equiv \left(\frac{1-\varepsilon}{\varepsilon}\frac{1-\omega}{\omega}\right)^{\eta/(\eta-1)}, & (12) \end{aligned}$$

and in addition, assume

$$h_m = a_h x_m. \tag{13}$$

The production equilibrium for $\mathcal{E}_p$ has price and allocation functions

$$\begin{aligned} h^*(x) &= a_h x, \qquad \text{all } x, & (14)\\ w(h) &= w_2 \left(\frac{h}{h_m}\right)^{1-\varepsilon}, \qquad \text{all } h, & (15)\\ y_F &= L_w N_p^{\Omega} p_0^{\rho} \phi(a_h, 1) \frac{\alpha_h}{\alpha_x} x_m, & (16) \end{aligned}$$

$$
\begin{aligned}
p(x) &= N_p^{\Omega} p_0 \left(\frac{x}{x_m}\right)^{-\varepsilon}, && \text{all } x, \\
y(x) &= \ell_2 \phi(a_h, 1) x \left(\frac{x}{x_m}\right)^{\alpha_x - \alpha_h}, && \text{all } x, \\
\ell(x) &= \ell_2 \left(\frac{x}{x_m}\right)^{\alpha_x - \alpha_h}, && \text{all } x, \\
\pi(x) &= \pi_2 \left(\frac{x}{x_m}\right)^{1-\zeta}, && \text{all } x,
\end{aligned} \tag{17}
$$

where

$$
\begin{aligned}
w_2 &\equiv \frac{\rho - 1}{\rho} L_w^{-1} y_F \frac{\alpha_x}{\alpha_h} p_0^{1-\rho}, \\
\ell_2 &\equiv L_w N_p^{-1} \frac{\alpha_h}{\alpha_x}, \\
\pi_2 &\equiv \frac{1}{\rho} N_p^{-1} y_F p_0^{1-\rho}, \\
p_0^{\rho-1} &\equiv \mathrm{E}_F \left(\frac{x}{x_m}\right)^{1-\zeta}.
\end{aligned} \tag{18}
$$

The shape parameters α_x and α_h need not be the same, but (9) puts a restriction on how different they can be. It implies that $1 - \varepsilon \in (0, 1)$ and $1 - \zeta \in (0, \rho - 1)$, so both the wage and profit functions are strictly increasing.

Pareto distributions for skill and technology imply that wages and profits also have Pareto distributions, with tail parameters $\alpha_w \equiv \alpha_h / (1 - \varepsilon)$ and $\alpha_\pi = \alpha_x / (1 - \zeta)$, respectively. Thus, α_w is increasing in α_x and is increasing or decreasing in α_h as $\rho > 1 + \alpha_x$ or $\rho < 1 + \alpha_x$, while α_π is increasing in α_h and decreasing in α_x. Employment has a Pareto distribution if and only if $\alpha_x > \alpha_h$, and in this case it has tail parameter $\alpha_\ell = \alpha_x / (\alpha_x - \alpha_h)$. If $\alpha_h = \alpha_x$, employment is uniform across firm types, and if $\alpha_x < \alpha_h$ employment declines with technology level. Thus, evidence on the size distribution of firms by employment suggests $\alpha_x > \alpha_h$ is the relevant case.

4. DYNAMICS

In this section the dynamic aspects of the model are described: investment decisions of incumbent producers and workers, the entry decisions of new firms, the evolution of the distribution functions for technology and skill, and the consumption/saving decisions of households. As in Perla and Tonetti (2014) investment is imitative, and it is a zero-one decision. The Pareto shape for the technology and skill distributions this investment technology requires for balanced growth fits well with the production environment here.

It is useful to start with a brief overview. Time is continuous and the horizon is infinite. At any date $t \geq 0$, there are three groups of firms: producers, process innovators and product innovators. A producer can at any time abandon its current technology and become a process innovator, attempting to acquire a new technology. The only cost is the opportunity cost: process innovators do not produce. Success is stochastic, with a fixed hazard rate, and conditional on success the process innovator receives a technology that is a random draw from those of current producers. Hence producers become process innovators if and only if their technology lies below an endogenously determined threshold.

New firms, product innovators, arrive at an endogenously determined rate. Each entrant chooses a one-time (sunk) investment level, which determines its hazard rate for success. After paying the sunk cost, product innovators are like process innovators except that their hazard rate is different.

Similarly, at any date the labor force has three groups: workers, retoolers and trainees. A worker can at any time become a retooler, attempting to acquire a new skill, and the only cost is an opportunity cost—retoolers do not work. Success is stochastic, with a fixed hazard rate, and conditional on success the retooler receives a skill that is a random draw from those of current workers. Hence workers become retoolers if and only if their skill lies below an endogenously determined threshold.

The workforce grows at an exogenously fixed rate υ. New entrants, trainees, are like retoolers except that their hazard rate for success may be different.

Note that there is an important asymmetry between firms and workers: the workforce grows at an exogenously fixed rate, while entry by new firms is endogenous, satisfying a free entry condition.

At date t, $N_p(t), N_i(t), N_e(t)$ are the numbers of producers, process innovators, and product innovators, with sum $N(t)$; $L_w(t), L_i(t), L_e(t)$ are the numbers of workers, retoolers, and entrants, with sum $L(t)$; $F(X,t)$, $\Psi(H,t)$ are the distribution functions for technology among producers and skill among workers; $W(H,t)$, $H^*(X,t)$, $P(X,t)$, $Y(X,t)$, $\mathfrak{L}(X,t)$, $\Pi(X,t)$, $Y_F(t)$, $t \geq 0$, are the wage function, skill allocation, and so on; and $r(t)$ is the interest rate. Note that only producers and workers are identified by a technology or skill level.

A. Firms: process and product innovation

Let $V^f(X,t)$ denote the value of a producer with technology X at date t. A firm that chooses to become a process innovator abandons its current technology and waits to acquire a new one. A process innovator pays no direct costs: there is only the opportunity cost of forgone profits. Abandoned technologies cannot be reclaimed, so all process innovators at date t are in the same position. Let $V_{fi}(t)$ denote their (common) value.

Success is stochastic, arriving at rate λ_{xi}, and conditional on success at date t, the innovator gets a new technology that is random draw from the distribution $F(\cdot,t)$ among current producers. Hence $V_{fi}(t)$ satisfies the Bellman equation

$$[r(t)+\delta_x]\,V_{fi}(t) = \lambda_{xi}\left\{\mathrm{E}_{F(\cdot,t)}[V^f(X,t)] - V_{fi}(t)\right\} + V'_{fi}(t), \qquad \text{all } t,$$

where the term in braces is the expected gain in value conditional on success, $r(t)$ is the interest rate, and $\delta_x > 0$ is an exogenous exit rate.

The value $V^f(X,t)$ of a producer is the expected discounted value of its future profit flows. Clearly V^f is nondecreasing in its first argument: a better technology can only raise the firm's value. Hence at any date t, producers with technologies below some threshold $X_m(t)$ become process innovators, while those with technologies above the threshold continue to produce. It follows that at date t, the value of a producer with technology X is $V_{fi}(t)$ if $X \leq X_m(t)$, and the irreversibility of investment means that $X_m(t)$ is nondecreasing. While a firm produces, its technology X grows (or declines) at a constant rate μ_x. Hence the value $V^f(X,t)$ of a producer, a firm with $X > X_m(t)$, satisfies the Bellman equation[1]

$$[r(t) + \delta_x] \, V^f(X,t) = \Pi\,(X,t) + \mu_x X V_X^f(X,t) + V_t^f(X,t), \qquad \text{all } t.$$

Value matching provides a boundary condition for this ODE, and the optimal choice about when to invest implies that smooth pasting holds. Hence

$$\begin{aligned} V^f[X_m(t),t] &= V_{fi}(t), \\ V_X^f[X_m(t),t] &= 0, \qquad\qquad \text{all } t. \end{aligned}$$

Entering firms—product innovators—have a similar investment technology, except that they make a one-time (sunk) investment I_e. The success rate of an innovator depends on his own investment I_e relative to the average spending $\overline{I}_e$ of others in his cohort, scaled by the ratio of new entrants to existing products. In particular, let

[1]At this stage, it would be easy to assume that the technology X of an incumbent evolves as a geometric Brownian motion. The cross-sectional distribution of technologies among initially identical firms, within each age cohort, would be lognormal, with a growing variance, and the overall distribution would be a mixture of lognormals. When the solution to the model is actually characterized in section 6, however, the argument relies on technologies across incumbents having a Pareto distribution. At that point the mixture of lognormals would be incompatible with the requirement of a Pareto distribution overall, and the variance term would have to be dropped.

$E(t)$ denote the flow of entrants at t, and define the entry rate $\epsilon(t) = E(t)/N_p(t)$. The success rate of an entrant who invests I_e is

$$\Lambda_{xe}(I_e/\overline{I}_e, \epsilon) = \frac{\phi_e}{\epsilon} \times \begin{cases} 0, & \text{if } I_e/\overline{I}_e < 1 - \varepsilon_x, \\ 1 - \left(1 - I_e/\overline{I}_e\right)/\varepsilon_x, & \text{if } I_e/\overline{I}_e \in [1 - \varepsilon_x, 1], \\ 1 + \varepsilon_x \left(I_e/\overline{I}_e - 1\right), & \text{if } I_e/\overline{I}_e > 1, \end{cases}$$

where $\phi_e > 0$ and where $\varepsilon_x > 0$ is small. Thus, $\Lambda_{xe}(., \epsilon)$ is kinked at $I_e/\overline{I}_e = 1$, reflecting a 'patent race' with intense competition among entrants, and scaling by $1/\epsilon$ reflects the reduced chances for success when the field is crowded. The value $V_{fe}(\cdot; \Lambda)$ of a product innovator with success rate Λ who is waiting for a technology at date τ, gross of the investment cost, satisfies the Bellman equation

$$[r(\tau) + \delta_x] V_{fe}(\tau; \Lambda) = \Lambda \left\{ \mathrm{E}_{F(\cdot,\tau)} \left[V^f(X, \tau) \right] - V_{fe}(\tau; \Lambda) \right\} + \frac{\partial V_{fe}(\tau; \Lambda)}{\partial \tau}, \quad \tau \geq 0.$$

Note that it does not depend on the investment date t.

An entrant takes $\overline{I}_e(t)$ and $\epsilon(t)$ as given, chooses I_e to solve

$$\max_{I_e} \left\{ V_{fe} \left[t; \Lambda_{xe}(I_e/\overline{I}_e, \epsilon) \right] - I_e \right\},$$

and is willing to enter if and only if the maximized value is nonnegative. Since Λ_{xe} diverges as $\epsilon \to 0$, in equilibrium there is positive entry at all dates, $E(t) > 0$. All entering firms choose the same investment level, and their common success rate is $\lambda_{xe}(t) = \phi_e/\epsilon(t)$. Free entry, together with the form of the function Λ_{xe}, imply that their common expenditure level is bid up to exhaust profits,

$$I_e(t) = V_{fe}(t; \lambda_{xe}(t)), \qquad \text{all } t,$$

and aggregate spending by entrants at t is $E(t)I_e(t)$.

B. Workers: investment in human capital

Workers invest to maximize the expected discounted value of their lifetime earnings. An individual who chooses to invest—a retooler—stops working, abandons his old skill and waits to acquire a new one. Let $V^w(H,t)$ denote the value of a worker with skill H at date t, and let $V_{wi}(t)$ denote the value of a retooler.

Success for retoolers is stochastic, arriving at rate $\lambda_{hi} > 0$, and conditional on success at date t, the individual gets a skill level drawn from the distribution $\Psi(\cdot,t)$ across current workers. Hence $V_{wi}(t)$ satisfies the Bellman equation

$$[r(t)+\delta_h]\,V_{wi}(t) = \lambda_{hi}\left\{\mathrm{E}_{\Psi(\cdot,t)}[V^w(H,t)] - V_{wi}(t)\right\} + V'_{wi}(t), \qquad \text{all } t,$$

where the term in braces is the expected gain in value conditional on success, and $\delta_h > 0$ is an exogenous exit rate.

Clearly $V^w(H,t)$ is nondecreasing in its first argument: higher human capital can only raise the worker's expected lifetime income. Hence at any date t, all individuals with skill below some threshold $H_m(t)$ become retoolers, while those with skill above the threshold continue working. It follows that at date t, the value of a worker with skill $H \le H_m(t)$ is $V_{wi}(t)$, and the irreversibility of investment implies that $H_m(t)$ is nondecreasing.

While an individual works, his human capital H grows (or declines) at a constant rate μ_h, which can be interpreted as on-the-job learning. Hence the value $V^w(H,t)$ for a worker with skill $H > H_m(t)$, satisfies the Bellman equation

$$[r(t)+\delta_h]\,V^w(H,t) = W(H,t) + \mu_h H V^w_H(H,t) + V^w_t(H,t), \qquad \text{all } t.$$

As for firms, value matching and smooth pasting hold at the threshold $H_m(t)$, so

$$\begin{aligned} V^w[H_m(t),t] &= V_{wi}(t), \\ V^w_H[H_m(t),t] &= 0, \qquad \text{all } t. \end{aligned}$$

New entrants into the workforce—trainees, have an investment technology like the one for retoolers, except that their hazard rate for success, call it λ_{he}, may be different. They pay no costs, so their value function $V_{we}(t)$ satisfies the Bellman equation

$$[r(t)+\delta_h]\,V_{we}(t) = \lambda_{he}\left\{\mathrm{E}_{\Psi(\cdot,t)}[V^w(H,t)] - V_{wi}(t)\right\} + V'_{we}(t), \qquad \text{all } t.$$

Trainees arrive at the rate $(\upsilon + \delta_h)\, N(t)$.

C. The evolution of technology and skill

Next consider the evolution of the group sizes (N_p, N_i, N_e) and (L_w, L_i, L_e) and the distribution functions F and Ψ.

The number of producers $N_p(t)$ grows because of success by innovators of both types, and declines because of exit and decisions to switch to process innovation. The producers that switch to innovating around date t are those with technologies $X(t)$ that are close enough to the threshold $X_m(t)$ so that growth in that threshold overtakes them. Since technologies for producers grow at the rate μ_x, there is a positive level of switching at date t if and only if

$$X'_m(t) - \mu_x X_m(t) > 0, \qquad \text{all } t, \tag{19}$$

and

$$\begin{aligned} N'_p(t) &= \lambda_{xi} N_i(t) + \lambda_{xe} N_e(t) - \delta_x N_p(t) \\ &\quad - \max\left\{0,\ \left[X'_m(t) - \mu_x X_m(t)\right] f[X_m(t), t] N_p(t)\right\}, \qquad \text{all } t. \end{aligned}$$

The number of process innovators N_i grows because producers switch to innovating, while the number of product innovators N_e grows because new entrants join. Each declines because of exit and success, so

$$\begin{aligned} N'_i(t) &= \max\left\{0,\ \left[X'_m(t) - \mu_x X_m(t)\right] f(X_m(t), t) N_p(t)\right\} - (\delta_x + \lambda_{xi})\, N_i(t), \\ N'_e(t) &= E(t) - (\delta_x + \lambda_{xe})\, N_e(t), \qquad \text{all } t. \end{aligned}$$

The distribution function F for technology among producers evolves because their technologies grow at rate μ_x, they exit at the rate δ_x, innovators of both types succeed, and firms at the threshold $X_m(t)$ switch to process innovations. As shown in the

Appendix, $F(X,t)$ satisfies

$$-F_t(X,t) = f(X,t)\mu_x X + [1 - F(X,t)]\left[-\frac{N_p'(t)}{N_p(t)} - \delta_x + \lambda_{xi}\frac{N_i(t)}{N_p(t)} + \lambda_{xe}\frac{N_e(t)}{N_p(t)}\right],$$
$$\text{all } X \geq X_m(t), \quad t \geq 0.$$

The dynamics for the labor force are analogous. Workers who switch to retooling around date t are those whose human capital $H(t)$ falls below the (moving) threshold $H_m(t)$, despite growth at the rate μ_h. Hence workers are switching at date t if and only if

$$H_m'(t) - \mu_h H_m(t) > 0, \qquad \text{all } t, \tag{20}$$

and

$$\begin{aligned}
L_w'(t) &= \lambda_{hi}L_i(t) + \lambda_{he}L_e(t) - \delta_h L_w(t) \\
&\quad - \max\{0,\ [H_m'(t) - \mu_h H_m(t)]\,\psi[H_m(t),t]L_w(t)\}, \\
L_i'(t) &= \max\{0,\ [H_m'(t) - \mu_h H_m(t)\psi[H_m(t),t]L_w(t)]\} - (\delta_h + \lambda_{hi})\,L_i(t), \\
L_e'(t) &= (\upsilon + \delta_h)\,L(t) - (\delta_h + \lambda_{he})\,L_e(t), \qquad \text{all } t,
\end{aligned}$$

and $\Psi(H,t)$ satisfies

$$-\Psi_t(H,t) = \psi(H,t)\mu_h H + [1 - \Psi(H,t)]\left[-\frac{L_w'(t)}{L_w(t)} - \delta_h + \lambda_{hi}\frac{L_i(t)}{L_w(t)} + \lambda_{he}\frac{L_e(t)}{L_w(t)}\right],$$
$$\text{all } H \geq H_m(t), \quad t \geq 0.$$

D. Consumption

Individuals are organized into a continuum of identical, infinitely lived households of total mass one, where each dynastic household comprises a representative cross-section of the population. New entrants to the workforce arrive at the fixed rate $\delta_h+\upsilon$, so each household grows in size at the constant rate $\upsilon \geq 0$, and total population at date t is $L(t) = L_0 e^{\upsilon t}$.

Members of the household pool their earnings and they own the profit streams from firms. The investment decisions of firms and workers, both incumbents and entrants, maximize, respectively, the expected discounted value of net profits and wages. Hence there are no further investment decisions at the household level. Since there is no aggregate uncertainty, the household faces no consumption risk.

The household's income consists of the wages of its workers plus the profits from its portfolio, which sum to output of the final good,

$$Y_F(t) = L_w(t)\mathrm{E}_{\Psi(\cdot,t)}\left[W(H,t)\right] + N_p(t)\mathrm{E}_{F(\cdot,t)}\left[\Pi(X,t)\right], \qquad \text{all } t.$$

That income is used for consumption and to finance the investment (entry) costs of new firms. Hence the household's net income at date t is $Y_F(t) - E(t)I_e(t)$.

All household members share equally in consumption, and the household has the constant-elasticity preferences

$$U = \int_0^\infty L_0 e^{\upsilon t} e^{-\hat{r}t} \frac{1}{1-\theta} c(t)^{1-\theta} dt,$$

where $\hat{r} > 0$ is the rate of pure time preference, $1/\theta > 0$ is the elasticity of intertemporal substitution, and $c(t)$ is per capita consumption.

The household chooses $c(t)$, $t \geq 0$, to maximize utility, subject to the budget constraint,

$$\int_0^\infty e^{-R(t)} \left\{ L_0 e^{\upsilon t} c(t) - \left[Y_F(t) - E(t)I_e(t)\right] \right\} dt \leq 0,$$

where

$$R(t) = \int_0^t r(s)ds, \qquad \text{all } t.$$

The condition for an optimum implies that per capita consumption grows at the rate

$$\frac{c'(t)}{c(t)} = \frac{1}{\theta}\left[r(t) - \hat{r}\right], \qquad \text{all } t,$$

with $c(0)$ determined by budget balance.

Final output is used for consumption and for the investment costs of entering firms. Hence market clearing for goods requires

$$Y_F(t) = L_0 e^{\upsilon t} c(t) + E(t) I_e(t), \qquad \text{all } t.$$

5. COMPETITIVE EQUILIBRIA, BGPS

This section provides formal definitions of a competitive equilibrium and a BGP. We start with the definition of a (dynamic) economy.

DEFINITION: An *economy* $\mathcal{E}$ is described by

i. parameters $(\rho, \chi, \omega, \eta, \theta, \hat{r}, \upsilon)$, with $\rho > 1$, $\chi \in (0, 1/\rho]$, $\omega \in (0, 1)$, $\eta \in (0, 1)$, $\theta > 0$, $\hat{r} > 0$, $\upsilon \geq 0$;

ii. parameters $\delta_j > 0, \lambda_{ji} > 0$ and μ_j, for $j = h, x$, and $\lambda_{he} > 0$;

iii. parameters $\phi_e > 0$ and $\varepsilon_x > 0$ for the function Λ_{xe};

iv. initial conditions $N_{p0}, N_{i0}, N_{e0} > 0$, $L_{w0}, L_{i0}, L_{e0} > 0$;

v. initial distribution functions $F_0(X)$ with continuous density $f_0(X)$ and lower bound X_{m0} on its support, and $\Psi_0(H)$ with continuous density $\psi_0(H)$ and lower bound $H_{m0} \geq 0$ on its support.

A. Competitive equilibrium

The definition of a competitive equilibrium is standard.

DEFINITION: A *competitive equilibrium* of an economy $\mathcal{E}$ consists of the following, for all $t \geq 0$:

a. the numbers of producers, process innovators, product innovators, workers, retoolers, and trainees, $[N_p(t), N_i(t), N_e(t), L_w(t), L_i(t), L_e(t)]$; and the inflow rate $E(t)$ of product innovators;

b. distribution functions $F(X;t), \Psi(H;t)$;

c. prices and allocations $[W(H;t), P(X,t), H^*(X;t), Y(X,t), \mathfrak{L}(X,t), \Pi(X,t), Y_F(t)]$;

d. value functions $\left[V^f(X,t),\ V_{fi}(t)\right]$ for producers and process innovators and an investment threshold $X_m(t)$; and a value function $V_{fe}(t;\Lambda)$, investment level $I_e(t)$, success rate, $\lambda_{xe}(t)$ for product innovators;

e. value functions $[V^w(H;t), V_{wi}(t), V_{we}(t)]$, for each workers, retooler, and trainees, and an investment threshold $H_m(t)$ for retoolers;

f. per capita consumption $c(t)$, and the interest rate $r(t)$;

such that for all $t \geq 0$,

i. $[W, P, H^*, Y, \mathfrak{L}, \Pi, Y_F]$, is a production equilibrium, given $[N_p, L_w, F, \Psi]$;

ii. $\left[V^f, X_m\right]$ solve the investment problem of producers, given $[r, \Pi, V_{fi}]$; V_{fi} and V_{fe} are consistent with $\left[r, V^f, F\right]$; I_e satisfies the optimization and entry conditions; and the success rate for product innovators is $\lambda_{xe} = \Lambda_{xe}(1, E/N_p)$;

iii. $[V^w, H_m]$ solve the investment problem of workers, given $[r, W, V_{wi}]$; and $[V_{wi}, V_{we}]$ are consistent with $[r, V^w, \Psi]$;

iii. $[N_p, N_i, N_e, F]$ are consistent with $[X_m, E]$, and the initial conditions $[N_{p0}, N_{i0}, N_{e0}, F_0]$; and $X_m(0) = X_{m0}$;

v. $[L_w, L_i, L_e, \Psi]$ are consistent with H_m and the initial conditions $[L_{w0}, L_{i0}, L_{e0}, \Psi_0]$; and $H_m(0) = H_{m0}$;

vi. c solves the consumption/savings problem of households, given $[r, Y_F - EI_e]$; and

vii. the goods market clears.

B. Balanced growth

The rest of the analysis focuses on balanced growth paths, competitive equilibria with the property that quantities grow at constant rates, and the normalized distributions of technology and skill are time invariant.

Let $Q(t) \equiv \mathrm{E}_{F(\cdot,t)}(X)$, $t \geq 0$, denote average technology at date t. On a BGP Q grows at a constant rate, call it g, and the distributions of relative technology

$x = X/Q(t)$ and relative human capital $h = H/Q(t)$ are constant. By assumption total population L grows at the fixed rate υ. On a BGP the number of firms N also grows at a constant rate, call it n, and the shares of firms and individuals in each category, $[N_p/N, N_i/N, N_e/N]$ and $[L_w/L, L_i/L, L_e/L]$ are constant. The growth rates g and n are endogenous.

It follows from Lemma 2 that on a BGP the labor allocation in terms of relative technology and relative skill is time invariant. The growth rates for wages, prices, output levels, and so on are then described by Lemma 3, where Ω, defined in (8), measures the impact of variety growth. In particular, average product price grows at rate Ωn, average output per firm at rate $g + \upsilon - n$, and average employment per firm at rate $\upsilon - n$. Aggregate output, g_Y, the average wage, g_w, and the average profit per firm, g_π, grow at rates

$$
\begin{aligned}
g_Y &= g + \Omega n + \upsilon, \\
g_w &= g + \Omega n \qquad\qquad = g_Y - \upsilon, \\
g_\pi &= g + (\Omega - 1)\, n + \upsilon = g_Y - n,
\end{aligned}
\tag{21}
$$

Aggregate consumption grows at rate g_Y, as do total investment costs $Ei_e\mathrm{E}_F[\Pi]$, and per capita consumption grows at rate g_w. If $\Omega > 1$, then love of variety is so strong that an increase in the number of producers actually raises the profits of each incumbent. In a dynamic model with free entry, this feature poses obvious problems. In the rest of the analysis we will assume that $\rho \geq 2$, which implies $\Omega < 1$.

These observations lead to the following definition.

Definition: A competitive equilibrium for $\mathcal{E}$ is a *balanced growth path* (BGP) if for some $g > 0$ and n, with $g_Y,\ g_w$ and g_π as in (21), the equilibrium has the property that for all $t \geq 0$:

a. the numbers of firms and individuals satisfy

$$
N_p(t) = e^{nt}N_{p0}, \qquad N_i(t) = e^{nt}N_{i0}, \qquad N_e(t) = e^{nt}N_{e0},
$$

$$L_w(t) = e^{\upsilon t}L_{w0}, \qquad L_i(t) = e^{\upsilon t}L_{i0} \qquad L_e(t) = e^{\upsilon t}L_{e0};$$

and for some $E_0 > 0$, the flow of entrants satisfies

$$E(t) = e^{nt}E_0;$$

b. for $Q_0 \equiv \mathrm{E}_{F_0}[X]$, average technology satisfies

$$Q(t) \equiv \mathrm{E}_{F(\cdot,t)}[X] = e^{gt}Q_0;$$

and for some $\left[\hat{F}(x), \hat{\Psi}(h)\right]$, the distribution functions satisfy

$$\begin{aligned} F(X,t) &= \hat{F}(X/Q(t)), \qquad \text{all } X, \\ \Psi(H,t) &= \hat{\Psi}(H/Q(t)), \qquad \text{all } H; \end{aligned}$$

c. for some $[w, p, h^*, y, \ell, \pi, y_F]$, the production equilibria satisfy

$$\begin{aligned} W(H;t) &= e^{g_w t}Q_0 w(H/Q(t)), \qquad \text{all } H; \\ P(X,t) &= e^{\Omega n t}Q_0 p(X/Q(t)), \\ H^*(X;t) &= e^{gt}Q_0 h^*(X/Q(t)), \\ Y(X,t) &= e^{(g+\upsilon-n)t}Q_0 y(X/Q(t)), \\ \mathfrak{L}(X,t) &= e^{(\upsilon-n)t}\ell(X/Q(t)), \\ \Pi(X,t) &= e^{g_\pi t}Q_0\pi(X/Q(t)), \qquad \text{all } X; \\ Y_F(t) &= e^{g_Y t}Q_0 y_F; \end{aligned}$$

d. for some $[v_{fp}(x), v_{fi}, x_m]$, the value function and optimal policy for producers and the value for process innovators satisfy

$$\begin{aligned} V^f(X,t) &= e^{g_\pi t}Q_0 v_{fp}(X/Q(t)), \qquad \text{all } X, \\ X_m(t) &= e^{gt}Q_0 x_m, \\ V_{fi}(t) &= e^{g_\pi t}Q_0 v_{fi}; \end{aligned}$$

for some $[v_{fe}(\lambda), i_e]$, the value function and investment level for product innovators satisfy

$$\begin{aligned} V_{fe}(\lambda;t) &= e^{g_\pi t} Q_0 v_{fe}(\lambda), \\ I_e(t) &= e^{g_\pi t} Q_0 i_e, \end{aligned}$$

and their success rate

$$\lambda_{xe}(t) = \lambda_{xe0} = \Lambda_{xe}\left(1, \frac{E_0}{N_{p0}}\right)$$

is constant;

e. for some $[v_w(h), v_{wi}, v_{we}, h_m]$, the values and optimal policies for individuals satisfy

$$\begin{aligned} V^w(H;t) &= e^{g_w t} Q_0 v_w(H/Q(t)), \qquad \text{all } H, \\ V_{wi}(t) &= e^{g_w t} Q_0 v_{wi}, \\ V_{we}(t) &= e^{g_w t} Q_0 v_{we}, \\ H_m(t) &= e^{g t} Q_0 h_m; \end{aligned}$$

f. for some $c_0 > 0$ aggregate consumption satisfies

$$C(t) = e^{g_Y t} Q_0 L_0 c_0;$$

and the interest rate satisfies

$$r(t) = r \equiv \hat{r} + \theta g_w.$$

BGPs arise—if at all—only for initial conditions $[N_{p0}, N_{i0}, N_{e0}, L_{w0}, L_{i0}, L_{e0}, F_0(X), \Psi_0(H)]$ that satisfy certain restrictions. The rest of the analysis focusses on a class of economies for which BGPs exist, and studies the determinants of the growth rates g and n of TFP and variety

6. CONDITIONS FOR BALANCED GROWTH

In this section we will show that if an economy $\mathcal{E}$ has initial distribution functions F_0 and Ψ_0 that are Pareto, with shape and location parameters that satisfy the requirements of Proposition 4, then the normalized value functions $v_{fp}(x)$ and $v_w(h)$ for producers and workers inherit the isoelastic forms of the normalized profit and wage functions, and simple closed form solutions can be found. Moreover, the growth rates g and n, as well as other features of the BGP, can be solved for explicitly. The arguments are summarized in Proposition 5, which provides sufficient conditions for existence and uniqueness of a BGP.

A. Production equilibrium

Suppose the initial distributions F_0 and Ψ_0 are Pareto, with parameters (α_x, X_{m0}) and (α_h, H_{m0}). Assume (9) holds, define ε, ζ and a_h by (10)-(12), and assume that $H_{m0} = a_h X_{m0}$. Average technology under the initial distribution is

$$Q_0 \equiv \mathrm{E}_{F_0}[X] = \frac{\alpha_x}{\alpha_x - 1} X_{m0}. \tag{22}$$

Use Q_0 to define the normalized distribution functions

$$\begin{aligned} \hat{F}(X/Q_0) &\equiv F_0(X), \qquad \text{all } X \geq X_m, \\ \hat{\Psi}(H/Q_0) &\equiv \Psi_0(H), \qquad \text{all } H \geq H_m. \end{aligned} \tag{23}$$

By construction $\mathrm{E}_{\hat{F}}(x) = 1$, and the location parameters for $\hat{F}$ and $\hat{\Psi}$ are

$$x_m = \frac{X_{m0}}{Q_0} = \frac{\alpha_x - 1}{\alpha_x}, \qquad h_m \equiv \frac{H_{m0}}{Q_0} = a_h x_m. \tag{24}$$

Hence the hypotheses of Proposition 4 hold for $\hat{F}, \hat{\Psi}$, and given N_{p0}, L_{w0}, (14)-(17) describe the production equilibrium $[w, p, h^*, y, \ell, \pi, y_F]$.

B. Firms, investment in technology

Next we will characterize the normalized value functions $[v_{fp}(x), v_{fi}, v_{fe0}]$ and investment cost i_{e0} for entrants as functions of (g, n), and derive one additional equation relating (g, n). Recall that a BGP requires positive process innovation, which in turn requires $g > \mu_x$, so that the investment threshold grows faster than producers' technologies. We will assume that this condition holds.

Consider the investment decision and value of a producer. As shown in the Appendix, if Π and V^f have the forms required for a BGP, and $\pi(x)$ has the isoelastic form in Proposition 4, then the normalized value function $v_{fp}(x)$ for a producer satisfies

$$(r + \delta_x - g_\pi)\, v_{fp}(x) = \pi_2 \left(x/x_m\right)^{1-\zeta} + (\mu_x - g)\, x v'_{fp}(x), \qquad x \geq x_m,$$

where π_2 is as defined in (18), with $L_w = L_{w0}$ and $N_p = N_{p0}$. This equation is a first-order ODE. Value matching provides a boundary condition, yielding the solution

$$v_{fp}(x) = B_x \pi_2 \left(x/x_m\right)^{1-\zeta} + (v_{fi} - B_x \pi_2)\left(x/x_m\right)^{R_x}, \qquad x \geq x_m, \tag{25}$$

where the constant $B_x > 0$ and characteristic root $R_x < 0$ are known functions of the parameters. The first term in (25) is the value of a producer who operates forever, never investing. The second term represents the additional value from the option to invest in process innovation. The smooth pasting condition

$$v_{fi} = B_x \pi_2 \left(1 - \frac{1-\zeta}{R_x}\right), \tag{26}$$

determines v_{fi}, the normalized value of a process innovator, from the optimal choice of the investment threshold by a producer switching to process innovation.

But from the perspective of a firm that has already switched and is waiting for a new technology, its value v_{fi} is the expected discounted value across current producers, adjusted for exit, growth, and waiting time. That is, on a BGP v_{fi} also satisfies

$$(r + \delta_x - g_\pi)\, v_{fi} = \lambda_{xi} \left\{ \mathrm{E}_{\hat{F}}\left[v_{fp}(x)\right] - v_{fi} \right\}.$$

To simplify this condition, substitute for $\mathrm{E}_{\hat{F}}\left[v_{fp}(x)\right]$ and v_{fi} from (25) and (26) and factor out $\pi_2 B_x$ to get

$$r+\delta_x-g_\pi=\frac{-R_x\left(1-\zeta\right)\lambda_{xi}}{\left(\alpha_x-1+\zeta\right)\left(\alpha_x-R_x\right)}. \tag{27}$$

Since r, g_π and R_x involve g and n, while all of the other parameters in this expression are exogenous, (27) is an equation in the pair (g,n), a restriction that the Bellman equation for process innovators places on the ratio $\mathrm{E}_{\hat{F}}\left[v_{fp}(x)\right]/v_{fi}$.

For any success rate λ, on a BGP the normalized value $v_{fe}(\lambda)$ of a product innovator, gross of the sunk cost, satisfies

$$\left(r+\delta_x-g_\pi+\lambda\right)v_{fe}(\lambda)=\lambda\mathrm{E}_{\hat{F}}\left[v_{fp}(x)\right], \tag{28}$$

which determines the function v_{fe}. The flow of new entrants is determined by the rate of variety growth, $E=\left(n+\delta_x\right)N$, so the entry rate is $\epsilon=\left(n+\delta_x\right)N/N_p$. As shown below, the ratio N/N_p is constant on a BGP, so the entry rate $\epsilon>0$ is also constant. In any equilibrium $i_e/\bar{i}_e=1$, so the success rate for product innovators is also constant,

$$\lambda_{xe0}=\Lambda_{xe}(1,\epsilon)=\phi_e/\epsilon>0.$$

From (28), the value of an entrant is then

$$v_{fe0}=\frac{\lambda_{xe0}}{r+\delta_x-g_\pi+\lambda_{xe0}}\mathrm{E}_{\hat{F}}\left[v_{fp}(x)\right], \tag{29}$$

and expected profits just cover entry costs if

$$i_{e0}=v_{fe0}. \tag{30}$$

We must also check the incentive to invest. Given the function $v_{fe}(\cdot)$ in (28), the entry rate ϵ, and the average investment $\bar{i}_e$ of others in its cohort, the normalized problem of a product innovator is

$$\max_{i_e}\left\{v_{fe}[\Lambda_{xe}(i_e/\bar{i}_e,\epsilon)]-i_e\right\},$$

As shown in the Appendix, the optimum is at $i_e/\bar{i}_e = 1$ if and only if

$$\varepsilon_x \leq \frac{r+\delta_x - g_\pi}{\left(r+\delta_x - g_\pi + \lambda_{xe0}\right)^2}\frac{\phi_e}{\epsilon} \leq \frac{1}{\varepsilon_x}, \tag{31}$$

which holds for $\varepsilon_x > 0$ sufficiently small.

C. Workers, investment in skill

The argument for the labor force is analogous except that entry of trainees is exogenous, as is their success rate λ_{he}. Hence the normalized value function v_w for a worker satisfies

$$\left(r+\delta_h - g_w\right) v_w(h) = w_2 \left(h/h_m\right)^{1-\varepsilon} + \left(\mu_h - g\right) h v_w'(h), \qquad h \geq h_m.$$

Suppose $g > \mu_h$, so there is positive retooling, as required on a BGP. Using value matching for the boundary condition, the solution to this ODE is

$$v_w(h) = B_h w_2 \left(h/h_m\right)^{1-\varepsilon} + \left(v_{wi} - B_h w_2\right)\left(h/h_m\right)^{R_h}, \qquad h \geq h_m, \tag{32}$$

where the constant $B_h > 0$ and characteristic root $R_h < 0$ are known functions of the parameters. The first term in (32) is the value of a worker who never invests, and the second represents the additional value from the option to retool. The value of a retooler v_{wi} is determined by the smooth pasting condition

$$v_{wi} = B_h w_2 \left(1 - \frac{1-\varepsilon}{R_h}\right). \tag{33}$$

The value of a retooler also satisfies the Bellman equation

$$\left(r+\delta_h - g_w\right) v_{wi} = \lambda_{hi} \left\{ \mathrm{E}_\Psi \left[v_w(h)\right] - v_{wi} \right\}.$$

Using (32) and (33) to substitute for $\mathrm{E}_{\hat{\Psi}}\left[v_w\right]$ and v_{wi}, and factoring out $w_2 B_h$, gives

$$r+\delta_h - g_w = \frac{-R_h \left(1-\varepsilon\right)\lambda_{hi}}{\left(\alpha_h - 1 + \varepsilon\right)\left(\alpha_h - R_h\right)}, \tag{34}$$

a second equation in the pair (g, n), a restriction that the Bellman equation for retoolers places on the ratio $\mathrm{E}_{\Psi}[v_w(h)]/v_{wi}$.

The value v_{we} of a trainee is determined by

$$(r + \delta_h - g_\pi + \lambda_{he})\, v_{we} = \lambda_{he} \mathrm{E}_{\hat{\Psi}}[v_w(h)]. \tag{35}$$

D. Flows of firms and workers, the evolution of technology and skill

On a BGP the number of firms grows at a constant rate n, and the shares of firms engaged in production and the two kinds of innovation are constant. Hence the flow of new entrants at any date is

$$E = (n + \delta_x)\, N, \tag{36}$$

and the entry rate is constant,

$$\epsilon = \frac{E}{N_p} = (n + \delta_x)\frac{N}{N_p}.$$

The laws of motion for N_p, N_i and N_e determine the ratios of process and product innovators to producers,

$$\begin{aligned} \frac{N_i}{N_p} &= \frac{\alpha_x (g - \mu_x)}{n + \delta_x + \lambda_{xi}}, \\ \frac{N_e}{N_p} &= \frac{n + \delta_x}{\lambda_{xe}} \left[1 + \frac{\alpha_x (g - \mu_x)}{n + \delta_x + \lambda_{xi}}\right]. \end{aligned} \tag{37}$$

The success rate for product innovators is

$$\lambda_{xe0} = \Lambda_{xe}(1, \epsilon) = \frac{\phi_e}{n + \delta_x}\frac{N_p}{N}, \tag{38}$$

where N_p/N is determined by (37).

The shares of the labor force in each group are also constant, and the laws of motion for L_w, L_i, and L_e imply that the ratios of retoolers and trainees to workers are

$$\begin{aligned} \frac{L_i}{L_w} &= \frac{\alpha_h (g - \mu_h)}{\upsilon + \delta_h + \lambda_{hi}}, \\ \frac{L_e}{L_w} &= \frac{\upsilon + \delta_h}{\lambda_{he}} \left[1 + \frac{\alpha_h (g - \mu_h)}{\upsilon + \delta_h + \lambda_{hi}}\right]. \end{aligned} \tag{39}$$

It is easy to check that if X_m and H_m grow at rate g, as required on a BGP, then the distribution functions $F(\cdot, t)$ and $\Psi(\cdot, t)$ evolve as required.

E. Consumption, the interest rate

On a BGP per capita consumption grows at the rate g_w, so the interest rate is

$$r = \hat{r} + \theta g_w. \tag{40}$$

Aggregate income grows at the rate g_Y, so its present discounted value is finite if and only if $r > g_Y$, or

$$\hat{r} > g_Y - \theta g_w = \upsilon + (1 - \theta)(g + \Omega n).$$

Using (30) for i_{e0}, market clearing for goods determines c_0,

$$y_F = L_0 c_0 + E_0 i_{e0}. \tag{41}$$

F. Existence of BGPs

The growth rates (g, n) are determined by (27) and (34). Substitute for g_w, g_π, r, and the roots R_x and R_h, to get a pair of linear equations in the two unknowns,

$$\begin{aligned} g &= \frac{1}{\xi_x}\left[\upsilon - n + \alpha_x \mu_x + \frac{1-\zeta}{\alpha_x - 1 + \zeta}\lambda_{xi} - \delta_x - \hat{r} - (\theta - 1)\,\Omega n\right], \qquad (42)\\ g &= \frac{1}{\xi_h}\left[\alpha_h \mu_h + \frac{1-\varepsilon}{\alpha_h - 1 + \varepsilon}\lambda_{hi} - \delta_h - \hat{r} - (\theta - 1)\,\Omega n\right], \end{aligned}$$

where

$$\xi_x \equiv \alpha_x - 1 + \theta > 0, \qquad \xi_h \equiv \alpha_h - 1 + \theta > 0,$$

and the signs follow from the fact that $\alpha_x, \alpha_h > 1$.

Propositions 5 and 6 both use the assumption $\rho \geq 2$, which implies $\Omega < 1$, so love of variety is not too strong. One additional joint restriction on ρ, α_h is also imposed if $\theta < 1$. Although stronger than required for existence, it will be needed for the comparative statics results.

Proposition 5: Let $\mathcal{E}$ be an economy with:

a. $\rho \geq 2$, and $\rho > \alpha_h / (\alpha_h - 1)$ if $\theta < 1$;

b. initial distributions F_0, Ψ_0 that are Pareto, with shape and location parameters (α_x, X_{m0}), (α_h, H_{m0}) satisfying the hypotheses of Proposition 4.

Define ε and ζ by (10) and (11). Then the pair of equations in (42) has a unique solution (g, n), and there are unique $[Q_0, \hat{F}, \hat{\Psi}]$, $[w, p, h^*, y, \ell, \pi, y_F]$ satisfying conditions (b)-(c) for a BGP; $[v_{fp}(x), x_m, v_{fi}, v_{fe}(\lambda), \lambda_{xe0}, i_{e0}]$ satisfying (d); $[v_w(h), h_m, v_{wi}, v_{we}]$ satisfying (e); and $[c_0, r]$ satisfying (f).

If in addition:

c. the initial ratios $[N_{i0}/N_{p0}, N_{e0}/N_{p0}]$ and $[L_{i0}/L_{w0}, L_{e0}/L_{w0}]$ satisfy (37) and (39); and

d. $g > \mu_x$, $g > \mu_h$, $\hat{r} > g_Y - \theta g_w$, ε_x satisfies (31), and $c_0 > 0$,

then $\mathcal{E}$ has a unique competitive equilibrium that is a BGP.

Proof: For existence and uniqueness of a solution to (42), it suffices to show that the two equations are not collinear. Here we will prove a slightly stronger result, that

$$\frac{1}{\xi_x}\left[1 + (\theta - 1)\,\Omega\right] > \frac{1}{\xi_h}(\theta - 1)\,\Omega,$$

or

$$\alpha_h > (\theta - 1)\left[(\alpha_x - \alpha_h)\,\Omega - 1\right]. \tag{43}$$

Since $\chi \in (0, 1/\rho]$ implies $\Omega \in [0, 1/(\rho - 1))$, and (9) implies $\alpha_x - \alpha_h \in (-1, \rho - 1)$, it follows that

$$(\alpha_x - \alpha_h)\,\Omega - 1 \in \left(-\frac{\rho}{\rho - 1},\ 0\right).$$

Hence if $\theta \geq 1$, the term on the right in (43) is zero or negative. If $\theta < 1$, then by assumption $\alpha_h > \rho/(\rho - 1)$. In either case (43) holds, and there exists a unique (g, n) satisfying (42).

Define $[Q_0, \hat{F}, \hat{\Psi}]$, x_m, h_m by (22)-(24). Since (24) implies (13) holds, by Proposition

4 the normalized production equilibrium $[w, p, h^*, y, \ell, \pi, y_F]$ is described by (14)-(17), so the price and allocation functions are isoelastic.

Then (25), (26) and (28) determine $[v_{fp}, v_{fi}, v_{fe}]$; (29) determines v_{fe0} as a function of λ_{xe0}; (30) determines i_{e0}; (32), (33) and (35) determine $[v_w, v_{wi}, v_{we}]$; (36) determines E_0; (38) determines λ_{xe0}; (40) and (41) determine r and c_0; and (d) implies that the the required inequalities hold. Hence the solution describes a BGP. ■

7. GROWTH RATES ON THE BGP

In this section we will show how various parameters affect the growth rates g and n. We will start with special cases, where $\theta = 1$ or $\Omega = 0$, or where $\alpha_x = \alpha_h$, and provide a numerical example. We will then study the general case, and look at subsidy policies that affect the two growth rates.

A. Special cases

If $(\theta - 1)\,\Omega n = 0$, then the second equation in (42), Bellman equation for retoolers, does not involve n. Hence that equation alone determines g.

If preferences are logarithmic, $\theta = 1$, then in addition $\xi_h = \alpha_h$, and the second equation in (42) simplifies to

$$g = \mu_h + \frac{1}{\alpha_h}\left[\frac{1-\varepsilon}{\alpha_h - 1 + \varepsilon}\lambda_{hi} - \delta_h - \hat{r}\right].$$

In this case g is a weighted sum of the skill parameters $(\mu_h, \lambda_{hi}, \delta_h)$ and the rate of time preference $\hat{r}$, and the technology parameters $(\mu_x, \lambda_{xi}, \delta_x)$ do not enter. Faster on-the-job skill growth μ_h raises g, as does a higher success rate λ_{hi} for retoolers. A higher exit rate δ_h or discount rate $\hat{r}$ reduces g.

The weights depend on the elasticity parameters α_h and $1 - \varepsilon$. A higher value for $1 - \varepsilon$ increases the elasticity of the wage with respect to skill, increasing the returns to investment and increasing g. But $1 - \varepsilon$ is in turn decreasing in $1/\rho$ and α_x, and increasing in α_h. Hence an increase in monopoly power $1/\rho$ reduces g. An increase in α_x, which decreases the mean of the technology distribution and makes the Pareto tail thinner, also reduces g. An increase in α_h decreases the mean of the skill distribution, and the direct effect is to reduce g. The indirect effect, through ε, is in the reverse direction, but presumably smaller.

If $\chi = 1/\rho$, so variety is not valued, then again $(\theta - 1)\,\Omega n = 0$. The solution is qualitatively like the logarithmic case, except that $\xi_h \neq \alpha_h$, so the weights on $(\mu_h, \lambda_{hi}, \delta_h)$ and $\hat{r}$ depend on θ as well as α_h.

Finally, if $\alpha_x = \alpha_h$, so the two Pareto distributions have the same shape parameter, then $\varepsilon = 1/\rho$ and $\zeta = \varepsilon$, and subtracting the second line in (42) from the first gives

$$n = \upsilon + \alpha_h\left(\mu_x - \mu_h\right) + \frac{\rho - 1}{\rho\left(\alpha_h - 1\right) + 1}\left(\lambda_{xi} - \lambda_{hi}\right) - \left(\delta_x - \delta_h\right).$$

In this case variety growth n is equal to the rate of population growth, adjusted for differences in the rates of on-the-job growth, the success rates of incumbent investors, and the exit rates for firms and workers. Faster population growth, υ, a larger gap in the on-the-job growth rates $\mu_x - \mu_h$, or a larger gap in the success rates $\lambda_{xi} - \lambda_{hi}$, increases the rate of variety growth, while a larger gap in the exit rates $\delta_x - \delta_h$, decreases it.

For a numerical example that satisfies the hypotheses of Proposition 5, let

$$\begin{aligned}
&\mu_x = \mu_h = 0.0075, && \lambda_{xi} = \lambda_{hi} = 0.05, && \delta_x = \delta_h = 0.05,\\
&\alpha_x = \alpha_h = 1.2, && \upsilon = 0.012, && \hat{r} = 0.04,\\
&\rho = 5, && \chi = 0.18, && \theta = 1.
\end{aligned}$$

Then

$$\Omega = 0.025; \quad \varepsilon = \zeta = 0.20,$$

and the growth rates are

$$\begin{aligned}
&g = 0.015833; && n = \upsilon = 0.012;\\
&g_w = 0.016133; && g_Y = 0.028133; \quad r = 0.056133.
\end{aligned}$$

The growth rates satisfy $g > \mu_x$, $g > \mu_h$, and $\hat{r} > g_Y - \theta g_w = \upsilon$. Clearly ε_x can be chosen so (31) holds, and as shown in the Appendix $c_0 > 0$.

B. General case

Proposition 6 extends these comparative statics results to the general case.

PROPOSITION 6: Let $\mathcal{E}$ be as in Proposition 5. Then

a. an increase in μ_h, λ_{hi} or a decrease in δ_h raises g and reduces n;

b. an increase in $\upsilon, \mu_x, \lambda_{xi}$ or a decrease in δ_x raises n, and

—raises g if $(\theta - 1)\,\Omega < 0$,

—has no effect on g if $(\theta - 1)\,\Omega = 0$, and

—reduces g if $(\theta - 1)\,\Omega > 0$;

c. a decrease in $\hat{r}$ raises g if $(\theta - 1)\,\Omega \leq 0$, and has otherwise ambiguous effects.

PROOF: First we will show that the equations in (42), plotted in n-g−space, are as shown Figure 1: the line defined by the first equation is downward sloping; the line defined by the second equation has a positive, zero, or negative slope as $(\theta - 1)\,\Omega < 0$, $= 0$, or > 0; and in all case the second line crosses the first from below.

For the first claim, note that $\rho \geq 2$ implies $\Omega \in [0, 1)$, so $[(\theta - 1)\,\Omega + 1] > 0$. The second claim is obvious, and the third follows from (43).

Then claims (a) - (c) follow directly from Figure 1. As shown in panel (a), an increase in μ_h or λ_{hi}, or a decrease in δ_h, shifts the second line upward, increasing g and decreasing n. As shown in panel (b), an increase in υ, μ_x or λ_{xi}, or a decrease in δ_x, shifts the first line to the right, increasing n. The effect on g depends on the slope of the second line. A decrease in $\hat{r}$ does both, as shown in panel (c). Hence it raises g if $(\theta - 1)\,\Omega \leq 0$, and otherwise the effects depend on the relative slopes of the two lines. ■

Changes in the initial population size and number firms, L_0 and N_0, do not affect the growth rates, although they do affect the levels for wages, profits and the distribution of employment across technologies. In particular, since N_{p0} and L_{w0} are directly proportional to N_0 and L_0, it follows from the definitions of w_2, ℓ_2, and π_2, that wages are proportional to N_0^{Ω}, employment is proportional to L_0. and profits per firm are proportional to $L_0 N_0^{\Omega-1}$.

C. Policies to increase growth

Competitive equilibria in the model here are inefficient. Investments by incumbent producers/workers have positive external effects, since they improve the pools from which later investors—both incumbents and entrants draw their technologies/skills. Since this positive externality is not taken into account by individual producers or workers, the competitive equilibrium has too little investment compared with the efficient level, as in Perla and Tonetti (2014, Propositions 3 and 4). Subsidies to investment are obvious policies to overcome this inefficiency.

A complete analysis of optimal policies, which would require characterizing the transition path between old and new BGPs, is beyond the scope of this paper. But it is easy to assess the long-run impact of a small subsidy to either type of investment on the rates of TFP and variety growth.

Consider a subsidy to process innovators at a fixed rate σ_x, scaled by the average profits of current producers, and a subsidy to retoolers at a fixed rate σ_h, scaled by the average wage of current workers. Under such a policy the normalized Bellman equations for process innovators and retoolers are

$$
\begin{aligned}
(r+\delta_x)\, v_{f0} &= \sigma_x \mathrm{E}_F\left[\pi(x)\right] + \lambda_{xi}\left\{\mathrm{E}_F[v^f(x)] - v_{f0}\right\} + g_\pi v_{f0}, \\
(r+\delta_h)\, v_{w0} &= \sigma_h \mathrm{E}_\Psi\left[w(h)\right] + \lambda_{hi}\left\{\mathrm{E}_\Psi[v^w(x)] - v_{w0}\right\} + g_w v_{w0},
\end{aligned}
$$

and the pair of equations in (42) becomes

$$g = \frac{1}{\xi_x}\left[\upsilon - n + \alpha_x\mu_x + \frac{1-\zeta}{\alpha_x(1-\sigma_x)-1+\zeta}\lambda_{xi} - \delta_x - \hat{r} - (\theta-1)\,\Omega n\right],$$
$$g = \frac{1}{\xi_h}\left[\alpha_h\mu_h + \frac{1-\varepsilon}{\alpha_h(1-\sigma_h)-1+\varepsilon}\lambda_{hi} - \delta_h - \hat{r} - (\theta-1)\,\Omega n\right].$$

The subsidies increase the coefficients on the hazard rates λ_{xi} and λ_{hi}. Hence by Proposition 6 a subsidy $\sigma_h > 0$ to retoolers raises g and reduces n, while a subsidy $\sigma_x > 0$ to process innovators increases n and increases, leaves unchanged, or decreases g as $(\theta-1)\,\Omega < 0$, $= 0$, or > 0.

If preferences are logarithmic, $\theta = 1$, or if there is no love of variety, $\Omega = 0$, a subsidy to retoolers raises g, and the negative effect on n can be offset by an appropriate subsidy to process innovators.

8. NO-GROWTH THEOREMS FOR SINGLE FACTORS

In the model here, investments in technology and skill are complementary. In this section we will show that without continued growth in technology (in skill), the incentive to invest in skill (in technology) vanishes.

Consider an economy where $\lambda_{xi} = 0$, so process innovation is impossible. Then the technology distribution is constant over time. Suppose $H_{m0}/X_{m0} > a_h$, so the skill distribution is shifted to the right relative to its position for a BGP. Proposition 7 shows that if H_{m0}/X_{m0} is sufficiently large, the unique competitive equilibrium features no retooling and just enough product innovation (entry) to offset exit and population growth.

Similarly, consider an economy where $\lambda_{hi} = 0$, so retooling is impossible. Then the skill distribution is constant over time,and for $H_{m0}/X_{m0} < a_h$ sufficiently small, the unique competitive equilibrium features no process innovation and just enough product innovation to offset exit and population growth.

For simplicity, we will assume that variety is not valued, and that there is no learning-by-doing for firms or on-the-job learning for workers.

PROPOSITION 7: Let $\mathcal{E}$ be an economy with:

a. $\chi = 1/\rho$ and $\mu_x = \mu_h = 0$;

b. initial distributions F_0, Ψ_0 that are Pareto, with shape and location parameters (α_x, X_{m0}), (α_h, H_{m0}) satisfying all of the hypotheses of Proposition 4 except (13); and

c. initial conditions satisfying $N_{i0}/N_{p0} = 0$, $N_{e0}/N_{p0} = (\upsilon + \delta_x)/\lambda_{xe}$, $L_{i0}/L_{w0} = 0$, and $L_{e0}/L_{w0} = (\upsilon + \delta_h)/\lambda_{he}$.

i. Suppose $\lambda_{xi} = 0$, so process innovation is impossible. Then for all $H_{m0}/X_{m0} > a_h$ sufficiently large, the unique competitive equilibrium features product innovation at the rate $\delta_x + \upsilon$ and no retooling.

ii. Alternatively, suppose instead that $\lambda_{hi} = 0$, so retooling is impossible. Then for all all $H_{m0}/X_{m0} < a_h$ sufficiently small, the unique competitive equilibrium features product innovation at the rate $\delta_x + \upsilon$ and no process innovation.

In the economy with $\lambda_{xi} = 0$, if the initial value for the ratio H_{m0}/X_{m0} is not large enough, the equilibrium features positive retooling for a finite period. During this time H_{m0} grows and X_{m0} is constant. In finite time the ratio hits the the required threshold and retooling ceases. An analogous result hold in the economy with $\lambda_{hi} = 0$.

COROLLARY TO PROPOSITION 7: i. In the economy with $\lambda_{xi} = 0$, for any initial H_{m0}/X_{m0}, retooling ceases in finite time.

ii. In the economy with $\lambda_{hi} = 0$, for any initial H_{m0}/X_{m0}, process innovation ceases in finite time.

9. WAGE AND EMPLOYMENT/REVENUE DYNAMICS

This section looks at the empirical predictions of the model for wage growth for individuals and growth in revenue and employment for firms. Growth for either type of agent has two components, continuous growth while working/producing, and jumps from successful investment. Since the jumps are hard to match with data, we will focus on age cohorts of individuals and firms.

Each age cohort of individuals has a mix of workers, retoolers, and individuals waiting for their initial draw for skill, with proportions that change as the cohort ages. Only workers receive wages. As shown in the Appendix, the share of workers among survivors in the cohort at age $a \geq 0$, call it $\sigma_w(a)$, is zero at $a = 0$ and grows monotonically as the cohort ages, converging to $\lambda_{hi}/[\alpha_h (g - \mu_h) + \lambda_{hi}]$.

The workers in any age cohort have average wages equal to the economy-wide average. Since average wages grow at the rate g_w, average earnings across all survivors at age a are

$$e_{Av}(a) = e^{g_w a}\sigma_w(a)W_0, \qquad a \geq 0,$$

where W_0 is the average wage in the economy when the cohort entered. Hence average earnings among survivors grows monotonically as a function of age, at a rate that declines toward g_w as a grows without bound. The number of survivors declines over time through exit, so total earnings for the cohort are $e_{Av}(a)$ scaled by $e^{-\delta_x a}$.

The argument for firms is analogous. Each age cohort of firms has a mix of producers, process innovators, and product innovators, where the last group consists of firms that are still waiting for their first technology draw. Only producers have revenue and employees. The share of producers among survivors, call it $\sigma_p(a)$, is zero when the cohort enters, and grows monotonically as the cohort ages, converging to $\lambda_{xi}/\left[\alpha_x (g - \mu_x) + \lambda_{xi}\right]$.

At any date average revenue and employment are the same among producers in any age cohort, growing at rates g_π (profits) and $\upsilon - n$ (employment). Hence average revenue and employment among survivors at age a are

$$R_{Av}(a) \;=\; e^{g_\pi a}\sigma_p(a)R_0,$$

$$\bar{\ell}_{Av}(a) \;=\; e^{(\upsilon-n)a}\sigma_p(a)\bar{\ell}_0, \qquad a \geq 0,$$

where R_0 and $\bar{\ell}_0$ are average revenue and employment across all firms when the cohort entered. Average revenue among survivors grows monotonically as a function of age, at a rate that declines toward g_π in the long run. Average employment among survivors grows when the cohort is young. It continues to grow in the long run if and only if population growth exceeds variety growth, if and only if $\upsilon - n > 0$. The number of survivors in the cohort declines over time as firms exit, so cohort totals for revenue and employment are scaled by $e^{-\delta_x a}$.

9. CONCLUSION

This paper develops a model in which both technological change and human capital accumulation are required for long run growth. The main results are to provide conditions for the existence of a BGP, and to show how the rates of TFP and variety growth depend on various parameters of the model.

On a BGP, skill and technology grow at a common rate. Nevertheless, the parameters governing skill accumulation are more important than those governing technological change in determining that rate. The parameters for skill and technology enter more symmetrically—but with opposite signs—in determining growth in product variety. Thus improvements in the parameters for technological change encourage entry, while improvements in the parameters for skill accumulation encourage investment in both skill and technology, but discourage growth in variety.

In equilibrium, continued investment in either factor remains worthwhile only because the other grows. There is no 'race' between technology and skill: they grow together. Although transitional dynamics are not studied, the results here suggest that if one factor started out with a distribution that was 'ahead' of the other, investment in that factor would slow down—and perhaps would cease altogether, while the incentive to invest in the lagging factor would be exceptionally strong. In this general sense, the analysis suggests that the system would converge to a BGP for any initial distributions that are Pareto, with shape parameters that are not too different.

In the model here TFP growth comes from imitation by incumbents workers and producers, so investments in skill and technology have positive external effects, improving the distribution offered to later investors. Hence the competitive equilibrium is likely to be inefficient: as in Perla and Tonetti (2014), investment is probably too low, for at least one factor and perhaps for both.

The model suggests a number of questions for further work. In terms of theory, it could be used to analyze the effects of a change in the (exogenous) rate of population growth or the transition path for an economy that received an inflow of technology from outside. On the empirical side, the implications for wages and revenue, sketch in section 9, could be further developed. In particular, it could be used to look at the transition paths in rapidly developing countries, where technology is often imported from abroad.

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[Incomplete]

APPENDIX A: PRODUCTION AND PRICES

A. Production equilibrium

PROOF OF PROPOSITION 1: Use (5) to write labor demand as

$$\ell(x) = N_p^{-\rho\chi}\left(\frac{\rho-1}{\rho}\right)^{\rho}\frac{\phi(h^*(x),x)^{\rho-1}}{w(h^*(x))^{\rho}}y_F, \qquad \text{all } x \geq x_m, \tag{44}$$

and differentiate (7) to write labor market clearing as

$$L_w\psi(h^*(x))h^{*\prime}(x) = N_p\ell(x)f(x), \qquad \text{all } x \geq x_m, \tag{45}$$

$$L_w = N_p\int_{x_m}^{\infty}\ell(\xi)f(\xi)d\xi. \tag{46}$$

Then (4) and (45) are a pair of ODEs in $w(h)$ and $h^*(x)$, with $\ell(x)$ given by (44). The price normalization (2) serves as a boundary condition for w, and (6) is the boundary condition for h^*, with y_F determined by (46). The other equations in (5) determine p, y, π. ■

PROOF OF PROPOSITION 4: The functions in (14)-(17) must satisfy (2), (4)-(6) and (45)-(46).

For any wage function of the form $w(h) = w_2(h/h_m)^{1-\varepsilon}$, as in (15), the linear function $h^*(x) = a_hx$ in (14), with a_h in (12), satisfies (4). Moreover, for $w(h)$ of this form the first line in (5) implies that

$$p(x) = \frac{\rho}{\rho-1}\frac{w_2}{x_m}\frac{(x/x_m)^{-\varepsilon}}{\phi(a_h,1)}, \qquad \text{all } x \geq x_m.$$

Then (2) implies w_2 is as in (18), with p_0 as claimed.

Then from the second and third lines of (5), $y(x)$ and $\ell(x)$ have the form

$$\begin{aligned} y(x) &= y_2 \left(\frac{x}{x_m}\right)^{\rho\varepsilon}, && \text{all } x \geq x_m. \\ \ell(x) &= \ell_2 \left(\frac{x}{x_m}\right)^{\rho\varepsilon - 1}, && \text{all } x \geq x_m. \end{aligned}$$

Using the Pareto distribution for F, (46) holds if and only if

$$\begin{aligned} L_w &= N_p \ell_2 \mathrm{E}_F \left[\left(\frac{x}{x_m}\right)^{\alpha_x - \alpha_h} \right] \\ &= N_p \ell_2 \frac{\alpha_x}{\alpha_h}, \end{aligned}$$

so ℓ_2 is as claimed. Then $y(x)$ and y_F follow from the production technologies, and $\pi(x)$ is straightforward from (5). The ODE in (45) requires

$$\psi(a_h x) a_h = \frac{\alpha_h}{\alpha_x} \left(\frac{x}{x_m}\right)^{\alpha_x - \alpha_h} f(x), \qquad \text{all } x \geq x_m,$$

which holds for the Pareto densities ψ and f, and clearly (13) implies (6), completing the proof. ■

B. The evolution of technology

The distribution function for technology among producers evolves as follows. As noted above, $X_m(t)$ is nondecreasing. Let $\Delta_t > 0$ be a small time increment. For any $t \geq 0$ and any $X \geq X_m(t + \Delta_t)$, the number of producers with technology above X at $t + \Delta_t$ consists of incumbents at t, adjusted for exit, plus successful innovators of both types, selected to include only those with technology greater than $(1 - \mu_x \Delta_t) X$ at date t,

$$\begin{aligned} & [1 - F(X, t + \Delta_t)] \, N_p(t + \Delta_t) \\ \approx \; & \{1 - F[(1 - \mu_x \Delta_t) X, t]\} \left[(1 - \delta_x \Delta_t) N_p(t) + \lambda_{xi} \Delta_t N_i(t) + \Lambda_{xe}(t) \Delta_t N_e(t) \right]. \end{aligned}$$

Taking a first-order approximation gives

$$
\begin{aligned}
&[1-F(X,t)]\left[N_p(t)+N_p'(t)\Delta_t\right]-F_t(X,t)\Delta_t N_p(t)\\
&\qquad\approx\ [1-F(X,t)]\left[(1-\delta_x\Delta_t)\,N_p(t)+\lambda_{xi}\Delta_t N_i(t)+\Lambda_{xe}(t)\Delta_t N_e(t)\right]\\
&\qquad\quad +f(X,t)\mu_x\Delta_t X N_p(t).
\end{aligned}
$$

Collecting terms and dividing by $\Delta_t N_p(t)$ gives the equation in the text.

APPENDIX B: BGPS WITH PARETO DISTRIBUTIONS

A. Firms: process and product innovation

If Π and V^f have the forms required for a BGP, then factoring out $e^{g_\pi t}Q_0$, the Bellman equation for a producing firm is

$$
\begin{aligned}
(r+\delta_x)\,v_{fp}(X/Q(t)) &= \pi(X/Q(t))+\mu_x\frac{X}{Q(t)}v_{fp}'(X/Q(t))\\
&\quad +g_\pi v_{fp}(X/Q(t))-v_{fp}'(X/Q(t))\frac{X}{Q(t)}\frac{\dot{Q}(t)}{Q(t)},
\end{aligned}
$$

or

$$
(r+\delta_x-g_\pi)\,v_{fp}(x)=\pi(x)+(\mu_x-g)\,xv_{fp}'(x),
$$

where $x=X/Q$ and $\dot{Q}/Q=g$. For π as in (17), the normalized Bellman equation is as claimed. Define

$$
\begin{aligned}
B_x &\equiv \left[(r+\delta_x-g_\pi)+(g-\mu_x)\,(1-\zeta)\right]^{-1}>0, \qquad (47)\\
R_x &\equiv -\frac{r+\delta_x-g_\pi}{g-\mu_x}<0,
\end{aligned}
$$

where R_x is the characteristic root of the ODE. It is straightforward to verify that $v_P(x)=B_x\pi_2\,(x/x_m)^{1-\zeta}$ is a particular solution, and $v_H(x)=c_x x^{R_x}$ is the homogeneous solution. The coefficient c_x is determined by the value matching condition,

$$
c_x=x_m^{-R_x}\,(v_{fi}-B_x\pi_2)>0,
$$

and $v_{fp}(x)$ is as in (25). Differentiate (25) to get the smooth pasting condition, $v'_{fp}(x_m) = 0$, so v_{fi} is as in (26).

On a BGP, the Bellman equation for v_{fi} is as claimed. To obtain (27), substitute from (26) and take the expectation in (25) to get

$$\mathrm{E}_{\hat{F}}\left[v_{fp}(x)\right] = B_x \pi_2 \left(\frac{\alpha_x}{\alpha_x - 1 + \zeta} - \frac{1-\zeta}{R_x} \frac{\alpha_x}{\alpha_x - R_x} \right). \tag{48}$$

Use this expression and (26) in the Bellman equation for v_{fi} to get (27),

$$\begin{aligned} r + \delta_x - g_\pi &= \lambda_{xi} \left(\frac{\mathrm{E}_{\hat{F}}\left[v_{fp}(x)\right]}{v_{fi}} - 1 \right) \\ &= \frac{\lambda_{xi} R_x}{R_x - 1 + \zeta} \left(\frac{\alpha_x}{\alpha_x - 1 + \zeta} - \frac{1-\zeta}{R_x} \frac{\alpha_x}{\alpha_x - R_x} - 1 + \frac{1-\zeta}{R_x} \right) \\ &= \frac{\lambda_{xi} R_x}{R_x - 1 + \zeta} \left(\frac{1-\zeta}{\alpha_x - 1 + \zeta} - \frac{1-\zeta}{\alpha_x - R_x} \right) \\ &= \frac{\lambda_{xi} R_x (1-\zeta)}{R_x - 1 + \zeta} \frac{-R_x + 1 - \zeta}{(\alpha_x - 1 + \zeta)(\alpha_x - R_x)} \\ &= \frac{-\lambda_{xi} R_x (1-\zeta)}{(\alpha_x - 1 + \zeta)(\alpha_x - R_x)}. \end{aligned}$$

To show that the optimal investment for a product innovator is at $i_e/\bar{i}_e = 1$, use the definition of v_{fe} in (28) and the definition of Λ_{xe} in the neighborhood of $i_e/\bar{i}_e = 1$ to find that

$$\begin{aligned} v'_{fe}(\lambda) &= \frac{r + \delta_x - g_\pi}{(r + \delta_x - g_\pi + \lambda)^2} \mathrm{E}_{\hat{F}}\left[v_{fp}(x)\right], \\ \frac{\partial}{\partial i_e} \Lambda_{xe}(i_e/\bar{i}_e, \epsilon) &= \frac{\phi_e}{\epsilon} \frac{1}{\bar{i}_e} \cdot \begin{cases} 1/\varepsilon_x, & \text{if } i_e/\bar{i}_e < 1, \\ \varepsilon_x, & \text{if } i_e/\bar{i}_e > 1. \end{cases} \end{aligned}$$

Hence an optimum at $i_e/\bar{i}_e = 1$, for $\bar{i}_e = \mathrm{E}_{\hat{F}}\left[v_{fp}(x)\right]$, requires (31).

For any $q < \alpha_x$, integrating w.r.t. the density $f(x) = \alpha_x x_m^{\alpha_x} x^{-\alpha_x - 1}$ gives $\mathrm{E}_{\hat{F}}\left[(x/x_m)^q\right] = \alpha_x / (\alpha_x - q)$.

B. Workers: investment in human capital on the BGP

The analysis for the labor force is analogous, except that trainees pay no entry cost. Hence (32)-(35) have the same form as (25)-(28), with

$$\begin{aligned} B_h &\equiv \left[(r+\delta_h-g_w)+(g-\mu_h)(1-\varepsilon)\right]^{-1}>0, \\ R_h &\equiv -\frac{r+\delta_h-g_w}{g-\mu_h}<0. \end{aligned} \tag{49}$$

C. Flows of firms, the DF for technology

On a BGP $X_m(t)$ grows at the rate g; $N_p(t), N_i(t)$, and $N_e(t)$ grow at the rate n; and there is strictly positive process innovation, so (19) holds. Hence the law of motion for N_p requires

$$\begin{aligned} nN_p &= \lambda_{xi}N_i+\lambda_{xe}N_e-\delta_x N_p-(g-\mu_x)\frac{X_m(t)}{Q(t)}f(x_m)N_p \\ &= \lambda_{xi}N_i+\lambda_{xe}N_e-\left[\delta_x+\alpha_x(g-\mu_x)\right]N_p, \end{aligned}$$

where the second line the Pareto density for $f(x)$. Hence

$$\left[n+\delta_x+\alpha_x(g-\mu_x)\right]N_p=\lambda_{xi}N_i+\lambda_{xe}N_e.$$

The laws of motion for N_i and N_e require

$$\begin{aligned} (n+\delta_x+\lambda_{xi})N_i &= \alpha_x(g-\mu_x)N_p, \\ (n+\delta_x+\lambda_{xe})N_e &= E. \end{aligned}$$

Sum the three laws of motion to get (36), which determines the entry rate E. The population shares for firms are

$$\begin{aligned} \frac{N_p}{N} &= \frac{n+\delta_x+\lambda_{xi}}{(n+\delta_x+\lambda_{xi})+\alpha_x(g-\mu_x)}\frac{\lambda_{xe}}{n+\delta_x+\lambda_{xe}}, \\ \frac{N_i}{N} &= \frac{\alpha_x(g-\mu_x)}{(n+\delta_x+\lambda_{xi})+\alpha_x(g-\mu_x)}\frac{\lambda_{xe}}{n+\delta_x+\lambda_{xe}}, \\ \frac{N_e}{N} &= \frac{n+\delta_x}{n+\delta_x+\lambda_{xe}}, \end{aligned}$$

and the ratios N_i/N_p and N_e/N_p are as in (37).

If $F(X,t)$ has the form required for a BGP and $Q(t)$ grows at the rate g, then

$$\begin{aligned} f(X,t) &= \hat{f}(X/Q(t))/Q(t), \\ -F_t(X,t) &= \hat{f}\left(X/Q(t)\right) gX/Q(t), \qquad \text{all } X \geq X_m(t), \quad \text{all } t. \end{aligned}$$

Use these expression and (**??**) in the law of motion for F to get

$$\begin{aligned} &\hat{f}\left(X/Q(t)\right) gX/Q(t) = \hat{f}(X/Q(t))\mu_x X/Q(t) \\ &\quad + \left[1 - \hat{F}(X/Q(t))\right] \alpha_x \left(g - \mu_x\right), \quad \text{all } X \geq X_m(t), \quad \text{all } t, \end{aligned}$$

so the required condition is

$$\left(g - \mu_x\right) x\hat{f}\left(x\right) = \left(g - \mu_x\right) \alpha_x \left[1 - \hat{F}\left(x\right)\right], \qquad \text{all } x \geq x_m,$$

which holds since F is a Pareto distribution with parameters $\left(\alpha_x, x_m\right)$.

The arguments for the workforce are analogous.

D. Proof that $c_0 > 0$ in the example

Note that $c_0 > 0$ if and only if $E_0 i_{e0}/y_F < 1$. Use (29) and (30) to find that

$$i_{e0} = \frac{\lambda_{xe0}}{r + \delta_x - g_\pi + \lambda_{xe0}} \mathrm{E}_{\hat{F}}\left[v_{fp}(x)\right].$$

Moreover, since $\left(\mu_x - g\right) x v'_{fp}(x) < 0$, the Bellman equation for v_{fp} implies

$$\begin{aligned} \mathrm{E}_{\hat{F}}\left[v_{fp}(x)\right] &< \frac{1}{r + \delta_x - g_\pi} \pi_2 \mathrm{E}_{\hat{F}}\left[\left(\frac{x}{x_m}\right)^{1-\zeta}\right] \\ &= \frac{1}{r + \delta_x - g_\pi} \frac{1}{\rho} \frac{y_F}{N_{p0}}, \end{aligned}$$

where the second line uses the fact that $1/\rho$ is the factor share of profits in income. Use these two expression and $E_0 = \left(n + \delta_x\right) N_0$ to get

$$\frac{E_0 i_{e0}}{y_F} < \frac{\lambda_{xe0}}{r + \delta_x - g_\pi + \lambda_{xe0}} \frac{1}{r + \delta_x - g_\pi} \frac{1}{\rho} \frac{\left(n + \delta_x\right) N_0}{N_{p0}}. \tag{50}$$

Hence $E_0 i_{e0}/y_F < 1$ if the term on the right in (50) is less than unity. For the parameter values in the example, $r - g_\pi = \hat{r}$ and $r - \theta g_w = \hat{r}$, so the condition is

$$\begin{aligned}\frac{\lambda_{xe0}}{\hat{r}+\delta_x+\lambda_{xe0}}\frac{n+\delta_x}{\hat{r}+\delta_x}\frac{1}{\rho} &< \frac{N_{p0}}{N_0} \\ &= \frac{n+\delta_x+\lambda_{xi}}{(n+\delta_x+\lambda_{xi})+\alpha_x(g-\mu_x)}\frac{\lambda_{xe}}{n+\delta_x+\lambda_{xe}},\end{aligned}$$

where the second line uses N_p/N from above. Since $\hat{r} > n$, it suffices if

$$\alpha_x(g-\mu_x) < (\rho-1)(n+\delta_x+\lambda_{xi}),$$

which holds for the parameters in the example..

E. No-growth result

Proof of Proposition 7: (i) First we will show that there is a competitive equilibrium with the stated features. Then we will show that it is unique.

(Existence) Let $[w_0, p_0, h_0^*, y_0, \ell_0, \pi_0, y_{F0}]$ be the production equilibrium at $t = 0$. In the hypothesized equilibrium $N_p(t) = e^{\upsilon t}N_{p0}$ and $N_e(t) = e^{\upsilon t}N_{e0}$,the technology distribution is$F(X,t) = F_0(X)$, all X, t, is constant., and the skill distribution $\Psi(H,t) = \Psi_0(H)$ is also constant. Hence by Lemma 3, the allocation and prices are also constant

$$\begin{aligned}&w(H,t) = w_0(H), \quad p(X,t) = p_0(X), \\ &h^*(X,t) = h_0^*(X), \quad y(X,t) = y_0(X), \\ &\ell(X,t) = \ell_0(X), \quad \pi(X,t) = \pi(X), \qquad \text{all } X, H, \quad \text{all } t.\end{aligned}$$

and final output grows like population, $y_F(t) = e^{\upsilon t}y_{F0}$.

Consider the value functions for workers and (potential) retoolers. It suffices to show that $V_{wi}(0) < V^w(H_{m0}, 0)$, so even labor with the lowest skill chooses to work rather than retool at date $t = 0$. Since

$$V^w(H,0) = \frac{1}{r+\delta_h}w_0(H), \qquad \text{all } H,$$

$$V_{wi}(0) = \frac{\lambda_{hi}}{r+\delta_h+\lambda_{hi}} \mathrm{E}_\Psi \left[v_0^w(H) \right],$$

it suffices to show that

$$\frac{\lambda_{hi}}{r+\delta_h+\lambda_{hi}} \mathrm{E}_\Psi \left[w_0(H) \right] < w_0(H_m). \tag{51}$$

The equilibrium allocation and prices can be constructed as follows. Fix any employment function $\ell(x)$ satisfying overall clearing in the labor market

$$\frac{L_x}{N_p} = \alpha_x x_m^{\alpha_x} \int_{x_m}^{\infty} \ell(x) x^{-\alpha_x - 1} dx.$$

Then markets clear for each skill level if and only if h^* satisfies

$$\begin{aligned} h_m &= h^*(x_m), \\ \frac{L_w}{N_p} \left(\frac{h^*(x)}{h_m} \right)^{-\alpha_h} &= \alpha_x x_m^{\alpha_x} \int_x^{\infty} \ell(v) v^{-\alpha_x - 1} dv, \qquad \text{all } x \geq x_m. \end{aligned}$$

To construct wages and prices, use the ODE (4) and the markup equation in (5),

$$\begin{aligned} \frac{d \ln w(h^*)}{d \ln h^*} &= \left[1 + \frac{1-\omega}{\omega} \left(\frac{h^*}{x} \right)^{(1-\eta)/\eta} \right]^{-1}, \qquad \text{all } h^* > h_m, \\ p(x) &= \frac{\rho}{\rho - 1} \frac{w(h^*(x))}{\phi(h^*(x), x)}, \qquad \text{all } x \geq x_m, \end{aligned}$$

with the multiplicative constant determined by the price normalization $p_F = 1$.

For output levels, use ℓ and h^* in the production function for differentiated goods. Then construct y_F from the production function for the final good. Demands depend on prices and y_F, and markets clear for all differentiated goods if and only if

$$\ell(x)\phi(h^*(x), x) = y_F N_p^{-\rho\chi} \left(\frac{\rho}{\rho - 1} \frac{w(h^*(x))}{\phi(h^*(x), x)} \right)^{-\rho}, \qquad \text{all } x \geq x_m.$$

If market clearing fails for some goods, adjust $\ell(x)$ according to excess demands.

[To be completed. It suffices to show that the equilibrium satisfies (51).]

APPENDIX C: WAGE AND FIRM SIZE DYNAMICS

As a function of age, the shares of firms of various types satisfy

$$\begin{pmatrix} \sigma_p'(a) \\ \sigma_i'(a) \\ \sigma_e'(a) \end{pmatrix} = \begin{pmatrix} -\alpha_x(g-\mu_x) & \lambda_{xi} & \lambda_{xe} \\ \alpha_x(g-\mu_x) & -\lambda_{xi} & 0 \\ 0 & 0 & -\lambda_{xe} \end{pmatrix} \begin{pmatrix} \sigma_p(a) \\ \sigma_i(a) \\ \sigma_e(a) \end{pmatrix}.$$

It is straightforward to solve this linear system, and find that

$$\begin{aligned} \sigma_p(a) &= c_1\lambda_{xi} + e^{-b_x a}(c_1+c_3)\alpha_x(g-\mu_x) + e^{-\lambda_{xe}a}c_3(\lambda_{xi}-\lambda_{xe}), \\ \sigma_i(a) &= c_1\alpha_x(g-\mu_x) - e^{-b_x a}(c_1+c_3)\alpha_x(g-\mu_x) + e^{-\lambda_{xe}a}c_3\alpha_x(g-\mu_x), \\ \sigma_e(a) &= 0 + 0 + e^{-\lambda_{xe}a}, \end{aligned}$$

where $b_x \equiv \alpha_x(g-\mu_x)+\lambda_{xi}$, $c_1 = 1/b_x$, and $c_3 = -1/(b_x-\lambda_{xe})$. Hence

$$\sigma_p(a) = \frac{\lambda_{xi}}{b_x} - e^{-b_x a}\frac{\lambda_{xe}}{b_x}\frac{b_x-\lambda_{xi}}{b_x-\lambda_{xe}} - e^{-\lambda_{xe}a}\frac{\lambda_{xi}-\lambda_{xe}}{b_x-\lambda_{xe}}, \qquad a \geq 0.$$

The argument for workers is analogous.

Firm size and wage dynamics

Normalize the entering size of a cohort to unity. Then on a BGP, the number of producers, process innovators, and product innovators in the cohort, as a function of age, $m_p(a), m_i(a), m_e(a)$, $a \geq 0$, satisfy

$$\begin{pmatrix} m_p'(a) \\ m_i'(a) \\ m_e'(a) \end{pmatrix} = \begin{pmatrix} -\delta_x - \alpha_x(g-\mu_x) & \lambda_{xi} & \lambda_{xe} \\ \alpha_x(g-\mu_x) & -\delta_x - \lambda_{xi} & 0 \\ 0 & 0 & -\delta_x - \lambda_{xe} \end{pmatrix} \begin{pmatrix} m_p(a) \\ m_i(a) \\ m_e(a) \end{pmatrix}.$$

where $m_p(0) = m_i(0) = 0$, and $m_e(0) = 1$. Let $\sigma_p(a), \sigma_i(a), \sigma_e(a)$, $a \geq 0$, denote the *shares* of each type in the cohort at age a. All types exit at rate δ_x, so

$$\begin{aligned} m_z(a) &= \sigma_z(a)e^{-\delta_x a}, \\ m_z'(a) &= \sigma_z'(a)e^{\delta_x a} - \delta_x m_z(a), \qquad z = p, i, e. \end{aligned}$$

Hence the shares, as a function of age, satisfy

$$\begin{pmatrix} \sigma_p'(a) \\ \sigma_i'(a) \\ \sigma_e'(a) \end{pmatrix} = \begin{pmatrix} -\alpha_x(g-\mu_x) & \lambda_{xi} & \lambda_{xe} \\ \alpha_x(g-\mu_x) & -\lambda_{xi} & 0 \\ 0 & 0 & -\lambda_{xe} \end{pmatrix} \begin{pmatrix} \sigma_p(a) \\ \sigma_i(a) \\ \sigma_e(a) \end{pmatrix}.$$

Since the three equations are linearly dependent, one characteristic root is $R_1 = 0$. Another is in the lower right corner, $R_3 = -\lambda_{xe}$. The third satisfies

$$\begin{aligned} 0 &= \det \begin{pmatrix} -R - \alpha_x(g-\mu_x) & \lambda_{xi} \\ \alpha_x(g-\mu_x) & -R - \lambda_{xi} \end{pmatrix} \\ &= R^2 + R\left[\alpha_x(g-\mu_x) + \lambda_{xi}\right], \end{aligned}$$

so $R_2 = -b_x$, where $b_x \equiv \alpha_x(g-\mu_x) + \lambda_{xi}$.

The eigenvectors satisfy $R_j v_j = M v_j$, where M is the matrix above, or

$$\begin{aligned} \left[R_j + \alpha_x(g-\mu_x)\right] v_{j1} &= \lambda_{xi} v_{j2} + \lambda_{xe} v_{j3}, \\ (R_j + \lambda_{xi}) v_{j2} &= \alpha_x(g-\mu_x) v_{j1}, \\ (R_j + \lambda_{xe}) v_{j3} &= 0, \qquad\qquad j = 1, 2, 3. \end{aligned}$$

Hence

for $R_1 = 0$, the associated vector is $v_1 = [\lambda_{xi}, \alpha_x (g - \mu_x), 0]$;

for $R_2 = -b_x$, it is $v_2 = (1, -1, 0)$; and

for $R_3 = -\lambda_{xe}$, it is

$$v_3 = [\lambda_{xi} - \lambda_{xe},\ \alpha_x (g - \mu_x),\ -(b_x - \lambda_{xe})],$$

where $\Delta_x = \lambda_{xi} - \lambda_{xe}$. Then the shares at age a are

$$\sigma(a) = e^{R_1 a} c_1 v_1 + e^{R_2 a} c_2 v_2 + e^{R_3 a} c_3 v_3,$$

where the initial conditions require $\sigma(0) = (0, 0, 1)'$, or

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} \lambda_{xi} \\ \alpha_x (g - \mu_x) \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} \lambda_{xi} - \lambda_{xe} \\ \alpha_x (g - \mu_x) \\ -(b_x - \lambda_{xe}) \end{pmatrix}.$$

Clearly $c_3 = -1/(b_x - \lambda_{xe})$. Sum the first two rows and substitute for c_3 to find that $0 = c_1 b_x - 1$, so $c_1 = 1/b_x$. From the second line

$$\begin{aligned} c_2 &= (c_1 + c_3) \alpha_x (g - \mu_x) \\ &= \left(\frac{1}{b_x} + \frac{1}{b_x - \lambda_{xe}} \right) \alpha_x (g - \mu_x) \\ &= \frac{-\lambda_x}{b_x (b_x - \lambda_{xe})} \alpha_x (g - \mu_x). \end{aligned}$$

Sum to get $\sigma(a) = 1$. In the long run, the shares converge to $[c_1 \lambda_{xi}, 1 - c_1 \lambda_{xi}, 0]$.

COST MINIMIZATION

Final good firms solve

$$\min_y N \int p(x)y(x)f(x)dx \qquad \text{s.t. } 1 = \left[N^{1-\chi}\int y(x)^{(\rho-1)/\rho}f(x)dx\right]^{\rho/(\rho-1)}.$$

The FOC is

$$Np = \lambda N^{(1-\chi)\rho/(\rho-1)}\left[\int y(x)^{(\rho-1)/\rho}f(x)dx\right]^{1/(\rho-1)} y^{-1/\rho}$$

$$1 = N^{(1-\chi)/(\rho-1)}\left[\int y(x)^{(\rho-1)/\rho}f(x)dx\right]^{1/(\rho-1)}$$

or

$$\begin{aligned} y^{1/\rho} &= \lambda p^{-1}N^{-1}N^{(1-\chi)\rho/(\rho-1)}N^{-(1-\chi)/(\rho-1)} \\ &= \lambda p^{-1}N^{-\chi} \end{aligned}$$

$$y(x) = N^{-\rho\chi}\lambda^{\rho}p^{-\rho}.$$

Then from the output constraint, λ satisfies

$$\begin{aligned} 1 &= N^{-\rho\chi}\lambda^{\rho}\left[N^{1-\chi}\int p(x)^{(1-\rho)}f(x)dx\right]^{\rho/(\rho-1)} \\ \lambda &= N^{\chi}\left[N^{1-\chi}\int p(x)^{(1-\rho)}f(x)dx\right]^{1/(1-\rho)} \end{aligned}$$

$$\begin{aligned} p_F &= N^{1-\rho\chi}\lambda^{\rho}\int p(x)^{1-\rho}f(x)dx \\ &= N^{\chi-\rho\chi}\lambda^{\rho}\left[N^{1-\chi}\int p(x)^{1-\rho}f(x)dx\right] \\ &= N^{\chi}\left[N^{1-\chi}\int p(x)^{1-\rho}f(x)dx\right]^{1+\rho/(1-\rho)} = \lambda \end{aligned}$$

$$y(x) = N^{-\rho\chi}\left(\frac{p}{p_F}\right)^{-\rho}.$$

Various junk

Here is n in terms of primitives, for the general case,

$$\begin{aligned}\left[1+(\theta-1)\,\Omega\left(1-\frac{\xi_x}{\xi_h}\right)\right]n &= \upsilon+\left(\alpha_x\mu_x-\frac{\xi_x}{\xi_h}\alpha_h\mu_h\right)+\left(\frac{1-\zeta}{\alpha_x-1+\zeta}\lambda_x-\frac{\xi_x}{\xi_h}\frac{1-\varepsilon}{\alpha_h-1+\varepsilon}\lambda_h\right)\\ &\quad -\left(\delta_x-\frac{\xi_x}{\xi_h}\delta_h\right)-\left(1-\frac{\xi_x}{\xi_h}\right)\hat{r}.\end{aligned}$$

For $\alpha_x = \alpha_h$, this simplifies to (**??**). Use this solution in the second line of (42) to get g.

Example: $\theta = 1,\ \mu_x = \mu_h = 0,\ \alpha_x = \alpha_h$.—

Suppose $\theta = 1,\ \mu_x = \mu_h,\ \lambda_x = \lambda_h, \delta_x = \delta_h,\ \alpha_x = \alpha_h$. Then $\zeta = \varepsilon = 1/\rho$, and

$$\begin{aligned} g &= \mu_h+\frac{1}{\alpha_h}\left[\frac{1-\varepsilon}{\alpha_h-(1-\varepsilon)}\lambda_x-(\hat{r}+\delta_x)\right],\\ n &= \upsilon.\end{aligned}$$

The restrictions (**??**) and (**??**) require $g > \mu_h$, or

$$\frac{1-\varepsilon}{\alpha_h-(1-\varepsilon)}\lambda_x > \hat{r}+\delta_x.$$

or

$$\lambda_x > \left(\frac{\rho}{\rho-1}\alpha_h-1\right)(\hat{r}+\delta_x).$$

The gross entry rate is nonnegative if $\upsilon + \delta_x > 0$.

Figure 1a: comparative static (a)

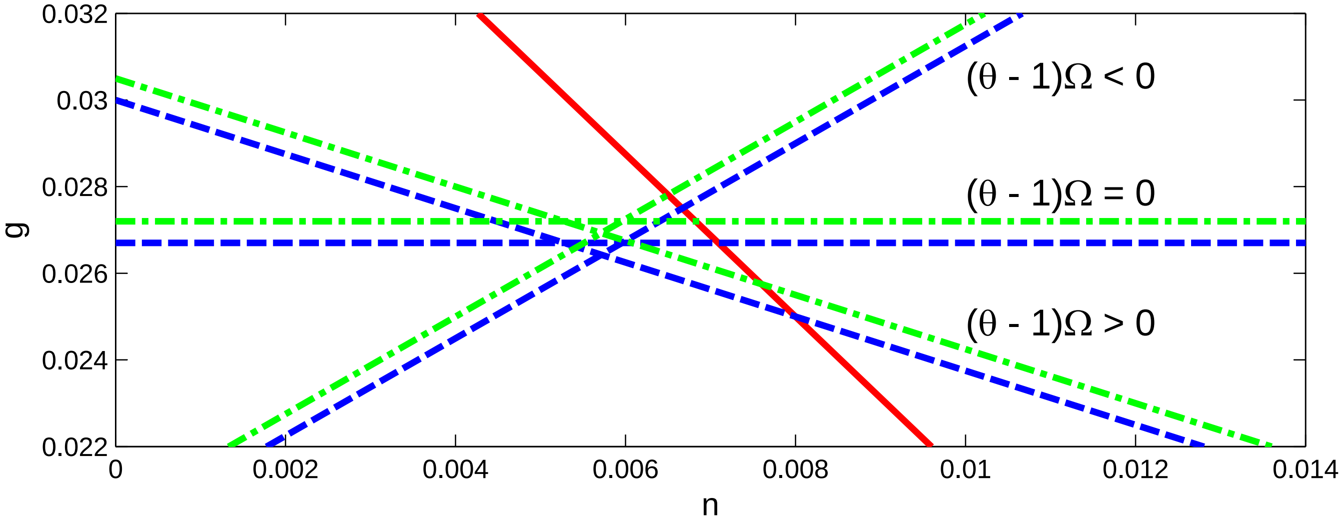

Figure 1b: comparative static (b)

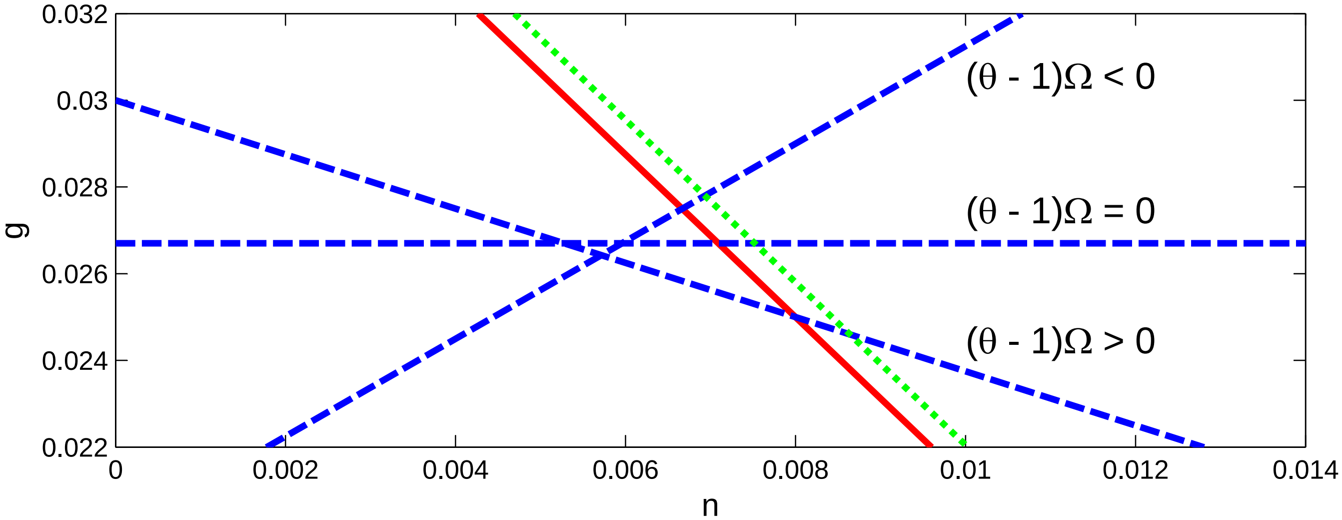

Figure 1c: comparative static (c)