Δ-Substitute Preferences and Equilibria with Indivisibilities

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June 2020

Abstract

Gross substitutes for quasi-linear preferences is characterized by the single improvement property, which says an agent can improve upon a sub-optimal bundle by adding or dropping a single item, or exchanging one item for another. We extend this notion in two ways: by allowing for non-quasi-linear preferences and the exchange of bundles up to Δ items. Our results connect the improvement property with the geometry of the choice correspondence. We derive prices at which the excess demand for each good is at most $\Delta - 1$ and provide applications to the design of pseudo-markets for allocating indivisible resources.

1 Introduction

Competitive equilibria (CE), when they exist are Pareto optimal and in the core. Under certain conditions, they satisfy fairness properties like equal treatment of equals and envy-freeness (see Hylland and Zeckhauser (1979), for example). With indivisible goods, however, CE need not exist. A well-known case that guarantees its existence is under quasi-linear, gross substitute preferences. Gul and Stacchetti (1999) show that this class of preference is

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characterized by single improvement property: adding, subtracting a good, or exchanging one good for another to improve utility. Informally, the marginal rate of technical substitution between two products evaluated at any bundle is either zero or one. Even though limited one-for-one substitution can improve their utility, consumers might be better off exchanging *bundles* of goods. Examples include couples in resident matching who would like to have jobs in the same city; students in course allocation wanting courses that fit their schedule; and wireless service providers seeking spectrum licenses that do not interfere with each other.

Unfortunately, such preference complementarities rule out the existence of CE even in its simplest form (agents consuming two goods). Moreover, when preferences are non-quasi linear, income effects introduce additional difficulties. This leads market designers to eschew CE allocations of indivisible goods and focus on methods with less attractive efficiency and equity properties such as serial dictatorship (Pápai, 2001; Hatfield, 2009) or lotteries (Hylland and Zeckhauser, 1979; Bogomolnaia and Moulin, 2001).[1]

The contribution of this paper is a set of new results for the existence of what Arrow and Hahn (1971) (page 177) called "social-approximate" equilibria: a price vector and associated demands that 'approximately' clear the market in that the excess demand for goods is small in an appropriate sense. What is significant is that we do not require preferences to be quasi-linear. Furthermore, the mismatch between supply and demand is bounded by a parameter Δ that depends on the agent's preferences and *not* the size of the market. Δ is the maximum size of bundles in a utility-improving exchange. It can be interpreted as a measure of the degree of preference complementarity, hence, the name Δ-substitution.

Our framework applies to settings, where an agent's preferences are satiated outside of a finite region. Examples are Shapley and Shubik (1971), Dierker (1971), Quinzii (1984), where agents consume a single good; or and Budish (2011) in which there is an upper bound on the size of bundles. We also propose a generalization of gross substitutes for quasi-linear preferences that we call bundled gross substitutes. They resemble the preferences used in

[1] An exception is Budish (2011). However, this work assumes the sizes of consumable bundles are small.

Fox and Bajari (2013) to estimate bidder valuations in spectrum auctions. Spectrum licenses within the same geographical cluster complement each other but are gross substitutes across different clusters.

The first idea that helps us to overcome the difficulties of non-quasi linear preferences is that the existence of CE depends only on the one-dimensional faces (edges) of the convex hull of the demand correspondence. The edges of the convex hull of the demand correspondence capture the "local" movement between two demanding bundles, corresponding to a utility-improving exchange when prices are perturbed.

When the edges have at most two non-zero coordinates and these of opposite sign and absolute value 1 (or more generally belong to a unimodular vector set), then, the unimodular theorem in (Baldwin and Klemperer, 2019) implies the existence of CE. This result implies well-known existence results for gross substitutes preferences as well as M-natural concave valuations (Murota and Tamura (2003)). Existence, however, fails even when an edge has two nonzero entries of the same sign (i.e, 1,1). Our next innovation is an extension of the unimodular theorem to when the convex hull of the demand correspondence has edges that have ℓ_1 norm at most Δ. While a CE need not exist, we can bound the violation in market-clearing by at most $\Delta - 1$ good-by-good.

The violation of market-clearing is unavoidable but it opens up new possibilities for market-design using pseudo markets. The magnitude of the excess demand, $\Delta - 1$, is the 'shadow' cost of dividing the indivisible. If a 'planner' knows the excess demand a priori, they can withhold that amount to 'add back in' to ensure that each agent's demand is satisfied which amounts to 'burning' some of the supply to ensure feasibility.

We summarize the relationship between this paper and prior work next. Section 2 introduces the Δ-substitute preferences property and some of its properties. The subsequent section states the main results. It is followed by a section on applications of pseudo-markets. Section 5 consider a special case, called generalized single improvement, that guarantees the existence of CE. The appendix contains the proofs.

1.1 Prior Work

Prior work deals with the non-existence of CE in two distinct ways. First, by restricting agent's preferences, for example, quasi-linearity and gross substitutes (see Kelso Jr and Crawford (1982)). Subsequent work relaxed the quasi-linearity assumption, such as Echenique (2012) or focused on agent's Hicksian demands as in Baldwin et al. (2020) who introduce a condition called net substitutes that guarantees the existence of a CE in the presence of income effects (see Section 5.3 for a description). Depressingly, minor deviations from substitute preferences are incompatible with the existence of CE.

Second, determine prices that 'approximately' clear the market. These social-approximate equilibria are preference independent but rely on the economy growing to infinity to ensure that the excess demand is negligible. See, for example, Starr (1969), Dierker (1971), Mas-Colell (1977) and Azevedo et al. (2013).[2]

Our approach restricts preferences and links the restriction to the magnitude of excess demand in a social-approximate equilibrium. The restriction is called Δ-substitute preferences, characterized by a parameter, Δ, that can be interpreted as the degree of complementarity exhibited by the preferences. Gross substitutes preferences are a strict subset of Δ substitute preferences with $\Delta = 2$. The bound on excess demand (good by good) is at most $\Delta - 1 = 1$, is therefore tight in this case.

Our results apply to various resource allocation problems both with and without monetary transfers. For example, if agents' von Neuman-Morgenstern preferences satisfy Δ-substitutes, our methods allow one to implement a CE in probability shares as a lottery over allocations in which the excess demand good-by-good is at most $\Delta - 1$, while maintaining *any* linear ex-ante constraints including budget constraints. This generalizes Gul et al. (2019) which assumes preferences satisfy gross substitutes.

Even when transfers and lotteries are not allowed, our techniques can be applied to

[2] There is also a focus on finding an acceptable approximation of a CE outcome in terms of cardinal measures of welfare that scale slowly with the size of the economy. See for example Akbarpour and Nikzad (2020), Cole and Rastogi (2007) and Feldman and Lucier (2014).

construct pseudo-markets for resource allocation, which assign agents a budget of artificial money and implement a competitive equilibrium. One application is course allocation. The goods are courses with the number of seats in a course being the supply of that good. Agents are students who have preferences over *subsets* of courses. An advantage of our approach over a similar pseudo-market mechanism due to Budish (2011) is the bound on excess demand for any course is at most $\Delta - 1$, where Δ is an upper limit on the number of courses a student may take in the relevant period. In contrast, the bound in Budish (2011) is based on the ℓ_2 norm and as a result, the excess demand for seats in a popular course can depend on the number of courses offered. (See Section 4.1.)

2 Δ-substitute Preferences

We first define Δ-substitute preferences when each agent is interested in consuming at most one copy of each good (unit demand). Then, we extend the definition to the case of multi-unit demand. The reason for doing it this way is discussed in Section 2.2.

As discussed in the introduction, the core idea is to analyze preferences using the edges of the convex hull of the demand correspondence. Some terminology from convex geometry will be helpful. A **polyhedron** is defined by the intersection of a finite number of half-spaces. A bounded polyhedron is called a **polytope**, it is also the convex hull of a finite set of points in $\mathbb{R}^m$. A subset F of a polytope $Q \subseteq \mathbb{R}^m$ is called a **face** of Q if there is a hyperplane $\{x : h \cdot x = \gamma\}$ such that $F = Q \cap \{x : h \cdot x = \gamma\}$ and $Q \subseteq \{x : h \cdot x \leq \gamma\}$. A face of a polytope is itself a polytope. A zero dimensional face is called an **extreme point** and a face of dimension 1 is called an **edge** of the polytope. The polytope Q is a (trivial) face of itself, where $h = \vec{0}, \gamma = 0$. The **dimension** of a face F is denoted $dim(F)$. For any $y \in Q$, there is a unique face F of Q with lowest dimension that contains y. We call F the **minimal face** containing y.

2.1 Unit Demand

Let M denote a set of m indivisible goods. A bundle of goods is denoted by a binary vector $x \in \{0,1\}^m$whose i^{th} component, denoted x_i, indicates whether good $i \in M$ is in the bundle.

An agent's utility for a bundle x and transfer $t \in \mathbb{R}$ is denoted $U(x,t)$. Associated with each agent is a finite set of bundles $X \subset \{0,1\}^m$that they can feasibly consume, their feasible bundles. The bundle $\vec{0}$ is always assumed to be feasible. For each feasible bundle $x \in X$, $U(x,t)$ is assumed continuous and non-increasing in t with $U(\vec{0},0) = 0$. If x is infeasible for the agent, then, $U(x,t) = -\infty$ for all t.

Quasi-linear preferences, where $U(x,t) = v(x) - t$ for some valuation function $v(\cdot)$ satisfy these conditions. Budget constraints can be incorporated by allowing $U(x,t)$ to approach $-\infty$ as t approaches the budget. For example, $U(x,t) = v(x) + \log(b-t)$ where b is the budget.

Let $p \in \mathbb{R}^m_+$ be a price vector where p_i is the unit price of good $i \in M$. The utility of an agent for bundle x at price p will be $U(x, p \cdot x)$. In fact, all our results hold even in the more general case where the utility is a function of the bundle and the *vector* of prices, i.e., $U(x,p)$. Given a price vector p, an agent's **choice correspondence**, denoted $Ch(p)$, is defined as follows:

$$Ch(p) = \arg\max\{U(x, p \cdot x) : x \in X\}.$$

Denote the convex hull of $Ch(p)$ by $conv(Ch(p))$. Notice, not every interior point of $conv(Ch(p))$ is contained in $Ch(p)$.

Let $(x-y)^+$ denote the vector whose i^{th} component is $\max\{x_i - y_i, 0\}$. The ℓ_1 norm of vectors will play an important role, to see why, consider two bundles x and y. Then,

$$||x-y||_1 = \vec{1} \cdot (x-y)^+ + \vec{1} \cdot (y-x)^+.$$

The term on the right hand side can be interpreted as the total number of items that must be swapped to get from bundle x to bundle y.

Definition 1 *When agents consume at most one unit of each good, a utility function $U(x, p \cdot x)$ satisfies the* <u>*Δ-substitutes*</u> *property if for any price vector p, the ℓ_1 norm of each edge of $conv(Ch(p))$ is at most Δ.*

We give two examples of Δ-substitutes preferences.

Example 2.1 *If an agent is satiated outside of a finite region, then, agent's preferences have the Δ-substitutes property for some Δ. Examples are Shapley and Shubik (1971), Dierker (1971)), Quinzii (1984), and Budish (2011).*

Example 2.2 *We introduce a generalization of gross substitutes for quasi-linear preferences that we call* **bundled gross substitutes**. *Each agent is interested in at most one copy of each good. Associated with each agent is a partition $P_1, P_2 \ldots, P_k$ of M such that $|P_r| \leq \Omega$ for all $r = 1, \ldots, k$. The partitions can vary across agents. If x is a bundle, let $x|_{P_r}$ denote the sub-bundle consisting only of components in P_r. Assume x is not a utility maximizing bundle under price p, and that there is a better bundle to be had by either adding a new sub-bundle, subtracting a sub-bundle or replacing $x|_{P_r} \neq 0$ with a different bundle in $P_{r'}$, where $r, r' \in \{1, .., k\}$ and are not necessarily different.*

If $\Omega = 1$ this corresponds exactly to the single improvement property of gross substitutes (adding a new item, subtracting a new item or replacing an item with another). When $\Omega > 1$, this improvement property includes changing the bundle inside each P_r arbitrarily. Thus, within P_r, there can be complementarities among the goods. Fox and Bajari (2013) suggests that spectrum preferences resemble bundled substitutes. Spectrum licenses within the same geographical cluster complement each other, but are substitutes across different clusters. Proposition 2.2 of Section 2.3 shows that bundled gross substitutes satisfies 2Ω-substitutes using a characterization of Δ-substitutes presented in Subsection 2.3.

2.2 Multi-unit Demand

Now suppose a bundle x can be any vector in Z_+^m. Thus, x_i represents the number of copies of good $i \in M$. It would seem natural to define Δ-substitute preferences in the same way, i.e., the the ℓ_1 norm of each demand type is at most Δ. However, unlike the unit demand case, $conv(Ch(p)$ can contain bundles that are not in $Ch(p)$. For this reason we define Δ-substitute preferences for the multi-unit case by reduction to the single unit case.

When a bundle contains multiple copies of the same good, one can think of each copy of the good as being a separate good. For example, if the bundle contained 3 oranges, we represent that as 3 distinct objects called orange copy #1, orange copy #2 and orange copy #3. Therefore, any vector $x \in Z_+^m$ can be represented as a 0-1 vector.

Let $C \in \mathbb{Z}_+$ be a constant at least as large as the maximum number of copies of a good that an agent consumes. We make C copies of each good. Let $y \in \{0,1\}^{C \cdot m}$ be a binary representation of a bundle. The total number of copies of good $i \in M$ contained in y is

$$T_i(y) = \sum_{k=C\cdot(i-1)+1}^{C\cdot i} y_k. \tag{1}$$

Thus, y is a binary representation of the bundle $(T_1(y), .., T_m(y))$.

A bundle $x \in Z_+^m$ can have multiple binary representations. Hence, to each bundle $x \in Z_+^m$ we associate a set $B(x)$ in $\{0,1\}^{C \cdot m}$ of all possible binary representations of x. Formally,

$$B(x) := \{y \in \{0,1\}^{C.m} | T_i(y) = x_i \ \forall i \in M\}. \tag{2}$$

The following is the definition of Δ-substitutes for the multi unit demand case.

Definition 2 *We say that multi-unit preferences satisfy the Δ-substitutes property if the ℓ_1 norm of each edge of the convex hull of $\cup_{x \in Ch(p)} B(x)$ is at most Δ.*

Notice the restriction is on $\cup_{x \in Ch(p)} B(x)$ rather than on $conv(Ch(p))$ itself. The advantage of defining preferences with respect to $\cup_{x \in Ch(p)} B(x)$ rather than $conv(Ch(p))$ is that one does not need to specify whether an integer vector in the interior of $conv(Ch(p))$ lies in $Ch(p)$.

From now on we will not restrict ourselves to unit demand.

2.3 Improvement Property

In this section, we relate Δ-substitute preferences to how demand changes for small price changes. By way of motivation, we recall the definition of gross substitutes.

DEFINITION 3 *A utility function satisfies gross substitutes if for every pair of price vectors p and q such that $p \leq q$ and for all $x \in Ch(p)$, there exists $y \in Ch(q)$ such that $y_i \geq x_i$ for all $i \in M$ such that $q_i = p_i$.*

Gross substitutes for quasi-linear preferences are characterized by a property called single improvement. There are two versions of it. The first was introduced in Gul and Stacchetti (1999), call it *weak* single improvement. Under quasi-linearity, weak single improvement characterizes gross substitute preferences only when agents wish to consume at most one copy of any good. The second, introduced in Murota and Tamura (2003), is the version we describe. It characterizes gross substitutes under quasi-linearity and multi-unit demand.

DEFINITION 4 *The quasi-linear utility function $v(x) - p \cdot x$ satisfies the single improvement property if for any price vector p and any two bundles x and y such that $v(x) - p \cdot x < v(y) - p \cdot y$, there exist non-negative vectors $a \leq (x-y)^+$ and $b \leq (y-x)^+$ such that $\max\{\vec{1} \cdot a, \vec{1} \cdot b\} \leq 1$ and*

$$v(x - a + b) + p \cdot a - p \cdot b > v(x).$$

To motivate the Δ-improvement property, observe that under single improvement, the improving bundle is near by, as measured by the ℓ_1 norm from the initial bundle. In particular,

$$||(x - a + b) - x||_1 \leq \vec{1} \cdot a + \vec{1} \cdot b \leq 2.$$

Relaxing this so that the ℓ_1 distance is at most Δ, leads to the following definition of Δ-improvement.

Definition 5 *The quasi-linear utility function $v(x) - p \cdot x$ satisfies the* *Δ-improvement* *property if for any price vector p and any two bundles x and y such that $v(x)-p\cdot x < v(y)-p\cdot y$ there exist integer vectors a and b such that $0 \leq a \leq (x-y)^+$ and $0 \leq b \leq (y-x)^+$ such that $\vec{1} \cdot a + \vec{1} \cdot b \leq \Delta$ and*

$$v(x - a + b) + p \cdot a - p \cdot b > v(x).$$

The Δ-improvement property for $\Delta = 2$ *contains* the class of substitutes preferences.

Definition 5 requires the improvement property to hold for *any* pair of bundles where one utility dominates the other. This is stronger than what is needed to prove the existence of a CE. We will show that it suffices for the improvement property to hold for pairs of bundles that reside in the same choice correspondence. This motivates the following extension of the Δ-improvement property to non-quasi-linear preferences. In particular, to satisfy the Δ-improvement property, when an agent is indifferent between two bundles x and y at price vector p, for every price *change* such that the cost of x increases more than the cost of y, there is a bundle $x - a + b \in Ch(p)$ that is in between them and 'close' to x in the sense that $||(x - a + b) - x||_1 \leq \Delta$. Formally, we have the following definition.[3]

Definition 6 *A utility function $U(x, p\cdot x)$ satisfies the* *generalized Δ-improvement* *property if for any price vector p, any two bundles $x, y \in Ch(p)$ and any price change $\delta p \in \mathbb{R}^m$ satisfying $\delta p \cdot x > \delta p \cdot y$, there exist $a \leq (x-y)^+$ and $b \leq (y-x)^+$ such that*

1. *$\vec{1} \cdot a + \vec{1} \cdot b \leq \Delta$,*

2. *$\delta p \cdot a > \delta p \cdot b$, and*

[3] Fujishige and Yang (2003) also propose a generalization of single improvement, but it is limited to quasi-linear preferences and is designed to cover the different versions of gross substitutes.

3. $x - a + b \in Ch(p)$.

Proposition 2.1 *Definition 5 is a special case of Definition 6.*

Proof. Suppose in definition 6 that U is quasi-linear, i.e. $U(x, p \cdot x) = v(x) - p \cdot x$. It suffices to show that $v(y) - p' \cdot y > v(x) - p' \cdot x$. We know that $v(y) - p \cdot y = v(x) - p \cdot x$. Given $(p' - p) \cdot y < (p' - p) \cdot x$ it follows that $v(y) - v(x) = p \cdot (y - x) > p' \cdot (y - x)$. ■

The next result identifies the connection between the improvement property and Δ-substitutes.

Theorem 1 *A preference satisfies the Δ-substitutes property, if and only if it satisfies the generalized Δ-improvement property.*

Proof. See Section A.1. ■

The following illustrate the usefulness of Theorem 1.

Proposition 2.2 *The preferences described in Example 2.2 is 2Ω-substitutes.*

Proof. Example 2.2 describes the improvement property of Definition 5. The ℓ_1 distance of this improvement is at most 2Ω. Thus, by Proposition 2.1 and Theorem 1, this preference is 2Ω-substitutes. ■

3 Main Results

Let N denote the set of agents. Each agent $j \in N$ is equipped with utility function $U_j(x, t)$ and feasible set X_j of bundles it can consume. Suppose $n = |N|$ and let $s_i \in \mathbb{Z}_+$ denote the supply of good $i \in M$ and $s \in \mathbb{Z}_+^m$ the supply vector. An economy is the collection $\{\{U_j\}_{j \in N}, s\}$. Given price vector p, agent j's choice correspondence is denoted by $Ch_j(p)$.

DEFINITION 7 *A competitive equilibrium for the economy $\{\{U_j\}_{j \in N}, s\}$ is a non-negative price vector p and demands $x^j \in Ch_j(p) \subset \mathbb{Z}_+^m$ for all $j \in N$ such that $\sum_{j \in N} x^j \leq s$ with equality for each $i \in M$ for which $p_i > 0$.*

While a CE might not exist, we show that one can perturb capacity of resource to guarantee the existence of CE. Our main result is the following.

THEOREM 2 *If all agent's preferences possess the Δ-substitutes property, for every supply vector s there exists s' such that $|s_i - s'_i| \leq \Delta - 1$ for all $i \in M$ and the economy $\{\{U_j\}_{j \in N}, s'\}$ has a competitive equilibrium.*

The idea behind Theorem 2 is to consider a pseudo-equilibrium (see Milgrom and Strulovici (2009)). A pseudo-equilibrium is a relaxation of competitive equilibrium where agent's preferences are 'convexified' by replacing each agent $j \in N$'s choice correspondence with its convex hull, $conv(Ch_j(p))$. Each $x \in conv(Ch_j(p))$, because it is a convex combination of the bundles in $Ch_j(p)$, is interpreted as a lottery over those bundles and x itself is the 'expected' bundle. A pseudo equilibrium, therefore, allocates to each agent probability shares in bundles. The key idea is that when preferences possess the Δ-substitutes property, this lottery can be implemented such that the ex-post violation of any resources constraint is bounded by $\Delta - 1$.

DEFINITION 8 *A price vector $p \geq 0$ and $x^j \in conv(Ch_j(p))$ for all $j \in N$ is called a pseudo-equilibrium if the supply vector $s \geq \sum_{j \in N} x^j$ with equality for every good $i \in M$ with $p_i > 0$.*

The next theorem shows the existence of a pseudo-equilibrium.

THEOREM 3 *Let $\vec{0} \in X_j$ denote the finite set of bundles that agent $j \in N$ can feasibly consume. Each agent j's utility function $U_j(x, p \cdot x)$ satisfies*

- $U_j(0, 0) = 0$,

- $U_j(x, p \cdot x) = -\infty$ *for* $x \notin X_j$

- $U_j(x, p \cdot x)$ *is continuous in* $p \in \mathbb{R}^m_+$ *for each* $x \in X_j$ *and*

- *there exists* $B > 0$ *such that if* $p_i \geq B$*, and* $x_i > 0$*, then,* $U_j(x, p \cdot x) < 0$.

Then, there exists a pseudo-equilibrium.

Proof. See Appendix A.2.

The following gives an approximate implementation of a pseudo-equilibrium, which implies Theorem 2 as a corollary.

THEOREM 4 *Let* $(p, \{x^j\}_{j=1}^n)$ *be a pseudo-equilibrium of an economy* $\{\{U_j\}_{j \in N}, s\}$ *in which each* U_j *satisfies the* Δ*-substitutes property. Then, the allocation vector* $(x^1, \ldots, x^n)$ *can be implemented as a lottery over a set of allocations* $\mathcal{A}$*, such that for every* $a = (a^1, \ldots, a^n) \in \mathcal{A}$ *there exists a perturbed supply vector* s' *satisfying* $\max_{i \in M} |s_i - s'_i| \leq \Delta - 1$*, and* $(p, \{a^j\}_{j=1}^n)$ *is a competitive equilibrium of the economy* $\{\{U_j\}_{j \in N}, s'\}$.

In the next section, we will provide the proof of this result.

3.1 Efficiency

It is well known that when preferences violate non-satiation that competitive equilibrium allocations need not be Pareto optimal. This is because some agents do not purchase their least expensive optimal bundle. To ensure efficiency, attention has focused on equilibria with slack (eg. McLennan (2017)) or 'paper money' (eg. Kajii (1996)). The first allows for reallocating unspent wealth (slack). The second interprets the slacks (or the dividends) as paper money, which is allocated to the consumers before the market takes place. If the conditions for the existence of these efficient equilibria hold, they can be used in Theorem 4. The resulting lottery will be over allocations that are efficient equilibria of $\{\{U_j\}_{j \in N}, s'\}$.

3.2 Technical Ideas

Call a polytope P **binary** if it is the convex hull of 0-1 vectors. Denote the set of P's extreme points by $ext(P)$. Recall that the edges of P are its 1 dimensional faces (formal definitions may be found in Section 2) and are vectors of the form $v - u$ for some (not all) pairs $v, u \in ext(P)$. The edges of a binary polytope P have components in $\{0, \pm 1\}$. Call a binary polytope P **Δ-uniform** if the ℓ_1 norm of each of its edges is at most Δ. The key technical result is the following.

THEOREM 5 *Let $P_1, \ldots, P_n$ be a collection of binary Δ-uniform polytopes in $\mathbb{R}^m$. Let $y = (y^1, \ldots, y^n) \in P_1 \times \ldots \times P_n$ such that $\sum_{j=1}^n y^j$ is integral. Then, there exist vectors $z^j \in ext(P_j)$ for all j such that $||\sum_{j=1}^n z^j - y||_\infty \leq \Delta - 1$.*

Proof. See Appendix A.3

We now describe how Theorem 4 follows from Theorem 5.

Proof of Theorem 4 First, we identify a pseudo-equilibrium price p and corresponding allocation $\{x^1, \ldots, x^n\}$ such that $x^j \in conv(Ch_j(p))$ for all $j \in N$. It is technically convenient to assume that $\sum_{j \in N} x^j = s$. This is without loss because we can add dummy agents who demand all goods. Let P_j denote the binary polytope $\cup_{x \in Ch_j(p)} B(x)$, where $B(x)$ is a binary presentation of x, as defined in (2).

For each $x^j \in conv(Ch_j(p))$ there is at least one corresponding $y^j \in P_j$. Furthermore, $\sum_{j \in N} x^j = s$ implies that $\sum_{j=1}^n T_i(y^j) = s_i$ for all $i \in M$. Hence, all the conditions needed to invoke Theorem 5 are satisfied. This implies Theorem 4. ∎

Theorem 5 is related to the Shapley-Folkman-Starr lemma which states that the non-convexities in an aggregate of non-convex sets diminishes with the number of sets making up the aggregate. If these sets are the choice correspondences of agents, then, relative to the size of the aggregate economy, their individual non-convexities become negligible. We state a version of the lemma due to Cassels (1975) that is restricted to binary polytopes.

Theorem 6 (Shapley-Folkman-Starr) *Let $P_1, \ldots, P_n$ be a collection of binary polytopes in $\mathbb{R}^m$ with $n > m$. Let $y = (y^1, \ldots, y^n) \in P_1 \times \ldots \times P_n$ such that $\sum_{j=1}^n y^j$ is integral. Then, there exist vectors $z^j \in ext(P_j)$ for all j such that $|| \sum_{j=1}^n z^j - y||_\infty \leq m$.*

Starr (1969) and Broome (1972) use Theorem 6 to deliver an approximate competitive equilibrium result of the following kind: there is a price vector p and a feasible allocation such that at least $n - m$ agents receive a utility maximizing bundle. If one were to permit the upto m agents who are rationed, to consume a bundle from their choice correspondence, the excess demand for some goods could be quite large. In contrast, the bound in Theorem 5 is independent of m.

Another variant of the Shapley-Folkman-Starr lemma involves the ℓ_2 norm (used in Budish (2011)). Here is a formulation due to Budish and Reny (2020).

Theorem 7 *Let $P_1, \ldots, P_n$ be a collection of binary polytopes in $\mathbb{R}^m$ with $n > m$. Suppose the diameter of each polytope is at most D. Let $y = (y^1, \ldots, y^n) \in P_1 \times \ldots \times P_n$ such that $y = \sum_{j=1}^n y^j$ is integral. Then, there exist vectors $z^j \in ext(P_j)$ for all j and $|| \sum_{j=1}^n z^j - y||_2 \leq \frac{D\sqrt{m}}{2}$.*

The bounds in Theorem 6 and 7 are incomparable with the one in Theorem 5 because they involve the ℓ_2 norm. When applied to obtain social approximate equilibria, it means that the excess demand for a small set of goods may be large. Our result, on the other hand, guarantees that the excess demand for every good is small.

The bound in Theorem 7 depends on the diameter of the polytopes. The maximum diameter binary polytope is a hypercube. In $\mathbb{R}^m$ it has a diameter $\sqrt{m}$, so the bound in Theorem 7 can be as large as $\frac{m}{2}$. The edge lengths of a hypercube are 1, so the bound on each component delivered by Theorem 5 is zero! This example illustrates that even though the diameter of a polytope is large, its edge lengths can be small. Our results, therefore, provide a more refined analysis by connecting the edge length of the convex hull of choice correspondence with a natural property of the preferences.

4 Application: Pseudo-Markets

Pseudo-market mechanisms use artificial currency rather than real currency either because of fairness concerns or a desire to limit certain kinds of trades. In course allocation, for example, all students would be considered to have an equal claim on available classes and in this case, the goal of the pseudo-market mechanism would be to allocate the available slots fairly (see Budish (2011)). In the context of food banks (Prendergast (2017)), the goal is to reallocate food so as to reduce waste. Artificial currency ensures that donated food remains within the food bank network rather than sold to the 'outside'.

Hylland and Zeckhauser (1979) was the first to propose the use of competitive equilibrium allocations of a market with *equal* artificial currency endowments and goods that are the probabilities of being assigned to each object. Trade in probability shares of each good rather than the good itself solved the indivisibility problem but raised two problems in its stead. First, trade in probability shares required that agents have preferences over lotteries rather than goods, i.e., von Neuman-Morgenstern utilities. Second, how does one implement the equilibrium allocation of probability shares as a lottery over deterministic allocations?[4] Except for the unit demand case, considered by Hylland and Zeckhauser (1979), this is generally impossible.[5]

To maintain ordinal preferences, Budish (2011) proposes randomizing over endowments of the artificial currency instead of allocations. In expectation, each agent receives the same endowment of artificial currency but ex-post, they receive different but roughly equal amounts. Using the realized endowments of artificial currency a social approximate CE is computed. We discuss this in Section 4.1 below and how our main result can be used to improve upon the mechanism in Budish (2011).

In section 4.2 we show how our main result can be used to implement a CE in probability shares as a lottery over approximately feasible allocations. We contrast this with

[4]See Pratt and Zeckhauser (1990) for a discussion of an actual application.

[5]Budish et al. (2013) and Echenique et al. (2019) examine extensions but these are limited.

4.1 Ordinal Preferences

When preferences are ordinal, Budish (2011) proposed randomizing over endowments of the artificial currency instead of allocations. In expectation, each agent receives the same endowment of artificial currency but ex-post, they receive different but roughly equal amounts. Using the realized endowments of artificial currency a CE is computed which results in excess demand for some goods. Budish (2011) bounds the excess demand in terms of the *Euclidean* distance between the supply vector and the vector of the number of goods allocated by $\frac{\sqrt{\min\{2\Delta, m\}m}}{2}$, where Δ is the size of a maximum bundle that an agent is interested in consuming and m is the number of goods.

The mechanism has been implemented to assign students to courses at the Wharton School (see Budish et al. (2017)). Agents are students and objects are courses with the number of seats in a course being the supply of that course. The supply is upper bounded by Fire Safety regulations. Every semester students take about 5 courses, thus $\Delta = 5$. Because the bound on excess demand is in terms of Euclidean distance, there is no guarantee that the number of students assigned to a course will exceed the regulated limit. For this reason, the mechanism needs to be rerun a number of times with reduced capacities to ensure feasibility. Nguyen et al. (2016) propose an alternative based on the probabilistic serial mechanism (see Bogomolnaia and Moulin (2001)). It enjoys different efficiency and fairness properties than the mechanism in Budish (2011). However, it results in an allocation of students to courses in which the excess demand for each course is at most $\Delta - 1$. It has been implemented at the Technical University of Munich (see Bichler et al. (2018)). The advantage of this mechanism is that one knows a-priori how many seats in each class to 'withhold' to ensure feasibility. Here we show that one can get the same fairness and efficiency guarantees as in Budish (2011) but with a course-by-course bound of $\Delta - 1$ on excess demand. Furthermore, this result holds beyond the case where agents can only consume small bundles. In Budish (2011), each student j has a feasible set of bundles (course schedules) X_j and a budget of 1.[6]

[6] The approach trivially extends to the case where agent's have differing budgets.

The size of a maximum feasible bundle is Δ. Let $\succeq_j$ denote the ordinal preference of agent j and let $v_j(x)$ be the utility of consuming x. We assume $0 \in X_j$ and $v_j(0) = 0$. Without loss we can assume that each $v_j(x)$ is a rank score of the bundles, $r_j(x)$, consistent with $\succeq_j$ and $r_j(0) = 0$.

We first define the following auxiliary utility function for each student. Let

$$U_j^\epsilon(x,t) := v_j(x) + \min\{0, \log \frac{1-t}{\epsilon}\} \text{ for an arbitrarily small, but positive } \epsilon. \tag{3}$$

Let $Ch_j(p)$ denote the choice correspondence of this auxiliary utility function. We first show that for any bundle in the choice correspondence of the auxiliary utility function, there exists a way to perturb the budget so that under the original ordinal preference, the bundle continues to be the optimal choice. Formally we have the following.

Claim 4.1 *Let $x \in Ch_j(p)$, then there exists a new budget b such that $1 - \epsilon \leq b < 1$ and*

$$x = max_{(\succeq_j)}\{x' \in X_j \text{ and } p \cdot x' \leq b\}.$$

Proof. Let $t := p \cdot x$, because of the form of the utility, we know that $t < 1$.

Case 1: If $1 - \epsilon < t < 1$, define the new budget to be $b := t$. Since $x \in Ch_j(p)$, for any feasible bundle x' that $v_j(x') > v_j(x)$ we have $p \cdot x' > p \cdot x = b$. Thus, under the new budget, the agent cannot afford x'.

Case 2: If $t \leq 1 - \epsilon$, then $U_j^\epsilon(x,t) = v_j(x)$. Define the new budget to be $b := 1 - \epsilon$. Because $x \in Ch_j(p)$, for any feasible bundle x' that $v_j(x') > v_j(x)$, $p \cdot x' > 1 - \epsilon$ otherwise x' will give a strictly better payoff than x. ■

To formally state our result, we modify the notion of approximate competitive equilibrium with equal budget to account for the ℓ_∞-norm instead of the ℓ_2 norm as in Budish (2011).

Definition 9 *Fix an economy, the allocation $(x^1, .., x^n)$, budget $(b_1, .., b_n)$ and item prices $(p_1, .., p_m)$ constitute an (Δ, ϵ)-approximate competitive equilibrium with equal budget if the* *following hold:*

- $x^j = max_{(\succeq_j)}\{x \in X_j \text{ and } p \cdot x \leq b_j\}$ *for all* $j \in N$
- $1 - \epsilon \leq b_j \leq 1$ *for all* $j \in N$
- $\max_i |z_i| \leq \Delta$, *where* $z = (z_1, .., z_m)$ *and*

  a) $z_i = \sum_j x_i^j - s_i$ *if* $p_i > 0$

  b) $z_i = \max\{0, \sum_j x_i^j - s_i\}$ *if* $p_i = 0$

Formally, we have the following result, which is a direct application of Theorem 2.

Proposition 4.1 *For every* $\epsilon > 0$, *if* U_j^ϵ *defined as in* (3) *is* Δ*-substitutes, there exists a* $(\Delta - 1, \epsilon)$*-approximate competitive equilibrium with equal budget.*

Now, each U_j^ϵ satisfies 2Δ-substitutes because no agent is interested in consuming a bundle whose size exceeds Δ. In words, we have an approximate CE with approximately equal incomes in which the excess demand for each good with a positive price is at most $2\Delta - 1$. In the case no agent is interested in consuming a bundle whose size exceeds Δ, we can actually reduce the bound of $2\Delta - 1$ to $\Delta - 1$ using a more specialized argument (see Nguyen et al. (2016)).

An advantage of our approach over Budish (2011) is that we guarantee that no capacity constraint will be violated by more than $\Delta - 1$. This means that $\Delta - 1$ seats can be withheld from each course. An approximate competitive equilibrium for the economy where the supply of each course is reduced by $\Delta - 1$ is computed. The seats withheld can be added back to ensure that the allocation is feasible. Thus, a single equilibrium computation suffices.

4.2 von Neuman-Morgenstern Preferences

Hylland and Zeckhauser (1979) were able to implement the CE outcome (with exogenous budgets) in probability shares as a lottery over feasible allocations precisely because of the unit demand assumption. Gul et al. (2019) extended this to the case where each agent demands multiple goods but no more than one copy of any good and satisfies gross substitute. We extend this in two ways. First, to any finite collection of ex-ante linear constraint on the shares an agent can consume not just a budget constraint. Second, to Δ-substitutes preferences in the sense that a CE in probability shares can be implemented as a lottery over approximately feasible allocations.

Let X_j denote the set of feasible bundles that agent $j \in N$ can consume. Let $v_j(x)$ be the value of agent j for bundle $x \in X_j$ and b_j her exogenously given budget. We assume $0 \in X_j$ and $v_j(0) = 0$.

Consider the quasi-linear choice correspondence associated with $v_j(x)$:

$$Ch_j(p) := \arg\max\{v_j(x) - p \cdot x \ , x \in X_j\}.$$

Let $\mathcal{L}(X_j)$ be the set of lotteries over bundles in X_j. For $y \in \mathcal{L}(X_j)$, let $v_j(y)$ be its expected utility. Denote by $\overline{y}$, the average consumption of the lottery y, thus $p \cdot \overline{y}$ is the expected price of the lottery y. Consider the choice correspondence that allows agents to consume lotteries subject to an (expected) budget constraint:

$$Ch^*_j(p) := \arg\max\{v_j(y) : y \in \mathcal{L}(X_j), p \cdot \overline{y} \leq b_j\}.$$

Note, $Ch^*_j(p)$ is convex and compact, furthermore, the value of $v_j(y)$ for $y \in Ch^*_j(p)$ is continuous in p. By standard arguments there exists a competitive equilibrium allocation of lotteries.

Claim 4.2 *There exists a price vector p and a vector of lotteries $(y^1, .., y^n)$ such that $y^j \in Ch^*_j(p)$ and $\overline{y}_i \leq s_i$ for all $i \in M$ and $\overline{y}_i = s_i$ if $p_i > 0$, where $\overline{y}$ denotes the average aggregate consumption.*

If the quasi-linear utility $v_j(x) - p \cdot x$ satisfies Δ-substitutes, we show how to implement

$(y^1, .., y^n)$ as a lottery over (approximately) feasible allocations. At the heart of the argument is the following claim due to Gul et al. (2019).

Proposition 4.2 *For every price vector p, there exist non-negative numbers $\{\beta_j\}_{j\in N}$ such that $Ch_j^*(p) = conv(Ch_j(\beta_j p))$ for all $j \in N$.*

If a quasi-linear preference satisfies Δ-substitutes, then, if the prices are scaled by a constant, Δ-substitutes is maintained. It is clear that Proposition 4.2 together with Lemma 1 and Theorem 4, imply the following:

Proposition 4.3 *If $v_j(x) - p \cdot x$ satisfies the Δ-improvement property, the equilibrium of Claim 4.2 can be implemented as a lottery over approximately-feasible allocations, where the violation for the approximately feasible allocations is at most $\Delta - 1$ good-by-good.*

We give a new proof of Proposition 4.2 that permits a generalization.

Proof of Proposition 4.2 $Ch_j^*(p)$ can be written as the set of optimal solutions to a linear program. For each $x \in X_j$ let w_x denote the fraction of bundle x selected. Then, the problem of selecting a utility maximizing lottery over X_j can be represented as follows:

$$\begin{aligned}
\max \quad & \sum_{x\in X_j} w_x \cdot v_j(x) \\
\text{s.t} \quad & \sum_{x\in X_j} w_x \leq 1 \\
& \sum_{x\in X_j} w_x \cdot p \cdot x \leq b_j \\
& w_x \geq 0 \ \forall x \in X_j.
\end{aligned}$$

Here we assume $v(0) = 0$, that's why we can have $\sum_x w_x \leq 1$ instead of $\sum_x w_x = 1$.

Let $\alpha_j, \beta_j \geq 0$ be the dual variables associated with this linear program. Dual feasibility and complementary slackness imply:

$$v_j(x) - \beta_j \cdot p \cdot x \leq \alpha_j \text{ for all } x \in X_j$$

$$\text{If } w_x > 0, \text{ then } v_j(x) - \beta_j \cdot p \cdot x = \alpha_j.$$

Hence, if $w_x > 0$, then, $x \in \arg\max\{v_j(x) - \beta_j \cdot p \cdot x\} = Ch_j(\beta_j p)$.

If $\alpha_j > 0$, then $\sum_x w_x = 1$ and thus any optimal solution of the linear program is a lottery over $Ch_j(\beta_j p)$.

If $\alpha_j = 0$ any solution of the linear program with $\sum_x w_x < 1$ can be extended to a lottery over $Ch_j(\beta_j p)$ by setting $w_0 = 1 - \sum_x w_x > 0$. Then, $0 \in Ch_j(\beta_j p)$. Furthermore, by complementary slackness, any lottery over $Ch_j(\beta_j p)$ is an optimal solution of the linear program. ■

Our proof of Proposition 4.2 extends to the case where the budget constraint is replaced by *any* finite collection of linear constraints. These constraints could, for example, represent bounds on the probability of receiving bundles from a certain category This would be relevant in an online advertising setting where advertisers prefer to diversify the audiences they reach. Such constraints could be baked into the utility function but not without violating Δ-substitutes. Thus, we extend Gul et al. (2019) beyond just budget constraints. Section 6 of Akbarpour and Nikzad (2020) also contains an approximate competitive equilibrium result which accommodates additional constraints on agents choices such as we have here. However, the quality of the approximation is in terms of agent utility and relies on a large market assumption.

Claim 4.3 *For every price vector p, and for every agent $j \in N$, let A_j be a non-negative matrix of size $k \times m$, and $b^j \in \mathbb{R}^k_+$ (both A_j and b^j can be functions of p). Let*

$$Ch_j^{**}(p) := \arg\max\{v_j(y) : y \in \mathcal{L}(X_j), A_j \cdot \overline{y} \leq b^j\}.$$

Then there exists a price vector p' such that $Ch_j^{**}(p) = conv(Ch_j(p'))$.

5 Generalized Single Improvement

In this section, we focus on a special case of the generalized 2-improvement property, which guarantees the existence of CE in a non-quasi-linear setting.

We apply this result to trading networks and also discuss the connection to the notion of demand types introduced in Baldwin and Klemperer (2019) and the notion of net substitutability introduced in Baldwin et al. (2020).

5.1 Generalized Single Improvement

Definition 4 requires the single improvement property to hold for *any* pair of bundles where one utility dominates the other. Below is our extension of the single-improvement property to non-quasi-linear preferences.

Definition 10 *A utility function $U(x, p \cdot x)$ satisfies the generalized single-improvement property if for any price vector p, two bundles $x, y \in Ch(p)$ and any price change $\delta p \in \mathbb{R}^m$ satisfying $\delta p \cdot x > \delta p \cdot y$, there exist integer vectors a and b such that $0 \leq a \leq (x - y)^+$ and $0 \leq b \leq (y - x)^+$ such that*

1. $\max\{\vec{1} \cdot a, \vec{1} \cdot b\} \leq 1$,
2. $\delta p \cdot a > \delta p \cdot b$, *and*
3. $x - a + b \in Ch(p)$.

The generalized single improvement property is a strict sub-case of generalized 2-improvement. This is because the generalized 2-improvement can include adding, subtracting two items, while Definition 10 rules these out.

Example 5.1 *A straightforward instance of the generalized single improvement is when each agent is interested in consuming bundles of size at most one. This arises in the housing model, see Quinzii (1984).*

Similar to Theorem 1, we have the following relationship between generalized single-improvement and the edges of the convex hull of the choice correspondence. The proof of this theorem is analogous to Theorem 1 and is omitted.

Call a binary polytope P, if all edge vectors of have at most 2 non-zero components and these being of opposite sign, we call P **special**.

THEOREM 8 *A preference, with choice correspondence $Ch(p)$, satisfies the generalized single improvement if and only if the convex hull of $\cup_{x \in Ch(p)} B(x)$ is special for every price p.*

Using Theorem 8, in what follows, we give an example of a preference satisfying generalized single-improvement but not gross substitutes.

Example 5.2 *Suppose two goods and a soft 'budget' of 1. In particular, let*

$$U(x_1, x_2, p \cdot x) = v(x_1, x_2) + \min\{0, \log(\frac{1 - p_1 x_1 - p_2 x_2}{\epsilon})\} \text{ for } \epsilon < 0.01.$$

Suppose $v(0,0) = 0, v(1,0) = 1, v(0,1) = 2$ and $v(1,1) = 3$. Notice, if $p_1 x_1 + p_2 x_2 < 1 - \epsilon$, the utility does not depend on prices.

Proposition 5.1 *The utility function defined in Example 5.2 satisfies the generalized single-improvement but not gross substitutes as described in definition 3*

Proof. For any $(p_1, p_2) \geq 0$ the agent's choice correspondence will never contain $(0,0)$ and $(1,1)$ unless $p_1 + p_2 < 1$. But then, $\min\{p_1, p_2\} \leq 0.5$ in which case the optimal choice would be $(1,0)$ or $(0,1)$. If for any price vector p where the choice correspondence contains more than one bundle, it can only take on one of the following values:

1. $\{(0,0), (1,0)\}$
2. $\{(0,0), (0,1)\}$
3. $\{(0,0), (1,0), (0,1)\}$
4. $\{(1,0), (1,1)\}$
5. $\{(0,1), (1,1)\}$
6. $\{(1,0), (0,1), (1,1)\}$

If the choice correspondence contains a single bundle, then its convex hull is trivially a special polytope.

Consider the polytopes associated with the convex hulls of each of the choice correspondences listed in example 5.2. The edges of each of them is a $0, \pm 1$ vector with at most two non-zero entries of opposite sign. Hence, they are special. By Theorem 8, this utility satisfies the generalized single improvement property.

Gross substitutes requires that if the price of a single good increases, the demand for the other goods does not decline. Suppose $p_1 = p_2 = 0.4$. The unique utility maximizing bundle is $(1, 1)$. Now increase p_2 to 0.6. The unique utility maximizing bundle is $(0, 1)$. ■

The following result shows the existence of CE in this environment.

THEOREM 9 *An economy $\{\{U_j\}_{j \in N}, s\}$ in which each U_j satisfies the generalized single-improvement property has a competitive equilibrium.*

Proof. See Appendix A.4.

5.2 Application: Trading Networks

Trades based on bilateral (indivisible) contracts can be represented by a network. Vertices correspond to agents while arcs represent the non-price elements of a bilateral contract. An arc's orientation identifies which agent is the "buyer" and which the "seller". The model allows an agent to be a buyer in some trades and a seller in others. Given prices for each arc/trade, an agent chooses the subset of incident arcs that maximize her utility.

Assuming quasi-linearity and a *full substitutability* condition on agents' preferences, Hatfield et al. (2013) have shown that a competitive equilibrium exists. Full substitutability is equivalent to valuations over contracts being $M^\sharp$ concave (Fujishige and Yang (2003), Murota and Tamura (2003). Fleiner et al. (2019) extends these existence results to non-quasi-linear preferences but maintaining the substitutability condition.[7] Theorem 9 allows one to establish the existence of a competitive equilibrium in this model with non-quasi-linear preferences and the generalized single improvement property. Recall, the generalized single improvement

[7]See also Schlegel (2018).

property is not implied by substitutes (Example 5.2). We follow Baldwin and Klemperer (2019) in showing how the trading networks model can be folded into the current setting.

A trading network is represented by a directed multigraph $G = (N, E)$ where N is the set of vertices and E the set of arcs. Each vertex $i \in N$ corresponds to an agent and each arc $e \in E$ corresponds to the non-price elements of trade between the incident pair of vertices. Let $\delta_+(i)$ and $\delta_-(i)$ be the outgoing and incoming arcs incident to vertex $i \in N$, and $\delta(i) = \delta_+(i) \cup \delta_-(i)$. So as to avoid having to distinguishing between a trade involving one unit of a good and two units of the same good, we allow for integral 'intensities' of trade. Formally, an **outcome** is any vector $x \in \mathbb{Z}^E_+$ where x_e denotes the number of copies of trade e that were executed. Suppose an integer s_e such that $x_e \leq s_e$ for all $e \in E$, i.e., an upper limit on the number of copies of trade e. We define a price vector $p \in \mathbb{R}^E$, where p_e is the price associated with the trade that corresponds to the arc e.

It is more convenient to represent the volume of trade associated with arc e between agents i and j using two variables rather than one: $y^i_e \geq 0$ and $y^j_e \geq 0$. If $e \in \delta_+(i)$ then, $s_e - y^i_e$ represents the number of copies of trade e that agent i sells. In other words, y^i_e is the number of copies of trade e that agent i 'retains'. If retaining a trade avoids a cost, one can think of y^i_e as counting the number of units a 'good'. As $e \in \delta_-(j)$, the variable y^j_e represents the number of copies of trade e that agent j acquires. Call the y variables **net-flows**. Every outcome $x \in \mathbb{Z}^E_+$ corresponds to a net-flow. For any arc e between agents i and j we have $|y^i_e| = |y^j_e| = x_e$ and $s_e - y^i_e - y^j_e = 0$. A net-flow $y = (y^1, \ldots, y^n)$ is called **feasible** if it corresponds to an outcome, i.e., for every arc e we have $y^i_e + y^j_e = s_e$ whenever $e \in \delta_+(i) \cap \delta_-(j)$. If y^i is the incidence vector of trades in which i is a part of, her utility for y^i at price vector p is

$$U_i(y^i, p) = U_i(y^i, -\sum_{e \in \delta_+(i)} (s_e - y^i_e)p_e + \sum_{e \in \delta_-(i)} y^i_e p_e).$$

Definition 11 *A feasible net-flow $y = (y^1, \ldots, y^n)$ along with a price vector $p \in \mathbb{R}^E$ is a*

competitive equilibrium *if, for all* $i \in N$, $y^i \in Ch_i(p)$.

Proposition 5.2 *Suppose each agent's utility function satisfies the generalized single improvement property. Suppose also that each agent strictly prefers to retain its zero priced trades in* $\delta_+(i)$ *and each agent strictly prefers to consume zero priced trades in* $\delta_-(i)$.[8] *The, the trading network has a competitive equilibrium.*

Proof. By Corollary 9, there is a price vector $p \in \mathbb{R}^E$ such that following holds:

1. $y^i \in Ch_i(p)$ for all $i \in N$
2. For all $i, j \in N$ and $e \in \delta_+(i) \cap \delta_-(j)$ with $p_e > 0$ we have $y^i_e + y^j_e = s_e$.
3. For all $i, j \in N$ and $e \in \delta_+(i) \cap \delta_-(j)$ with $p_e = 0$ we have $y^i_e + y^j_e \leq s_e$.

By the assumption about preferences at zero prices, the third item on this list is ruled out. Therefore, we have a competitive equilibrium. ∎

5.3 Relation to Net substitutes and Demand types

Baldwin et al. (2020), among other results, accommodate non-quasi-linear preferences by focusing on Hicksian demand. In addition to the assumptions on $U(x, t)$ we impose, they *also* require U to be *strictly* decreasing in the transfer t. Given a price vector p and a target level of utility u, the Hicksian demand at (p, u) is

$$D_H(p, u) = \arg\min\{px : x \in X, U(x, p \cdot x) \geq u\}.$$

Baldwin et al. (2020) introduce an analog of gross substitutes for Hicksian demand called (strong) net substitutes. For all utility levels u, price vectors p and $\lambda > 0$ whenever $D_H(p, u) = x$ and $D_H(p + \lambda e^i, u) = x'$ where e^i is the i^{th} unit vector, we have that $x'_k \geq x_k$

[8] Alternatively one could assume strict monotonicity in utility as is done in Schlegel (2018).

for all $k \neq i$. If all preferences satisfy net substitutes, Baldwin et al. (2020), show that a competitive equilibrium exists.

The generalized single improvement property is not implied by the net substitutes condition because it doesn't require the utility to be strictly monotone in transfers as example 5.2 illustrates. Furthermore, net substitutes require the relevant property to hold for *all* prices and utility levels. The generalized single improvement property depends on prices only.

We now contrast Theorem 9 with a similar existence result in Baldwin and Klemperer (2019) for quasi-linear preferences. They proposed that the preferences of an agent over bundles of indivisible goods be characterized in terms of demand changes in response to a small generic price change.[9] The set of vectors that summarize the possible demand changes is called the **demand type** of an agent. They give a variety of definitions that under quasi-linearity are equivalent. One that doesn't depend upon quasi-linearity involves the edges of $conv(Ch(p))$. Scale the edges of $conv(Ch(p))$ so that their greatest common divisor is 1 and call them **primitive edge directions**. The demand type of an agent is the set of primitive edge directions of $Ch(p)$ for all price vectors p. Baldwin and Klemperer (2019) show that if utilities are quasi-linear, concave and the demand types of all agents are from a unimodular vector system, then, a CE exists.

The column vectors of a network matrix, which is a $0, \pm 1$ matrix with at most two non-zero entries in each column and these being of opposite sign, is a unimodular vector system. When the matrix of vectors in the demand type is a network matrix, the underlying preferences are quasi-linear and gross substitutes. Hence, Baldwin and Klemperer (2019) is an extension of Kelso Jr and Crawford (1982) but maintaining quasi-linearity.

Unlike Baldwin and Klemperer (2019), our preferences depend on the edges of a different polytope: the convex hull of the choice correspondence in its binary presentation. The polytopes coincide when agents demand at most one unit of each good. Considering the binary presentation allows us to extend Kelso Jr and Crawford (1982) to a non-quasi linear

[9]This idea is implicit in Danilov et al. (2001).

setting, while Baldwin and Klemperer (2019) cannot do so without further assumptions.

6 Conclusion

This paper extends the single improvement property in two directions: (i) generalizing it to non-quasi-linear preferences and (ii) allowing complementarities. Our first result identifies a social approximate equilibrium where the mismatch between supply and demand depends on preferences rather than the size of the economy. Our second result yields a new sufficient condition for the existence of competitive equilibrium, generalizing gross substitute preferences to non quasi-linear settings. The usefulness of this approximation was illustrated in the context of trading networks and pseudo markets.

Acknowledgements

We thank Elizabeth Baldwin, Ravi Jagadeesan, Alex Teytelboym, and Ngoc Tran for useful conversations and suggestions. This research is supported by DARPA Grant HR001118S0045.

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Appendix

A Proofs

A.1 Proof of Theorem 1

Let P be the convex hull of $\cup_{x \in Ch(p)} B(x)$. Assuming preferences satisfy Δ-improvement we show that P is Δ-uniform. Let $C \geq \max_{i \in M} x_i$ for all $x \in Ch(p)$. Let $\bar{y}$ be an extreme point of P, let $\bar{z}^1, \ldots, \bar{z}^K \in P$ be *all* extreme points of P such that $||\bar{z}^r - y||_1 \leq \Delta$. Note, as P is binary, every integer vector in P is an extreme point.

Suppose, for a contradiction that $\bar{w} - \bar{y}$ is an edge of P such that $||\bar{w} - \bar{y}||_1 > \Delta$. Then, $\bar{w}$ cannot lie in the cone generated by $\{\bar{z}^1 - \bar{y}, \ldots, \bar{z}^K - \bar{y}\}$, i.e., there do not exist non-negative numbers $\{\lambda_i\}_{r=1}^K$ such that

$$\bar{w} - \bar{y} = \sum_{r=1}^{K} \lambda_i (\bar{z}^r - \bar{y}).$$

By Farkas' lemma, there exists a vector β such that $\beta \cdot (\bar{w} - \bar{y}) < 0$ and $\beta \cdot (\bar{z}^r - \bar{y}) \geq 0$ for all $r = 1, \ldots, K$.

Without loss of generality, we order the copies of each good i in non-decreasing order in β, i.e.,

$$\beta_{C\cdot(i-1)+1} \leq \beta_{C\cdot(i-1)+2} \leq \dots \leq \beta_{C\cdot i}.$$

We now argue that for each good i we can assume that

$$\bar{y}_{C\cdot(i-1)+1} \geq \bar{y}_{C\cdot(i-1)+2} \geq \dots \geq \bar{y}_{C\cdot i}.$$

If not, there exist two copies of good i: i_1 and i_2 such that $\beta_{i_1} < \beta_{i_2}$ and $\bar{y}_{i_1} = 0; \bar{y}_{i_2} = 1$. Consider the bundle $\bar{y}'$ obtained from $\bar{y}$ by switching the values of $\bar{y}_{i_1}$ and $\bar{y}_{i_2}$. Note $\bar{y}' \in P$ and $\beta \cdot (\bar{y}' - \bar{y}) = \beta_{i_1} - \beta_{i_2} < 0$. Hence, $\bar{y}'$ cannot lie in the cone generated by $\{\bar{z}^1 - y, \dots, \bar{z}^K - y\}$, However, $2 = ||\bar{y}' - \bar{y}||_1 \leq \Delta$ which is a contradiction.

We now use the extreme point $\bar{w} \in P$ to construct a vector $\bar{w}^* \in P$ such that for each good i:

$$\bar{w}^*_{C\cdot(i-1)+1} \geq \bar{w}^*_{C\cdot(i-1)+2} \geq \dots \geq \bar{w}^*_{C\cdot i}$$

and $\bar{w}^*$ is not in the cone generated by $\{\bar{z}^1 - \bar{y}, \dots, \bar{z}^K - \bar{y}\}$.

If there exist two copies of good i: i_1 and i_2 such that $\beta_{i_1} < \beta_{i_2}$ and $\bar{w}_{i_1} = 0; \bar{w}_{i_2} = 1$, switch the values of $\bar{w}_{i_1}$ and $\bar{w}_{i_2}$ to obtain the bundle $\bar{w}' \in P$. Then, $\beta \cdot (\bar{w}' - \bar{w}) = \beta_{i_1} - \beta_{i_2} < 0$. Therefore,

$$\beta \cdot (\bar{w}' - \bar{y}) = \beta \cdot (\bar{w}' - \bar{w}) + \beta \cdot (\bar{w} - \bar{y}) < 0.$$

Hence, $\bar{w}'$ is not in the cone generated by $\{\bar{z}^1 - \bar{y}, \dots, \bar{z}^K - \bar{y}\}$.

Repeat this step until we terminate in a vector $\bar{w}^*$ where for each j, the components in the interval $[C \cdot (i-1) + 1, C \cdot i]$ are arranged in non-increasing order. Note $\beta \cdot (\bar{w}^* - \bar{y}) < 0$. Hence, $\bar{w}^* \in P$ cannot lie in the cone generated by $\{\bar{z}^1 - \bar{y}, \dots, \bar{z}^K - \bar{y}\}$ and so $||\bar{w}^* - \bar{y}||_1 > \Delta$.

Let x be such that $\bar{y} \in B(x)$ and x^* such that $\bar{w}^* \in B(x^*)$. Because $||\bar{w}^* - \bar{y}||_1 > \Delta$ and the components of $\bar{w}^*$ and $\bar{y}$ in each interval $[C \cdot (i-1) + 1, C \cdot i]$ are both arranged in non-increasing order, $||x^* - x||_1 > \Delta$.

Consider good i. If $x^*_i = x_i$ set $\tilde{\beta}_i = 0$. If $x^*_i > x_i$, set $\tilde{\beta}_i$ to be the average of the β-s of the copies of good i in $(\bar{w}^* - \bar{y})$. Similarly, if $x^*_i < x_i$, then let $\tilde{\beta}_j$ be the average of the β-s of the copies of good j in $(\bar{y} - \bar{w}^*)$. Hence,

$$\tilde{\beta} \cdot (x^* - x) = \beta \cdot (\bar{w}^* - \bar{y}) < 0.$$

We now invoke the definition of Δ-improvement. Interpret $\tilde{\beta}$ as a price change, i.e. $\delta p = \tilde{\beta}$. Then, $\delta p \cdot x > \delta p \cdot x^*$. Hence, there exist non-negative integer vectors $a \leq (x - x^*)^+$ and $b \leq (x^* - x)^+$ such that $\vec{1} \cdot a + \vec{1} \cdot b \leq \Delta$, $z^* := x - a + b \in Ch(p)$, $||z^* - x||_1 \leq \Delta$ and $(p' - p) \cdot x > (p' - p) \cdot z^*$, i.e. $\tilde{\beta} \cdot (z^* - x) < 0$.

Choose $\bar{z} \in B(z^*)$ such that

$$\bar{z}_{C\cdot(i-1)+1} \geq \bar{z}_{C\cdot(i-1)+2} \geq \ldots \geq \bar{z}_{C\cdot i}.$$

Because of this ordering

$$||\bar{z} - \bar{y}||_1 = ||z^* - x||_1 \leq \Delta.$$

As $\bar{z}$ is a binary representation of z^* and $\bar{y}$ is a binary representation of x it follows that $\beta \cdot (\bar{z} - \bar{y}) = \tilde{\beta} \cdot (z^* - x) < 0$, which is a contradiction.

We now prove the converse. Denote the convex hull of $\cup_{x\in Ch(p)} B(x)$ by P. We will show that if P is Δ-uniform, the corresponding preferences satisfy the Δ-improvement property. For economy of exposition only, suppose $C = 1$, i.e., the agent wishes to consume at most one copy of each good. In this case the binary presentation of a vector is itself: $B(x) = x$, $\cup_{x\in Ch(p)} B(x) = Ch(p)$ and P is the convex hull of $Ch(p)$. Fix a price vector p and two bundles $x, y \in Ch(p)$. Consider a price change $\delta p \in \mathbb{R}^m$ satisfying $\delta p \cdot x > \delta p \cdot y$.

As all binary vectors of binary polytope are extreme points, $x, y \in ext(P)$. Now, $y - x$ is in the cone generated by the edges of P adjacent to x. This means that there exit a set of extreme points $\{z^1, \ldots, z^K\}$ adjacent to x and a set of *positive* numbers $\{\lambda_1, .., \lambda_K\}$ such that

$$y - x = \sum_{r=1}^{K} \lambda_r (z^r - x). \tag{4}$$

Multiplying both sides of equation (4) by δp, yields

$$0 > \delta p \cdot (y - x) = \sum_{r=1}^{K} \lambda_r \cdot \delta p \cdot (z^r - x).$$

Hence, there exists $r \in \{1, .., K\}$ such that $\delta p \cdot (z^r - x) < 0$.

Let $a := (x - z^r)^+$ and $b := (z^r - x)^+$. Because P is Δ-uniform, $\vec{1} \cdot a + \vec{1} \cdot b \leq \Delta$.

It is left to show that $a \leq (x - y)^+$ and $b \leq (y - x)^+$. Let I_0 and I_1 be the set of coordinates in which both x, y are 0 and 1, respectively. Notice $(z_i^r - x_i) \geq 0$ while $(y_i - x_i) = 0$ for all $i \in I_0$, thus, because of (4), $(z_i^r - x_i) = 0$ for all $r \in \{1, \ldots, K\}$ and $i \in I_0$. Similarly, $(z_j^r - x_j) \leq 0$ while $(y_j - x_j) = 0$ for all $i \in I_1$, and thus because of (4), $(z_j^r - x_j) = 0$ for all $r \in \{1, \ldots, K\}, j \in I_1$. This shows that $(x - z^r)^+ \leq (x - y)^+$ and $(z^r - x)^+ \leq (y - x)^+$ for all $r \in \{1, \ldots, K\}$. ■

A.2 Proof of Theorem 3: Existence of a Pseudo Equilibrium

Proof. Denote by X_j the set of feasible bundles available to agent $j \in N$. Let $\mathcal{L}(X_j)$ be the set of lotteries over X_j. We construct a correspondence

$$f : [0, B]^m \times \mathcal{L}(X_1) \times \ldots \times \mathcal{L}(X_n) \rightrightarrows [0, B]^m \times \mathcal{L}(X_1) \times \ldots \times \mathcal{L}(X_n)$$

and use Kakutani's fixed point theorem to show that it has a fixed point, $(p, x^1, .., x^n)$. This fixed point will correspond to a pseudo-equilibrium.

Given $\{x^j \in \mathcal{L}(X_j)\}_{j=1}^n$, let $\overline{x} = \sum_{j=1}^n x^j$ be the aggregate consumption. The excess demand vector is $\overline{x} - s$. Let

$$g(p, x^1, .., x^n) := (p + \overline{x} - s)^+.$$

Notice, for $p \in \mathbb{R}^m_+$, $g(p, x^1, .., x^n) = p$ implies that for all $i \in M$, if $p_i > 0$, then $\overline{x}_i = s_i$ and if $p_i = 0$, then $\overline{x}_i \leq s_i$. This is exactly the condition of that price and excess demand must meet to be considered a pseudo equilibrium.

To apply Kakutani's Theorem we need the function g to be bounded, so we modify it. As B is the bound on the willingness to pay of each agent, let

$$z_i(p, x^1, .., x^n) := \min\{(p_i + \overline{x}_i - s_i)^+, B\} \text{ for all } i \in M.$$

Each z_i is bounded. Define the following correspondence

$$f(p, x^1, .., x^n) := (z, conv(Ch_1(p)), .., conv(Ch_n(p)).$$

It is easy to see that f satisfies all the conditions of Kakutani's Theorem. Let $(p, x^1, .., x^n)$ be a fixed point of the correspondence f.

At this fixed point $x^j \in conv(Ch_j(p))$. Furthermore, because of the bounded willingness to pay assumption, if $p_i = B$, then for all $x^j \in conv(Ch_j(p))$, $x^j_i = 0$. Thus,

$$\min\{(p_i + \overline{x}_i - s_i)^+, B\} = (B - s_i)^+ < B.$$

Therefore, at the fixed point $p = (p + \overline{x} - s)^+ < B$. This implies that $(p, x^1, .., x^n)$ is a pseudo-equilibrium. ∎

A pseudo-equilibrium is an equilibrium with respect to the 'convexified' choice correspondences. As these may violate non-satiation, some competitive equilibria may be Pareto inefficient (with respect to the convexified choice correspondences). To avoid this, one can, when possible, select an **efficient disposal** equilibrium as defined in McLennan (2017).

A.3 Proof of Theorem 5

The following basic result is needed for our lottery implementation result.

Lemma A.1 *Given an $y \in \mathbb{R}^m$, and some property $(*)$, y can be expressed as a lottery over a set of vectors satisfying (*) if and only if for every weight vector $w \in \mathbb{R}^m$, there exists $z \in \mathbb{R}^m$ satisfying $(*)$ and $w \cdot z \geq w \cdot y$.*

Proof. Let Z be the set of all vectors satisfying $(*)$. Now, $y \notin conv(Z)$ if and only if there exists a hyperplane separating y from $conv(Z)$. This means that there exists $w \in \mathbb{R}^m$ such that $w \cdot y > w \cdot z$ for all $z \in Z$.

Furthermore, any algorithm to find a vector z satisfying $(*)$ and $w \cdot z \geq w \cdot y$, can be used to express y as a convex combination of vectors in Z. See Nguyen et al. (2016). ∎

Using Lemma A.1, given a vector y and a weight vector w, we will need to find a vector z satisfying $w \cdot z \geq w \cdot y$. This is proved using the algorithm described in Figure 1.

First we need some terminology. Given a binary polytope $Q \in \mathbb{R}^{Cmn}$, a coordinate $i \in \{1, \ldots, Cmn\}$ is called **fixed with respect to** Q if $x_i = y_i$ for all $x, y \in Q$. Otherwise it is called **free**. In other words, if a coordinate i is fixed with respect to Q it means that there is a constant θ such that $x_i = \theta$ for all $x \in Q$.

We now show that the algorithm is correct and terminates in the desired solution z. At each iteration of the algorithm, we solve a linear program, (5) with the objective given by the weight vector w and one less supply constraint than the iteration before. Thus, the linear program (5) is feasible at each iteration and the optimal objective function value is non-decreasing. Thus, the terminal solution z will satisfy $w \cdot z \geq w \cdot y$.

Claim A.1 *At each iteration of the algorithm Q is Δ uniform and the minimal face $F \subset Q$ containing the extreme point solution of (5) satisfies $dim(F) \leq |S|$.*

Proof. At the beginning of the algorithm $Q = P_1 \times .. \times P_n$. As each P_i is Δ uniform, so is Q. In the subsequent iterations, Q is replaced by one of its faces. The edges of a face are a subset of the edges of the corresponding polytope, thus Q remains Δ uniform.

To show $dim(F) \leq |S|$, rewrite the constraints of the linear program (5) in the matrix form:$\{Az \leq a, Bz = b\}$. Here $Az \leq a$ represents $z \in Q$ and $Bz = b$ represents $z \in R$. In

Figure 1: ALGORITHM

Input: $\{P_j\}_{j=1}^n$ Δ-uniform binary polytopes, $y = (y^1, \ldots, y^n) \in P_1 \times \ldots \times P_n$, $\sum_{j=1}^n T_i(y^j) = s_i \in \mathbb{Z}$, and weight vector $w \in \mathbb{R}^{Cmn}$.

Output: $z \in ext(P_1) \times \ldots \times ext(P_n)$ such that $w \cdot z \geq w \cdot y$ and for all $i \in M$, $|T_i(\sum_j z^j) - s_i| \leq \Delta - 1$.

Step 0: Initiate $Q = P_1 \times \ldots \times P_n$, and let $S := M$ to be the set of active supply constraints.

Step 1: Let

$$R := \{z = (z^1, \ldots, z^n) | z^i \in [0,1]^{Cm}; \sum_{j=1}^n T_i(z^j) = s_i \text{ for all } i \in S\}.$$

Solve

$$\{\max w \cdot z | z \in Q \cap R\}. \tag{5}$$

Let $z = (z^1, \ldots, z^n)$ be an optimal extreme point solution and $F \subset Q$ be the minimal face of Q containing z.

Step 2a: If $dim(F) = 0$, that is F contains a single element, call it $(z^1, \ldots, z^n)$ and STOP.

Step 2b: Else, each constraint i, which is $\sum_{j=1}^n T_i(z^j) = s_i$, can be written as $\alpha(i) \cdot z = s_i$, where $\alpha(i)$ is a $\{0,1\}$ vector.

Let a_i be the number of non-zero coordinates of $\alpha(i)$ that are free with respect to F. Among the active constraints in S, choose the constraint $i \in S$ with smallest a_i.

Update $Q := F$ and $S := S \setminus \{i\}$.

Step 3: Return to **Step 1**.

an extreme point solution, the number of independent binding constraints is equal to the number of coordinates of z, which is Cmn. B has $|S|$ rows, thus, the number of independent binding constraints in $Az \leq a$ is at least $Cmn - |S|$. This shows that the minimal face containing z is of dimension at most $|S|$. ■

Claim 2 shows that when there are no more supply constraints left to delete, $dim(F) = 0$ and the algorithm terminates. Hence, the algorithm terminates in at most m iterations.

To show that the violation in each supply constraint cannot exceed $\Delta - 1$, we need the following.

Claim A.2 *At each iteration, the number of free coordinates with respect to Q is at most $\Delta \cdot dim(Q)$.*

Proof. Fix an extreme point v of Q and let E_v be a maximal linearly independent set of edges of Q that are incident to v. Recall, that the dimension of the space spanned by the edges incident to any extreme point is equal to the dimension of the polytope. Thus, $|E_v| = dim(Q)$.

By claim A.1, Q is binary and Δ-uniform, the components of each of the vectors in E_v belongs to $\{-1, 0, 1\}$ and the number of non-zero components in each of them is at most Δ.

By the definition of free coordinates, a non-zero component of an edge can only be at a free coordinate of Q. The reverse is also true. If j is a free coordinate, then there exists $v' \in Q$ such that the j^{th} coordinate of $v' - v$ is not 0. But $v' - v$ is in the span of E_v, thus, there must be a vector in E_v, whose j^{th} coordinate is not 0.

There are $dim(Q)$ vectors in E_v, each vector has at most Δ nonzero components. Therefore, the number of free coordinates of Q is at most $\Delta \cdot dim(Q)$. ■

Claim A.3 *Let $z^* = (z^{*1}, .., z^{*n})$ be the algorithm's output. Then, for every good $i \in M$*

$$|\sum_{j=1}^{n} T_i(z^{*j}) - s_i| \leq \Delta - 1.$$

Proof. Fox some iteration of the algorithm where S is the set of active supply constraints, z is the extreme point solution of (5), and $F \subset Q$ is the minimal face of Q containing z.

The coordinates of the non-zero components of distinct supply constraints are disjoint. By Claim A.2 there are at most $\Delta \cdot dim(F)$ free coordinates with respect to F. By Claim A.1, $dim(F) \leq |S|$. Thus, the supply constraint selected for deletion in Step (2b) of the algorithm contains at most Δ free coordinates with respect to F.

The outcome of the algorithm belongs to a polytope that shrinks in dimension at every step of the algorithm. Thus, as the output of the algorithm, $z^* \in F$. Consider the supply constraint i to be deleted at this iteration. Because constraint i contains at most Δ free coordinates with respect to the binary polytope F and $z \in F$ as well, it follows that

$$|\sum_{j=1}^{n} T_i(z^{*j}) - \sum_{j=1}^{n} T_i(z^j)| \leq \Delta.$$

Equality can only occur if constraint i contains *exactly* Δ free coordinates and the values of these coordinates in z are either all 0 or all 1, while the opposite is true for z^*. However, this is impossible because, for example, if the values of these coordinates in z are all 0 then, these coordinates will be fixed (at 0) with respect to the minimal face containing $\{z, z^*\}$, which contradicts the fact that they are free coordinates.

Now, because $\sum_{j=1}^{n} T_i(z^j) = s_i$ and $\sum_{j=1}^{n} T_i(z^{*j})$ are both integral, we have.

$$|\sum_{j=1}^{n} T_i(z^{*j}) - \sum_{j=1}^{n} T_i(z^j)| = |\sum_{j=1}^{n} T_i(z^{*j}) - s_i| \leq \Delta - 1.$$

■

A.4 Proof of Theorem 9

Similar to the proof of Theorem 4, we will need to round the allocation of a pseudo equilibrium to an integral allocation. In particular, to prove Corollary 9 it suffices to show that for any weight vector w, the following linear program, when feasible, has an integral solution.

$$\begin{aligned} \max \quad & w \cdot z \\ \text{s.t.} \quad & z = (z^1, .., z^n) \in P_1 \times .. \times P_n & (6) \\ & \sum_{j=1}^{n} T_i(z^j) = s_i, \ \forall i \in M. & (7) \end{aligned}$$

Let Q denote the binary polytope determined by (7). i.e.,

$$Q := \{(z^1, .., z^n) | z^i \in [0,1]^{Cm} \text{ and } \sum_{j=1}^{n} T_i(z^j) = s_i \text{ for all } i \in M\}.$$

We show that Q is special.

From equation (1), the i^{th} constraint that defines Q has the form:

$$\sum_{j=1}^{n} \sum_{k=C\cdot(i-1)+1}^{C\cdot i} z_k^j = s_i.$$

From this we see that the non-zero coefficients in constraint i and $i' \neq i$ do not overlap. Hence, the matrix of coefficients associated with the linear system $\sum_{j=1}^{n} T_i(z^j) = s_i$ for all $i \in M$ has all 0-1 entries with exactly one non-zero entry in each column. This is precisely the description of the partition matroid polytope. Matroid polytopes are well known to be special (see Theorem 3 of Frank et al. (2014)).

As each of the P_i are special binary polytopes, it is straightforward to see that $P = P_1 \times .. \times P_n$ is also a special binary as well. Standard arguments (see Frank et al. (2014), for example) tell us that the intersection of two special binary polytopes has integral extreme points. For completeness, we include the argument here.

Claim A.4 *If P and Q are two special binary polytopes in $\mathbb{R}^{Cmn}$ with $P \cap Q \neq \emptyset$, then, the extreme points of $P \cap Q$ are integral.*

Proof. By the fundamental theorem of Linear Programming, it suffices to show that for any $w \in \mathbb{R}^{Cmn}$, $\{\max w \cdot z : z \in P \cap Q\}$ has an integral solution. Let z be an extreme point solution of this program. Let F_1, F_2 be the minimal faces of P and Q, respectively, that contain z. Because z is an extreme point solution, the linear spaces spanned by the edges of F_1 and F_2 are independent.[10]

[10] This is because the faces F_1 and F_2 correspond to the binding constraints which must be linearly independent at an optimal solution.

Let v_i be an extreme point of F_i and E_i the set of its edges for $i = 1, 2$. As $z \in F_1 \cap F_2$, there exist $\alpha_e \in \mathbb{R}$ for $e \in E_1 \cup E_2$ such that

$$z = v_1 + \sum_{e \in E_1} \alpha_e \cdot e = v_2 + \sum_{e \in E_2} \alpha_e \cdot e. \tag{8}$$

It remains to show that $\sum_{e \in E_1} \alpha_e \cdot e$ is integral as this implies that z is integral.

Equation (8) implies

$$v_1 - v_2 = \sum_{e \in E_2} \alpha_e \cdot e - \sum_{e \in E_1} \alpha_e \cdot e. \tag{9}$$

Because F_1, F_2 are each special polytopes, their edges $E_1 \cup E_2$ are $\{0, \pm 1\}$ vector with at most 2 nonzero components and these are of opposite signs. Hence, the matrix each of whose columns is in $E_1 \cup E_2$ is a network matrix and hence is totally unimodular. Combined with the fact that $v_1 - v_2$ is integral implies that there exists $\beta_e \in \mathbb{Z}$ for $e \in E_1 \cup E_2$ such that

$$v_1 - v_2 = \sum_{e \in E_2} \beta_e \cdot e - \sum_{e \in E_1} \beta_e \cdot e. \tag{10}$$

Equations (9) and (10) imply

$$\sum_{e \in E_1} (\alpha_e - \beta_e) \cdot e = \sum_{e \in E_2} (\alpha_e - \beta_e) \cdot e. \tag{11}$$

Now, the linear spaces spanned by E_1, E_2 are independent. Therefore, either side of equation (11) must be zero 0. Thus, $\sum_{e \in E_1} \alpha_e \cdot e = \sum_{e \in E_1} \beta_e \cdot e$, which is an integral vector as desired.

■