Bargaining with Evolving Private Information*

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Abstract

I study how the arrival of new private information affects bargaining outcomes. A seller makes offers to a buyer. The buyer is privately informed about her valuation, and the seller privately observes her stochastically changing cost of delivering the good. The seller's time-varying private information gives rise to new dynamics. Prices fall gradually at the early stages of negotiations, and trade is inefficiently delayed. Inefficiencies persist even when gains from trade are common knowledge. Privately observed costs lead to lower welfare, higher seller revenue and lower buyer surplus (especially for high value buyers) relative to a setting with publicly observed costs.

Keywords: bargaining, inefficient delay, Coase conjecture, evolving private information, two-sided private information.

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1 Introduction

In many bargaining settings, new private information may arrive as negotiations proceed. Consider, for instance, a producer of a new intermediate good negotiating a sale with a potential industrial buyer. Since the good for sale is new, production costs are likely to be initially high. Over time, costs may fall as the seller privately becomes more efficient. The goal of this paper is to study how the arrival of new private information affects bargaining dynamics.

I study a bargaining game in which a seller makes offers to a privately informed buyer.[1] The seller's cost of producing the good (or, equivalently, the opportunity cost of selling it) changes stochastically over time, and is privately observed by the seller. For simplicity, I assume that the seller's cost can take two values, high or low, and that it evolves over time as a Markov chain. For most of the analysis, I focus on separating Perfect Bayesian Equilibria (PBE), under which the seller's price each period reveals her cost.[2] These equilibria are intuitive, tractable, and provide a natural point of comparison with prior papers in the literature (e.g. Cho, 1990, Ortner, 2017).

The analysis delivers three main results. First, I provide a characterization of the set of separating PBE. In any separating equilibrium, buyer and seller trade at a slow rate when the seller's cost is high, and prices fall gradually. When the seller's cost falls, equilibrium becomes Coasian: buyer and seller trade very fast at a low price. Market dynamics under separating PBE are broadly consistent with dynamics typically observed in markets for new durable goods, where prices fall gradually during the early stages, and market penetration raises slowly (Conlon, 2012). Moreover, without loss, separating PBE can be taken to be weakly stationary.

The key drivers of these equilibrium dynamics are the information revelation constraints

[1] As usual, this bargaining model is mathematically equivalent to a setting in which a durable good monopolist sells to a population of heterogenous buyers.

[2] Section 7 discusses other equilibria.

that arise as a result of the seller's evolving private information. In any separating equilibrium, a seller whose cost just fell must not gain by mimicking a high cost seller and posting a high price. The slow rate at which buyer and seller trade when costs are high makes this deviation unprofitable, since a low cost seller has a stronger incentive to trade fast. An implication is that information revelation constraints lead to inefficiencies relative to the first-best outcome.

While all equilibria of the game are inefficient, the environment that I study admits under certain conditions an efficient mechanism satisfying individually rationality, incentive compatibility and budget balance.[3] This contrasts with prior bargaining games with two-sided private information (e.g. Cho, 1990), in which efficient equilibria exist if and only if the environment admits an efficient mechanism satisfying IR, IC and budget balance.[4]

The second main result studies the frequent-offers limit of (most efficient) separating equilibria. I show that this limit is characterized by a system of differential equations, which specifies how prices and probability of trade change over time while the seller's cost is high. This tractable characterization allows me to derive several comparative statics. An increase in the seller's high cost increases equilibrium prices, and lowers the speed with which buyer and seller trade. An increase in the distribution of buyer values, or in the rate at which costs fall, have similar effects on bargaining dynamics. Lastly, seller's profits become negligible as the buyer's lowest valuation converges to zero, as in classic Coasian bargaining games (Fudenberg et al., 1985, Gul et al., 1985). The difference, however, is that this fall in seller profits comes together with a drop in social welfare.

The third main result compares equilibrium outcomes in this model with a model in which the evolution of the seller's cost is publicly observed, as in Ortner (2017). Stationary equilibria of the game with publicly changing costs retain several features of the Coasian

[3]See Online Appendix B. This appendix also shows that all equilibria of the game (separating or not) are inefficient.

[4]Indeed, the equilibria in Cho (1990) are inefficient only when gains from trade are not common knowledge. We know from Myerson and Satterthwaite (1983) that such environments do not admit an efficient mechanism satisfying IR, IC and budget balance (and hence do not admit efficient equilibria).

model. In the frequent-offers limit, the equilibrium outcome is efficient. Moreover, the seller is unable to extract rents from high value buyers: her limiting profits are exactly what she would earn if it was common knowledge that the buyer had the lowest possible value. This contrasts sharply with the model with privately observed costs, in which trade is inefficiently delayed, and the seller extracts rents. An implication is that privately observed costs lead to lower social welfare, higher seller revenues and lower buyer surplus (especially for high value buyers) relative to settings with public costs.

Related literature. This paper fits into the literature on dynamic bargaining with private information. Early contributions in this literature illustrate how, in settings with one-sided private information, the uninformed party's inability to commit to future offers severely limits the rents she can extract (Bulow (1982), Fudenberg et al. (1985), Gul et al. (1985), Gul and Sonnenschein (1988)). Stationary equilibria satisfy the Coase conjecture when offers are frequent (Coase, 1972): the seller's initial price is very low, and buyer and seller reach an immediate agreement.

Several papers have identified economic forces that push towards inefficient bargaining outcomes within the one-sided private information framework. Bargaining inefficiencies can arise when bargainers strategically delay trade to signal their types (Admati and Perry, 1987), when bargainers use non-stationary strategies (Ausubel and Deneckere, 1989), when the seller faces capacity constraints (Kahn, 1986, McAfee and Wiseman, 2008), or when values are interdependent (Evans, 1989, Vincent, 1989, Deneckere and Liang, 2006, Gerardi and Maestri, 2017). Costly delays can also arise in the presence of deadlines (Güth and Ritzberger, 1998, Hörner and Samuelson, 2011, Fuchs and Skrzypacz, 2013), when bargainers have outside options (Board and Pycia, 2014), or when bargainers seek to build a reputation for being tough (Myerson, 2013, Abreu and Gul, 2000).[5]

A smaller literature studies how delays and inefficiencies arise when there is two-sided

[5] See also Abreu and Pearce (2007), Fanning (2016, 2018), Sanktjohanser (2017).

private information (Cramton, 1984, 1992, Chatterjee and Samuelson, 1987, 1988, Cho, 1990, Ausubel and Deneckere, 1992). The current paper adds to this literature by analyzing a model in which one of the bargaining sides receives new private information over time. The analysis illustrates how such evolving private information affects bargaining outcomes, and gives rise to new distortions.

The current paper also relates to Ortner (2017), who studies a continuous-time durable goods monopoly model in which the seller's cost is publicly observed, and changes stochastically over time.[6] Ortner (2017) shows that time-varying costs allow the seller to extract rents when buyer values are discrete. With a continuum of buyer types (as in the current paper), the seller is unable to extract rents, and the market outcome is efficient. The results for public costs in the current paper show that the conclusions in Ortner (2017) also hold in the frequent-offers limit of discrete-time games.

Fuchs and Skrzypacz (2010) and Daley and Green (2020) study bargaining games with one-sided private information in which parties may receive public news while negotiating. Their results shed light into how the arrival of public information affects bargaining outcomes and can lead to costly delays and inefficiencies. In contrast, the current paper highlights the inefficiencies generated by the arrival of new private information.

Hwang (2018) studies how the arrival of new private information affects trading dynamics between a long-run seller and a sequence of short-term buyers. I instead study how new private information affects bargaining dynamics between two long-run agents. Kennan (2001) studies a repeated bargaining game with imperfectly persistent one-sided private information, and shows that this may give rise to path-dependent bargaining outcomes.

Lastly, several papers construct models to rationalize sales in durable goods markets. Conlisk et al. (1984) and Sobel (1984, 1991) propose theories of sales driven by entry of new consumers. Board (2008), Board and Skrzypacz (2016) and Dilmé and Li (2019) show

[6]See also Acharya and Ortner (2017), who study how public shocks affect equilibrium dynamics in environments with perfectly persistent private information.

that sales can be part of an optimal selling scheme when demand is time-varying. Dilmé and Garrett (2017) show that sellers might extract additional rents by offering random price discounts. The current paper adds to this literature by providing a theory of sales driven by changes in the seller's cost of production.

The paper proceeds as follows. Section 2 introduces the model. Section 3 characterizes the efficient outcome and the commitment solution. Section 4 characterizes the set of separating PBE. Section 5 studies the frequent-offers limit of welfare maximizing separating PBE and derives several comparative statics. Section 6 compares this model to a model in which costs are publicly observed. Section 7 discusses extensions and other equilibria. Proofs are collected in the Appendix and the Online Appendix.

2 Model

A seller with the technology to deliver a good faces a buyer. The buyer's valuation for the seller's good, v, is her private information, and is drawn from distribution F with support $[\underline{v}, \overline{v}]$ and continuous density $F' = f$ satisfying $f(v) > 0$ for all $v \in [\underline{v}, \overline{v}]$. I assume that $\underline{v} > 0$. Time is discrete, with $t \in T(\Delta) = \{0, \Delta, 2\Delta, ..., \infty\}$.

The seller's cost of delivering the good (or, equivalently, her opportunity cost of selling it) changes over time. The seller's cost can take two values: $c_H > 0$ or $c_L = 0$. At $t = 0$, the seller's cost c_0 is c_H with probability $q \in (0, 1)$ and c_L with probability $1 - q$. For all times $t \in T(\Delta)$, $\text{prob}(c_{t+\Delta} = c_H | c_t = c_H) = e^{-\lambda\Delta}$ with $\lambda > 0$, and $\text{prob}(c_{t+\Delta} = c_L | c_t = c_L) = 1$. The assumption that low cost c_L is absorbing simplifies the exposition, but is not necessary; Section 7 shows how the results generalize when c_L is not absorbing. The seller is privately informed about her production cost: she privately observes her current cost realization at the start of each period $t \in T(\Delta)$.

The timing of moves within each period t is as follows. At $t = 0$, the buyer privately learns her valuation and the seller privately learns her initial cost. Then, the seller offers price $p_0 \in \mathbb{R}_+$, and the buyer chooses to accept or reject this price. At any time $t > 0$, if the buyer hasn't yet accepted a price, the seller first privately observes current cost c_t. After observing c_t, the seller offers price $p_t \in \mathbb{R}_+$, and the buyer chooses to accept or reject this price. If the buyer accepts the seller's offer at time t, trade happens and the game ends, with the buyer obtaining payoff $e^{-rt}(v - p_t)$ and the seller obtaining payoff $e^{-rt}(p_t - c_t)$, where $r > 0$ is the common discount rate.

Histories and strategies. At any period t before agreement is reached, the seller's history $h_t^S = \{c_s, p_s\}_{s<t}$ records all previous cost realizations and all previous prices, and the buyer's history $h_t^B = \{v, \{p_s\}_{s<t}\}$ records her valuation and all previous prices. A (pure) strategy for the seller $\sigma_S : h_t^S \times c_t \mapsto p_t$ maps seller's histories h_t^S and current cost c_t into a price. A (pure) strategy for the buyer $\sigma_B : h_t^B \times p_t \mapsto \{accept, reject\}$ maps buyer's histories h_t^B and the seller's current price p_t into a decision of whether or not to accept price p_t.

Solution concept. For most of the paper, I focus on *separating* Perfect Bayesian Equilibrium (PBE) under which, at every seller history, the seller's price reveals her current cost.[7] Formally, let (σ, μ) be a PBE, where $\sigma = (\sigma_S, \sigma_B)$ are the players' strategies, and $\mu = (\mu_S, \mu_B)$ are players' beliefs: $\mu_S(h_t^S)$ is the seller's beliefs over the buyers' type after history h_t^S, and $\mu_B(h_t^B \sqcup p_t)$ the buyer's beliefs over the seller's cost at time t after history $h_t^B \sqcup p_t$. I look for PBE (σ, μ) with the property that, for every seller history h_t^S, $\mathsf{supp}\, \sigma^B(h_t^S)(c_H) \cap \mathsf{supp}\, \sigma^B(h_t^S)(c_L) = \emptyset$. That is, for every history h_t^S, the seller charges a different price if her cost at time t is c_H than if it is c_L. As a result, for every on-path buyer history $h_t^B \sqcup p_t$, $\mu_B(h_t^B \sqcup p_t)$ assigns probability 1 to either $c_t = c_L$ or $c_t = c_H$.

I impose one additional restriction on the buyer's beliefs: if at any history h_t^B the buyer assigns probability 1 to the seller's current cost being c_L, then I require that for all histories that follow h_t^B, the buyer continues to assign probability 1 to the seller's cost being c_L. This

[7] Section 7 discusses other equilibria.

restriction is natural, since cost c_L is absorbing. Let $\Sigma^{\mathsf{S}}(\Delta)$ denote the set of PBE satisfying these conditions, under which the seller uses a pure action while her costs are c_H.[8]

Successive skimming. Any PBE must satisfy the *skimming property*: if at time t a buyer with valuation $v \in [\underline{v}, \overline{v})$ finds it optimal to accept the current price p_t, then a buyer with valuation $v' > v$ also finds it optimal to accept p_t. The reason for this is that it is more costly for high-value buyers to delay trade.[9] The skimming property implies that, for any period t, there exists a cutoff $\kappa_{t+\Delta}$ such that a buyer with valuation $v > \kappa_{t+\Delta}$ accepts the current offer p_t, and a buyer with valuation $v < \kappa_{t+\Delta}$ rejects the offer. Moreover, if the buyer rejects all of the seller's offers $\{p_s\}_{s\leq t}$ up to time t, the seller believes that the buyer's valuation is distributed according to $\text{prob}(v \leq \hat{v}) = \frac{F(\hat{v})}{F(\kappa_{t+\Delta})}$ for all $\hat{v} \in [\underline{v}, \kappa_{t+\Delta}]$.

3 Benchmarks

This section derives two benchmarks: (i) the first-best outcome, and (ii) the seller's optimal commitment outcome.

First-best. Define $\rho(\Delta) \equiv \frac{e^{-r\Delta}(1-e^{-\lambda\Delta})}{1-e^{-(r+\lambda)\Delta}}$ to be the expected discounted time until costs fall to c_L, provided current cost is c_H. Define $v^*(\Delta)$ to be the solution to $v^*(\Delta) - c_H = \rho(\Delta)v^*(\Delta)$. Under the first-best outcome, the seller sells to a buyer with valuation $v \geq v^*(\Delta)$ at $t = 0$, regardless of the initial cost, and sells to a buyer with valuation $v < v^*(\Delta)$ the first time costs fall to c_L. Define $\tau_L \equiv \min\{t \in T(\Delta) : c_t = c_L\}$ to be the random time at which costs fall to c_L. The following proposition summarizes the first-best outcome.

Proposition 1 (First best)**.** *Under the first-best, a buyer with valuation $v \geq v^*(\Delta)$ buys at $t = 0$, and a buyer with valuation $v < v^*(\Delta)$ buys at time τ_L.*

Throughout the paper, I maintain the following assumption:

[8] In Appendix A, I briefly study equilibria in which the seller mixes while her cost are c_H.
[9] See Lemma 1 in Fudenberg et al. (1985) for a formal proof.

Assumption 1. $v^*(\Delta) \in (\underline{v}, \overline{v})$.

Commitment solution. Suppose next that the seller can commit to a mechanism at time $t = 0$, after learning her initial cost c_0. Let $\phi(v) \equiv v - \frac{1-F(v)}{f(v)}$ denote the buyer's virtual valuation. The following result holds.[10]

Proposition 2 (Commitment solution)**.** *Suppose $\phi(\cdot)$ is strictly increasing. Then, under the commitment solution, the seller sells to buyers with $\phi(v) \geq v^*(\Delta)$ at $t = 0$, and to buyers with $\phi(v) \in [0, v^*(\Delta)]$ at time τ_L. Buyers with $\phi(v) < 0$ never buy.*

The proof of Proposition 2 is in Online Appendix A. As is standard in screening models, under the commitment solution the seller inefficiently delays trade with lower value buyers to reduce the informational rents of higher value buyers. These inefficiencies appear in two ways. First, a buyer with value $v \in [v^*(\Delta), \phi^{-1}(v^*(\Delta))]$ trades inefficiently late. Second, a buyer with value $v \in [\underline{v}, \phi^{-1}(v^*(\Delta))]$ never trades.

4 Separating Equilibria

This section characterizes equilibrium set $\Sigma^{\mathsf{S}}(\Delta)$. I start with a few preliminary observations. Note that in any PBE in $\Sigma^{\mathsf{S}}(\Delta)$, when costs fall to c_L the buyer's beliefs about the seller's cost remain concentrated at c_L at all future periods. Hence, the continuation game is strategically equivalent to the one-sided incomplete information game in Fudenberg et al. (1985) and Gul et al. (1985). This game has a unique equilibrium (since $\underline{v} > c_L = 0$), which is weakly stationary: the buyer's acceptance rule at histories at which the current price is the lowest among all past prices depends solely on her valuation (see Fudenberg et al. (1985) and Gul et al. (1985)). For any $\kappa \in [\underline{v}, \overline{v}]$, let $p^L(\kappa)$ denote the price that a seller posts in the one-sided incomplete information game when her belief cutoff is κ, and let $U^L(\kappa)$ denote the seller's equilibrium continuation profits given belief cutoff κ.

[10] The proof of Proposition 2 shows that the same result holds if the seller can commit to a mechanism prior to learning her initial cost.

Consider next equilibrium behavior at periods at which costs are high. By the skimming property, for any $(\sigma, \mu) \in \Sigma^{\mathsf{S}}(\Delta)$, on-path behavior at times t with $c_t = c_H$ is characterized by sequences $\{p_t^H, \kappa_t^H\}_{t \in T(\Delta)}$ such that: p_t^H is the price that the seller charges at time t if $c_t = c_H$, and κ_t^H is the seller's belief cutoff at the start of time t if her cost last period was c_H. Therefore, along the equilibrium path, at any time $t \in T(\Delta)$ with $c_t = c_H$, the buyer accepts the seller's price if her valuation lies in $[\kappa_{t+\Delta}^H, \kappa_t^H)$; and the conditional probability with which buyer and seller trade is $\frac{F(\kappa_t^H) - F(\kappa_{t+\Delta}^H)}{F(\kappa_t^H)}$.

For any sequence $\{p_\tau^H, \kappa_\tau^H\}$, and for all times t, let $U_t^H(\{p_\tau^H, \kappa_\tau^H\})$ be the seller's continuation payoff if $c_t = c_H$, when play is given by $\{p_\tau^H, \kappa_\tau^H\}$:

$$
\begin{aligned}
U_t^H(\{p_\tau^H, \kappa_\tau^H\}) = (p_t^H - c_H)\frac{F(\kappa_t^H) - F(\kappa_{t+\Delta}^H)}{F(\kappa_t^H)} + e^{-(r+\lambda)\Delta}\frac{F(\kappa_{t+\Delta}^H)}{F(\kappa_t^H)} U_{t+\Delta}^H(\{p_\tau^H, \kappa_\tau^H\}) \\
+ e^{-r\Delta}(1 - e^{-\lambda\Delta})\frac{F(\kappa_{t+\Delta}^H)}{F(\kappa_t^H)} U^L(\kappa_{t+\Delta}^H).
\end{aligned}
$$

The following result holds.

Theorem 1. *(i) Suppose sequences $\{p_\tau^H, \kappa_\tau^H\}$ are induced by an equilibrium $(\sigma, \mu) \in \Sigma^{\mathsf{S}}(\Delta)$. Then, $\{\kappa_\tau^H\}$ is decreasing, and for all $t \in T(\Delta)$:*

$$
\kappa_{t+\Delta}^H - p_t^H = e^{-(r+\lambda)\Delta}(\kappa_{t+\Delta}^H - p_{t+\Delta}^H) + e^{-r\Delta}(1 - e^{-\lambda\Delta})(\kappa_{t+\Delta}^H - p^L(\kappa_{t+\Delta}^H)), \tag{1}
$$

$$
\frac{F(\kappa_t^H) - F(\kappa_{t+\Delta}^H)}{F(\kappa_t^H)} p_t^H \leq U^L(\kappa_t^H) - e^{-r\Delta}\frac{F(\kappa_{t+\Delta}^H)}{F(\kappa_t^H)} U^L(\kappa_{t+\Delta}^H), \tag{2}
$$

$$
U_t^H(\{p_\tau^H, \kappa_\tau^H\}) \geq \rho(\Delta) U^L(\kappa_t^H). \tag{3}
$$

(ii) There exists $\overline{\Delta} > 0$ such that, if $\Delta \leq \overline{\Delta}$, for any sequences $\{p_\tau^H, \kappa_\tau^H\}$ satisfying (1)-(3) *with $\{\kappa_\tau^H\}$ decreasing, there exists an equilibrium $(\sigma, \mu) \in \Sigma^{\mathsf{S}}(\Delta)$ that induces $\{p_\tau^H, \kappa_\tau^H\}$.*

Theorem 1 emphasizes three properties of any equilibrium in $\Sigma^{\mathsf{S}}(\Delta)$. First, by equation (1), for all periods t with $c_t = c_H$ the marginal buyer $\kappa_{t+\Delta}^H$ is indifferent between trading at the current price p_t^H, or waiting and trading at time $t + \Delta$.

Second, by inequality (2), the probability $(F(\kappa_t^H) - F(\kappa_{t+\Delta}^H))/F(\kappa_t^H)$ with which buyer and seller trade at a period t with $c_t = c_H$ cannot be too large. As a result, equilibrium trade is slow relative to the first-best outcome. To see why (2) holds, suppose that the seller's belief cutoff at t is κ_t^H, and that her cost falls from c_H to $c_L = 0$ at this period. The seller's profit from posting price $p^L(\kappa_t^H)$ and revealing that her cost is c_L is $U^L(\kappa_t^H)$. The seller's profit from mimicking a high cost seller for one period, and revealing her cost at $t+\Delta$, is

$$\frac{F(\kappa_t^H) - F(\kappa_{t+\Delta}^H)}{F(\kappa_t^H)} p_t^H + e^{-r\Delta} \frac{F(\kappa_{t+\Delta}^H)}{F(\kappa_t^H)} U^L(\kappa_{t+\Delta}^H).$$

Inequality (2) guarantees that this deviation is not profitable.

Third, equation (3) shows that the seller's equilibrium payoff when her cost is c_H must be at least as large as what she would get by delaying trade until her cost falls to c_L, and playing the continuation equilibrium from that point onwards.

Proposition 3. *Suppose sequences $\{p_\tau^H, \kappa_\tau^H\}$ are induced by an equilibrium $(\sigma, \mu) \in \Sigma^{\mathsf{S}}(\Delta)$. Then, for all $t \in T(\Delta)$, $\kappa_t^H \geq v^*(\Delta)$.*

Proposition 3 shows that any inefficiency takes the form of too much delay: since $\kappa_t^H \geq v^*(\Delta)$ for all t, a buyer with value below cutoff $v^*(\Delta)$ only trades when seller's cost is c_L.

Equilibrium existence. For all $\Delta > 0$, there always exist sequences $\{p_\tau^H, \kappa_\tau^H\}$ satisfying the conditions in Theorem 1(i). For instance, sequences $\{p_\tau^H, \kappa_\tau^H\}$ with $\kappa_\tau^H = \overline{v}$ and $p_\tau^H = p$ for all τ, with p satisfying

$$\overline{v} - p = e^{-(r+\lambda)\Delta}(\overline{v} - p) + e^{-r\Delta}(1 - e^{-\lambda\Delta})(\overline{v} - p^L(\overline{v}))$$

satisfy (1)-(3). Hence, by Theorem 1(ii), $\Sigma^{\mathsf{S}}(\Delta)$ is non-empty whenever Δ is small enough. Under this PBE, the seller posts a very high price while her costs are high, which all buyer types reject. Buyer and seller only trade when seller's costs fall to c_L. Note, however, that the game also admits more efficient separating equilibria, under which buyer and seller trade with positive probability while $c_t = c_H$.

Stationary equilibria. Appendix A shows that, when looking for equilibria in $\Sigma^{\mathsf{S}}(\Delta)$, it is without loss to focus on equilibria that are weakly stationary; i.e., equilibria in which, at histories where the current price is the lowest among all past prices posted (given buyer's current beliefs), the buyer's acceptance rule depends on her value and her beliefs about the seller's cost. In particular, the arguments in the proof of Theorem 1 imply that, for all $\Delta < \overline{\Delta}$ and for any $(\sigma, \mu) \in \Sigma^{\mathsf{S}}(\Delta)$, there exists a weakly stationary equilibrium $(\sigma^{\mathsf{s}}, \mu^{\mathsf{s}}) \in \Sigma^{\mathsf{S}}(\Delta)$ that induces the same outcome as (σ, μ).

Welfare maximizing equilibria. In any equilibrium, the probability with which buyer and seller trade while seller's cost is high is bounded by inequality (2). This delayed trade is socially costly. Therefore, under the most efficient equilibrium in $\Sigma^{\mathsf{S}}(\Delta)$, constraint (2) binds at (almost) all periods t with $\kappa_t^H > \kappa_{t+\Delta}^H$.

5 Frequent-offer Limit

This section studies the frequent-offer limit of welfare maximizing equilibria. For each $\Delta > 0$, let $(\sigma^\Delta, \mu^\Delta)$ be an equilibrium in $\Sigma^{\mathsf{S}}(\Delta)$ achieving the largest social welfare (among equilibria in $\Sigma^{\mathsf{S}}(\Delta)$). Let $\{p_t^H(\Delta), \kappa_t^H(\Delta)\}$ denote the prices and belief cutoffs induced by $(\sigma^\Delta, \mu^\Delta)$ at periods at which the seller's costs are c_H.

Recall that, when the seller's costs fall to c_L, continuation play under any equilibrium in $\Sigma^{\mathsf{S}}(\Delta)$ is equivalent to the continuation equilibrium in a game with one-sided private information. By Fudenberg et al. (1985) and Gul et al. (1985), as $\Delta \to 0$, the seller's price converges to $\underline{v}$ (regardless of her belief cutoff), the buyer buys immediately at this price, and the seller obtains profits $\underline{v}$.

Define $\hat{v} \equiv \lim_{\Delta \to 0} v^*(\Delta) = \frac{r+\lambda}{r} c_H$. The following result holds.

Theorem 2. *There exists functions* $p^H : \mathbb{R}_+ \to \mathbb{R}$ *and* $\kappa^H : \mathbb{R}_+ \to [\underline{v}, \overline{v}]$ *such that, for all* $t \in T(\Delta)$, $\lim_{\Delta\to 0} p_t^H(\Delta) = p^H(t)$ *and* $\lim_{\Delta\to 0} \kappa_t^H(\Delta) = \kappa^H(t)$.

Functions $p^H(t)$ *and* $\kappa^H(t)$ *satisfy*

$$-\frac{dp^H(t)}{dt} = r(\kappa^H(t) - p^H(t)) + \lambda(\underline{v} - p^H(t)), \tag{4}$$

$$-\frac{d\kappa^H(t)}{dt} = \frac{F(\kappa^H(t))}{f(\kappa^H(t))} \frac{r\underline{v}}{(p^H(t) - \underline{v})}, \tag{5}$$

for all $t \leq \hat{t} = \inf\{t \geq 0 : \kappa^H(t) = \hat{v}\}$, *with boundary conditions* $\kappa^H(0) = \overline{v}$ *and* $p^H(\hat{t}) = c_H + \frac{\lambda}{r+\lambda}\underline{v}$. *For all* $t > \hat{t}$, $\frac{dp^H(t)}{dt} = \frac{d\kappa^H(t)}{dt} = 0$.

Theorem 2 shows that the frequent-offer limit of welfare maximizing equilibrium is characterized by a system of differential equations. Equation (4) has the following interpretation. The buyer's benefit from delaying her purchase for an instant at time t while $c_t = c_H$ is

$$-\frac{dp^H(t)}{dt} + \lambda(p^H(t) - \underline{v}).$$

Indeed, the seller's price falls at rate $\frac{dp^H(t)}{dt}$ if costs remain high, and drops from $p^H(t)$ to $\underline{v}$ if costs fall to c_L. By equation (4), this benefit must equal the cost $r(\kappa^H(t) - p^H(t))$ that the marginal buyer type $\kappa^H(t)$ incurs from delaying trade for an instant.

To see the intuition for (5), note that the equation can be written as

$$-\frac{d\kappa^H(t)}{dt} \frac{f(\kappa^H(t))}{F(\kappa^H(t))}(p^H(t) - \underline{v}) = r\underline{v}. \tag{6}$$

The left-hand side of equation (6) is the net benefit that a seller whose cost fell to $c_L = 0$ at time t obtains from pretending that her cost is c_H for an instant longer. Indeed, the seller makes a sell with instantaneous probability $-\frac{d\kappa^H(t)}{dt}\frac{f(\kappa^H(t))}{F(\kappa^H(t))}$ if she pretends to have cost c_H, and sells at price $p^H(t)$ instead of $\underline{v}$. The right-hand side of (6) is the cost in terms of delayed trade that the seller incurs by following such a mimicking strategy. The speed of trade $-\frac{d\kappa^H(t)}{dt}\frac{f(\kappa^H(t))}{F(\kappa^H(t))}$ under a welfare maximizing equilibrium is such that the net gain from pretending to have a high cost is equal to the cost of delayed trade.

Theorem 2 shows that, in the continuous-time limit, while costs are c_H the seller trades with the buyer until her belief cutoff reaches the efficient cutoff $\hat{v} = \frac{r+\lambda}{r}c_H$; i.e., until time $\hat{t} = \inf\{t \geq 0 : \kappa^H(t) = \hat{v}\}$. Price $p^H(\hat{t})$ at which a buyer with type $\hat{v}$ trades leaves this buyer indifferent between buying at $\hat{t}$, or waiting and buying at price $\underline{v}$ when costs fall to c_L:

$$\hat{v} - p^H(\hat{t}) = \frac{\lambda}{r+\lambda}(\hat{v} - \underline{v}) \iff p^H(\hat{t}) = c_H + \frac{\lambda}{r+\lambda}\underline{v},$$

where the second equality uses $\hat{v} = \frac{r+\lambda}{r}c_H$.

The dynamics in Theorem 2 are broadly consistent with dynamics typically observed in markets for new durable goods. During the early stages, prices typically fall gradually, but at a faster rate than costs, and market penetration raises slowly (Conlon, 2012).

Relation to models with two-sided private information. Theorem 2 allows for a comparison between the current model and separating stationary equilibria of models with two-sided private information. Cho (1990) shows that, in such models, separating stationary equilibria satisfy a version of the Coase conjecture: when gains from trade are common knowledge (i.e., seller's highest cost is lower than buyer's lowest value), bargaining outcomes are efficient, and the seller cannot extract rents from high value buyers.

In the current model, in contrast, bargaining inefficiencies persist even when gains from trade are common knowledge. Indeed, information revelation constraint (2) (or (6) in the continuous-time limit) bounds the rate at which buyer and seller trade while costs are high even when gains from trade are common knowledge (i.e., even when $c_H < \underline{v}$).

De-coupling equation (4)**.** The system of differential equations (4)-(5) in Theorem 2 is coupled. I now show how to transform it to get a de-coupled ODE for prices.

For each $\kappa \in [\hat{v}, \overline{v}]$, let $P^H(\kappa)$ denote the price at which a buyer with value κ trades when costs are c_H in the continuous-time limit; that is, for all $t \in [0, \hat{t}]$, $P^H(\kappa^H(t)) = p^H(t)$. Combining (4) and (5), and using $\frac{dp^H(t)}{dt} = \frac{dP^H(\kappa^H(t))}{d\kappa^H}\frac{d\kappa^H(t)}{dt}$, $P^H(\cdot)$ solves:

$$\forall \kappa \in [\hat{v}, \overline{v}], \quad \frac{dP^H(\kappa)}{d\kappa} = \left(r(\kappa - P^H(\kappa)) + \lambda(\underline{v} - P^H(\kappa))\right)\frac{f(\kappa)}{F(\kappa)}\frac{(P^H(\kappa) - \underline{v})}{r\underline{v}}, \tag{7}$$

with $P^H(\hat{v}) = c_H + \frac{\lambda}{r+\lambda}\underline{v}$.

Comparative statics. I now use Theorem 2 to study equilibrium properties and derive several comparative statics. The first result shows how prices and speed of trade change with changes in (i) cost c_H, (ii) value distribution F, and (iii) rate λ at which costs fall.

Proposition 4. *The following comparative statics hold:*

(i) as c_H increases, price $P^H(\kappa)$ increases for all $\kappa > \hat{v}$, and speed of trade $-\frac{d\kappa^H(t)}{dt}\frac{f(\kappa^H(t))}{F(\kappa^H(t))}$ falls for all $t < \hat{t}$.

(ii) as F increases in terms of its reverse hazard rate, price $P^H(\kappa)$ increases for all $\kappa > \hat{v}$, and speed of trade $-\frac{d\kappa^H(t)}{dt}\frac{f(\kappa^H(t))}{F(\kappa^H(t))}$ falls for all $t < \hat{t}$.

(iii) as λ increases, price $P^H(\kappa)$ increases for all $\kappa > \hat{v}$ close to $\hat{v}$, and speed of trade $-\frac{d\kappa^H(t)}{dt}\frac{f(\kappa^H(t))}{F(\kappa^H(t))}$ falls for all t close to $\hat{t}$.

The first part of Proposition 4 shows that the prices at which the different buyer types trade when costs are high increase with an increase in c_H.[11] Since prices are now higher, by equation (6) the rate at which buyer and seller trade when costs are high must be adjusted downwards to deter a low cost seller from pretending to have a high cost. The second and third parts of Proposition 4 establish similar results for changes in the value distribution and in the rate at which the seller's cost falls.

The last result in this section studies equilibrium outcomes as the buyer's lowest value $\underline{v}$ becomes small. Recall that $q = \text{prob}(c_0 = c_H)$.

[11]Note that, although cost c_H does not appear in the ODE (7), it does appear in the boundary condition $P(\hat{v}) = P\left(\frac{\lambda+r}{r}c_H\right) = c_H + \frac{\lambda}{r+\lambda}\underline{v}$.

Proposition 5. *In the limit as $\underline{v} \to 0$,*

(i) speed of trade $-\frac{d\kappa^H(t)}{dt}\frac{f(\kappa^H(t))}{F(\kappa^H(t))}$ *converges to zero for all t;*

(ii) seller's equilibrium profits converge to zero; and

(iii) total equilibrium surplus converges to $(q\frac{\lambda}{r+\lambda} + 1 - q)\mathbb{E}[v]$.

Proposition 5 follows from equation (6): as $\underline{v} \to 0$, the speed at which buyer and seller trade while costs are c_H must converge to zero to deter a low cost seller from pretending to have a high cost.

Proposition 5 allows for further comparisons between the current model and previous models in the literature. When the seller's production cost is fixed and publicly known, the seller's profits converge to zero as the buyer's lowest valuation $\underline{v}$ converges to zero (Fudenberg et al., 1985, Gul et al., 1985). But the limiting equilibrium outcome is efficient: all buyers trade immediately at price equal to marginal cost.

For models with two-sided private information and with time-invariant costs, the results in Cho (1990) imply that, in any separating stationary equilibrium, the seller's profits also converge to zero as the buyer's lowest value converges to zero. However, inefficiencies "explode" in this limit: only the seller with the lowest possible cost makes sales.[12]

Proposition 5 illustrates how these results generalize when the seller is privately informed about her time-varying production cost. As in the two cases described above, the seller's profits go to zero as the buyer's lowest value $\underline{v}$ goes to zero. Moreover, as in Cho (1990), inefficiencies also grow in this "gapless" limit. The difference, however, is that seller and buyer eventually trade with probability 1 in this model, when costs fall to c_L.

[12] A related result is proved in Ausubel and Deneckere (1992), Theorem 1.

6 Publicly observable costs

This section compares equilibrium outcomes described above with equilibrium outcomes of a model in which the seller's changing cost is publicly observable, as in Ortner (2017). Let $\Sigma^{\mathsf{pub}}(\Delta)$ denote the set of weakly stationary PBE of the game with publicly observable costs.[13] Note that, in any $(\sigma, \mu) \in \Sigma^{\mathsf{pub}}(\Delta)$, continuation play when costs are c_L is identical to continuation play in a model with one-sided private information. In particular, as $\Delta \to 0$, the seller's price when costs are c_L converges to $\underline{v}$, and the buyer accepts this price immediately.

For each $\Delta > 0$, let $(\sigma^\Delta, \mu^\Delta) \in \Sigma^{\mathsf{pub}}(\Delta)$ be a weakly stationary equilibrium of the game with public costs, and let $U^{\mathsf{pub}}(\sigma^\Delta, \mu^\Delta; \Delta)$ denote the seller's profits under $(\sigma^\Delta, \mu^\Delta)$ at $t = 0$, conditional on her initial cost being c_H. Recall that τ_L is the first time the seller's cost falls to c_L. The following result holds.

Theorem 3. *Suppose the seller's costs are publicly observable. Then, as $\Delta \to 0$:*

(i) the equilibrium outcome under $(\sigma^\Delta, \mu^\Delta)$ converges to the first-best outcome;

(ii) if $c_0 = c_H$, the seller's initial price under $(\sigma^\Delta, \mu^\Delta)$ converges to $c_H + \frac{\lambda}{r+\lambda}\underline{v}$; and

(iii) if $c_0 = c_H$, seller's profits $U^{\mathsf{pub}}(\sigma^\Delta, \mu^\Delta; \Delta)$ converge to $\frac{\lambda}{r+\lambda}\underline{v}$.

Theorem 3 generalizes the classic Coase conjecture (Coase, 1972, Fudenberg et al., 1985, Gul et al., 1985) to settings in which production costs publicly change over time. As the time period vanishes, the equilibrium outcome becomes efficient, and the seller is unable to extract rents from high value buyers: her profits when $c_0 = c_H$ are $\frac{\lambda}{r+\lambda}\underline{v}$, which is exactly what she would obtain if the buyer's value was $\underline{v}$ with probability 1.[14]

Interestingly, even if the seller earns exactly what she would earn if the buyer had the lowest value $\underline{v}$, different types of buyers trade at different times, and at different prices: when

[13] Existence of weakly stationary PBE can be proved by extending arguments in Gul et al. (1985).

[14] Indeed, if the buyer's valuation was $\underline{v}$ with probability 1, the seller would optimally wait until time τ_L, would charge price $\underline{v}$ at that point, and her expected profits would be $\frac{\lambda}{r+\lambda}\underline{v}$.

initial costs are c_H, buyers with value above $\hat{v}$ buy at time $t = 0$ at price $c_H + \frac{\lambda}{r+\lambda}\underline{v} > \underline{v}$, and buyers with value below $\hat{v}$ buy at time τ_L at price $\underline{v}$. However, the profit margin that the seller makes on a high value buyer is exactly equal to the expected discounted profit margin from selling to a low value buyer when costs fall to c_L.

Theorem 3 contrasts sharply with the results in Theorems 1 and 2: the seller is able to extract rents from high value buyers when she privately observes her cost, and the market outcome is inefficient. Indeed, with privately observed costs, trade is inefficiently delayed to satisfy the information revelation constraint in (2). As in the commitment solution in Proposition 2, this inefficiently delayed trade reduces the rents of high value buyers, and allows the seller to obtain larger profits.

Theorem 3, together with Theorems 1 and 2, imply that the buyer is worse-off when the seller privately observes her evolving cost of production. However, this evolving private information affects different buyer types differently: buyers with valuation above $\hat{v}$ are strictly worse-off when the seller privately observes her costs, whereas buyers with valuation below $\hat{v}$ are indifferent (since, in both cases, they trade at time τ_L at price $\underline{v}$).

It is worth highlighting that the model with public costs cannot rationalize the price dynamics typically observed in markets for new durable goods. In those markets, during the early stages prices tend to fall at a faster pace than costs, leading to falling profit margins. In contrast, Theorem 3 shows that prices fall in tandem with costs when costs are public, and profit margins increase over time (from $\frac{\lambda}{r+\lambda}\underline{v}$ when $c = c_H$, to $\underline{v}$ when $c = c_L = 0$).

Relation to Ortner (2017). Ortner (2017) studies a durable goods monopoly model in which the seller's costs are publicly observed and change stochastically over time. The key results in that paper are: (i) time-varying costs allow the seller to extract rents when buyer values are discrete; (ii) when there is a continuum of buyer types (as in the current model), the seller is unable to extract rents and the market outcome is efficient, as in Theorem 3.

The key difference is that Ortner (2017) casts the model directly in continuous time, and introduces a new equilibrium notion to get around well-known difficulties in analyzing continuous-time games with observable actions. Hence, Theorem 3 is a new result, which shows that the conclusions in Ortner (2017) still hold in the continuous-time limit of discrete-time games.[15]

7 Discussion

I end the paper with a discussion of three issues: (i) conditions under which the model admits an efficient bargaining mechanism; (ii) how the results extend when low cost c_L is not absorbing, and (iii) other types of equilibria.

An efficient mechanism. Online Appendix B shows that, under certain conditions, the framework I study admits a mechanism that is budget balance, incentive compatible and individual rational, and that implements the first-best outcome. This is true in spite of the fact that both players have private information at the start of the game. Online Appendix B also shows that the game does not admit an efficient equilibrium (separating or not).

The existence of an efficient mechanism satisfying IC, IR and budget balance distinguishes the current model from prior bargaining games with two-sided asymmetric information. For instance, the equilibria in Cho (1990) are inefficient only when the distribution of buyer values and the distribution of seller costs overlap. But we know from Myerson and Satterthwaite (1983) that such a framework does not admit an efficient mechanism satisfying IC, IR and budget balance.

Increasing costs. The model assumes that cost c_L is absorbing. This assumption greatly simplifies the analysis and exposition. Indeed, it implies that in any equilibrium in $\Sigma^{\mathsf{S}}(\Delta)$, continuation play when the seller's cost falls is equivalent to equilibrium play in a model

[15]Another difference is that, in Ortner (2017), the seller's cost evolves as a geometric Brownian motion.

with one-sided private information. Hence, in any separating equilibrium, as $\Delta \to 0$ the seller's price converges to $\underline{v}$ when costs are low, and the buyer trades immediately.

In Online Appendix C, I study the case in which the cost evolution satisfies $\text{prob}(c_{t+\Delta} = c_H | c_t = c_H) = e^{-\lambda\Delta}$ and $\text{prob}(c_{t+\Delta} = c_L | c_t = c_L) = e^{-\gamma\Delta}$. I show that, in any weakly stationary separating equilibrium, the seller's price also converges to $\underline{v}$ as $\Delta \to 0$ whenever her cost is c_L, and trade is immediate. Indeed, when costs are c_L, the Coasian incentive to accelerate trade constraints the seller's ability to extract rents from high value buyers.

With increasing costs, while $c_t = c_H$ prices must still satisfy equation (1). Online Appendix C shows that, in this setting, the seller's IC constraint (2) becomes:

$$\frac{F(\kappa_t^H) - F(\kappa_{t+\Delta}^H)}{F(\kappa_t^H)} p_t^H \leq U^L(\kappa_t^H) - e^{-(r+\gamma)\Delta} \frac{F(\kappa_{t+\Delta}^H)}{F(\kappa_t^H)} U^L(\kappa_{t+\Delta}^H) - e^{-r\Delta}(1 - e^{-\gamma\Delta}) \frac{F(\kappa_{t+\Delta}^H)}{F(\kappa_t^H)} U^H(\kappa_{t+\Delta}^H). \tag{8}$$

The last term in (8) takes into account the possibility that the seller's cost increases next period if her current cost is c_L, in which case the seller gets a continuation profit $U^H(\kappa_{t+\Delta}^H)$.[16] Lastly, in any weakly stationary PBE, it must be that, $\forall \kappa,\ U^H(\kappa) \geq \rho(\Delta) U^L(\kappa)$.

Other equilibria. Throughout the paper, I focused on separating equilibria, under which the seller's price each period perfectly reveals her current cost realization. Such equilibria are intuitive, tractable, and help rationalize observed pricing dynamics in markets for new durable goods. Moreover, such equilibria represent a natural point of comparison to prior papers in the literature, like Cho (1990) and Ortner (2017)

However, the game admits many other equilibria. First, the game admits semi-separating equilibria, in which a seller with $c_t = c_H$ posts price p_t^H, and a seller whose cost fell to c_L posts price p_t^H with probability $1 - \alpha_t$ and price $p^L(\kappa_t^H)$ with probability $\alpha_t \in (0, 1)$.[17] Under

[16] Appendix C briefly shows how Theorem 2 extends to this environment.

[17] Under such an equilibrium, once the seller posts price $p^L(\kappa_t^H)$, she reveals that her cost is c_L, and the continuation equilibrium is as in Gul et al. (1985), Fudenberg et al. (1985).

such an equilibrium, equation (2) holds with equality at all periods t in which $\alpha_t \in (0,1)$. Hence, under a semi-separating equilibrium the speed of trade is also slow relative to the first-best. On the other hand, equation (1) becomes

$$\kappa^H_{t+\Delta} - p^H_t = e^{-r\Delta}(\mu^B_{t+\Delta}e^{-\lambda\Delta} + (1-\mu^B_{t+\Delta})\alpha_{t+\Delta})(\kappa^H_{t+\Delta} - p^H_{t+\Delta}) \\ + e^{-r\Delta}(\mu^B_{t+\Delta}(1-e^{-\lambda\Delta}) + (1-\mu^B_{t+\Delta}))(1-\alpha_{t+\Delta})(\kappa^H_{t+\Delta} - p^L(\kappa^H_{t+\Delta})), \quad (9)$$

where μ^B_τ is the probability that the buyer assigns to the seller's cost being c_H at the beginning of period τ (i.e., before observing the seller's price at time τ).

The game also admits pooling equilibria. For instance, the game admits equilibria in which: (i) both types of sellers post the same price at times $t = 0, ..., \tau - \Delta$; and (ii) buyer and seller play a continuation equilibrium in Σ^{S} from time τ onwards.

When $q = \text{prob}(c_0 = c_H)$ is small, pooling equilibria are less efficient than separating equilibrium. Indeed, when $q \approx 0$, separating equilibria are approximately efficient in the limit as $\Delta \to 0$. However, since pooling equilibria don't need to satisfy information revelation constraint (2) during the pooling periods, when q is large there are pooling equilibria that are more efficient than separating equilibria.

Appendix

A Proofs of Theorem 1 and Proposition 3

In any PBE in Σ^{S}, when costs falls to c_L, continuation play coincides with equilibrium play in the one-sided incomplete information game in Fudenberg et al. (1985) and Gul et al. (1985).[18] Hence, I focus on characterizing equilibrium behavior at periods t with $c_t = c_H$.

By the skimming property, any PBE in Σ^{S} induces a decreasing sequence of belief cutoffs

[18]For ease of exposition, throughout Appendix A I drop the dependence on time period Δ.

$\{\kappa_t^H\}$ such that along the path of play, at any time t with $c_t = c_H$, (i) the seller believes that the buyer's type lies in $[\underline{v}, \kappa_t^H]$, and (ii) the buyer buys at time t if and only if her valuation lies in $[\kappa_{t+\Delta}^H, \kappa_t^H)$.

Lemma A.1. *Fix a PBE $(\sigma, \mu) \in \Sigma^{\mathsf{S}}$. Consider a seller history h_t^S with $c_s = c_H$ for all $s < t$ such that the seller's belief cutoff κ_t at time t is strictly larger than $\underline{v}$. Let p_t^H be the price that the seller charges under (σ, μ) at history h_t^S if $c_t = c_H$, and let $\kappa_{t+\Delta}$ be the highest consumer type that buys at time t when $c_t = c_H$. Then, κ_t and $\kappa_{t+\Delta}$ satisfy*

$$p_t^H \frac{F(\kappa_t) - F(\kappa_{t+\Delta})}{F(\kappa_t)} \le U^L(\kappa_t) - e^{-r\Delta} \frac{F(\kappa_{t+\Delta})}{F(\kappa_t)} U^L(\kappa_{t+\Delta}). \tag{10}$$

Proof. Consider a seller whose cost changed from c_H to c_L after history h_t^S. The profits that this seller obtains by revealing her cost are $U^L(\kappa_t)$. The profits that this seller would make by posting price p_t^H that she would have posted if $c_t = c_H$, and then from $t + \Delta$ onwards playing the continuation strategy with common knowledge cost c_L and belief cutoff $\kappa_{t+\Delta}$ are $p_t^H \frac{F(\kappa_t) - F(\kappa_{t+\Delta})}{F(\kappa_t)} + e^{-r\Delta} \frac{F(\kappa_{t+\Delta})}{F(\kappa_t)} U^L(\kappa_{t+\Delta})$. A seller whose cost changed to c_L at period t has an incentive to reveal her cost only if (10) holds. ■

Recall that $\rho = \frac{e^{-r\Delta}(1-e^{-\lambda\Delta})}{1-e^{-(r+\lambda)\Delta}}$. Fix a PBE in Σ^{S}, and consider a seller history h_t^S with $c_s = c_H$ for all $s < t$ leading to belief cutoff $\kappa_t^H = \kappa$. Note that at such a history, a seller with cost $c_t = c_H$ can obtain a payoff equal to $\rho U^L(\kappa)$ by posting prices above κ at all periods until her costs fall to c_L, and then playing her continuation strategy. Hence, the seller's continuation profits at this history under (σ, μ) cannot be lower than $\rho U^L(\kappa)$.

Lemma A.2. *Fix a PBE $(\sigma, \mu) \in \Sigma^{\mathsf{S}}$, and consider a seller history h_t^S with belief cutoff κ_t. If $c_t = c_H$, then $\kappa_{t+\Delta} \ge \min\{\kappa_t, v^*\}$. In particular, if $\kappa_t \le v^*$ and $c_t = c_H$, the seller makes a sale with probability zero at time t.*

Proof. Towards a contradiction, suppose that $c_t = c_H$ and $\kappa_{t+\Delta} < \min\{\kappa_t, v^*\} \le v^*$. Let $\{\kappa_{t+\tau\Delta}\}_{\tau=0}^{\infty}$ be a weakly decreasing sequence such that for all $\tau \ge 0$, if the seller's cost is c_H at time $t+\tau\Delta$, under (σ,μ) the seller sells to the buyer when her valuation is in $[\kappa_{t+(\tau+1)\Delta},\kappa_{t+\tau\Delta})$. Let $\{p^H_{t+\tau\Delta}\}_{\tau=0}^{\infty}$ denote the sequence of prices that the seller charges at each time $t+\tau\Delta$ if $c_{t+\tau\Delta}=c_H$. Recall that $p^L(\kappa)$ is the price that the seller charges if her cutoff belief is κ and her costs are c_L. By Fudenberg et al. (1985) and Gul et al. (1985), $p^L(\kappa)$ is weakly increasing in κ.

Note first that, for all $\tau\geq 0$, it must be that

$$\kappa_{t+(\tau+1)\Delta}-p^H_{t+\tau\Delta}\geq\rho(\kappa_{t+(\tau+1)\Delta}-p^L(\kappa_{t+(\tau+1)\Delta})). \tag{11}$$

Indeed, a buyer with value $\kappa_{t+(\tau+1)\Delta}$ can guarantee a payoff of at least $\rho(\kappa_{t+(\tau+1)\Delta}-p^L(\kappa_{t+(\tau+1)\Delta}))$ by delaying her purchase until the seller's cost falls to c_L. Note further that,

$$\kappa_{t+(\tau+1)\Delta}-c_H<\rho\kappa_{t+(\tau+1)\Delta},$$

where the inequality follows since $\kappa_{t+(\tau+1)\Delta}\leq\kappa_{t+\Delta}<v^*$ and since $v^*-c_H=\rho v^*$. Combining this inequality with inequality (11),

$$p^H_{t+\tau\Delta}\leq(1-\rho)\kappa_{t+(\tau+1)\Delta}+\rho p^L(\kappa_{t+(\tau+1)\Delta})<c_H+\rho p^L(\kappa_{t+(\tau+1)\Delta}) \tag{12}$$

Equation (12) implies that the profit margin $p^H_{t+\tau\Delta}-c_H$ that the seller earns from selling to consumers with value $v\in[\kappa_{t+(\tau+1)\Delta},\kappa_{t+\tau\Delta})$ when her costs are c_H is strictly lower than the expected discounted profit margin $\rho p^L(\kappa_{t+(\tau+1)\Delta})$ that the seller would earn if she waited until her costs fell to $c_L=0$ and then charged price of $p^L(\kappa_{t+(\tau+1)\Delta})$.

For all $s\in T(\Delta)$, let U^H_s denote the seller's on-path continuation payoff at time s if $c_s=c_H$ under equilibrium (σ,μ). For all $\kappa\in[\underline{v},\overline{v}]$, recall that $U^L(\kappa)$ is the seller's continuation payoff under (σ,μ) at a history with belief cutoff κ and at which her costs are c_L, respectively. I now use equation (12) to show that $U^H_t<\rho U^L(\kappa_t)$. This implies that (σ,μ) cannot be an equilibrium, since at time t the seller can earn $\rho U^L(\kappa_t)$ by waiting until her costs fall to c_L

and then playing the continuation equilibrium from that point onwards.

Note that, for all $\tau \geq 0$,

$$
\begin{aligned}
U^H_{t+\tau\Delta} &= (p^H_{t+\tau\Delta} - c_H)\frac{F(\kappa_{t+\tau\Delta}) - F(\kappa_{t+(\tau+1)\Delta})}{F(\kappa_{t+\tau\Delta})} + e^{-(r+\lambda)\Delta}\frac{F(\kappa_{t+(\tau+1)\Delta})}{F(\kappa_{t+\tau\Delta})}U^H_{t+(\tau+1)\Delta} \\
&\quad + e^{-r\Delta}\frac{F(\kappa_{t+(\tau+1)\Delta})}{F(\kappa_{t+\tau\Delta})}(1-e^{-\lambda\Delta})U^L(\kappa_{t+(\tau+1)\Delta}) \\
&< \rho p^L(\kappa_{t+(\tau+1)\Delta})\frac{F(\kappa_{t+\tau\Delta}) - F(\kappa_{t+(\tau+1)\Delta})}{F(\kappa_{t+\tau\Delta})} + e^{-(r+\lambda)\Delta}\frac{F(\kappa_{t+(\tau+1)\Delta})}{F(\kappa_{t+\tau\Delta})}U^H_{t+(\tau+1)\Delta} \\
&\quad + e^{-r\Delta}\frac{F(\kappa_{t+(\tau+1)\Delta})}{F(\kappa_{t+\tau\Delta})}(1-e^{-\lambda\Delta})U^L(\kappa_{t+(\tau+1)\Delta}) \\
&= \rho\left(p^L(\kappa_{t+(\tau+1)\Delta})\frac{F(\kappa_{t+\tau\Delta}) - F(\kappa_{t+(\tau+1)\Delta})}{F(\kappa_{t+\tau\Delta})} + \frac{F(\kappa_{t+(\tau+1)\Delta})}{F(\kappa_{t+\tau\Delta})}U^L(\kappa_{t+(\tau+1)\Delta})\right) \\
&\quad - e^{-(r+\lambda)\Delta}\rho\frac{F(\kappa_{t+(\tau+1)\Delta})}{F(\kappa_{t+\tau\Delta})}U^L(\kappa_{t+(\tau+1)\Delta}) + e^{-(r+\lambda)\Delta}\frac{F(\kappa_{t+(\tau+1)\Delta})}{F(\kappa_{t+\tau\Delta})}U^H_{t+(\tau+1)\Delta} \qquad (13)
\end{aligned}
$$

where the strict inequality follows from (12), and the last equality uses $\rho = e^{-r\Delta}(1-e^{-\lambda\Delta}) + e^{-(r+\lambda)\Delta}\rho$. Note next that, for all $\tau \geq 0$,

$$
U^L(\kappa_{t+\tau\Delta}) \geq p^L(\kappa_{t+(\tau+1)\Delta})\frac{F(\kappa_{t+\tau\Delta}) - F(\kappa_{t+(\tau+1)\Delta})}{F(\kappa_{t+\tau\Delta})} + \frac{F(\kappa_{t+(\tau+1)\Delta})}{F(\kappa_{t+\tau\Delta})}U^L(\kappa_{t+(\tau+1)\Delta}). \qquad (14)
$$

Indeed, a seller with cost $c = c_L$ and with belief cutoff $\kappa_{t+\tau\Delta}$ can earn the right-hand side of (14) by posting price $p^L(\kappa_{t+(\tau+1)\Delta})$ and then playing her continuation strategy.[19] Combining (14) with (13), for all $\tau \geq 0$,

$$
U^H_{t+\tau\Delta} < \rho\left(U^L(\kappa_{t+\tau\Delta}) - e^{-(r+\lambda)\Delta}\frac{F(\kappa_{t+(\tau+1)\Delta})}{F(\kappa_{t+\tau\Delta})}U^L(\kappa_{t+(\tau+1)\Delta})\right) + e^{-(r+\lambda)\Delta}\frac{F(\kappa_{t+(\tau+1)\Delta})}{F(\kappa_{t+\tau\Delta})}U^H_{t+(\tau+1)\Delta}. \qquad (15)
$$

[19]This follows since the equilibrium of the game with one-sided incomplete information is weakly stationary (Gul et al., 1985).

Using equation (15) repeatedly for all $\tau \geq 0$ yields

$$U_t^H < \sum_{\tau=0}^{\infty} e^{-(r+\lambda)\tau\Delta}\rho\left(\frac{F(\kappa_{t+\tau\Delta})}{F(\kappa_t)}U^L(\kappa_{t+\tau\Delta}) - e^{-(r+\lambda)\Delta}\frac{F(\kappa_{t+(\tau+1)\Delta})}{F(\kappa_t)}U^L(\kappa_{t+(\tau+1)\Delta})\right) = \rho U^L(\kappa_t).$$

But this cannot be, since a seller whose cost is c_H at time t can obtain $\rho U^L(\kappa_t)$ by waiting until her costs fall to $c_L = 0$ and then playing her continuation strategy. ■

For any equilibrium $(\sigma, \mu) \in \Sigma^{\mathsf{S}}$, let

$$\kappa^{(\sigma,\mu)} = \inf\{\kappa \in [\underline{v}, \overline{v}] : \exists \text{ on-path history } (h_t^S \sqcup c_H) \text{ at which type } \kappa \text{ buys under } (\sigma,\mu).\}$$

Note that $\kappa^{(\sigma,\mu)}$ is the lowest valuation at which the buyer buys when costs are c_H under equilibrium (σ, μ). By Lemma A.2, $\kappa^{(\sigma,\mu)} \geq v^*$ for all $(\sigma, \mu) \in \Sigma^{\mathsf{S}}$.

Fix a PBE $(\sigma, \mu) \in \Sigma^{\mathsf{S}}$. Let $\{\kappa_t^H\}$ be the sequence of beliefs cutoffs induced by (σ, μ) at histories at which the seller's costs is c_H. Under (σ, μ), a high cost seller stops selling whenever her cutoff beliefs about the buyer's valuation reach $\kappa^{(\sigma,\mu)}$, so $\kappa_t^H \geq \kappa^{(\sigma,\mu)}$ for all t.

Let $\hat{t}$ denote the time at which a high cost seller sells to a buyer with valuation $\kappa^{(\sigma,\mu)}$, provided that $\hat{t}$ is finite, and let $\kappa_{\hat{t}+\Delta}^H = \kappa^{(\sigma,\mu)}$. Note that, for all periods $t \geq \hat{t} + \Delta$ a high cost seller does not make sales. Hence $\kappa_t^H = \kappa_{\hat{t}+\Delta}^H$ for all $t \geq \hat{t} + \Delta$ (if $\hat{t}$ is infinite, this is vacuous).

Let $\{p_t^H\}_{t=0}^{\hat{t}}$ be the prices that the seller charges at times $t \leq \hat{t}$ under (σ, μ) at histories at which her cost is high. For all $t \leq \hat{t} - \Delta$, these prices satisfy:

$$\kappa_{t+\Delta}^H - p_t^H = e^{-(r+\lambda)\Delta}(\kappa_{t+\Delta}^H - p_{t+\Delta}^H) + e^{-r\Delta}(1 - e^{-\lambda\Delta})(\kappa_{t+\Delta}^H - p^L(\kappa_{t+\Delta}^H)). \tag{16}$$

Indeed, prices $\{p_t^H\}_{t=0}^{\hat{t}}$ are such that a buyer with valuation $\kappa_{t+\Delta}^H$ is indifferent between buying at time t or waiting and buying at period $t + \Delta$.

For all $\kappa \in [\underline{v}, \overline{v}]$, define $\hat{p}(\kappa) \equiv \kappa(1 - \rho) + \rho p^L(\kappa)$. Price $\hat{p}(\kappa)$ is such that a buyer with valuation κ is indifferent between buying at $\hat{p}(\kappa)$ when costs are c_H and waiting until costs fall to c_L and buying at price $p^L(\kappa)$. Note that $\hat{p}(\kappa)$ is increasing in κ (since $p^L(\kappa)$ is increasing in κ). Note further that, if $\hat{t}$ is finite, it must be that $p^H_{\hat{t}} = \hat{p}(\kappa^{(\sigma,\mu)}) = \kappa^{(\sigma,\mu)}(1-\rho) + \rho p^L(\kappa^{(\sigma,\mu)})$. If $\hat{t}$ is finite, it is without loss to set $p^H_t = p^H_{\hat{t}}$ for all $t \geq \hat{t} + \Delta$.

Given sequences $\{p^H_t, \kappa^H_t\}$, for all times s let $U^H_s(\{p^H_t, \kappa^H_t\})$ be continuation profits that a seller obtains if $c_s = c_H$, when play is given by $\{p^H_t, \kappa^H_t\}$:

$$U^H_s(\{p^H_t, \kappa^H_t\}) = (p^H_s - c_H)\frac{F(\kappa^H_s) - F(\kappa^H_{s+\Delta})}{F(\kappa^H_s)} + e^{-(r+\lambda)\Delta}\frac{F(\kappa^H_{s+\Delta})}{F(\kappa^H_s)}U^H_{s+\Delta}(\{p^H_t, \kappa^H_t\})$$
$$+ e^{-r\Delta}(1 - e^{-\lambda\Delta})\frac{F(\kappa^H_{s+\Delta})}{F(\kappa^H_s)}U^L(\kappa^H_{s+\Delta}).$$

If an equilibrium $(\sigma, \mu) \in \Sigma^{\mathsf{S}}$ induces sequences $\{p^H_t, \kappa^H_t\}$, it must be that

$$\forall s, \quad U^H_s(\{p^H_t, \kappa^H_t\}) \geq \rho U^L(\kappa^H_s). \tag{17}$$

Indeed, a seller whose cost is high by time s and whose belief cutoff is κ^H_s can obtain a payoff of $\rho U^L(\kappa^H_s)$ by waiting until her costs fall to c_L and then playing the continuation equilibrium from that point onwards.

Proof of Theorem 1. The arguments above imply that conditions (1)-(3) must hold in any PBE $(\sigma, \mu) \in \Sigma^{\mathsf{S}}$.

I now turn to the proof of part (ii) of the Theorem. Fix sequences $\{p^H_\tau, \kappa^H_\tau\}$, with $\{\kappa^H_\tau\}$ decreasing, satisfying conditions (1)-(3). I now show show that there exists $\overline{\Delta} > 0$ such that, for all $\Delta \leq \overline{\Delta}$, there exists a PBE $(\sigma, \mu) \in \Sigma^{\mathsf{S}}$ that induces $\{p^H_\tau, \kappa^H_\tau\}$.

Let $\underline{\kappa} = \lim_{t\to\infty} \kappa^H_t$. By Lemma A.2, $\underline{\kappa} \geq v^*$. For all $\kappa \in [\underline{\kappa}, \overline{v}]$, let $\overline{p}^H(\kappa)$ denote the price at which a buyer with type κ buys under $\{p^H_t, \kappa^H_t\}$. For all $\kappa \in [\underline{v}, \overline{\kappa})$, let $\overline{p}^H(\kappa) = \overline{p}^L(\kappa)$, where $\overline{p}^L(\kappa)$ is the price that a buyer with type κ is willing to pay in the game with one-sided private information. The buyer's strategy under the proposed equilibrium (σ, μ) is as follows. For all histories $h^B_t \sqcup p_t$ with $\mu^B(h^B_t \sqcup p_t) = \text{prob}(c_t = c_H | h^B_t \sqcup p_t) = 1$, a buyer with type κ

buys iff $p_t \leq \overline{p}^H(\kappa)$. For all other histories, a buyer with type κ buys iff $p_t \leq \overline{p}^L(\kappa)$.

Buyer's beliefs under (σ, μ) are as follows. If at all periods $s \leq t$ the seller offered price p_s^H, the buyer at time t believes that the seller's cost is c_H with probability 1. In any other case, the buyer at time t believes that the seller's cost is c_L with probability 1.

The seller's strategy is as follows. On the equilibrium path, for all t with $c_t = c_H$, she charges price p_t^H. For all off-path histories $h_t^S \sqcup c_H$, the seller posts a price higher than $\overline{v}$ (and no buyer type buys). For all t with $c_t = c_L$, the seller plays the continuation equilibrium of the game with one-sided private information.

Since $\{p_\tau^H, \kappa_\tau^H\}$ satisfies (16), optimal buyer behavior induces belief cutoffs $\{\kappa_\tau^H\}$, given the seller's strategy. Hence, the buyer's strategy is sequentially rational at histories at which she believes that the seller's cost is high. Moreover, buyer's strategy is sequentially rational at histories at which she believes that the seller's cost is low (since, at such histories, buyer uses the equilibrium strategy of the game with one-sided private information; and since seller uses the equilibrium strategy of the game with one-sided private information whenever her cost is c_L).

I now show that, for Δ small enough, the seller's strategy is also sequentially rational. Note first that, since $\{p_t^H, \kappa_t^H\}$ satisfy (10), the seller does not find it optimal to deviate at a period t such that $c_{t-1} = c_H$ and $c_t = c_L$. Moreover, she doesn't find it optimal to deviate at a period t with $c_{t-1} = c_L$ and $c_t = c_L$ (since, at such histories, buyer and seller are using the equilibrium strategy of the game with one-sided private information).

By the Coase conjecture (Gul et al., 1985), for every $\eta > 0$ there exists $\overline{\Delta}_\eta > 0$ such that, for all $\Delta \leq \overline{\Delta}_\eta$, price $p^L(\kappa)$ that the seller charges when costs are $c = c_L = 0$ is strictly smaller than $\underline{v} + \eta$ for all κ. Pick $\eta' > 0$ such that $\underline{v} + \eta' - c_H < \rho \underline{v}$; since $\underline{v} < v^* = \frac{c_H}{1-\rho}$ (by Assumption 1), such an η' exists. Let $\overline{\Delta} = \overline{\Delta}_{\eta'}$, and suppose $\Delta \leq \overline{\Delta}$. Note that if at a period s with $c_s = c_H$ the seller posts a price different from p_s^H, the highest profit she can obtain is $\rho U^L(\kappa_s^H)$.[20] Since $\{p_t^H, \kappa_t^H\}$ satisfies (17), the seller finds it optimal to post price

[20]This follows since $p^L(\kappa) \in [\underline{v}, \underline{v}+\eta']$ for all $\kappa \in [\underline{v}, \overline{v}]$ whenever $\Delta \leq \overline{\Delta}$, and since $\underline{v}+\eta' - c_H < \rho \underline{v}$. Hence,

p^H_s. ■

Proof of Proposition 3. Follows from Lemma A.2. ■

Mixed strategy equilibria. Theorem 1 characterizes equilibria under which the seller uses a pure action while her costs are c_H.

The game also admits separating equilibria under which the seller mixes while her costs are c_H. In any such equilibrium, the (now random) sequence $\{p^H_t, \kappa^H_t\}$ must still satisfy (10) and (16). Indeed, Lemma A.1 applies to mixed strategy separating equilibria as well. And inequality (17) must hold in any separating equilibrium, pure or mixed. In addition to these conditions, if the seller mixes at some period t with $c_t = c_H$, she must be indifferent among any price that she posts with positive probability.

Welfare maximizing equilibria. Let (σ, μ) be an equilibrium in Σ^{S} that delivers the largest social surplus (among all equilibria in Σ^{S}). Under (σ, μ), constraint (10) must be satisfied with equality at (almost) all times t. As a result, there exists a finite period $\hat{t}$ at which, under (σ, μ), a buyer with value $\kappa^{(\sigma,\mu)}$ buys if $c_{\hat{t}} = c_H$; (and so $\kappa^H_{\hat{t}+\Delta} = \kappa^{(\sigma,\mu)}$).

Moreover, under (σ, μ), the price $p^H_{\hat{t}}$ at which the seller sells at time $\hat{t}$ if $c_{\hat{t}} = c_H$ must be equal to $\hat{p}(\kappa^{(\sigma,\mu)}) = (1-\rho)\kappa^{(\sigma,\mu)} + \rho p^L(\kappa^{(\sigma,\mu)})$. Indeed, if the buyer rejects price $p^H_{\hat{t}}$, buyer and seller don't trade until costs fall to c_L. Price $\hat{p}(\kappa^{(\sigma,\mu)})$ is the price that leaves consumer $\kappa^{(\sigma,\mu)}$ indifferent between buying at time $\hat{t}$ with $c_{\hat{t}} = c_H$, or waiting until costs fall to c_L and buying at that point (at price $p^L(\kappa^{(\sigma,\mu)})$).

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the seller's profit margin $p - c_H$ from any sale she makes while costs are high following such a deviation is strictly smaller than $\rho\underline{v}$. Since $p^L(\kappa) \geq \underline{v}$ for all κ, the seller's most profitable deviation is to wait until costs fall to c_L and then play the continuation equilibrium, obtaining a payoff of $\rho U^L(\kappa^H_s) \geq \rho\underline{v}$.

B Proof of Theorem 2

For each $\Delta > 0$, let $(\sigma^\Delta, \mu^\Delta)$ be an equilibrium in $\Sigma^{\mathsf{S}}(\Delta)$ achieving the largest social welfare. Let $\{p_t^H(\Delta), \kappa_t^H(\Delta)\}_{t\in T(\Delta)}$ denote the prices and belief cutoffs induced by $(\sigma^\Delta, \mu^\Delta)$ at periods at which the seller's costs are c_H, and let $\kappa^{(\sigma^\Delta,\mu^\Delta)}$ be the lowest value buyer who trades while costs are c_H under $(\sigma^\Delta, \mu^\Delta)$.

Lemma B.1. $\kappa^{(\sigma^\Delta,\mu^\Delta)} - v^*(\Delta) \to 0$ *as* $\Delta \to 0$.

Proof. Towards a contradiction, suppose the result is false. Hence, there exists a sequence $\{\Delta^n\} \to 0$ and an $\epsilon > 0$ such that $\lim_{n\to\infty} \kappa^{(\sigma^{\Delta^n},\mu^{\Delta^n})} - v^*(\Delta^n) > \epsilon$.

For each n, let $\hat{t}_n$ be the time at which a buyer with value $\kappa_n \equiv \kappa^{(\sigma^{\Delta^n},\mu^{\Delta^n})}$ buys under $(\sigma^{\Delta^n}, \mu^{\Delta^n})$ if $c_t = c_H$ for all $t \leq \hat{t}_n$. The price at which a buyer with value κ_n buys under $(\sigma^{\Delta^n}, \mu^{\Delta^n})$ when costs are c_H is $\hat{p}(\kappa_n) = (1 - \rho(\Delta^n))\kappa_n + \rho(\Delta^n)p^L(\kappa_n)$.

For each n, fix $\hat{\kappa}_n \in (v^*(\Delta^n), \kappa_n)$ such that

$$\hat{p}(\hat{\kappa}_n)\left(\frac{F(\kappa_n) - F(\hat{\kappa}_n)}{F(\kappa_n)}\right) \leq U^L(\kappa_n) - e^{-r\Delta^n}\frac{F(\hat{\kappa}_n)}{F(\kappa_n)}U^L(\hat{\kappa}_n).$$

Let $\{\tilde{\kappa}_t^H(\Delta^n)\}$ be such that, for all $t \leq t_n + \Delta^n$, $\tilde{\kappa}_t^H(\Delta^n) = \kappa_t^H(\Delta^n)$, (where $\{\kappa_t^H(\Delta^n)\}$ is the sequence of belief cutoffs under $(\sigma^{\Delta^n}, \mu^{\Delta^n})$) and for all $t \geq \hat{t} + 2\Delta^n$, $\tilde{\kappa}_t^H(\Delta^n) = \hat{\kappa}_n$. Let $\{\tilde{p}_t^H(\Delta^n)\}$ be such that $\tilde{p}_t^H(\Delta^n) = \hat{p}(\hat{\kappa}_n)$ for all $t \geq t_n + \Delta^n$, and such that, for all $t < \hat{t}_n + \Delta^n$,

$$\begin{aligned}\tilde{\kappa}_{t+\Delta^n}^H(\Delta^n) - \tilde{p}_t^H(\Delta^n) &= e^{-(r+\lambda)\Delta^n}(\tilde{\kappa}_{t+\Delta^n}^H(\Delta^n) - \tilde{p}_{t+\Delta^n}^H(\Delta^n)) \\ &\quad + e^{-r\Delta^n}(1 - e^{-\lambda\Delta^n})(\tilde{\kappa}_{t+\Delta^n}^H(\Delta^n) - p^L(\tilde{\kappa}_{t+\Delta^n}^H(\Delta^n))). \qquad (18)\end{aligned}$$

That is, $\{\tilde{p}_t^H(\Delta^n), \tilde{\kappa}_t^H(\Delta^n)\}$ satisfies (16). Note that the inefficiencies under $\{\tilde{p}_t^H(\Delta^n), \tilde{\kappa}_t^H(\Delta^n)\}$ are smaller than under $\{p_t^H(\Delta^n), \kappa_t^H(\Delta^n)\}$, since trade is delayed by less under the former. The rest of the proof shows that, for n large enough, $\{\tilde{p}_t^H(\Delta^n), \tilde{\kappa}_t^H(\Delta^n)\}$ can be supported by an equilibrium in $\Sigma^{\mathsf{S}}(\Delta^n)$. This leads to a contradiction, since $(\sigma^{\Delta^n}, \mu^{\Delta^n})$ was assumed to be a welfare maximizing equilibrium in $\Sigma^{\mathsf{S}}(\Delta^n)$.

As a first step, I show that $\tilde{p}^H_t < p^H_t$ for all $t \leq \hat{t}_n$. Since sequences $\{\kappa^H_t(\Delta^n), p^H_t(\Delta^n)\}$ satisfy (10) for all $t \leq \hat{t}_n$, and since $\tilde{\kappa}^H_t(\Delta_n) = \kappa^H_t(\Delta_n)$ for all $t \leq \hat{t}_n + \Delta^n$, $\tilde{p}^H_t < p^H_t$ for all $t \leq \hat{t}_n$ implies that sequences $\{\tilde{\kappa}^H_t(\Delta^n), \tilde{p}^H_t(\Delta^n)\}$ satisfy (10).

Note that

$$\begin{aligned}
\tilde{\kappa}^H_{\hat{t}_n+\Delta^n} - \tilde{p}^H_{\hat{t}_n} &= e^{-(r+\lambda)\Delta^n}(\tilde{\kappa}^H_{\hat{t}+\Delta^n} - \tilde{p}^H_{\hat{t}_n+\Delta^n}) + e^{-r\Delta^n}(1-e^{-\lambda\Delta^n})(\tilde{\kappa}^H_{\hat{t}_n+\Delta^n} - p^L(\tilde{\kappa}^H_{\hat{t}_n+\Delta^n}))\\
&> e^{-(r+\lambda)\Delta^n}(\tilde{\kappa}^H_{\hat{t}_n+\Delta^n} - \hat{p}(\tilde{\kappa}^H_{\hat{t}_n+\Delta_n})) + e^{-r\Delta^n}(1-e^{-\lambda\Delta^n})(\tilde{\kappa}^H_{\hat{t}_n+\Delta^n} - p^L(\tilde{\kappa}^H_{\hat{t}_n+\Delta^n}))\\
&= e^{-(r+\lambda)\Delta^n}\rho(\tilde{\kappa}^H_{\hat{t}_n+\Delta^n} - p^L(\tilde{\kappa}^H_{\hat{t}_n+\Delta^n})) + e^{-r\Delta^n}(1-e^{-\lambda\Delta^n})(\tilde{\kappa}^H_{\hat{t}_n+\Delta^n} - p^L(\tilde{\kappa}^H_{\hat{t}_n+\Delta^n}))\\
&= \rho(\Delta^n)(\tilde{\kappa}^H_{\hat{t}_n+\Delta^n} - p^L(\tilde{\kappa}^H_{\hat{t}_n+\Delta^n})),
\end{aligned}$$

where the strict inequality uses $\tilde{p}^H_{\hat{t}_n+\Delta^n} = \hat{p}(\tilde{\kappa}^H_{\hat{t}_n+2\Delta^n}) < \hat{p}(\tilde{\kappa}^H_{\hat{t}_n+\Delta^n})$, the second equality uses $\hat{p}(\tilde{\kappa}^H_{\hat{t}_n+\Delta^n}) = \tilde{\kappa}^H_{\hat{t}_n+\Delta^n}(1-\rho(\Delta^n)) + \rho(\Delta^n)p^L(\tilde{\kappa}^H_{\hat{t}_n+\Delta^n})$, and the last equality uses $\rho(\Delta) = e^{-r\Delta}(1-e^{-\lambda\Delta}) + \rho e^{-(r+\lambda)\Delta}$. Since $\tilde{\kappa}^H_{\hat{t}_n+\Delta^n}(\Delta^n) = \kappa^H_{\hat{t}_n+\Delta^n}(\Delta_n)$, and since $p^H_{\hat{t}_n}(\Delta^n) = \hat{p}(\kappa_n) = \hat{p}(\kappa^H_{\hat{t}_n+\Delta^n}(\Delta^n))$, it follows that $\tilde{p}^H_{\hat{t}_n}(\Delta^n) < p^H_{\hat{t}_n}(\Delta^n)$.

I now use this to show that $\tilde{p}^H_t(\Delta^n) < p^H_t(\Delta^n)$ for all $t < \hat{t}_n$. For all $t \leq \hat{t}_n$, prices $\{p^H_t(\Delta^n)\}$ satisfy

$$\begin{aligned}
\kappa^H_{t+\Delta^n}(\Delta^n) - p^H_t(\Delta^n) &= e^{-(r+\lambda)\Delta^n}(\kappa^H_{t+\Delta^n}(\Delta^n) - p^H_{t+\Delta^n}(\Delta^n))\\
&\quad + e^{-r\Delta^n}(1-e^{-\lambda\Delta^n})(\kappa^H_{t+\Delta^n}(\Delta^n) - p^L(\kappa^H_{t+\Delta^n}(\Delta^n))).
\end{aligned}$$

Combining this equation with (18), for all $t < \hat{t}_n$,

$$p^H_t(\Delta^n) - \tilde{p}^H_t(\Delta^n) = e^{-(r+\lambda)\Delta^n}(p^H_{t+\Delta^n}(\Delta^n) - \tilde{p}^H_{t+\Delta^n}(\Delta^n)),$$

where I used $\tilde{\kappa}^H_t(\Delta^n) = \kappa^H_t(\Delta^n)$ for all $t \leq \hat{t}_n + \Delta^n$. Since $\tilde{p}^H_{\hat{t}_n} < p^H_{\hat{t}_n}$, it follows that $p^H_t(\Delta^n) > \tilde{p}^H_t(\Delta^n)$ for all $t < \hat{t}_n$. Hence, $\{\tilde{\kappa}^H_t(\Delta^n), \tilde{p}^H_t(\Delta^n)\}$ satisfies (10).

I now show that, for n sufficiently large, $\{\tilde{\kappa}_t^H(\Delta^n), \tilde{p}_t^H(\Delta^n)\}$ also satisfies (17). I start by showing that $\tilde{p}_t^H(\Delta^n) > \tilde{p}_{t+\Delta^n}^H(\Delta^n)$ for all $t < \hat{t}_n + \Delta^n$, so prices $\tilde{p}_t^H(\Delta^n)$ are decreasing. This implies that $\tilde{p}_t^H(\Delta^n) > \tilde{p}_{\hat{t}_n+\Delta^n}^H(\Delta^n) = \hat{p}(\hat{\kappa}_n)$ for all $t \leq \hat{t}_n$. Since $\hat{p}(\hat{\kappa}_n) = (1-\rho(\Delta^n))\hat{\kappa}_n + \rho(\Delta)p^L(\hat{\kappa}_n)$, $\hat{\kappa}_n > v^*(\Delta^n) = \frac{c_H}{1-\rho(\Delta^n)}$, and $p^L(\hat{\kappa}_n) \geq \underline{v}$, this further implies that $\hat{p}(\hat{\kappa}_n) - c_H > \rho(\Delta^n)\underline{v}$. Hence, if prices $\tilde{p}_t^H(\Delta^n)$ are decreasing, then $\tilde{p}_t^H(\Delta^n) - c_H > \rho(\Delta^n)\underline{v}$ for all $t \leq \hat{t}_n + \Delta^n$.

Recall that

$$\begin{aligned}
&\tilde{p}_{\hat{t}_n+\Delta^n}^H = \hat{p}(\tilde{\kappa}_{\hat{t}_n+2\Delta^n}^H(\Delta^n)) = (1-\rho(\Delta^n))\tilde{\kappa}_{\hat{t}_n+2\Delta^n}^H(\Delta^n) + \rho(\Delta^n)p^L(\tilde{\kappa}_{\hat{t}_n+2\Delta^n}^H(\Delta^n)) \\
&\quad \Longleftrightarrow \tilde{\kappa}_{\hat{t}_n+2\Delta^n}^H - \tilde{p}_{\hat{t}_n+\Delta^n}^H = \rho(\Delta^n)(\tilde{\kappa}_{\hat{t}_n+2\Delta^n}^H - p^L(\tilde{\kappa}_{\hat{t}_n+2\Delta^n}^H)) \\
&\quad \Longleftrightarrow \tilde{\kappa}_{\hat{t}_n+2\Delta^n}^H - \tilde{p}_{\hat{t}_n+\Delta^n}^H = e^{-(r+\lambda)\Delta^n}(\tilde{\kappa}_{\hat{t}_n+2\Delta^n}^H - \tilde{p}_{\hat{t}_n+\Delta^n}^H) + e^{-r\Delta^n}(1-e^{-\lambda\Delta^n})(\tilde{\kappa}_{\hat{t}_n+2\Delta^n}^H - p^L(\tilde{\kappa}_{\hat{t}_n+2\Delta^n}^H)),
\end{aligned} \tag{19}$$

where the last line uses $\rho(\Delta) = \frac{e^{-r\Delta}(1-e^{-\lambda\Delta})}{1-e^{-(r+\lambda)\Delta}}$. Moreover, $\tilde{p}_{\hat{t}_n}^H$ satisfies (18), and so

$$\tilde{\kappa}_{\hat{t}_n+\Delta^n}^H - \tilde{p}_{\hat{t}_n}^H = e^{-(r+\lambda)\Delta^n}(\tilde{\kappa}_{\hat{t}_n+\Delta^n}^H - \tilde{p}_{\hat{t}_n+\Delta^n}^H) + e^{-r\Delta^n}(1-e^{-\lambda\Delta^n})(\tilde{\kappa}_{\hat{t}_n+\Delta^n}^H - p^L(\tilde{\kappa}_{\hat{t}_n+\Delta^n}^H))$$

Combining this with (19) yields

$$\tilde{p}_{\hat{t}_n}^H - \tilde{p}_{\hat{t}_n+\Delta^n}^H = (1-e^{-r\Delta^n})(\tilde{\kappa}_{\hat{t}_n+\Delta^n}^H - \tilde{\kappa}_{\hat{t}_n+2\Delta^n}^H) + e^{-r\Delta^n}(1-e^{-\lambda\Delta^n})(p^L(\tilde{\kappa}_{\hat{t}_n+\Delta^n}^H) - p^L(\tilde{\kappa}_{\hat{t}_n+2\Delta_n}^H)) > 0,$$

where the strict inequality follows since $\tilde{\kappa}_{\hat{t}_n+\Delta^n}^H > \kappa_{\hat{t}_n+2\Delta^n}^H$ and $p^L(\cdot)$ is weakly increasing.

Towards an induction, suppose that $\tilde{p}_{t'}^H > \tilde{p}_{t'+\Delta^n}^H$ for all $t' = t+\Delta^n, ..., \hat{t}_n$. I now show that $\tilde{p}_t^H > \tilde{p}_{t+\Delta^n}^H$. Since $\tilde{p}_t^H$ and $\tilde{p}_{t+\Delta^n}^H$ satisfy (18), it follows that

$$\begin{aligned}
\tilde{p}_t^H - \tilde{p}_{t+\Delta^n}^H = (1-e^{-r\Delta^n})(\tilde{\kappa}_{t+\Delta^n}^H - \tilde{\kappa}_{t+2\Delta^n}^H) + e^{-(r+\lambda)\Delta^n}(\tilde{p}_{t+\Delta^n}^H - \tilde{p}_{t+2\Delta^n}^H) \\
+ e^{-r\Delta^n}(1-e^{-\lambda\Delta^n})(p^L(\tilde{\kappa}_{t+\Delta^n}^H) - p^L(\tilde{\kappa}_{t+2\Delta^n}^H)) > 0.
\end{aligned}$$

By the Coase conjecture, for all κ, $U^L(\kappa) \to \underline{v}$ as $\Delta \to 0$; i.e., the seller earns a profit margin of $\underline{v}$ on each sale she makes when her costs are c_L. Since the profit margin $(\tilde{p}_t^H - c_H)$ that she earns on each sale when her cost is c_H is larger than $\rho\underline{v}$, in the limit as $n \to \infty$ the seller's profits from selling when her costs are c_H are larger than what she would get by waiting until her costs fall to c_L and then playing the continuation equilibrium. Hence, constraint (17) is satisfied under sequences $\{\tilde{p}_t^H(\Delta^n), \tilde{\kappa}_t^H(\Delta^n)\}$ when n is sufficiently large.

The arguments above show that, for n large enough, $\{\tilde{\kappa}_t^H(\Delta^n), \tilde{p}_t^H(\Delta^n)\}$ satisfies all the conditions in Theorem 1(ii). Hence, for n large enough, $\{\tilde{\kappa}_t^H(\Delta^n), \tilde{p}_t^H(\Delta^n)\}$ can be supported by an equilibrium in $\Sigma^{\mathsf{S}}(\Delta^n)$. But this contradicts the fact that, for all n, $(\sigma^{\Delta^n}, \mu^{\Delta^n})$ is a welfare maximizing equilibrium in $\Sigma^{\mathsf{S}}(\Delta^n)$ (recall that inefficiencies under $\{\tilde{p}_t^H(\Delta^n), \tilde{\kappa}_t^H(\Delta^n)\}$ are smaller than under $\{p_t^H(\Delta^n), \kappa_t^H(\Delta^n)\}$). Therefore, $\kappa^{(\sigma^\Delta, \mu^\Delta)} - v^*(\Delta) \to 0$ as $\Delta \to 0$. ■

For all $\kappa \in [\underline{v}, \overline{v}]$ and $\Delta > 0$, let $U^L(\kappa; \Delta)$ be the seller's continuation profits when her cost is c_L and her belief cutoff is κ. Define $\pi^L(\kappa; \Delta) \equiv F(\kappa) U^L(\kappa; \Delta)$.

Lemma B.2 (no atoms)**.** *Fix a sequence $\{\Delta^n\} \to 0$. For each n, let $(\sigma^{\Delta^n}, \mu^{\Delta^n})$ be a welfare maximizing equilibrium in $\Sigma^{\mathsf{S}}(\Delta^n)$, and let $\{\kappa_t^H(\Delta^n), p_t^H(\Delta^n)\}$ be the sequences of prices and belief cutoffs induced by $(\sigma^{\Delta^n}, \mu^{\Delta^n})$. There exists $B > 0$ such that, for all $t \in T(\Delta^n)$,*

$$\lim_{n \to \infty} \frac{F(\kappa_t^H(\Delta^n)) - F(\kappa_{t+\Delta^n}^H(\Delta^n))}{\Delta^n} \leq B.$$

Hence, for all $t \in T(\Delta^n)$, $\kappa_t^H(\Delta^n) - \kappa_{t+\Delta^n}^H(\Delta^n) \to 0$ as $n \to \infty$.

Proof. Note first that, for all n, there exists $\hat{t}_n$ such that $\frac{F(\kappa_t^H(\Delta^n)) - F(\kappa_{t+\Delta^n}^H(\Delta^n))}{\Delta^n} = 0$ for all $t > \hat{t}_n$; i.e., $\hat{t}_n$ is the last period at which the seller makes sells when costs are high.

Consider next $t \leq \hat{t}_n$. By Lemma A.1, and using $\pi^L(\kappa;\Delta) = F(\kappa)U^L(\kappa;\Delta)$,

$$(F(\kappa_t^H(\Delta^n)) - F(\kappa_{t+\Delta^n}^H(\Delta^n)))p_t^H(\Delta^n) \leq \pi^L(\kappa_t^H(\Delta^n);\Delta^n)(1-e^{-r\Delta^n}) \\ + e^{-r\Delta^n}(\pi^L(\kappa_t^H(\Delta^n);\Delta^n) - \pi^L(\kappa_{t+\Delta^n}^H(\Delta^n);\Delta^n)). \tag{20}$$

Let $p^L(\kappa;\Delta)$ be the price that a low cost seller would charge when her cutoff beliefs are κ in a setting with time period Δ. Note that, since $p^L(\kappa;\Delta) \in [\underline{v}, p^L(\overline{v};\Delta)]$ for all κ,

$$\pi^L(\kappa_t^H(\Delta^n);\Delta^n) - \pi^L(\kappa_{t+\Delta^n}^H(\Delta^n);\Delta^n) \leq p^L(\overline{v};\Delta^n)(F(\kappa_t^H(\Delta^n)) - F(\kappa_{t+\Delta^n}^H(\Delta^n))).$$

Combining this with (20),

$$\frac{F(\kappa_t^H(\Delta^n)) - F(\kappa_{t+\Delta^n}^H(\Delta^n))}{\Delta^n}(p_t^H(\Delta^n) - e^{-r\Delta^n}p^L(\overline{v};\Delta^n)) \leq \pi^L(\kappa_t^H(\Delta^n);\Delta^n)\frac{1-e^{-r\Delta^n}}{\Delta^n}. \tag{21}$$

Next, recall from the proof of Lemma B.1 that prices $p_t^H(\Delta^n)$ are decreasing: for all $t < \hat{t}_n$, $p_t^H(\Delta^n) > p_{\hat{t}_n}^H(\Delta^n) = \hat{p}(\kappa^{(\sigma^{\Delta^n},\mu^{\Delta^n})}) \geq \hat{p}(v^*(\Delta^n)) = (1-\rho(\Delta^n))v^*(\Delta^n) + \rho(\Delta^n)p^L(v^*(\Delta^n);\Delta^n)$. Since $\lim_{\Delta\to 0}\rho(\Delta) = \frac{\lambda}{r+\lambda}$, $\lim_{\Delta\to 0} v^*(\Delta) = \frac{r+\lambda}{r}c_H$ and $\lim_{\Delta\to 0} p^L(\overline{v};\Delta) = \underline{v}$, it follows that

$$\lim_{n\to\infty} p_t^H(\Delta^n) - e^{-r\Delta^n}p^L(\overline{v};\Delta^n) \geq c_H + \frac{\lambda}{r+\lambda}\underline{v} - \underline{v} = c_H - \frac{r}{r+\lambda}\underline{v} > 0.$$

The strict inequality holds since, by Assumption 1, $v^*(\Delta) \in (\underline{v}, \overline{v})$, and so $\lim_{\Delta\to 0} v^*(\Delta) = \frac{r+\lambda}{r}c_H > \underline{v}$.

Using this in inequality (21)

$$\lim_{n\to\infty}\frac{F(\kappa_t^H(\Delta^n)) - F(\kappa_{t+\Delta}^H(\Delta^n))}{\Delta^n} \leq \lim_{n\to\infty}\frac{1}{p_t^H(\Delta^n) - e^{-r\Delta^n}p^L(\overline{v};\Delta^n)}\pi^L(\kappa_t^H(\Delta^n);\Delta^n)\frac{1-e^{-r\Delta^n}}{\Delta^n} \\ \leq \frac{r+\lambda}{(r+\lambda)c_H - r\underline{v}}r\underline{v},$$

where the last inequality uses $\lim_{\Delta\to 0}\pi^L(\kappa;\Delta) = F(\kappa)\underline{v} \le \underline{v}$. ■

Proof of Theorem 2. Note first that, by (16), sequences $\{\kappa_t^H(\Delta), p_t^H(\Delta)\}_{t\in T(\Delta)}$ are such that, for all $t < \hat{t}$,

$$\kappa_{t+\Delta}^H(\Delta) - p_t^H(\Delta) = e^{-(r+\lambda)\Delta}(\kappa_{t+\Delta}^H(\Delta) - p_{t+\Delta}^H(\Delta)) \\ + e^{-r\Delta}(1-e^{-\lambda\Delta})(\kappa_{t+\Delta}^H(\Delta) - p^L(\kappa_{t+\Delta}^H(\Delta);\Delta)). \tag{22}$$

For each $t \in [0,\infty)$, let $p^H(t) = \lim_{\Delta\to 0} p_t^H(\Delta)$ and $\kappa^H(t) = \lim_{\Delta\to 0}\kappa_t^H(\Delta)$ (if needed, take a convergent subsequence, which exists by Helly's Selection Theorem). Dividing both sides of (22) by Δ and rearranging,

$$\frac{p_t^H(\Delta) - p_{t+\Delta}^H(\Delta)}{\Delta} = \kappa_{t+\Delta}^H(\Delta)\frac{(1-e^{-r\Delta})}{\Delta} - p_{t+\Delta}^H(\Delta)\frac{(1-e^{-(r+\lambda)\Delta})}{\Delta} \\ + e^{-r\Delta}\frac{(1-e^{-\lambda\Delta})}{\Delta}p^L(\kappa_{t+\Delta}^H(\Delta);\Delta). \tag{23}$$

Taking limits on both sides of (23) as $\Delta\to 0$ and using $\lim_{\Delta\to 0} p^L(\kappa,\Delta) = \underline{v}$ and $\lim_{\Delta\to 0}\kappa_t^H(\Delta) - \kappa_{t+\Delta}^H(\Delta) = 0$ (Lemma B.2),

$$\lim_{\Delta\to 0}\frac{p_t^H(\Delta) - p_{t+\Delta}^H(\Delta)}{\Delta} = -\frac{dp^H(t)}{dt} = r\kappa^H(t) - (r+\lambda)p^H(t) + \lambda\underline{v}.$$

Under the most efficient equilibrium, it must be that inequality (10) holds with equality for almost $t \in T(\Delta)$. Using $\pi^L(\kappa;\Delta) = F(\kappa)U^L(\kappa;\Delta)$,

$$p_t^H(\Delta)(F(\kappa_t^H(\Delta)) - F(\kappa_{t+\Delta}^H(\Delta))) = \pi^L(\kappa_t^H(\Delta);\Delta) - \pi^L(\kappa_{t+\Delta}^H(\Delta);\Delta) \\ + (1-e^{-r\Delta})\pi^L(\kappa_{t+\Delta}^H(\Delta);\Delta). \tag{24}$$

Note next that, for all $\kappa, \kappa' \in [\underline{v}, \overline{v}]$ with $\kappa > \kappa'$, the following inequalities hold:

$$\pi^L(\kappa;\Delta) - \pi^L(\kappa';\Delta) \geq \underline{v}(F(\kappa) - F(\kappa'))$$
$$\pi^L(\kappa;\Delta) - \pi^L(\kappa';\Delta) \leq p^L(\overline{v};\Delta)(F(\kappa) - F(\kappa')).$$

The inequalities follow since, for all belief cutoffs $\tilde{\kappa}$, $p^L(\tilde{\kappa};\Delta) \in [\underline{v}, p^L(\overline{v};\Delta)]$. Combining these inequalities with (24), and dividing through by Δ, yields

$$\begin{aligned}
&\underline{v}\frac{F(\kappa_t^H(\Delta)) - F(\kappa_{t+\Delta}^H(\Delta))}{\Delta} + \frac{1-e^{-r\Delta}}{\Delta}\pi^L(\kappa_{t+\Delta}^H(\Delta);\Delta) \\
\leq &p_t^H(\Delta)\frac{F(\kappa_t^H(\Delta)) - F(\kappa_{t+\Delta}^H(\Delta))}{\Delta} \\
\leq &p^L(\overline{v};\Delta)\frac{F(\kappa_t^H(\Delta)) - F(\kappa_{t+\Delta}^H(\Delta))}{\Delta} + \frac{1-e^{-r\Delta}}{\Delta}\pi^L(\kappa_{t+\Delta}^H(\Delta);\Delta).
\end{aligned}$$

Taking the limit as $\Delta \to 0$ and using $\lim_{\Delta\to 0} p^L(\overline{v};\Delta) = \underline{v}$ and $\lim_{\Delta\to 0}\pi^L(\kappa;\Delta) = \underline{v}F(\kappa)$,

$$\begin{aligned}
\lim_{\Delta\to 0} p_t^H(\Delta)\frac{F(\kappa_t^H(\Delta)) - F(\kappa_{t+\Delta}^H(\Delta))}{\Delta} &= \lim_{\Delta\to 0}\underline{v}\frac{F(\kappa_t^H(\Delta)) - F(\kappa_{t+\Delta}^H(\Delta))}{\Delta} + r\underline{v}F(\kappa^H(t)) \\
\iff \lim_{\Delta\to 0}\frac{F(\kappa_t^H(\Delta)) - F(\kappa_{t+\Delta}^H(\Delta))}{\Delta} &= -\frac{d\kappa^H(t)}{dt}f(\kappa^H(t)) = \frac{r\underline{v}F(\kappa^H(t))}{p^H(t) - \underline{v}}.
\end{aligned}$$

The boundary condition for $\kappa^H(\cdot)$ is $\kappa^H(0) = \overline{v}$. To derive the boundary condition for $p^H(\cdot)$, let $\hat{v} = \lim_{\Delta\to 0} v^*(\Delta) = \frac{r+\lambda}{r}c_H$. By Lemma B.1, belief cutoffs $\kappa^H(t)$ reaches $\hat{v} = \frac{r+\lambda}{r}c_H$ at finite time $\hat{t} = \inf\{t \geq 0 : \kappa^H(t) = \hat{v}\}$. The price at which the seller sells to a buyer with valuation $\hat{v}$ must be such that this buyer is indifferent between buying now, or waiting until costs fall to c_L and getting the good at price $\underline{v}$. Hence, $p^H(\hat{t}) = \frac{r}{r+\lambda}\hat{v} + \frac{\lambda}{r+\lambda}\underline{v} = c_H + \frac{\lambda}{r+\lambda}\underline{v}$. ■

C Proofs of Propositions 4 and 5

Proof of Proposition 4. I start by showing that $\kappa - P(\kappa) > \frac{\lambda}{r+\lambda}(\kappa - \underline{v})$ for all $\kappa > \hat{v}$. Since $P(\kappa^H(t)) = p^H(t)$ for all $t \leq \hat{t}$, this is equivalent to showing that $\kappa^H(t) - p^H(t) > \frac{\lambda}{r+\lambda}(\kappa^H(t) - \underline{v})$ for all $t < \hat{t}$, or that

$$\forall t < \hat{t}, \qquad D(t) \equiv r(\kappa^H(t) - p^H(t)) + \lambda(\underline{v} - p^H(t)) > 0.$$

Using equation (4),

$$\begin{aligned} D'(t) &= r\frac{d\kappa^H(t)}{dt} - (r+\lambda)\frac{dp^H(t)}{dt} \\ &= r\frac{d\kappa^H(t)}{dt} + (r+\lambda)[r(\kappa^H(t) - p^H(t)) + \lambda(\underline{v} - p^H(t))] \\ &= r\frac{d\kappa^H(t)}{dt} + (r+\lambda)D(t). \end{aligned} \tag{25}$$

Note that $p^H(\hat{t}) = \hat{v} - \frac{\lambda}{r+\lambda}(\hat{v} - \underline{v}) = \kappa^H(\hat{t}) - \frac{\lambda}{r+\lambda}(\kappa^H(\hat{t}) - \underline{v})$, and so $D(\hat{t}) = 0$. Since $\frac{d\kappa^H(t)}{dt} < 0$ for all $t \leq \hat{t}$, it follows that $D'(\hat{t}) < 0$. Hence, $D(t) > 0$ for all $t < \hat{t}$ close to $\hat{t}$. Let $\tilde{t} = \inf\{t < \hat{t} : D(t) > 0\}$. Towards a contradiction, suppose that $\tilde{t} \geq 0$. Since $D(t)$ is continuous, $D(\tilde{t}) = 0$. Moreover, since $D(t) > 0$ for all $t \in (\tilde{t}, \hat{t})$, it must be that $D'(\tilde{t}) \geq 0$. Using (25), and noting that $\frac{d\kappa^H(t)}{dt}|_{t=\tilde{t}} < 0$ and $D(\tilde{t}) = 0$,

$$D'(\tilde{t}) = r\frac{d\kappa^H(t)}{dt}|_{t=\tilde{t}} + (r+\lambda)D(\tilde{t}) < 0,$$

a contradiction. Hence, $D(t) > 0$ for all $t < \hat{t}$. And so $\kappa - P(\kappa) > \frac{\lambda}{r+\lambda}(\kappa - \underline{v})$ for all $\kappa > \hat{v}$.

I now show part (i). For each c_H, let $\hat{v}(c_H) = \frac{\lambda+r}{r}c_H$ be the efficient cutoff for cost c_H, and let $P^H(\kappa; c_H)$ denote the solution to (7) and boundary condition for cost c_H.

Fix $c_H' > c_H$, so $\hat{v}(c_H') > \hat{v}(c_H)$. Note that $\hat{v}(c_H) - P^H(\hat{v}(c_H); c_H) = \frac{\lambda}{r+\lambda}(\hat{v}(c_H) - \underline{v})$. By the arguments above, $\kappa - P(\kappa; c_H) > \frac{\lambda}{r+\lambda}(\kappa - \underline{v})$ for all $\kappa > \hat{v}(c_H)$; in particular, $\hat{v}(c_H') - P^H(\hat{v}(c_H'); c_H) > \frac{\lambda}{r+\lambda}(\hat{v}(c_H') - \underline{v}) = \hat{v}(c_H') - P(\hat{v}(c_H'); c_H')$, and so $P^H(\hat{v}(c_H'); c_H') >$

$P^H(\hat{v}(c'_H); c_H)$.

I now show that $P^H(\kappa; c'_H) > P^H(\kappa; c_H)$ for all $\kappa \in [\hat{v}(c'_H), \overline{v}]$. Towards a contradiction, suppose the result is not true, and let $\tilde{\kappa} = \inf\{\kappa \in [\hat{v}(c'_H), \overline{v}] : P^H(\kappa; c'_H) \leq P^H(\kappa; c_H)\}$. Since $P^H(\kappa; c'_H)$ and $P^H(\kappa; c_H)$ are continuous and since $P^H(\hat{v}(c'_H); c'_H) > P^H(\hat{v}(c'_H); c_H)$, it must be that $\tilde{\kappa} > \hat{v}(c'_H)$ and $P^H(\tilde{\kappa}; c'_H) = P^H(\tilde{\kappa}; c_H)$. But then, $P^H(\cdot; c'_H)$ and $P^H(\cdot; c_H)$ both solve ODE (7), with $P^H(\tilde{\kappa}; c'_H) = P^H(\tilde{\kappa}; c_H)$; and so $P^H(\cdot; c'_H) = P^H(\cdot; c_H)$, a contradiction. Hence, $P^H(\kappa; c'_H) > P^H(\kappa; c_H)$ for all $\kappa \in [\hat{v}(c'_H), \overline{v}]$. Finally, by equation (5), the speed of trade falls when prices $p^H(t) = P^H(\kappa^H(t))$ increase.

I now turn to part (ii). Fix distributions F_1 and F_0 such that F_1 dominates F_0 in terms of the reverse hazard-rate. Let $P^H(\kappa; F_i)$ denote the solution to (7) and boundary condition under distribution F_i.

I start by showing that $P^H(\kappa; F_1) > P^H(\kappa; F_0)$ for all $\kappa > \hat{v}$. Note first that $P^H(\hat{v}; F_i) = c_H + \frac{\lambda}{r+\lambda}\underline{v} = \hat{v} - \frac{\lambda}{r+\lambda}(\hat{v} - \underline{v})$ for $i = 0, 1$. Using (7), for $i = 0, 1$,

$$\frac{dP^H(\kappa; F_i)}{d\kappa}|_{\kappa=\hat{v}} = 0,$$
$$\frac{d^2P^H(\kappa; F_i)}{d\kappa^2}|_{\kappa=\hat{v}} = r\frac{f_i(\hat{v})}{F_i(\hat{v})}\frac{P^H(\hat{v}) - \underline{v}}{r\underline{v}}.$$

Since $\frac{f_1(v)}{F_1(v)} > \frac{f_0(v)}{F_0(v)}$ for all v, $\frac{d^2P^H(\kappa;F_1)}{d\kappa^2}|_{\kappa=\hat{v}} > \frac{d^2P^H(\kappa;F_0)}{d\kappa^2}|_{\kappa=\hat{v}}$. Hence, there exists $\tilde{v} > \hat{v}$ such that $P^H(\kappa; F_1) > P^H(\kappa; F_0)$ for all $\kappa \in (\hat{v}, \tilde{v})$.

Towards a contradiction, suppose that the result is not true, and let $\tilde{\kappa} = \inf\{\kappa > \hat{v} : P^H(\kappa; F_1) \leq P^H(\kappa; F_0)\}$. Since $P^H(\kappa; F_1)$ and $P^H(\kappa; F_0)$ are continuous, $P^H(\tilde{\kappa}; F_1) = P^H(\tilde{\kappa}; F_0)$. Since $P^H(\kappa; F_1) > P^H(\kappa; F_0)$ for all $\kappa \in (\hat{v}, \tilde{\kappa})$, it must be that $\frac{dP^H(\kappa;F_1)}{d\kappa}|_{\kappa=\tilde{\kappa}} \leq \frac{dP^H(\kappa;F_0)}{d\kappa}|_{\kappa=\tilde{\kappa}}$. But $P^H(\tilde{\kappa}; F_1) = P^H(\tilde{\kappa}; F_0)$ and $\frac{f_1(\tilde{\kappa})}{F_1(\tilde{\kappa})} > \frac{f_0(\tilde{\kappa})}{F_0(\tilde{\kappa})}$, together with ODE (7) implies $\frac{dP^H(\kappa;F_1)}{d\kappa}|_{\kappa=\tilde{\kappa}} > \frac{dP^H(\kappa;F_0)}{d\kappa}|_{\kappa=\tilde{\kappa}}$, a contradiction. Therefore, $P^H(\kappa; F_1) > P^H(\kappa; F_0)$ for all $\kappa > \hat{v}$. Lastly, since prices are higher under F_1 than under F_0, by equation (6) the rate at which the seller makes sells is slower under F_1 than under F_0.

I now turn to part (iii). For each λ, let $\hat{v}(\lambda) = \frac{\lambda+r}{r}c_H$, and let $P^H(\kappa; \lambda)$ denote the solution to (7) and boundary condition for λ. Note that $\frac{dP^H(\kappa;\lambda)}{d\kappa}\mid_{\kappa=\hat{v}(\lambda)}=0$. Note further that

$$\frac{d}{d\lambda}P^H(\hat{v}(\lambda);\lambda)=\frac{\partial}{\partial\lambda}\left(c_H+\frac{\lambda}{r+\lambda}\underline{v}\right)=\frac{r}{(\lambda+r)^2}\underline{v}>0.$$

Hence, for all $\lambda'>\lambda$ close enough to λ, it must that $P^H(\hat{v}(\lambda');\lambda')>P^H(\hat{v}(\lambda');\lambda)$. Since $P^H(\cdot;\lambda')$ and $P^H(\cdot;\lambda)$ are continuous, there exists $\tilde{\kappa}>\hat{v}(\lambda')$ such that $P^H(\kappa;\lambda')>P^H(\kappa;\lambda)$ for all $\kappa\in(\hat{v}(\lambda'),\tilde{\kappa})$. Next, note that by equation (5), the speed of trade falls when prices $p^H(t)=P^H(\kappa^H(t))$ increase. Hence, for all t with $\kappa^H(t)\in(\hat{v}(\lambda'),\tilde{\kappa})$, the speed of trade is lower under λ'. ■

Proof of Proposition 5. Part (i) follows from equation (6) and the fact that, for all $t\leq\hat{t}$, $p^H(t)\geq p^H(\hat{t})=c_H+\frac{\lambda}{r+\lambda}\underline{v}>\underline{v}$ (since, by Assumption 1, $\underline{v}<\hat{v}=\frac{r+\lambda}{r}c_H$). For part (ii), note that in the limit as $\underline{v}\to 0$ the seller only trades with the buyer once costs are c_L, at price $\underline{v}$. Hence, seller's profits go to zero as $\underline{v}\to 0$. Since all types of buyers trade at time τ_L in the limit as $\underline{v}\to 0$, the total equilibrium surplus converges to $(q\frac{\lambda}{r+\lambda}+1-q)\mathbb{E}[v]$, establishing part (iii) (recall that $q=\text{prob}(c_0=c_H)$). ■

D Proof of Theorem 3

I start with some preliminary observations. Note that, in any equilibrium $(\sigma^\Delta,\mu^\Delta)\in\Sigma^{\mathsf{pub}}(\Delta)$ of the game with public costs, when the seller's costs fall to c_L the continuation equilibrium is the same as in the game in which it is common knowledge that the seller's costs are c_L.

This observation implies that Lemma A.2 continues to hold when costs are public. In particular, in any $(\sigma^\Delta,\mu^\Delta)\in\Sigma^{\mathsf{pub}}(\Delta)$, the seller sells with probability zero at any period t with $c_t=c_H$ and $\kappa_t\leq v^*(\Delta)$. The reason for this twofold. First, in any $(\sigma^\Delta,\mu^\Delta)\in\Sigma^{\mathsf{pub}}(\Delta)$ and for any cutoff beliefs κ, the seller's profits when costs are c_H cannot be lower than $\rho\pi^L(\kappa)$, since the seller can wait until costs fall to c_L and then play the continuation equilibrium. Second, price p_t^H at which a buyer with valuation $\kappa_{t+\Delta}^H$ buys when costs are high must be such that $\kappa_{t+\Delta}^H - p_t^H \geq \rho(\kappa_{t+\Delta}^H - p^L(\kappa_{t+\Delta}^H))$. Indeed, a buyer with valuation $\kappa_{t+\Delta}^H$ can get a payoff at least as large as $\rho(\kappa_{t+\Delta}^H - p^L(\kappa_{t+\Delta}^H))$ by waiting until costs fall to c_L and buying at that time. With these two observations, the proof of Lemma A.2 goes through as is. I summarize this discussion in the following Lemma.

Lemma D.1. *Fix a PBE $(\sigma, \mu) \in \Sigma^{pub}(\Delta)$, and consider a period t with belief cutoff κ_t. If $c_t = c_H$, then $\kappa_{t+\Delta} \geq \min\{v^*(\Delta), \kappa_t\}$. In particular, if $\kappa_t \leq v^*(\Delta)$ and $c_t = c_H$, the seller makes a sale with probability zero at time t.*

For each $\Delta > 0$, let $(\sigma^\Delta, \mu^\Delta) \in \Sigma^{\mathsf{pub}}(\Delta)$. For any on-path belief cutoff κ, let $p^H(\kappa; \Delta)$ denote the price that the seller charges under $(\sigma^\Delta, \mu^\Delta)$ when costs are c_H and the seller's belief cutoff is κ. Let $U^H(\kappa; \Delta)$ denote the seller's continuation profits under $(\sigma^\Delta, \mu^\Delta)$ when costs are c_H and the seller's belief cutoff is κ. Lastly, let $\{\kappa_t^H(\Delta), p_t^H(\Delta)\}_{t \in T(\Delta)}$ denote the sequence of belief cutoffs and prices induced by $(\sigma^\Delta, \mu^\Delta)$ on the equilibrium path when the seller's costs are c_H.

Lemma D.2 (Efficiency)**.** *As $\Delta \to 0$, the equilibrium outcome of $(\sigma^\Delta, \mu^\Delta)$ converges to the efficient outcome: for all $t > 0$, $\lim_{\Delta \to 0} \kappa_t^H(\Delta) = \hat{v}$.*

Proof. The proof adapts arguments in Liu (2015) to the current setting. Consider a period t with belief cutoff $\kappa_t^H(\Delta)$ and with $c_t = c_H$. For each $v \in [\underline{v}, \kappa_t^H(\Delta)]$, let $\tau^\Delta(v)$ denote the random time at which a buyer with valuation v buys under $(\sigma^\Delta, \mu^\Delta)$. The seller's continuation profits $U^H(\kappa_t^H(\Delta); \Delta)$ at this history satisfies:

$$F(\kappa_t^H(\Delta))U^H(\kappa_t^H(\Delta); \Delta) = \mathbb{E}\left[\int_{\underline{v}}^{\kappa_t^H(\Delta)} e^{-r\tau^\Delta(v)}(\phi_{\kappa_t^H(\Delta)}(v) - c_{\tau^\Delta(v)})f(v)dv | c_t = c_H\right],$$

where $\mathbb{E}[\cdot | c_t = c_H]$ is the expectation over future cost realizations, and where for each κ and $v \leq \kappa$, $\phi_\kappa(v) = v - \frac{F(\kappa) - F(v)}{f(v)}$ is the virtual valuation of a buyer with type v under truncated distribution $\frac{F(v)}{F(\kappa)}$. The reason why this expression holds is that, at any PBE, incentive compatibility must hold at every history. Note that, for every $s \in T(\Delta), s > t$,

$$\begin{aligned} F(\kappa_t^H(\Delta))U^H(\kappa_t^H(\Delta);\Delta) \geq & \int_{\kappa_s^H(\Delta)}^{\kappa_t^H(\Delta)} (\phi_{\kappa_t^H(\Delta)}(v) - c_H) f(v) dv \\ & + e^{-r\Delta}\mathbb{E}\left[\int_{\underline{v}}^{\kappa_s^H(\Delta)} e^{-r(\tau^\Delta(v)-s)}(\phi_{\kappa_t^H(\Delta)}(v) - c_{\tau^\Delta(v)}) f(v) dv \,|c_t = c_H\right] \end{aligned} \tag{26}$$

Indeed, the right-hand side of (26) is the profits that the seller would obtain if she accelerated trade and sold to all buyer types $v \in [\kappa_s^H(\Delta), \kappa_t^H(\Delta)]$ at time t and then played the continuation equilibrium.[21]

By Helly's Selection Theorem, there exists a sequence $\Delta_n \to 0$ and functions κ_t^H, p_t^H and $\tau(v)$ such that, as $n \to \infty$, $\kappa_t^H(\Delta_n)$ and $p_t^H(\Delta_n)$ converge pointwise to κ_t^H and p_t^H, and $\tau^{\Delta_n}(v)$ converges pointwise to $\tau(v)$. By Lemma D.1, $\kappa_t^H \geq \lim_{\Delta\to 0} v^*(\Delta) = \hat{v}$ for all $t \geq 0$. Since κ_t^H is decreasing in t, to establish the result it suffices to show that $\kappa_{0^+}^H = \lim_{t\searrow 0} \kappa_t^H = \hat{v}$.

Towards a contradiction, suppose that the result is not true, so $\kappa_{0^+}^H > \hat{v}$. Let $\kappa_{s_n}^H$ be an increasing sequence converging to $\kappa_{0^+}^H$. By dominated convergence, and using (26), for all s_n it must be that

$$\begin{aligned} & \mathbb{E}\left[\int_{\underline{v}}^{\kappa_{0^+}^H} e^{-r\tau(v)}(\phi_{\kappa_{0^+}^H}(v) - c_{\tau(v)}) f(v) dv | c_{0^+} = c_H\right] \\ \geq & \int_{\kappa_{s_n}^H}^{\kappa_{0^+}^H} (\phi_{\kappa_{0^+}^H}(v) - c_H) f(v) dv + \mathbb{E}\left[\int_{\underline{v}}^{\kappa_{s_n}^H} e^{-r(\tau(v)-s_n)}(\phi_{\kappa_{0^+}^H}(v) - c_{\tau(v)}) f(v) dv \,| c_{0^+} = c_H\right]. \end{aligned} \tag{27}$$

Since $\kappa_{s_n}^H \nearrow \kappa_{0^+}^H > \hat{v}$, for all n large enough we have that $\phi_{\kappa_{0^+}^H}(v) = v - \frac{F(\kappa_{0^+}^H) - F(v)}{f(v)} > \hat{v}$ for all $v \in [\kappa_{s_n}^H, \kappa_{0^+}^H]$. It follows that, for all n large, for all $v \in [\kappa_{s_n}^H, \kappa_{0^+}^H]$ and for all random

[21] By stationarity of the equilibrium, such a deviation does not affect the prices that the different types of buyers are willing to accept.

times $\tau > 0$, $\phi_{\kappa^H_{0+}}(v) - c_H > \mathbb{E}[e^{-r\tau}(\phi_{\kappa^H_{0+}}(v) - c_\tau)|c_0 = c_H]$.[22] Since the seller's continuation payoff $\frac{1}{F(\kappa^H_{s_n})}\mathbb{E}\left[\int_{\underline{v}}^{\kappa^H_{s_n}} e^{-r\tau(v)}(\phi_{\kappa^H_{0+}}(v) - c_{\tau(v)})f(v)dv \,|c_{0^+} = c_H\right]$ at state $\kappa^H_{s_n}$ is non-negative,[23] it follows that for all n large enough

$$\int_{\kappa^H_{s_n}}^{\kappa^H_{0+}} (\phi_{\kappa^H_{0+}}(v) - c_H)f(v)dv + \mathbb{E}\left[\int_{\underline{v}}^{\kappa^H_{s_n}} e^{-r(\tau(v)-s_n)}(\phi_{\kappa^H_{0+}}(v) - c_{\tau(v)})f(v)dv \,|c_{0^+} = c_H\right]$$
$$> \mathbb{E}\left[\int_{\underline{v}}^{\kappa^H_{0+}} e^{-r\tau(v)}(\phi_{\kappa^H_{0+}}(v) - c_{\tau(v)})f(v)dv|c_{0^+} = c_H\right],$$

which violates (27). ■

Proof of Theorem 3. Part (i) follows from Lemmas D.1 and D.2, together with the fact that, when $c_0 = c_L = 0$, by the Coase conjecture all buyers buy immediately in the limit as $\Delta \to 0$.

For every Δ, let $\overline{\kappa}^H(\Delta)$ be such that $\{\kappa^H_t(\Delta)\}$ converges to $\overline{\kappa}^H(\Delta)$ as $t \to \infty$. That is, $\overline{\kappa}^H(\Delta)$ is the lowest valuation at which the buyer buys when the seller's costs are c_H and the time period is Δ. The price p at which the seller sells to a buyer with valuation $\overline{\kappa}^H(\Delta)$ when her cost is c_H must be such that $\overline{\kappa}^H(\Delta) - p = \rho(\Delta)(\overline{\kappa}^H(\Delta) - p^L(\overline{\kappa}^H(\Delta))) \iff p = \hat{p}(\overline{\kappa}^H(\Delta))$.

By Lemma D.2, $\overline{\kappa}^H(\Delta) \to \hat{v} = \frac{r+\lambda}{r}c_H$ as $\Delta \to 0$. Since $p^L(\kappa) \to \underline{v}$ for all κ as $\Delta \to 0$, it follows that $\lim_{\Delta\to 0}\hat{p}(\kappa^H(\Delta)) \to \frac{r\hat{v}+\lambda\underline{v}}{r+\lambda} = c_H + \frac{\lambda}{r+\lambda}\underline{v}$. Therefore, when $c_0 = c_H$ the limiting initial price is $c_H + \frac{\lambda}{r+\lambda}\underline{v}$, establishing part (ii).

The limiting profits that the seller makes when $c_0 = c_H$ are then

$$\lim_{\Delta\to 0} U^{\mathsf{pub}}(\sigma^\Delta, \mu^\Delta; \Delta) = (1 - F(\hat{v}))\left(c_H + \frac{\lambda}{r+\lambda}\underline{v} - c_H\right) + F(\hat{v})\frac{\lambda}{r+\lambda}\underline{v} = \frac{\lambda}{r+\lambda}\underline{v},$$

establishing part (iii). ■

[22] Indeed, for all v with $\phi_\kappa(v) > \hat{v}$, the solution to $\sup_\tau \mathbb{E}[e^{-r\tau}(\phi_\kappa(v) - c_\tau)|c_0 = c_H]$ is $\tau = 0$.

[23] The seller's continuation payoff $\frac{1}{F(\kappa^H_{s_n})}\mathbb{E}\left[\int_{\underline{v}}^{\kappa^H_{s_n}} e^{-r\tau(v)}(\phi_{\kappa^H_{0+}}(v) - c_{\tau(v)})f(v)dv \,|c_{0^+} = c_H\right]$ is bounded below by $\frac{\lambda}{r+\lambda}\underline{v} > 0$, which is the payoff from waiting until costs fall to $c_L = 0$ and posting price $\underline{v}$.

Online Appendix – Not for publication

A Proof of Proposition 2

Proof of Proposition 2. Suppose first that the seller can commit to a mechanism at time $t = 0^-$, before learning her initial cost. For each $v \in [\underline{v}, \overline{v}]$, let $\tau(v) \in \mathbb{R}_+ \cup \{\infty\}$ denote the random time at which a buyer with type v buys under this commitment solution, and let $p(v)$ denote the price at which consumer with value v buys.

The seller's expected profits are

$$\mathbb{E}\left[\int_{\underline{v}}^{\overline{v}} e^{-r\tau(v)}(p(v) - c_{\tau(v)})f(v)dv\right] = \mathbb{E}\left[\int_{\underline{v}}^{\overline{v}} e^{-r\tau(v)}(\phi(v) - c_{\tau(v)})f(v)dv\right],$$

where $\mathbb{E}[\cdot]$ denotes the expectation with respect to cost process $\{c_t\}$, and where the equality follows since, by incentive compatibility, $\mathbb{E}[e^{-r\tau(v)}p(v)] = \mathbb{E}\left[e^{-r\tau(v)}v - \int_{\underline{v}}^{v} e^{-r\tau(x)}dx\right]$ for all $v \in [\underline{v}, \overline{v}]$. Note that, for each v, the solution to $\max_\tau \mathbb{E}[e^{-r\tau}(\phi(v) - c_\tau)]$ is: $\tau = 0$ if $\phi(v) \geq v^*(\Delta)$; $\tau = \tau_L = \inf\{t : c_t = c_L = 0\}$ if $\phi(v) \in [0, v^*(\Delta)]$; and $\tau = \infty$ if $\phi(v) < 0$. Hence, if the seller can commit before learning her initial cost, a buyer with $\phi(v) \geq v^*(\Delta)$ buys at time $t = 0$, a buyer with $\phi(v) \in [0, v^*(\Delta)]$ buys at time $\tau = \tau_L = \inf\{t : c_t = c_L = 0\}$, and a buyer with $\phi(v) < 0$ never buys. Let v_H be such that $\phi(v_H) = v^*(\Delta)$ and let $v_L = \inf\{v \in [\underline{v}, \overline{v}] : \phi(v) \geq 0\}$. This commitment solution can be implemented with the following path of prices: the seller charges price $p_H = v_H - \rho(\Delta)(v_H - v_L)$ while her cost is c_H, and charges price $p_L = v_L$ when her cost reaches c_L.

Suppose next that the seller can only commit to a mechanism after learning her initial cost c_0. Under a direct mechanism: at $t = 0$ the buyer reports her value $v \in [\underline{v}, \overline{v}]$ and the seller reports her initial cost $c_0 \in \{c_L, c_H\}$; at any time $t > 0$, the seller reports her cost

$c_t \in \{c_L, c_H\}$.

Consider the following direct mechanism. If the seller's initial cost report is c_L: (i) buyer and seller trade immediately at price $p_L = v_L$ if buyer reported $v \geq v_L$; (ii) if buyer reported $v < v_L$, she never trades, and pays nothing to the seller. If the seller's initial cost report is c_H: (i) buyer and seller trade immediately at price p_H if buyer reported $v \geq v_H$; (ii) if buyer reported $v \in [v_L, v_H]$, she pays $\rho(\Delta)v_L$ to the seller at time $t = 0$, and then gets the good the first time the seller reports that her cost fell to $c_L = 0$, at additional price $c_L = 0$; (iii) if buyer reported $v < v_L$, she never trades, and pays nothing to the seller.

Note that, conditional on truth-telling, this mechanism gives the same expected payoff to all buyer types than the commitment solution discussed above; and hence it is incentive compatible for the buyer to report truthfully, conditional on the seller reporting truthfully. Moreover, it also gives the same expected revenues to the seller. I now show that it is incentive compatible for the seller to report truthfully. To see why, note first that a seller with $c_0 = c_H$ does not find it profitable to mimic a seller with $c_0 = c_L$. Moreover, a seller who reported $c_0 = c_H$ finds it weakly optimal to report her cost c_t truthfully for all $t > 0$.

A seller with $c_0 = c_L$ earns profits $v_L(1 - F(v_L))$ by reporting truthfully at $t = 0$. Note that, since $v_L = \inf\{v \in [\underline{v}, \overline{v}] : \phi(v) \geq 0\}$ and since $\phi(v)$ is strictly increasing, $v_L = \arg\max_v v(1 - F(v))$. If a seller with $c_0 = c_L$ reports cost c_H at $t = 0$, her profits are

$$p_H(1 - F(v_H)) + \rho(\Delta)v_L(F(v_H) - F(v_L)) = v_H(1 - \rho(\Delta))(1 - F(v_H)) + \rho(\Delta)v_L(1 - F(v_L)),$$

where I used $p_H = (1 - \rho(\Delta))v_H + \rho(\Delta)v_L$. Note that

$$\begin{aligned} & v_L(1 - F(v_L)) - (v_H(1 - \rho(\Delta))(1 - F(v_H)) + \rho(\Delta)v_L(1 - F(v_L))) \\ =& (1 - \rho(\Delta))(v_L(1 - F(v_L)) - v_H(1 - F(v_H)) \geq 0, \end{aligned}$$

where the inequality uses $v_L = \arg\max_v v(1 - F(v))$. Hence, a seller with initial cost c_L finds it optimal to report truthfully at $t = 0$. ■

B Mechanism design: efficient mechanism and inefficient equilibria

This Appendix has two results. First, it shows that, under certain conditions, there exists a direct mechanism satisfying IC, IR, and BB that attains the efficient outcome. Second, it shows that all PBE of the game are inefficient.

Consider the following direct mechanism, which I denote M^{FB}. At $t = 0$, buyer reports her type $v \in [\underline{v}, \overline{v}]$ and seller reports her initial cost $c_0 \in \{c_L, c_H\}$. If the seller reports $c_0 = c_L = 0$, then all types of buyers trade at $t = 0$ and pay price $\underline{v} > 0$ to the seller.

If the seller instead reports $c_0 = c_H$, then at $t = 0$: (i) all buyer types with $v \in [v^*(\Delta), \overline{v}]$ trade at $t = 0$ and pay price $c_H + \rho(\Delta)\underline{v}$; (ii) all buyer types with $v \in [\underline{v}, v^*(\Delta))$ pay the seller a price $\rho(\Delta)\underline{v}$ but don't trade yet. Then, at each period $t \in T(\Delta), t > 0$, the seller reports her cost $c_t \in \{c_L, c_H\}$. If at $t > 0$ the seller reports $c_t = c_H$, nothing happens. The first period $t > 0$ at which the seller reports $c_t = c_L$, all buyer types $v \in [\underline{v}, v^*(\Delta))$ trade, and pay price $c_L (= 0)$ to the seller at this point.

Proposition B.1. *Suppose that* $(1 - \rho(\Delta))\underline{v} \geq (1 - F(v^*(\Delta)))c_H$. *Then, mechanism* M^{FB} *is budget balance, satisfies IC and IR, and implements the first best outcome.*

Proof. It is easy to check that mechanism M^{FB}: (a) is budget balance, (b) satisfies IC for the buyer, (c) satisfies IR for buyer and seller, and (d) implements the efficient outcome under truthful reporting. I now show that the mechanism also satisfies IC for the seller. Consider first a seller who reported $c_0 = c_H$ at $t = 0$. Then, for all $t > 0$, the seller strictly prefers to report $c_t = c_H$ if her current cost is c_H, while she is indifferent between reporting c_L or c_H if her cost is c_L. Hence, truthful reporting is (weakly) optimal.

Consider next time $t = 0$. A seller with initial cost c_H obtains a payoff of $\rho(\Delta)\underline{v}$ from reporting truthfully, and gets a payoff of $\underline{v} - c_H$ from reporting $c_0 = c_L$. Recall than $v^*(\Delta) = \frac{c_H}{1-\rho(\Delta)} > \underline{v}$, where the inequality follows from Assumption 1. Hence, $\rho(\Delta)\underline{v} > \underline{v} - c_H$, so a seller with initial cost c_H strictly prefers to report truthfully.

A seller with initial cost c_L gets a payoff of $\underline{v}$ if she reports truthfully. Her payoff from reporting $c_0 = c_H$ is $(1 - F(v^*(\Delta)))(c_H + \rho(\Delta)\underline{v}) + F(v^*(\Delta))\rho(\Delta)\underline{v}$. Reporting truthfully is optimal when $(1 - \rho(\Delta))\underline{v} \geq (1 - F(v^*(\Delta)))c_H$. ■

I now show that every PBE of the game, or any limiting PBE as $\Delta \to 0$, is inefficient. Note that any $(\sigma, \mu) \in \Sigma(\Delta)$ (or any limit of equilibria $(\sigma^n, \mu^n) \in \Sigma(\Delta^n)$ with $\Delta^n \to 0$) induces an outcome $\tau : [\underline{v}, \overline{v}] \times \{c_L, c_H\} \to \mathbb{R}_+$ and $p : [\underline{v}, \overline{v}] \times \{c_L, c_H\} \to \mathbb{R}_+$, where $\tau(v, c_0)$ (resp. $p(v, c_0)$) is the possibly random time (resp. expected price) at which a buyer with value v buys when the seller's initial cost is c_0.

Since the seller is making all the offers, prices $p(v, c_0)$ must satisfy $p(v, c_0) \geq \underline{v}$ for all $v \in [\underline{v}, \overline{v}]$ and $c_0 \in \{c_L, c_H\}$: in any PBE, all buyer types accept a price $\underline{v}$ with probability 1.[24] This implies that, in any PBE, the profits of a seller with initial cost c_H are bounded below by $\rho\underline{v}$. Indeed, a seller with initial cost c_H can wait until her cost falls to c_L, charge price $\underline{v}$, and make a sale with probability 1, earning $\rho\underline{v}$.

Recall that the first-best outcome $\tau^{FB} : [\underline{v}, \overline{v}] \times \{c_L, c_H\} \to \mathbb{R}_+$ has $\tau^{FB}(v, c_L) = 0$ for all v, and $\tau^{FB}(v, c_H) = \mathbf{1}_{v<v^*}\tau_L$, where $\tau_L = \inf\{t : c_t = c_L\}$.

Proposition B.2. *Let $\tau : [\underline{v}, \overline{v}] \times \{c_L, c_H\} \to \mathbb{R}_+$ and $p : [\underline{v}, \overline{v}] \times \{c_L, c_H\} \to \mathbb{R}_+$ be an outcome induced by a PBE $(\sigma, \mu) \in \Sigma(\Delta)$, or the limiting outcome induced by a sequence of PBE $(\sigma^n, \mu^n) \in \Sigma(\Delta^n)$ with $\Delta^n \to 0$. Then, $\tau \neq \tau^{FB}$.*

Proof. Suppose by contradiction that the result is not true, so $\tau = \tau^{FB}$. Let $U(v)$ be the

[24]This follows from arguments in Lemma 1 in Gul et al. (1985), or Lemma S10 in Ortner (2017).

utility that a buyer with type v gets under this outcome:

$$U(v) = \mathbb{E}[qe^{-r\tau^{FB}(v,c_H)}(v - p(v, c_H)) + (1 - q)e^{-r\tau^{FB}(v,c_L)}(v - p(v, c_L))].$$

By incentive compatibility, $U(v)$ satisfies:

$$U(v) = U(\underline{v}) + \int_{\underline{v}}^{v} \mathbb{E}[qe^{-r\tau^{FB}(x,c_H)} + (1 - q)e^{-r\tau^{FB}(x,c_L)}]dx \tag{28}$$

for all $v \in [\underline{v}, \overline{v}]$. Since $p(v, c) \geq \underline{v}$ for all v, $U(\underline{v}) = 0$.

Consider first $v < v^*$, and note that

$$\begin{aligned} U(v) &= q\rho(v - p(v, c_H)) + (1 - q)(v - p(v, c_L)) \\ &= q\rho(v - \underline{v}) + (1 - q)(v - \underline{v}), \end{aligned} \tag{29}$$

where the first equality uses the properties of $\tau^{FB}(v, c_0)$ and the second follows from equation (28), using $U(\underline{v}) = 0$. Since $p(v, c_0) \geq \underline{v}$ for $c_0 \in \{c_L, c_H\}$ and for all v, equation (29) implies $p(v, c_L) = p(v, c_H) = \underline{v}$ for all $v < v^*$.

Consider next $v \geq v^*$, and note that

$$\begin{aligned} U(v) &= q(v - p(v, c_H)) + (1 - q)(v - p(v, c_L)) \\ &= q[\rho(v^* - \underline{v}) + (v - v^*)] + (1 - q)(v - \underline{v}), \end{aligned} \tag{30}$$

where again the first equality uses the properties of $\tau^{FB}(v, c_0)$ and the second follows from equation (28), using $U(\underline{v}) = 0$. Equation (30) implies that, for all $v \geq v^*$,

$$qp(v, c_H) + (1 - q)p(v, c_L) = q[v^* - \rho(v^* - \underline{v})] + (1 - q)\underline{v} = q(c_H + \rho\underline{v}) + (1 - q)\underline{v},$$

where the last equality uses $v^* - c_H = \rho v^*$. Since $p(v, c_L) \geq \underline{v}$ for all v, it follows that

$p(v, c_H) \leq c_H + \rho\underline{v}$. I now show that $p(v, c_L) = \underline{v}$ and $p(v, c_H) = c_H + \rho\underline{v}$ for almost all $v \geq v^*$. Suppose not, so there exists a positive measure of buyer types $v \geq v^*$ with $p(v, c_H) < c_H + \rho\underline{v}$. Since $p(v, c_H) = \underline{v}$ for all $v < v^*$, the profits of seller with $c_0 = c_H$ under outcome (τ, p) are

$$(1 - F(v^*))\mathbb{E}[p(v, c_H) - c_H | v \geq v^*] + F(v^*)\rho\underline{v} < \rho\underline{v}.$$

But this cannot be, since a seller with $c_0 = c_H$ can obtain $\rho\underline{v}$ by waiting until her costs fall to c_L and charging price $\underline{v}$. Hence, $p(v, c_H) = c_H + \rho\underline{v}$ and $p(v, c_L) = \underline{v}$ for almost all $v \geq v^*$.

By the arguments above, under outcome (τ, p) a seller with $c_0 = c_L$ earns profits $\underline{v}$. The profits that this seller can obtain by mimicking a seller with $c_0 = c_H$, and then playing as if her cost fell to c_L at time $s\Delta$ are $(1 - F(v^*))(c_H + \rho\underline{v}) + e^{-rs\Delta}F(v^*)\underline{v}$, which is strictly larger than $\underline{v}$ for all $s\Delta$ small enough (since, by Assumption 1, $v^* = \frac{c_H}{1-\rho} > \underline{v} \iff c_H + \rho\underline{v} > \underline{v}$), a contradiction. Hence, $\tau \neq \tau^{FB}$. ■

C Increasing costs

This appendix studies a version of the model in which the evolution of $\{c_t\}$ satisfies $\text{prob}(c_{t+\Delta} = c_H | c_t = c_H) = e^{-\lambda\Delta}$ and $\text{prob}(c_{t+\Delta} = c_L | c_t = c_L) = e^{-\gamma\Delta}$.

Recall that v^* is such that $v^* - c_H = \rho v^*$, where ρ is defined as in the main text. Note first that Proposition 1 continues to hold in this setting: under the first-best solution, the seller sells to a buyer with value $v \geq v^*$ at $t = 0$, and sells to a buyer with value $v < v^*$ the first time her cost falls to c_L. I maintain Assumption 1, so $v^* \in (\underline{v}, \overline{v})$.

I focus on PBE (σ, μ) that satisfy the following conditions. First, as in the main text, I restrict attention to separating equilibria: for all histories h_t^S, $\mathsf{supp}\, \sigma^B(h_t^S)(c_H) \cap \mathsf{supp}\, \sigma^B(h_t^S)(c_L) = \emptyset$. Second, I consider weakly stationary equilibria; i.e., equilibria in which the buyer's purchasing decision at histories at which the seller's current price offer is the lowest (given buyer's current beliefs) depends solely on her value and her beliefs about the seller's costs. With a slight abuse of notation, I let Σ^{S} denote the set of PBE satisfying these conditions.

For each PBE $(\sigma, \mu) \in \Sigma^{\mathsf{S}}$, let $U^H_{(\sigma,\mu)}(\kappa)$ and $U^L_{(\sigma,\mu)}(\kappa)$ denote, respectively, the seller's profits under (σ, μ) when her belief cutoff is κ and her cost is c_H and c_L. Note first that, in any PBE $(\sigma, \mu) \in \Sigma^{\mathsf{S}}$, it must be that $U^H_{(\sigma,\mu)}(\kappa) \geq \rho U^L_{(\sigma,\mu)}(\kappa)$. Indeed, under a weakly stationary equilibrium, a seller with a high cost can always delay trade until her cost falls, and earn continuation profits $U^L_{(\sigma,\mu)}(\kappa)$.

The following result generalizes Lemma A.1 to the current environment:

Lemma C.1. *Consider a seller history h^S_t such that the seller's belief cutoff κ_t at time t is strictly larger than $\underline{v}$. Let p^H_t be the price that the seller charges under $(\sigma, \mu) \in \Sigma^{\mathsf{S}}$ at history h^S_t if $c_t = c_H$, and let $\kappa_{t+\Delta}$ be the highest consumer type that buys at time t when $c_t = c_H$. Then, κ_t and $\kappa_{t+\Delta}$ satisfy*

$$p^H_t \frac{F(\kappa_t) - F(\kappa_{t+\Delta})}{F(\kappa_t)} \leq U^L_{(\sigma,\mu)}(\kappa_t) - e^{-(r+\gamma)\Delta} \frac{F(\kappa_{t+\Delta})}{F(\kappa_t)} U^L_{(\sigma,\mu)}(\kappa_{t+\Delta}) - e^{-r\Delta}(1 - e^{-\gamma\Delta}) \frac{F(\kappa_{t+\Delta})}{F(\kappa_t)} U^H_{(\sigma,\mu)}(\kappa_{t+\Delta}). \tag{31}$$

Proof. Consider a seller whose cost changed from c_H to c_L at time t, after history h^S_t. The profits that this seller obtains by revealing her cost are $U^L_{(\sigma,\mu)}(\kappa_t)$. The profits that this seller would make by posting price p^H_t that she would have posted if $c_t = c_H$, and then from $t + \Delta$ onwards playing the continuation strategy with belief cutoff $\kappa_{t+\Delta}$ are

$$p^H_t \frac{F(\kappa_t) - F(\kappa_{t+\Delta})}{F(\kappa_t)} + e^{-(r+\gamma)\Delta} \frac{F(\kappa_{t+\Delta})}{F(\kappa_t)} U^L_{(\sigma,\mu)}(\kappa_{t+\Delta}) + e^{-r\Delta}(1 - e^{-\gamma\Delta}) \frac{F(\kappa_{t+\Delta})}{F(\kappa_t)} U^H_{(\sigma,\mu)}(\kappa_{t+\Delta}).$$

A seller whose cost changed to c_L at period t has an incentive to reveal her cost only if (31) holds. ■

Fix $(\sigma, \mu) \in \Sigma^{\mathsf{S}}$. For each $\tilde{\kappa}$, let $p^H(\tilde{\kappa})$ (resp. $p^L(\tilde{\kappa})$) be the price that the seller charges when her belief cutoff is $\tilde{\kappa}$ and her cost is c_H (resp. c_L) under (σ, μ). Let h_t^S be a history in which the seller's belief cutoff is κ_t and her current cost is c_H, and let $\kappa_{t+\Delta} \leq \kappa_t$ be the lowest valuation that buys at price $p^H(\kappa_t)$. Note that the following equality must hold:

$$\kappa_{t+\Delta} - p^H(\kappa_t) = e^{-r\Delta}(1 - e^{-\lambda\Delta})(\kappa_{t+\Delta} - p^L(\kappa_{t+\Delta})) + e^{-(r+\lambda)\Delta}(\kappa_{t+\Delta} - p^H(\kappa_{t+\Delta})). \quad (32)$$

For each $\Delta > 0$, let $(\sigma^\Delta, \mu^\Delta)$ be an equilibrium in Σ^{S} in a game with time-period Δ. For each $\Delta > 0$ and each $t \in T(\Delta)$, let $\kappa_t(\Delta)$ denote the (random) sequence of belief cutoffs under $(\sigma^\Delta, \mu^\Delta)$ at time t.[25] Let $p^L(\kappa; \Delta)$ and $p^H(\kappa; \Delta)$ be the price that the seller offers under $(\sigma^\Delta, \mu^\Delta)$ when costs are c_L and c_H, respectively, and her belief cutoff is κ.

The next result shows that, in any equilibrium in Σ^{S}, as $\Delta \to 0$ buyer and seller trade immediately whenever the seller's cost are c_L. Let τ_L^Δ denote the first time in $T(\Delta)$ that the seller's costs are c_L,

Lemma C.2. *For all $t > \tau_L^\Delta$, $\lim_{\Delta\to 0} \kappa_t(\Delta) = \underline{v}$.*

Proof. Consider period τ_L^Δ with belief cutoff $\kappa_{\tau_L^\Delta} = \kappa$ and with $c_t = c_L$. For each $v \in [\underline{v}, \kappa]$, let $\tau^\Delta(v)$ denote the random time at which a buyer with valuation v buys. The seller's continuation profits $U^L(\kappa; \Delta)$ at this history satisfy:

$$F(\kappa)U^L(\kappa; \Delta) = \mathbb{E}\left[\int_{\underline{v}}^{\kappa_t^H(\Delta)} e^{-r\tau^\Delta(v)}(\phi_\kappa(v) - c_{\tau^\Delta(v)})f(v)dv | c_t = c_L\right],$$

where $\mathbb{E}[\cdot | c_t = c_H]$ is the expectation over future cost realizations, and where for each κ and $v \leq \kappa$, $\phi_\kappa(v) = v - \frac{F(\kappa)-F(v)}{f(v)}$ is the virtual valuation of a buyer with type v under truncated distribution $\frac{F(v)}{F(\kappa)}$. This expression holds since incentive compatibility must hold at every history.

[25]Note that $\kappa_t(\Delta)$ is random even if the seller uses a pure strategy, since behavior until time t might depend on past cost realizations.

For every $s \in T(\Delta), s > \tau_L^\Delta$, let $\kappa_s^L(\Delta)$ denote the seller's belief cutoff at time s under $(\sigma^\Delta, \mu^\Delta)$, conditional on $c_\tau = c_L$ for all $\tau \in [\tau_L^\Delta, s]$. Let $p^L(\kappa_s^L(\Delta), \Delta)$ be the price that the seller charges at time s under $(\sigma^\Delta, \mu^\Delta)$ conditional on $c_\tau = c_L$ for all $\tau \in [\tau_L^\Delta, s]$.

For all such $s \in T(\Delta), s > \tau_L^\Delta$, it must be that

$$F(\kappa)U^L(\kappa;\Delta) \geq \int_{\kappa_s^L(\Delta)}^{\kappa} (\phi_\kappa(v) - c_L)f(v)dv \\ + e^{-r\Delta}\mathbb{E}\left[\int_{\underline{v}}^{\kappa_s^L(\Delta)} e^{-r(\tau^\Delta(v)-s)}(\phi_\kappa(v) - c_{\tau^\Delta(v)})f(v)dv \,|c_t = c_L\right] \tag{33}$$

Indeed, the right-hand side of (33) is the profits that the seller would obtain if she accelerated trade and sold to all buyer types $v \in [\kappa_s(\Delta), \kappa]$ at time t and then played the continuation equilibrium.[26]

By Helly's Selection Theorem, there exists a sequence $\Delta_n \to 0$ and functions κ_t^L, p_t^L and $\tau(v)$ such that, as $n \to \infty$, $\kappa_t^L(\Delta_n)$ and $p_t^L(\Delta_n)$ converge pointwise to κ_t^L and p_t^L, and $\tau^{\Delta_n}(v)$ converges pointwise to $\tau(v)$. Since κ_t^L is decreasing in t, to establish the result it suffices to show that $\kappa_{\tau_L^+}^H = \lim_{s \searrow 0} \kappa_{\tau_L+s}^H = \underline{v}$.

Towards a contradiction, suppose that the result is not true, so $\kappa_{\tau_L^+}^L > \underline{v}$. Let $\kappa_{s_n}^L$ be an increasing sequence converging to $\kappa_{\tau_L^+}^L$. By dominated convergence, and using (33), for all s_n it must be that

$$\mathbb{E}\left[\int_{\underline{v}}^{\kappa_{\tau_L^+}^L} e^{-r\tau(v)}(\phi_{\kappa_{\tau_L^+}^L}(v) - c_{\tau(v)})f(v)dv|c_{0^+} = c_L\right] \\ \geq \int_{\kappa_{s_n}^L}^{\kappa_{\tau_L^+}^L} (\phi_{\kappa_{\tau_L^+}^L}(v) - c_L)f(v)dv + \mathbb{E}\left[\int_{\underline{v}}^{\kappa_{s_n}^L} e^{-r(\tau(v)-s_n)}(\phi_{\kappa_{\tau_L^+}^L}(v) - c_{\tau(v)})f(v)dv\,|c_{0^+} = c_L\right]. \tag{34}$$

Since $\kappa_{s_n}^L \nearrow \kappa_{\tau_L^+}^L > \underline{v}$, for all n large enough we have that $\phi_{\kappa_{\tau_L^+}^L}(v) = v - \frac{F(\kappa_{\tau_L^+}^L) - F(v)}{f(v)} >$

[26]By stationarity of the equilibrium, such a deviation does not affect the prices that the different types of buyers are willing to accept.

$\underline{v} > c_L = 0$ for all $v \in [\kappa^L_{s_n}, \kappa^L_{\tau^+_L}]$. It follows that, for all n large, for all $v \in [\kappa^L_{s_n}, \kappa^L_{\tau^+_L}]$ and for all random times $\tau > 0$, $\phi_{\kappa^L_{\tau^+_L}}(v) - c_L > \mathbb{E}[e^{-r\tau}(\phi_{\kappa^L_{\tau^+_L}}(v) - c_\tau)|c_0 = c_L]$.[27] Since the seller's continuation payoff $\frac{1}{F(\kappa^L_{s_n})}\mathbb{E}\left[\int_{\underline{v}}^{\kappa^L_{s_n}} e^{-r\tau(v)}(\phi_{\kappa^L_{\tau^+_L}}(v) - c_{\tau(v)})f(v)dv\,|c_{0^+} = c_L\right]$ at state $\kappa^L_{s_n}$ is non-negative, it follows that for all n large enough

$$\int_{\kappa^L_{s_n}}^{\kappa^L_{\tau^+_L}} (\phi_{\kappa^L_{\tau^+_L}}(v) - c_H)f(v)dv + \mathbb{E}\left[\int_{\underline{v}}^{\kappa^L_{s_n}} e^{-r(\tau(v)-s_n)}(\phi_{\kappa^L_{\tau^+_L}}(v) - c_{\tau(v)})f(v)dv\,|c_{0^+} = c_L\right]$$
$$> \mathbb{E}\left[\int_{\underline{v}}^{\kappa^L_{\tau^+_L}} e^{-r\tau(v)}(\phi_{\kappa^L_{\tau^+_L}}(v) - c_{\tau(v)})f(v)dv|c_{0^+} = c_L\right],$$

which violates (34). ■

The following result follows from Lemma C.2.

Corollary C.1. *For any $\kappa \in [\underline{v}, \overline{v}]$, $p^L(\kappa;\Delta) \to \underline{v}$ and $U^L(\kappa;\Delta) \to \underline{v}$ as $\Delta \to 0$.*

By arguments similar to those in the proof of Theorem 2, and using Corollary C.1, one can show that in the limit as $\Delta \to 0$, under a most efficient separating equilibrium, the evolution of prices when costs are c_H still satisfies equation (4). Moreover, the limiting speed of trade now satisfies:

$$-\frac{d\kappa^H(t)}{dt} = \frac{F(\kappa^H(t))}{f(\kappa^H(t))}\frac{(r+\gamma)\underline{v} - \gamma U^H(\kappa^H(t))}{(p^H(t) - \underline{v})},$$

where $U^H(\kappa^H(t))$ is the limiting payoff of a seller with cost c_H and belief cutoff $\kappa^H(t)$.

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[27] Indeed, for all v with $\phi_\kappa(v) > c_L$, the solution to $\sup_\tau \mathbb{E}[e^{-r\tau}(\phi_\kappa(v) - c_\tau)|c_0 = c_L]$ is $\tau = 0$.

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