A Macro-Finance Approach to Sovereign Debt Spreads and Returns

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Abstract

Foreign currency sovereign bond spreads tend to be higher than historical sovereign credit losses, and cross-country spread correlations are larger than their macro-economic counterparts. Foreign currency sovereign debt exhibits positive and time-varying risk premia, and standard linear asset pricing models using US-based factors cannot be rejected. I develop a quantitative and tractable continuous-time model of endogeneous sovereign default in order to account for these stylized facts. My framework leads to pseudo-closed form expressions for local asset pricing moments of interest, helping disentangle which of the model features influences credit spreads, expected returns and cross-country correlations. Standard pricing kernels used to explain properties of US equity returns can be nested into my quantitative framework in order to test the hypothesis that US-based bond investors are marginal in sovereign debt markets. I also show how to leverage my model to study the early 1980’s Latin American debt crisis, during which high short term US interest rates and floating rate dollar-denominated debt led to a wave of sovereign defaults.

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1 Introduction

Driven by low real interest rates, high commodity prices and easy credit, Latin American external debt grew significantly in the 1970s. The Volker shock, combined with debt contracts indexed to US short term rates, contributed to the subsequent debt crisis and the “lost decade” suffered by many Latin American countries in the 1980s. A quarter of a century later, in the fall 2008, the US subprime crisis morphed into a global financial crisis, leading to a shut down of emerging economies’ access to international credit markets and a violent widening of their sovereign spreads. Those two episodes highlight the central importance of the supply of capital for sovereign debt dynamics. However, a large component of the international macroeconomic literature on sovereign credit risk uses economic models where external creditors are risk-neutral, assuming away any possible link between investors’ attributes and government financing and default decisions. The modeling hypothesis of this line of research stems from its main focus on macroeconomic quantities (such as the current account balance and the debt-to-GDP ratio) as opposed to prices, and from the difficulty of adding one or several dimensions to already complex models of endogeneous default. Separately, the fixed income asset pricing literature on sovereign debt takes seriously investors’ risk attributes when explaining properties of sovereign credit spreads and returns, but it does so at the expense of modelling the underlying asset cashflows and their dynamic properties. Indeed, its primary objective is to use bond and CDS market prices in order to estimate hazard rate of default processes, without having the need to relate them to economic fundamentals.

My paper bridges the gap between these two seemingly disconnected literatures by offering a new framework of endogeneous sovereign default where the supply-side of capital takes on a prominent role, as supported by known stylized facts as well as new evidence I document in my empirical work. Thanks to its reduced dimensionality, the proposed framework remains tractable and allows me to obtain pseudo-closed form expressions for several macroeconomic and asset pricing local moments of interest. In addition, it facilitates the estimation and testing of the model, and an in-depth analysis of the government financing and default policies. It can then be used to answer numerous questions: how much of sovereign governments’ financing costs can be attributed to bond investors’ risk characteristics, and how much to country-specific macro-economic risks? Are sovereign debt return co-movements mostly due to correlated fundamentals, or the fact that a common bond buyer base is marginal in sovereign bond markets? Can supply-side shocks to capital markets rationalize the magnitude of current account reversals observed in the context of “sudden-stops”?

In the empirical section of my paper, I infer market-implied (sometimes called “risk-
neutral”) default intensities from sovereign credit-default swap (“CDS”) premia, and then compute returns on CDS contracts. Leveraging my constructed data-set, I document three sets of empirical facts that will not only guide my model but also will be used for estimation and testing.

First, I provide evidence that investors in sovereign debt markets do not behave risk-neutrally. To do so, I show that market-implied default intensities are significantly larger than historical default frequencies, and that sovereign CDS’ expected excess returns are positive. Together, these empirical properties of sovereign debt spreads and returns illustrate the two sides of the same coin: creditors require compensation for being exposed to a risk (the sovereign default risk) that co-moves with their pricing kernel. While these stylized facts have already been investigated by Broner, Lorenzoni, and Schmukler (2013) and Borri and Verdelhan (2011) in the context of foreign currency sovereign bonds, I contribute to the empirical debate by showing that this property of sovereign credit prices and returns also holds for CDS contracts.

Second, the data supports not only that sovereign debt investors are risk-averse, but also that their pricing of risk is time-varying and relates to measures of US credit and equity market risks. Indeed, the difference between market-implied and historical default intensities is time-varying and cannot be explained by time-varying country-specific macroeconomic risk factors. Using my constructed CDS return data, I then perform standard linear asset pricing tests, using either US equity or US credit market returns, and I fail to reject the hypothesis that a linear stochastic discount factor can price my set of excess returns. Finally, I show that cross-country CDS return correlations are significantly larger than their macroeconomic counterparts, suggesting that a common bond buyer base is marginal in hard currency sovereign debt markets.

While these facts, taken together, help us understand the required characteristics of a sovereign investors’ pricing kernel, they are silent on the type of mechanism leading to sovereign defaults, and how supply side factors may impact a sovereign government’s borrowing decisions. I speak to this question by illustrating a third set of facts, related to the term structure of market-implied default intensities and returns. First, I show that the term structure of default intensities is upward sloping for most countries, except for distressed countries, for which the term structure is downward sloping. Second, I show that holding period excess returns are increasing with the maturity of the CDS contract. Both facts are consistent with a “first hitting time” model, where a sovereign default is triggered by some –

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2I also argue that CDS are more appropriate instruments for computing risk-premia than sovereign bonds. Indeed, the JPMorgan EMBI index (used by most of the quantitative sovereign debt literature) includes mostly fixed rate bonds, meaning that a large component of sovereign debt returns inferred from the index is related to changes in long term US yields as opposed to changes in credit risk.
possibly endogeneous – mean-reverting fundamental variable exceeding a certain threshold.

What might this macroeconomic “fundamental” variable be? In my theoretical setup, it is the debt-to-GDP ratio. I leverage the canonical sovereign default model of Arellano (2008) and Aguiar and Gopinath (2006) and develop a quantitative continuous time model of sovereign debt issuances and defaults, in which a government uses non-state contingent debt sold to foreign creditors for the purpose of consumption smoothing and consumption tilting\(^3\). The government’s inability to commit to repay its debt leads to default risk. Following a default, the country suffers an instantaneous discrete drop in output and loses access to capital markets for an exponentially distributed time period. The country then re-enters financial markets with a lower debt burden, the result of an un-modeled renegotiation with its creditors. The sovereign debt-to-GDP ratio naturally arises as the fundamental state variable – a consequence of the homotheticity of the government’s objective function and the linearity of output and debt dynamics. I deviate from the canonical sovereign debt models along several dimensions. Since my model is cast in continuous-time, infinitesimally short term debt would not carry any default risk\(^4\); I thus model a government issuing long-term exponentially amortizing debt, a common tool in the corporate credit literature (Leland (1998) was the first paper to my knowledge to use this modeling assumption), and recently adopted by the sovereign default literature (Hatchondo and Martinez (2009), Chatterjee and Eyigungor (2015) for example). Some of the continuous time elements of my model are similar to Nuño and Thomas (2015), who study monetary policy and the trade-off between inflation and debt monetization, and to Cohen and Villemot (2013). Since my focus is on the supply of capital and its impact on sovereign bond prices and returns, I introduce investors, whose preferences and equilibrium consumption lead to a pricing kernel that features regime-dependent risk free rates and risk prices, in the spirit of Chen (2010). Those regimes act as a second – exogeneous and discrete – state variable that describes the international capital market environment.

My modeling ingredients lead to sovereign spreads that are greater than model-implied historical credit losses, as I document in the empirical part of the paper. For a panel of emerging market countries, I can then estimate the proportion of the average credit spread that can be attributed to (a) pure default risk and (b) the risk premium charged by international investors. In the model, spread volatilities stem not only from output shocks, but also from stochastic discount factor (“SDF”) shocks, and are thus close to spread volatilities in the data, a moment notoriously difficult to match with standard models (Aguiar et al. (2016)). For the same reason, cross-country sovereign spread correlations are larger than cross-country

\(^3\)As is typically the case in the international macroeconomic literature, the sovereign government will be more impatient than its creditors, providing an incentive to borrow in order to consume early.

\(^4\)See appendix for a more formal proof of this claim.
output correlations. By turning on and off those SDF shocks, I can then infer the proportion of such cross-country spread correlation that relates to correlated fundamentals, and the proportion that relates to pricing by a common stochastic discount factor.

In my model, the sovereign default decision features an optimal debt-to-GDP default boundary that depends on the specific pricing kernel regime. Consistent with the data, this characteristic of my model leads to upward sloping term structures of spreads and default intensities for countries whose economic fundamentals are not too bad and in environments where risk-prices are not too high. Transitions from a “good regime” (where prices of risk are low for example) to a “bad regime” (with higher prices of risk) might cause the sovereign to “jump-to-default”. Even if the sovereign government does not jump to default, it adjusts downwards its financing policy, switching from running a current account deficit to a current account surplus, and endogenously creating a sudden stop. Those regime transitions are also associated with inversions of the term structure of credit spreads, another feature of the data. When looking across multiple countries, transitions from “good regimes” to “bad regimes” lead to sudden increases in sovereign spreads as well as correlated defaults, a feature of several sovereign debt crisis – the Latin American crisis of the early 80’s or the Asian tiger crisis of the late 90’s. The jump-to-default risk induced by SDF shocks also leads to high short term credit spreads, another stylized fact I document in the empirical section of my paper.

The objective of focusing on the joint behavior of macroeconomic quantities, prices and returns makes the continuous time framework ideal. It facilitates the transition from physical probabilities (under which the government optimizes) to risk-neutral probabilities (under which creditors price the debt issued). It allows for pseudo-closed form expressions of local macro and asset pricing moments of interest, providing greater insight into the specific impact of the model assumptions on endogeneous quantities of focus. My model features only two state variables – the debt-to-GDP ratio of the country being considered (a continuous variable), and the SDF regime (a discrete variable). This low dimensionality of the state space makes the framework more tractable than alternative models that have been studied in the literature\textsuperscript{5}. It permits an estimation of the key parameters of the model using a panel of countries, and gives me the ability to test whether pricing kernels used to explain properties of US equity returns can also explain properties of emerging market sovereign bond

\textsuperscript{5}Other articles focused on sovereign spreads include Borri and Verdelhan (2011), which feature 4 state variables, and Aguiar et al. (2016), which feature 5 state variables; in order to find an equilibrium in such models, not only does the researcher have to find a global solution to the value function of the government (a function of all the state variables), but he also has to find the bond price schedule, which depends on both (i) the state variables and (ii) the amount of bonds that the government considers issuing. As will be clear in this paper, in continuous time the bond price schedule is no longer a function of the amount of bonds issued “in the next period”.

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returns. In my numerical applications, I test a long-run risk model (see for example Bansal and Yaron (2004)) as well as an external habit model (see Campbell and Cochrane (1995)).

I finally highlight the flexibility of my framework by testing two new ideas. First, I use a class of preferences new to the international macroeconomic literature on sovereign defaults in order to generate high equilibrium debt-to-GDP ratios. Indeed, this literature has historically found difficult to develop models where the government takes on large amounts of debt. Such failure has been investigated by Aguiar and Gopinath (2006), who conclude that the maximum supportable levels of government debt are related to welfare losses due to business cycle fluctuations: if those losses are small, it is not costly for the country to stay in financial autarky, giving the government an incentive to default at relatively low debt-to-GDP levels. Two modeling tools are usually introduced to address this issue: (a) long term debt, and (b) (potentially asymmetric) output losses upon default. I test a new mechanism for generating high debt-to-GDP ratios by studying a government that features recursive preferences (see Kreps and Porteus (1978) and Epstein and Zin (1989)), in which the country’s intertemporal elasticity of substitution is independent of its degree of risk-aversion. By using intertemporal elasticity of substitution levels consistent with the real business cycle literature and risk-aversion parameters consistent with the asset pricing literature, I show that welfare losses associated with consumption fluctuations are larger than in standard real business cycle models – a result that has been known by the asset pricing literature since Tallarini (2000). Thus, the theoretical level of sovereign debt supportable in equilibrium, as computed using a framework similar to Lucas (1987) or Aguiar and Gopinath (2006), could be higher than in standard sovereign default models. However, large levels of risk-aversion also give the government strong precautionary savings motives; this latter force turns out to dominate, and higher levels of risk-aversion in my model lead to lower debt issuance policies, and thus overall lower levels of government debt at default.

In a second exercise, I focus on the contractual structure of sovereign debt and study a government that issues floating rate – as opposed to fixed rate – debt. Until recently, most external sovereign debt issuances of emerging market economies were denominated in US dollars. While those bonds are nowadays mainly issued in fixed rate form, Latin American countries used floating rate debt in the 1970’s and early 80’s, since the funding came in the form of loans from US commercial banks. Given that my model features time-varying risk-free rates, I can investigate the impact of US monetary policy on sovereign default risk. In this paper, I show that a simple mechanism may have been at play during the Latin American sovereign debt crisis of the early 80’s: as the US monetary authorities increased short term rates to fight domestic inflation, Latin American government interest costs soared, leading to increases in sovereign spreads and ultimately the default of Mexico in 1982.
This paper is organized as follows. The first part of the paper focuses on some empirical facts of sovereign CDS premia and returns. I then develop a continuous time version of the canonical model of sovereign borrowing and default, and enhance it by introducing a Markov switching model of the stochastic discount factor used to price sovereign bonds. I estimate the model and perform a variety of exercises to illustrate the tractability of the framework.

2 Stylized Facts

2.1 Historical vs. Market Implied Default Intensities

I focus my empirical discussion on a set of 27 emerging market economies: Argentina, Brazil, Bulgaria, Chile, Colombia, Dominican Republic, Ecuador, Egypt, El Salvador, Hungary, Indonesia, Kazakhstan, Malaysia, Mexico, Pakistan, Panama, Peru, Philippines, Poland, Russia, Serbia, South Africa, Turkey, Ukraine, Uruguay, Venezuela, and Vietnam. For each of these countries, I collect bond spread data, CDS price data, bond issuance data, credit ratings data as well as macroeconomic data.

I first illustrate the fact that historical default frequencies are significantly smaller than default intensities implied by credit spreads (whether bond spreads or CDS), supporting the idea that creditors in foreign currency sovereign debt markets do not behave in a risk-neutral fashion. Given the rare-event nature of sovereign defaults, the task of estimating sovereign default frequencies is notoriously difficult. Tomz and Wright (2013) for example focus on 176 sovereign entities over a 200-year time-period, and estimate an unconditional default probability of 1.7% per year. A more informative measure of historical default frequency is a measure of conditional default frequency – in other words, the probability of default over a specific time horizon of a government, conditional on all the information available at a given

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6 The definition of “emerging market” varies vastly across the literature; here, I will loosely define emerging market an economy whose real GDP per capita is below a certain threshold.
7 The data on bond spreads comes from the JP Morgan EMBI Global Index. The credit spread of a given country is a weighted average of the country’s USD denominated bonds’ swap spreads (where a bond’s credit spread is computed as the difference between the bond’s yield to maturity and the relevant USD interest rate swap benchmark). Eligibility criteria for inclusion of a particular bond of a particular country into the EMBI Global Index are described in more details in Morgan (1999); at a high level, a sovereign bond is included in the index if its aggregate issuance is above $500 mm, and if its remaining life is above 2.5 years.
8 The CDS data comes from WRDS, which itself collects the data from Markit.
9 The data on bond issuances comes from downloading all available foreign currency bond issuances from Bloomberg.
10 I focus on Moody’s foreign currency issuer ratings, collected from Moody’s website.
11 The data on GDP and external debt comes from the Global Financial Development Database.
12 By restricting their sample to governments that defaulted at least once, the authors compute an unconditional default frequency of 3% per year for the time period 1945-1980, and 3.8% per year for the time period 1980-2012.
time. Rating agencies provide such measure of conditional default frequency. While different rating agencies use different methodologies, their analyses can be reduced to an assessment of a country’s expected default frequency conditional on all observables – such as the country’s debt-to-GDP ratio, its current account balance, the size of its foreign currency reserves, or its stock of foreign currency vs. local currency debt. In table 1, in columns “Moody’s Cum. Default Rates”, I reproduce calculations from Tudela et al. (2012). In their article, using a panel of 114 countries over the time period 1983 – 2012H1, the authors estimate issuer-weighted cumulative default frequencies over different time horizons of the sovereign issuers in their dataset, conditional on the credit rating. Next to each time-horizon \( T \), I calculate the equivalent yearly historical default intensity \( \lambda_T \), assuming a constant hazard rate:

\[
\Pr (\tau < T) = 1 - e^{-\lambda_T T}
\]

Using my data on bond spreads and CDS premia, I then construct time-series of weighted average market-implied default intensities for each rating category. I describe the procedure in details in section A.1.2. I plot the resulting market-implied default intensities in figure 12 and figure 13. From the plot, one notices that market implied default intensities – whether implied by bond prices or CDS premia – are consistently greater than Moody’s implied default intensities. Early 2007 is the only time period during which those two measures of default intensities almost coincide. Column “Bond-Implied” and “CDS-Implied” default intensities in table 1 show unconditional mean default intensities computed from bond and CDS prices. For example, bond-implied default intensities are between 1.46% and 7.77% greater than their Moody’s counterparts.

\(^{13}\) See for example Bhatia (2002) for a detailed evaluation of Moody’s, S&P and Fitch sovereign rating methodologies.

\(^{14}\) One may argue that the Moody’s sample goes further back in time than the time period for which my spread data is available: Moody’s sample starts in 1983, and no Moody’s rated sovereign bond defaults until

<table>
<thead>
<tr>
<th>Rating Category</th>
<th>Moody’s 5yr Cum. Rate</th>
<th>Moody’s 5yr Intensity</th>
<th>Moody’s 10yr Cum. Rate</th>
<th>Moody’s 10yr Intensity</th>
<th>Bond Implied Rate</th>
<th>Bond Implied Intensity</th>
<th>CDS Implied Rate</th>
<th>CDS Implied Intensity</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1.29%</td>
<td>0.26%</td>
<td>4.29%</td>
<td>0.44%</td>
<td>1.72%</td>
<td>1.20%</td>
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<tr>
<td>Baa</td>
<td>1.59%</td>
<td>0.32%</td>
<td>2.01%</td>
<td>0.20%</td>
<td>3.27%</td>
<td>2.17%</td>
<td></td>
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</tr>
<tr>
<td>Ba</td>
<td>6.14%</td>
<td>1.27%</td>
<td>14.37%</td>
<td>1.55%</td>
<td>5.81%</td>
<td>3.86%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>11.16%</td>
<td>2.37%</td>
<td>18.54%</td>
<td>2.05%</td>
<td>9.81%</td>
<td>8.43%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Caa-C</td>
<td>40.93%</td>
<td>10.53%</td>
<td>40.93%</td>
<td>5.26%</td>
<td>18.30%</td>
<td>15.04%</td>
<td></td>
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</tr>
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</table>

Table 1: Historical vs. Market Implied Default Rates
The second observation from these time series is that there is significant time variation in the spread between market-implied intensities and historical intensities, and that this variation is related to a variety of measures of global credit or equity market risks. To illustrate this time variation, I estimate the following panel regression:

\[
\hat{\lambda}_{it}(T) = \sum_r \beta_r^T 1\{r_{it}=r\} + \beta_s^T s_t + \epsilon_{it} \tag{2}
\]

\(\hat{\lambda}_{it}(T)\) is country \(i\)'s market implied spot default intensity (extracted from \(T\)-maturity CDS contracts) in quarter \(t\), \(r_{it}\) is the Moody’s rating category of sovereign \(i\) at time \(t\), and \(s_t\) is either the CDX\(^{15}\) or VIX index at time \(t\). I display the result of those linear regressions in table 6 for \(T = 5\) years. In column (1), only the credit rating categories are used as regressors, while I use the CDX index in column (2), and the VIX index in column (3). Column (1) indicates that sovereign market-implied default intensities do vary with country fundamentals, as summarized by their credit ratings: the worst the credit rating, the higher the market-implied default intensity. Even after controlling for the level of the CDX (or the VIX) index, estimated coefficients for credit ratings remain statistically and economically significant. However, factors not directly related to a sovereign’s fundamentals also seem to contribute to explaining the level of sovereign market-implied default intensities. For example, controlling for fundamentals, a 1bp increase in the level of US investment grade corporate credit spreads is accompanied by a 4.8bps increase in 5y CDS-implied default intensities. This observation suggests that creditors’ attitude towards risk may be time-varying: indeed, controlling for a given country’s fundamentals, the differential between market-implied and historical default intensities varies over time, and this variation is related to measures of global credit and equity market risks in a positive way: a deterioration of US credit markets (as reflected by a widening in CDX levels), or an increase in US equity uncertainty (as reflected by increases in the VIX index) widens the gap between market-implied and historical default intensities.

In the model developed in this paper, a country’s market implied default intensity and credit spread will be a function of (i) its debt-to-GDP ratio (the “fundamental” variable), as well as debt investors’ price of risk (loosely speaking, a measure of investors’ risk-aversion). This function will be increasing and convex in the country’s debt-to-GDP ratio and increasing

\(^{1998,\text{ while my CDS sample starts in 2001 and my bond spread sample starts in 1994; if I was to double the Moody's implied default intensities in order to correct for this potential bias, bond-implied default intensities would be between 1.20% and 5.07% greater than their Moody's counterparts, excluding “Caa” rated sovereign, for which the Moody’s implied default intensities, after correction, would be higher than the market-implied counterparts.}}\)

\(^{15}\text{The CDX index is a credit derivative contract referencing a basket of 125 single-name US investment grade corporate credits.}\)
in the market price of risk. To test the prediction of the model, I estimate the following panel regression:

\[ \varsigma_{it} = \alpha_i + \beta_1 x_{it} + \beta_2 x_{it}^2 + \beta_s s_t + \epsilon_{it} \]

\( \varsigma_{it} \) is the quarterly average spread of country \( i \) in quarter \( t \), \( \alpha_i \) is a country fixed effect, \( x_{it} \) is country \( i \)'s debt-to-GDP ratio, and \( s_t \) will be a measure of global market risk – either the CDX index or the VIX index. Since the data on CDS prices only goes back to 2003, I instead use EMBI spread data, which are available for most countries since 1994. Column (1) of table 7 shows that a 1% increase in a country’s debt-to-GDP ratio contributes to a 15bps increase in the country’s bond spreads. The hypothesis of convexity of the bond spread as a function of the debt-to-GDP ratio is however rejected – the estimate \( \hat{\beta}_2 \) turns out to be negative in all the specifications tested. Columns (3) and (4) suggests that levels of equity and credit market risks (used as proxy for the investor’s price-of-risk) also contribute positively in explaining sovereign bond spreads; Results in column (3) for example suggest that a 1bps increase in the CDX index contributes to a 3.1bps increase in the sovereign bond spreads, after controlling for the debt-to-GDP ratio. After establishing the main results of the model developed in the next sections, I will return to these model predictions and their data counterpart.

The third observation relates to short term market-implied default intensities, and the fact that they are statistically greater than zero. Table 8 shows estimates of equation (2) for \( T = 1 \) – in other words, using spot default intensities implied by 1-year CDS premia. Column (1) shows the outcome of regressing 1-year default intensities on rating category dummies, and column (2) includes a control for the CDX index. Those results are consistent with the regression results obtained using 5-year default intensities. Moody’s estimated historical default frequencies at the 1-year horizon (see Tudela et al. (2012)) are however negligible: 0% for rating categories of “Baa” and above, 0.64% for “Ba” rated countries and 2.72% for “B” rated countries. As will become clear in section 3, most sovereign default models, whether written in continuous time or in discrete time (and the time-period is taken to be infinitesimally short), lead to short term historical and market-implied default intensities that are negligible. I will discuss in section 3.6 model ingredients that can be used to produce non-zero short term default intensities.

My last observation on the level of default intensities relates to the slope of the default intensity term structure. In table 8, I regress (a) the difference between 5-year and 1-year

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\[^{16}\text{Note: it is likely that the data on 1-year CDS premia is polluted with measurement error, since the market for short dated CDS contracts is less liquid than the market for the more standard 5-year CDS contract. Any measurement error though would bias my estimates downwards.}\]
CDS-implied spot default intensities on (b) rating dummies and the CDX index.

\[ \hat{\lambda}_{it}(5) - \hat{\lambda}_{it}(1) = \sum_r \beta_r 1_{\{r_{it}=r\}} + \beta_s s_t + \epsilon_{it} \]  

Column (3) shows regression results without the CDX index; column (4) includes the CDX index as a regressor; column (5) excludes the rating dummies and instead uses country fixed-effects. The intensity slope exhibits a “tent” shape, as a function of credit rating: the slope is lowest (and negative) for distressed countries (rated “Caa” to “C”), or for countries with very good fundamentals (ratings “Aa” and “A”). It is the highest for countries that are neither distressed, nor with good fundamentals (countries rated “Baa”, “Ba” and “B”). The intuition for the negative default intensity slope of a “distressed” country is as follows: since the fundamentals of such country are bad, it is likely that it will default in the short term. However, conditional on such country surviving such period of bad fundamentals, its survival prospects improve, leading to a downwards sloping term structure of intensities. Loosely speaking, credit markets price a country’s sovereign debt as if its fundamentals were exhibiting some form of mean-reversion conditional on survival. Finally, specifications (4) and (5) illustrate an additional aspect of the term structure of default intensities: increases in measures of US risk (as represented by the CDX index) lead to decreases in the intensity slope. Reconciling these facts will be an important part of my modeling work.

2.2 Expected Excess Returns

I then turn my attention to foreign currency sovereign debt and sovereign CDS returns and excess returns. Let \( dR_{it}(T) - r_t^f dt \) be the instantaneous excess return of being invested into country \( i \)'s T-maturity CDS contract at time \( t \) for a \( dt \) time period\(^\text{17}\). In practice, I will focus my analysis on 1-week time periods.

My first empirical observation relates to the presence of expected excess returns in foreign currency sovereign credit markets. Table 10 shows unconditional average excess returns for (a) 5-year CDS contracts as well as (b) the basket of bonds in the JPMorgan EMBI index. Except for Ecuador, all 5y CDS excess returns’ unconditional averages are positive, with varying degrees of statistical significance\(^\text{18}\). At the same time, country-specific EMBI

\(^\text{17}\)The (excess) return realized by a protection seller between \( t \) and \( t+dt \) is equal to (a) the premium accrual (at rate \( CDS_{it}(T) \)) over the \( dt \) time period plus (b) the change in the price of the premium leg between \( t \) and \( t+dt \) minus (c) the change in the price of the loss leg between \( t \) and \( t+dt \). The “premium leg” refers to the value of receiving a premium stream equal to \( CDS_{it}(T) \) over an horizon \( \tau_i \wedge T \), and the “loss leg” refers to the value of receiving \( 1-R \) at time \( \tau_i \) if \( \tau_i < T \).

\(^\text{18}\)Note however that I cannot reject the hypothesis that expected excess returns are zero for a majority of those countries.
expected excess returns are also all statistically different from zero, some at the 1% confidence level, some at the 5% confidence level. It might come as a surprise to the reader that the EMBI expected excess returns are significantly larger than the excess returns computed from 5y CDS contract prices. This difference does not come from the different sample time periods – when I restrict the time period of EMBI returns to match the time period for which CDS prices are available, the large difference persists (those EMBI expected excess returns are showed in the second column of table 11). This difference is also unlikely to come either (a) from the bond-CDS basis (such basis stayed near zero before 2008, and only exceeded 2% per annum in 2009, as documented in Bai and Collin-Dufresne (2013)), or (b) the fact that the EMBI portfolios include bonds with durations that differ from the duration of 5-year CDS contracts (in an unpublished analysis, I obtain comparable return differentials when using 10-year CDS contracts). Instead, I argue that this difference comes from the fact that EMBI returns are computed using a portfolio of mostly fixed rate bonds – thus, those bonds are exposed not only to a sovereign’s default risk, but also to long-term US interest rates. Table 11 shows that the differential between EMBI excess returns and 5-year CDS excess returns is consistently between 4% and 5%. Over the same time period, 5-year US treasuries had average excess returns of 2.5%, while 10-year treasuries had average excess returns of 5%.

I also regress, for each country, the EMBI excess return onto (a) the 5-year CDS excess return (estimated regression coefficient $\hat{\beta}_{CDS}$), (b) the 5-year US zero coupon treasury excess return (estimated regression coefficient $\hat{\beta}_{ZC}$), and (c) a constant (estimated regression coefficient $\hat{\alpha}$). The point estimates and standard errors, indicated in table 11, show that the loading on the 5-year zero coupon US treasury excess return is in almost all cases statistically significantly different from zero at the 1% confidence level, and that the intercept is not statistically different from zero (meaning that once EMBI excess returns have been projected onto CDS and US treasury returns, no excess return is left unaccounted for). In other words, I suspect that a substantial portion of the expected excess returns computed by Borri and Verdelhan (2011) stem from term premia, as opposed to sovereign credit premia

I then illustrate the fact that expected foreign currency sovereign excess returns relate positively to the “riskiness” of a country, as encoded by such country’s Moody’s credit rating.

---

19Borri and Verdelhan (2011) look at portfolios of sovereign bonds, grouped by ratings and market beta. Using those portfolios, they compute expected excess returns from 3% for low beta low risk countries to 14% for high beta high risk countries. Broner, Lorenzoni, and Schmukler (2013) instead obtain lower expected excess returns since they use fixed rate sovereign bond returns but subtract comparable maturity US treasury returns in order to back-out excess returns; in this latter study, authors find excess returns between 2% and 3% for “stable” countries and between 2% and 7% for “volatile” countries.
In table 12, I run the following panel regressions:

\[
dR_{it}(T) - r_f^T dt = \sum_r \beta_r^T 1\{r_{it}=r\} + \epsilon_{it} \tag{4}
\]

I first use the excess return of the EMBI portfolios, and then use excess returns of 1-year, 5-year and 10-year sovereign CDS contracts. Results are displayed in table 12. Irrespective of the type of data used, the worse the Moody’s ratings (in other words, the worse a country’s fundamentals are), the higher the expected excess return. This empirical regularity will have a close theoretical counterpart. In my model, expected excess returns earned by investors buying the sovereign debt of a given country will be equal to the product of (a) a risk-exposure, and (b) a risk price. The closer the sovereign’s fundamentals are from an endogeneously-determined boundary, the greater the sovereign bond’s risk-exposure.

When varying the time-to-maturity of the CDS contract of interest, I also notice in table 12 that expected excess returns increase with the time horizon. For example, a creditor taking exposure to a “Baa”-rated country will be expected to earn 0.80% per annum for a 1-year credit exposure, 1.70% per annum for a 5-year credit exposure and 2.20% per annum for a 10-year credit exposure. My model will enable me to price CDS contracts of different maturities, and I will show that the longer the maturity of the CDS contract, the greater the risk-exposure – rationalizing the empirical fact that, for a given level of risk-prices, CDS expected excess returns increase with the time-to-maturity of such contract.

I end this section by focusing on potential stochastic discount factors that can price sovereign debt excess returns. I look at whether the US equity market excess returns \(dR_{US,t} - r_f^T dt\) can explain the cross-section of expected excess returns of sovereign bonds by running the following time-series regressions (one per country):

\[
dR_{it}(T) - r_f^T dt = \alpha_i + \beta_i \left( dR_{US,t} - r_f^T dt \right) + \epsilon_{it} \tag{5}
\]

Under the null hypothesis, the regression intercepts \(\alpha_i\) are equal to zero. As table 13 indicates, for each sovereign taken separately, I cannot reject the null hypothesis. I calculate a GRS test statistic (see Gibbons, Ross, and Shanken (1989)) of 11.2 (and Pr (\(\chi^2_{27} < 37.7\) = 0.95); I thus fail to reject the hypothesis that all the intercepts are jointly equal to zero. I also run a cross-sectional asset pricing test in order to assess whether different sovereign credit exposures to US equity market shocks can explain the variation in sovereign credit expected excess returns. To do this, I use the betas obtained from the time series regression equation (5), and then
run the cross-section regression:

\[
\frac{1}{N} \sum_{t=1}^{N} \left[ dR_{it}(T) - r_{ft} dt \right] = \beta_i \nu + \epsilon_i
\]

Both regressions are nested into a GMM estimation, as described in Cochrane (2009). The R-square of my second stage estimation is large (81%), while the pricing errors (i.e. the errors \( \epsilon_i \) in the second stage cross-section regression) are in the order of 1% per annum except for a handful of countries (Argentina and Pakistan having the largest pricing errors). The risk-price estimate \( \hat{\nu} = 14\% \), with a 90% confidence interval of [-6\%, 34\%], which prevents me from rejecting the hypothesis that the risk-price is zero. The chi-square test statistic for the second-stage pricing errors all equal to zero is 8.3, which does not allow me to reject the null that all the pricing errors are equal to zero. Figure 14 is a plot of predicted vs. realized (weekly) expected excess returns, using weekly 5y CDS excess returns and the US equity market returns as a factor. These results provide some supporting evidence that any stochastic discount factor pricing foreign currency sovereign debt must be directly or indirectly related to US equity market returns.

### 2.3 Spread and Return Comovements

I end my empirical work by focusing on the joint behavior of sovereign spreads (bond and CDS) and excess returns across countries. As highlighted in the past by several studies (see for example Augustin and Tédongap (2014), who perform a principal component analysis of the level of spreads or Longstaff et al. (2011), who focus on spread changes), there is a high degree of commonality in the level of spreads for my panel of countries of focus. More precisely, daily data for my panel of 27 countries, the first principal component of the level of CDS (resp. the level of EMBI bond spreads) accounts for 78.5% (resp. 81.7%) of the total variance in the data. Those principal components are also highly correlated with measures of US credit market risk, as well as measures of US equity market volatilities, as figure 15 illustrates: the first principal component of CDS for example has 82% correlation with the VIX index and 88% correlation with the CDX index. When I focus on credit risk returns, a similar picture emerges. The first principal component of 5y CDS excess returns (resp. EMBI bond returns) accounts for 60% (resp. 69%) of the total variance of the data, and such first principal component has a 66% (resp. 50%) correlation with US equity market returns. The theory I develop in section 3 will lead to formulas for cross-country spread correlations and return correlations that will highlight the model features necessary to obtain this degree of co-movement.
3 A Sovereign Default Model

3.1 The Government

While I focus my empirical and quantitative analyses on bonds (and CDS) issued by multiple sovereign governments, the theoretical section of this paper only deals with a single government “n”. Since there are no strategic interactions between different sovereign governments (aside from the fact that their bonds are invested into by a common set of investors), I will abstract from the identity of such government in my notation. The (benevolent) government of focus is endowed with real output \( Y_t \) per unit of time, which evolves according to a Markov modulated geometric Brownian motion:

\[
\frac{dY_t}{Y_t} = \mu_{s_t}dt + \sigma_{s_t} \cdot dB_t 
\]  

(6)

My notation will use bold letters for vectors. \( \{B_t\}_{t \geq 0} \) is an \( N_b \)-dimensional standard Brownian motion on the underlying probability space \((\Omega, \mathcal{F}, \mathbb{P})\). I use multi-dimensional Brownian shocks to be able to discuss how idiosyncratic, regional and global shocks affect spread and return properties of sovereign debt. \( \{s_t\}_{t \geq 0} \), taking values in \( \{1, \ldots, N_s\} \), is a discrete state Markov process with a generator matrix \( \Lambda = (\Lambda_{ij})_{1 \leq i,j \leq N_s} \) that is assumed to be conservative (in other words \( \sum_{j=1}^{N_s} \Lambda_{ij} = 0 \) for all \( i \)). I will assume that \( \{s_t\}_{t \geq 0} \) is recurrent, thus admitting a unique stationary distribution \( \pi \) (an \( N_s \times 1 \) real-valued vector) that solves \( \pi' \Lambda = 0 \), and whose elements sum to 1. I will note \( N_s^{(i,j)} \) the Poisson counting process for transitions from state \( i \) to state \( j \). I will refer to \( \mathbb{P} \) as the physical probability measure, and note \( \mathcal{F}_t \) the \( \sigma \)-algebra generated by the Brownian motion \( B_t \) and the discrete state Markov process \( s_t \).

The Markov state \( s_t \) is not essential for modeling the country’s output dynamics – in most of the quantitative applications of this model, I will in fact assume that the expected GDP growth rate and the GDP growth volatility do not depend on the regime \( s_t \). It could also be argued that the length of the GDP growth time series of the countries of interest is too limited to detect such regime shifts in the data. Instead, the discrete regime \( s_t \) will be the key variable describing the state of the creditors’ stochastic discount factor, as will be discussed in section 3.2. I keep the flexibility to model a country’s output dynamics as a Markov modulated geometric Brownian motion for two reasons. First, it allows me to deal with time-varying output growth volatility, a phenomenon empirically relevant for certain countries, as Seoane (2013) suggests when focusing on Greece, Italy, Spain and Portugal. Second, I argue in section A.1.3 that this stochastic growth model enables me to approximate the output process used by Aguiar and Gopinath (2006) and many other articles in the international...
macroeconomic literature\textsuperscript{20}. The government objective is to maximize the life-time utility function:

$$J_t = \mathbb{E} \left[ \int_t^{+\infty} \varphi (C_s, J_s) \, ds | \mathcal{F}_t \right]$$ \hspace{1cm} (7)

The notation $\mathbb{E}$ denotes expectations under the measure $\mathbb{P}$. The aggregator $\varphi$ takes the following form:

$$\varphi (C, J) := \delta \frac{1-\gamma}{1-\rho} J \left( \frac{C^{1-\rho}}{(1-\gamma)J} \right)^{\frac{1-\gamma}{1-\rho}} - 1$$ \hspace{1cm} (8)

Note that the formula for $J_t$ is a homogeneous of degree $1 - \gamma$ recursion. $\delta$ is the government rate of time preference, $1/\rho$ is the intertemporal elasticity of substitution, while $\gamma$ is the risk aversion coefficient. While the rest of the sovereign debt literature assumes standard iso-elastic time-separable preferences, I want to explore the separate impact of the intertemporal smoothing motive and the impact of risk aversion on the government financing and default policies. The standard iso-elastic time-separable preference specification corresponds to the parameter restriction $\gamma = \rho$. I will return to the rationale for studying this class of preferences after having introduced the full economic environment. If the government does not have any financial contracts at its disposal, its life-time utility is equal to:

$$J_{st}(Y_t) = K_{st} Y_t^{1-\gamma}$$ \hspace{1cm} (9)

Equation (9) as well as the $N_s$ constants $\{K_i\}_{i \leq N_s}$ are determined in section A.1.5. In order for equation (9) to be well defined, I need to impose a parameter restriction that will be assumed going forward.

**Assumption 1.** Let $\{A_i\}_{i \leq N_s}$ be the family of constants defined via:

$$A_i := \delta + (\rho - 1) (\mu_i - \frac{1}{2} \gamma |\sigma_i|^2)$$ \hspace{1cm} (10)

Then $(\delta, \rho, \gamma, \{\mu_i\}_{i \leq N_s}, \{\sigma_i\}_{i \leq N_s})$ are such that $A_i > 0$ for all $i$.

One can think of assumption 1 as a lower bound on the government’s rate of time preference. The government does not have a full set of Arrow-Debreu securities at its disposal. Instead, it can only use non-contingent long-term debt contracts, with aggregate face value $F_t$ and coupon rate $\kappa$. The incentive for the government to issue debt is two-fold: first,
it enables the government to smooth consumption, and to reduce the welfare losses associated with consumption volatility. Second, differences between the government’s rate of time preference and sovereign debt investors’ discount rates will enable the government to “tilt” consumption into the present. During each time period \((t, t + dt]\), a constant fraction \(mdt\) of the government’s total debt amortizes, which the government repays with \(mF_t dt\) units of output. This contract structure guarantees a constant debt average life of \(1/m\) years, and allows me to carry only one state variable \((F_t)\) as a descriptor of the government’s indebtedness, as opposed to the full history of past debt issuances. The long-term debt assumption is also essential in my continuous time framework in order to insure that an equilibrium with default can be supported: I show heuristically in section A.1.1 that the continuous sample paths of my output process preclude short term debt from being supportable in any sovereign default equilibrium. During each time period \((t, t + dt]\), the government can also decide to issue a dollar face amount \(I_t dt\) of bonds. I will require \(I_t\) to be an \(\mathcal{F}_t\)-progressively measurable process that satisfies the standard integrability condition:

\[
E \left[ \int_0^t |I_s| ds \right] < +\infty \quad \text{a.e.} (\forall t \geq 0)
\]

Note that this formulation of an admissible issuance policy prevents “lumpy” debt issuances, and results in a government face value process \(F_t\) that is absolutely continuous:

\[
dF_t = (I_t - mF_t) dt
\]

Per period flow consumption consists of (a) total per-period output, plus (b) proceeds (in units of consumption goods) raised from capital markets minus (c) debt interest and principal repayments due:

\[
C_t = Y_t + I_t D_t - (\kappa + m) F_t
\]

In the above, \(D_t\) is the endogenous debt price per unit of face value, and is determined in equilibrium. My formulation of the debt dynamics as well as the resource constraint for the government lead to a cumulative consumption process that is absolutely continuous; in other words, the government does not consume in “lumpy fashion”, but rather always in “flow” fashion. I can interpret the difference \(Y_t - C_t\) as the trade balance. The government cannot commit to repay its debt, which is thus credit risky. In other words, the government will choose a sequence of default times \(\{\tau_k\}_{k \geq 1}\) out of the set of sequences of stopping times \(\mathcal{T}\). Default leads to the following consequences. First, output jumps down, from \(Y_{\tau_k}\) to

\[\text{Note:}\] The continuous time setting of this model allows me to abstract from the specific timing assumption of the government bond auction. In discrete time models, Cole and Kehoe (1996), Aguiar and Amador (2013)
$Y_\tau = \alpha Y_{\tau-}$, with $\alpha < 1$. Second, the country is locked out of capital markets for a (random) time period $\tau_e$ that is exponentially distributed with parameter $\lambda$. Once the country emerges from financial autarky, it has an outstanding debt balance that is only a fraction of its pre-default debt balance, according to:

$$F_{\tau+\tau_e} = \theta \frac{Y_{\tau+\tau_e}}{Y_{\tau-}} F_{\tau-}$$  \hspace{1cm} (13)$$

One can think of the parameter $\theta$ as the outcome of a bargaining game between creditors and the sovereign government, once such government has elected to default. However, for simplicity and since the strategic interactions between the government in default and its creditors are not a focus of this paper, I elect to model the outcome of this renegotiation exogenously.$^{22}$

I end this section by revisiting my choice of recursive utility specification, and by showing that this choice may lead to equilibrium debt-to-GDP levels that are greater than those found in models with time-separable iso-elastic preferences.$^{23}$ Using a methodology similar to Aguiar and Gopinath (2006), I compare the welfare of (i) a government that has perfectly hedged its business cycle fluctuations via borrowings with (ii) a government that is in financial autarky forever. The incentives of a government to default are tightly linked to this calculation: if a government finds its autarky life-time utility to be greater than its fully-hedged life-time utility via rolling over financial claims, that government will elect to default on its debt.

I perform this calculation for the particular case where $\mu_i, \sigma_i$ are constant across discrete Markov states in order to simplify the exposition. In the case where the government has perfectly “hedged out” its consumption fluctuations via debt contracts, I assume that the interest cost of such debt contract is some exogenously given interest rate $r$. As detailed in section A.1.6, the approximate$^{24}$ break-even debt-to-GDP ratio $x$ that equates those quanti-

and Aguiar et al. (2016) (for example) all assume that the bond auction happens before the default decision is made by the government, while Aguiar and Gopinath (2006), Arellano (2008) and many other quantitative models of sovereign debt assume that the government makes its default decision before the bond auction takes place. The former timing convention allows, in discrete time, for the existence of potentially multiple equilibria, induced by the creditor’s self-fulfilling belief that the government will default immediately after debt has been issued, leading to a low auction debt price and a rational decision by the government to default. Those considerations are absent from the continuous time environment.

$^{22}$Note that the adjustment factor $\frac{Y_{\tau+\tau_e}}{Y_{\tau-}}$ in the debt face value post-restructuring is included for tractability purposes, since it will lead me to solve nested ordinary differential equations, as opposed to integro-differential equations. This feature is also used in Nuño and Thomas (2015).

$^{23}$The early quantitative sovereign debt models developed in Arellano (2008) or Aguiar and Gopinath (2006) featured maximum debt-to-GDP ratios below 20%. Models featuring long term debt, as in Chatterjee and Eyigungor (2010) or Hatchondo and Martinez (2009), managed to increase such maximum debt-to-GDP ratios to levels more consistent with the data.

$^{24}$This approximation is valid whenever the squared GDP volatility $|\sigma|^2$ is small.
ties is equal to:

\[ x \approx \frac{1}{2r \delta + (\rho - 1)\mu} \frac{\gamma|\sigma|^2}{\delta} \]  

(14)

This breakeven is declining as the interest rate \( r \) increases – a greater debt interest expense leads to a lower supportable debt-to-GDP ratio in equilibrium. Similarly, an impatient government will not be able to support as high a debt load as a more patient one. Finally, the theoretical supportable debt-to-GDP level is decreasing in the parameter \( \rho \) (the inverse of the inter-temporal elasticity of substitution) when \( \rho > 1 \) (the relevant case studied in the literature) but increasing in the level of risk-aversion \( \gamma \) – this is the theory I am interested in testing. As an illustration, I use standard parameter values as disclosed in table 2 from the business cycle literature and display those break-even debt-to-GDP ratios in table 3.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta )</td>
<td>20%</td>
<td>Sovereign rate of time preference</td>
</tr>
<tr>
<td>( \mu )</td>
<td>3%</td>
<td>GDP growth rate</td>
</tr>
<tr>
<td>( \sigma )</td>
<td>5%</td>
<td>GDP growth volatility</td>
</tr>
<tr>
<td>( r )</td>
<td>5%</td>
<td>“World” interest rate</td>
</tr>
</tbody>
</table>

Table 2: Calibration Parameters

<table>
<thead>
<tr>
<th>( \rho )</th>
<th>( \gamma = \rho )</th>
<th>( \gamma = 5 )</th>
<th>( \gamma = 10 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>.5</td>
<td>6.7%</td>
<td>65.9%</td>
<td>128.6%</td>
</tr>
<tr>
<td>1</td>
<td>12.5%</td>
<td>61.5%</td>
<td>121.1%</td>
</tr>
<tr>
<td>1.5</td>
<td>17.4%</td>
<td>57.7%</td>
<td>114.6%</td>
</tr>
</tbody>
</table>

Table 3: Break-even debt-to-GDP ratios

Those calculations illustrate the fact that the time-separable preference specification leads to a maximum sustainable debt-to-GDP ratio of around 17%. If instead I assume that the risk aversion parameter is meaningfully larger than the inverse of the intertemporal elasticity of substitution parameter, I obtain maximum sustainable debt-to-GDP levels that are more consistent with the data. A greater equilibrium debt level is however not guaranteed given that a higher risk-aversion parameter\(^{25}\) will also increase the precautionary savings motive of the government, mitigating the aforementioned effect.

\(^{25}\)In the standard international business cycle literature with CRRA preferences, the risk-aversion parameter is identical to the inverse of the intertemporal elasticity of substitution; the desire to keep the intertemporal elasticity of substitution at realistic levels of 0.5 or greater leads to artificially low levels of risk aversion – less than 2.
3.2 Creditors

External creditors purchase the debt issued by the government. I assume that the stochastic discount factor \( M_t \) that creditors use to discount cash-flows evolves according to:

\[
\frac{dM_t}{M_t} = -r_stdt - \nu_st \cdot dB_t + \sum_{st \neq st-} (e^{v(st-st)} - 1) \left( dN_t^{(st-st)} - \Lambda_{st-st} dt \right)
\]  

(15)

Conditioned on being in the discrete Markov state \( i \), creditors’ risk free rate \( r_i \) and the \( N_b \times 1 \) risk price vector \( \nu_i \) are constant. As section A.1.7 or Chen (2010) show, this stochastic discount factor can be obtained for example if creditors have iso-elastic time-separable or recursive preferences and an equilibrium consumption process that follows a Markov modulated geometric Brownian motion. This stochastic discount factor can also be obtained in a general equilibrium environment with a continuum of countries, by re-intrepreting \( C_t \) as spending by government \( n \), \( Y_t \) as the tax revenues of government \( n \), and introducing a “world investor” who can diversify away all countries’ idyosyncratic risks, as I show in section A.1.8. This latter interpretation has the benefit of tying the world interest rate and the world risk prices to the investor’s preferences and the countries’ endowment growth rates, but would not add any additional insight to the paper.

My formulation of the stochastic discount factor implicitly assumes that government \( n \)’s sovereign debt component of the creditor’s portfolio is negligible, and that government \( n \)’s sovereign debt cash-flows do not alter the equilibrium consumption of creditors. Using Girsanov’s theorem, equation (15) enables me to define a new probability measure \( Q \), such that \( \hat{B}_t := B_t + \int_0^t \nu_s du \) is a standard \( N_b \) dimensional Brownian motion under \( Q \), where \( \{s_t\} \) is a discrete state Markov process with generator matrix \( \hat{\Lambda} = (e^{v(i,j)} \Lambda_{ij})_{1 \leq i,j \leq N_s} \) under \( Q \). \( \hat{\Lambda} \) is also assumed to be conservative. I will use \( \hat{E} \) for the expectation operator under the physical measure \( P \), and will use \( \hat{E} \) for the expectation operator under the risk-neutral measure \( Q \). Any cash-flow stream \( A := \{A_t\}_{t \geq 0} \) will thus be priced according to:

\[
\text{Price}_t (A) = \hat{E} \left[ \int_0^{+\infty} \frac{M_{t+s}}{M_t} A_{t+s} ds \bigg| \mathcal{F}_t \right] = \hat{E} \left[ \int_0^{+\infty} e^{-\int_0^s r_t + u du} A_{t+s} ds \bigg| \mathcal{F}_t \right]
\]

Since most of the elements of the model have been introduced, I conclude this section by introducing two parameter restrictions. The first restriction guarantees that the risk-neutral value of a claim to the government’s output be finite.

**Assumption 2.** \( \{r_i\}_{i \leq N_s}, \{\nu_i\}_{i \leq N_s}, \{\mu_i\}_{i \leq N_s}, \{\sigma_i\}_{i \leq N_s} \) jointly satisfy the following param-
The second restriction insures that the government is impatient enough to front-load consumption in equilibrium. To be specific, when the government has neither debt nor assets outstanding, I need the government’s financing policy to be such that it wants to borrow, instead of save. While I do not provide an explicit restriction on the deep model parameters in order to satisfy such condition, I verify ex-post after solving the model that it is the case. Intuitively, this parameter restriction should insure that the rate of time-preference $\delta$ of the government is sufficiently greater than the level of interest rates at which the government can finance itself via debt issuances.

### 3.3 Debt Valuation, Government Problem and Equilibrium

The payoff-relevant variables for the sovereign government and creditors are $s_t$, $Y_t$ and $F_t$. I will focus on a Markovian setting. An admissible issuance policy $\{I_t\}_{t \geq 0}$ of the government is a progressively measurable process that is a function of the payoff-relevant variables (in other words where $I_t = I_{s_t}(Y_t, F_t)$, for a set of measurable functions $I := \{I_i\}_{i \leq N_s}$). An admissible default policy is an increasing sequence of stopping times (with respect to $\mathcal{F}_t$) noted $\tau = \{\tau_k\}_{k \geq 1}$, which can be written (for $k \geq 0$), $\tau_{k+1} = \inf\{t \geq \tau_k + \tau_{e,k} : (Y_t, F_t) \in \mathcal{O}_s\}$, for a finite number of Borel sets $\{\mathcal{O}_i\}_{i \leq N_s}$, where $\{\tau_{e,k}\}_{k \geq 1}$ is a sequence of i.i.d. exponentially distributed times (with parameter $\lambda$), and where I have set $\tau_0 = \tau_{e,0} = 0$. I will note $\mathcal{I}$ the set of admissible issuance policies, and $\mathcal{T}$ the set of admissible default policies. For a given admissible default policy $\tau \in \mathcal{T}$, I note $N^{(\tau)}_{d,t} := \max\{k \in \mathbb{N} : \tau_k \leq t\}$ (resp. $N^{(\tau)}_{e,t} := \max\{k \in \mathbb{N} : \tau_k + \tau_{e,k} \leq t\}$) the corresponding counting process for default events (resp. capital markets re-entry events). Let $1_{d,t}^{(\tau)}$ be the default indicator, equal to 1 when the sovereign government is in default, and zero otherwise:

$$1_{d,t}^{(\tau)} := 1_{t \in [\tau_N^{(\tau)}, \tau_N^{(\tau)} + \tau_{e,N}^{(\tau)}]}$$

Using this notation, the dynamic evolution of the controlled stochastic process $Y_t^{(\tau)}$ can be expressed as follows:

$$dY_t^{(\tau)} = Y_t^{(\tau)} \left( \mu_{s_t} dt + \sigma_{s_t} \cdot dB_t + (\alpha - 1)dN^{(\tau)}_{d,t} \right)$$
Similarly, the dynamic evolution of the controlled stochastic process $F^{(I,\tau)}$ can be expressed as follows:

$$
\begin{align*}
\frac{dF_t^{(I,\tau)}}{dt} &= \left( I_t - mF_t^{(I,\tau)} \right) \left( 1 - 1_{d,t} \right) dt \\
&\quad + \left( \theta \alpha \exp \left[ \int_{\tau_{d,t}}^t \left( \mu_{s_u} - \frac{1}{2} |\sigma_{s_u}|^2 \right) du + \sigma_{s_u} \cdot dB_u \right] - 1 \right) F_t^{(I,\tau)} dN_{\tau_{e,t}}^{(\tau)} \tag{18}
\end{align*}
$$

The drift term in the dynamic evolution of $F_t^{(I,\tau)}$ relates to issuances and debt redemptions when the government is performing under its debt obligations, while the jump term relates to reductions in the debt face value in connection with a restructuring and re-entry into capital markets post-default. Creditors price the sovereign debt rationally. If they anticipate that the government will follow admissible policy $(I, \tau) \in I \times T$, they will value one unit of debt of a government currently performing under its contractual obligations as follows:

$$
D_{d,i} (Y, F; (I, \tau)) := \hat{E}^{i,Y,F}_{\tau_{e},(I,\tau)} \left[ e^{-\int_{\tau_{e}}^{\tau} (r_{s_u} + m) du} (\kappa + m) dt \right]
$$

The stopping time $\tau$ in the equation above refers to the first element of the sequence of default times $\tau$. The superscript notation next to the expectation operator denotes the conditioning on the initial state. $D_{d,i} (\cdot, \cdot; (I, \tau))$ is the debt price in default, which satisfies:

$$
D_{d,i} (Y, F; (I, \tau)) := \hat{E}^{i,Y,F} \left[ e^{-\int_{\tau_{e}}^{\tau} (r_{s_u} + m) du} F_{\tau_{e}} (\alpha Y_{\tau_{e}}, F_{\tau_{e}}; (I, \tau)) \right] \tag{20}
$$

The stopping time $\tau_{e}$ in equation (20) refers to the first capital markets’ re-entry delay of the sequence $\{\tau_{e,k}\}_{k \geq 1}$. I use a notation that makes the dependence of the debt price functions on the anticipated issuance and default policies explicit. Creditors receive cash-flows $\kappa + m$ per unit of time on a debt balance that amortizes exponentially at rate $m$. Following a default, creditors receive no cash-flows for the exponentially distributed random time $\tau_{e}$, following which their claim face value suffers a haircut. The expectations are taken under the risk-neutral measure $Q$.

I then focus on the government life-time utility. For a given debt price schedule $D := \{D_i(\cdot, \cdot)\}_{i \leq N_{\tau}}$ that the government faces, admissible issuance and default policies $(I, \tau)$ (where $(I, \tau)$ might not necessarily be consistent with the debt prices $D$) lead to the following flow
consumption:

\[ C_t^{(I, \tau; D)} = \left[ Y_t^{(\tau)} + I_t D_{s_t} \left( Y_t^{(\tau)}, F_t^{(I, \tau)} \right) - (\kappa + m)F_t^{(I, \tau)} \right] \left( 1 - 1_{d,t}^{(\tau)} \right) + Y_t^{(\tau)} 1_{d,t}^{(\tau)} \]

The indicator functions in this expression highlight the fact that the government can smooth consumption via debt issuances and buy-backs when performing, while it is unable to do so in default. This leads to the following government life-time utility:

\[ J_i (Y, F; (I, \tau); D) = \mathbb{E}^{i,Y,F} \left[ \int_0^\infty \varphi \left( C_t^{(I, \tau; D)}, J_{s_t} \left( Y_t^{(\tau)}, F_t^{(I, \tau)}; (I, \tau); D \right) \right) dt \right] \quad (21) \]

The expectations are taken under the physical probability measure \( \mathbb{P} \), under which the debt face value and output processes evolve according to equations (17) and (18). The government takes as given the family of debt price functions \( D \) and \( D^d \) and chooses its issuance and default policies in order to solve the following problem:

\[ V_i (Y, F; D) := \sup_{(I, \tau) \in \mathcal{I} \times \mathcal{T}} J_i (Y, F; (I, \tau); D) \quad (22) \]

When choosing its issuance policy, the government takes into account the debt price schedule and the impact that such schedule has on flow consumption, via the resource constraint. Consistent with Maskin and Tirole (2001), I then define a Markov perfect equilibrium as follows.

**Definition 1.** A Markov perfect equilibrium is a set of Markovian issuance and default policies \((I^*, \tau^*) \in \mathcal{I} \times \mathcal{T}\) such that for any initial state \((i, Y, F)\),

\[(I^*, \tau^*) = \arg \max_{(I, \tau) \in \mathcal{I} \times \mathcal{T}} J_i (Y, F; (I, \tau); D (\cdot, \cdot; (I^*, \tau^*))) \]

For a given equilibrium \((I^*, \tau^*)\), I will note \( V_{si}(Y_t, F_t) \) the government’s equilibrium value function when performing, and \( V_{sd} (Y_{\tau-}, F_{\tau-}) \) the government’s equilibrium value function at default time \( \tau \), when the pre-default output is equal to \( Y_{\tau-} \) and the pre-default debt face value is equal to \( F_{\tau-} \). The following set of lemmas will help narrow down the class of Markov perfect equilibria I will be focusing on.

**Lemma 1.** If for each state \( i \leq N_s \), the debt price schedule \( D_i(\cdot, \cdot, \cdot) \) is homogeneous of degree zero and decreasing in \( F \), the life-time utility \( V_i (\cdot, \cdot; D) \) is strictly increasing in \( Y \) and strictly decreasing in \( F \). In such case, the optimal issuance policy is homogeneous of degree one and...
the optimal government default policy is a state-dependent barrier policy, in other words there exists a set of positive cutoffs \( \{\bar{x}_t\} \leq N_s \) such that \( \tau_{k+1} = \inf \{t \geq \tau_k + \tau_{e,k} : F_t \geq \bar{x}_{st} Y_t\} \) (with \( \tau_0 = \tau_{e,0} = 0 \)). Finally, the life-time utilities \( V_i(\cdot, \cdot; D) \) are homogeneous of degree \( 1 - \gamma \).

The proof of this lemma is detailed in section A.1.9. I then focus on the debt price schedule for specific types of issuance and default policies.

**Lemma 2.** If \( I \in \mathcal{I} \) is a homogeneous of degree 1 Markov issuance policy, and if \( \tau \in \mathcal{T} \) is a barrier default policy, the debt price functions \( D_i(\cdot, \cdot) \) are homogeneous of degree zero and decreasing in \( F \).

The proof can be found in section A.1.10. As discussed in the next section, by restricting the set of equilibria of focus, lemma 1 and lemma 2 will enable me to reduce the dimensionality of the state space and deal with only one continuous and one discrete state variables.

### 3.4 Equilibrium Debt Value

Using the previous observations, I look for an equilibrium of the model for which \( x_t := F_t/Y_t \) (the debt-to-output ratio) and \( s_t \) are the unique state variables, and for which the government follows a barrier policy: it defaults when the debt-to-output ratio \( x_t \) is at or above a state-dependent threshold \( \bar{x}_{st} \). In other words, the sovereign’s first time of default is \( \tau := \inf \{t \geq 0 : x_t \geq \bar{x}_{st}\} \). The government issuance policy can be re-written \( I_t = \iota_{st}(x_t)Y_t \), where \( \iota_{st}(x_t) \) represents the rate of debt issuance per unit of output, for a given debt-to-output ratio and when the discrete Markov state is \( s_t \). \( \iota > 0 \) means that the government is either decumulating net foreign assets (when \( x < 0 \)) or borrowing (when \( x > 0 \)), whereas \( \iota < 0 \) means that the government is buying back outstanding debt. The dynamic evolution of \( x_t \) under the measure \( \mathbb{P} \) is:

\[
\begin{align*}
    dx_t^{(\iota, \tau)} &= \left(1 - \frac{1}{\alpha} \right) \left[ \left( \mu_{st} - |\sigma_{st}|^2 \right) x_t^{(\iota, \tau)} \right] dt - x_t^{(\iota, \tau)} \sigma_{st} \cdot dB_t \\
    &+ x_t^{(\iota, \tau)} \left( \frac{1}{\alpha} - 1 \right) dN_d^{(\tau)} + x_t^{(\iota, \tau)} (\theta \alpha - 1) dN_e^{(\tau)}
\end{align*}
\]

The debt-to-GDP ratio jumps up by \( (\alpha - 1) \) percents at default given the fact that output jumps down and that the debt face value in autarky, by construction, stays constant at its default level. The debt-to-GDP ratio jumps down by \( (\alpha \theta - 1) \) percents at restructuring and exit from autarky. Under \( \mathbb{Q} \), following Girsanov’s theorem, the drift of \( x_t \) must be adjusted upward by \( \nu_{st} \cdot \sigma_{st} x_t \). Creditors take the government issuance policy \( \iota \) and the government
default policy as given when pricing a unit of sovereign debt. Finally, I will postulate (and verify) that in equilibrium, \( \iota_i(0) > 0 \) for all states \( i \leq N_s \). This means that when the government has neither financial assets nor financial liabilities, it finds it optimal to borrow and front-load consumption in all states \( i \leq N_s \). This also means that once the state \( x_t \) enters the interval \( (0, \max_i \bar{x}_i) \), it never leaves such interval, since the diffusion term in the stochastic differential equation for \( x_t \) vanishes and the drift term is strictly positive. I thus restrict the focus of my analysis to the state space \( (0, \max_i \bar{x}_i) \).

With an abuse of notation, I use \( D_i(x; (\iota, \tau)) \) (resp. \( D^d_i(x; (\iota, \tau)) \)) to denote the debt value (resp. the debt value in default) per dollar of face-value when the debt-to-output ratio is \( x \) and the SDF regime is \( i \). I will also omit the dependence of the debt price function on the government policies \( (\iota, \tau) \) whenever possible. First, note that the debt price cannot be greater than the price \( D_{rf}^i \) of a risk-free claim to sovereign debt cashflows, where \( D_{rf}^i \) verifies:

\[
D_{rf}^i = (\kappa + m) \left( \text{diag}_i (r_i + m) - \hat{\Lambda} \right)^{-1} 1
\]  

When \( s_t \) is in state \( i \) and when \( x \in [0, \bar{x}_i) \), the government is indebted, and the price \( D_i(\cdot; (\iota, \tau)) \) of defaultable sovereign debt verifies:

\[
D_i(x) = \hat{\mathbb{E}}^{i,x} \left[ \int_0^\tau e^{-\int_0^t (r_s + m) du} (\kappa + m) dt + e^{-\int_0^\tau (r_s + m) du} D_{s_t}^d(x_{\tau-}) \right]
\]  

Using Feynman-Kac, it is immediate to show that \( D_i \) is twice differentiable and satisfies the following HJB equation for \( x \in (0, \bar{x}_i) \):

\[
(r + m) D_i(x) = \kappa + m + \hat{L}_i^{(t)} D_i(x) + \sum_{j=1}^{N_s} \hat{\Lambda}_{ij} D_j(x)
\]  

For ease of notation, I have introduced the infinitesimal operator \( \hat{L}_i^{(t)} \) as follows:

\[
\hat{L}_i^{(t)} := [\iota_i(x) - (\mu_i + m - |\sigma_i|^2 - \nu_i \cdot \sigma_i) x] \frac{\partial}{\partial x} + \frac{1}{2} |\sigma_i|^2 x^2 \frac{\partial^2}{\partial x^2}
\]

2 \( \times N_s \) boundary conditions are required in order to solve this set of \( N_s \) nested second order ordinary differential equations. They are as follows, for \( 1 \leq i \leq N_s \):

\[
D_i(\bar{x}_i) = D^d_i(\bar{x}_i)
\]  

\[
(r_i + m) D_i(0) = \kappa + m + \iota_i(0) D'_i(0) + \sum_{j=1}^{N_s} \hat{\Lambda}_{ij} D_j(0)
\]
For each state $i$, the first boundary condition is a value matching condition, which says that the debt price at the default boundary $x = \bar{x}_i$ is equal to the price of a claim on the defaulted debt, $D^d_i(\bar{x}_i)$ (which will be calculated later on). The second boundary condition is a Robin boundary condition; it relates the value of the function $D_i$ at the origin to its first derivative at the origin. It can be obtained by simply taking a limit of the HJB equation satisfied by $D_i$ at $x = 0$. I need to compute the debt price in default $D^d_i(x)$, for $x \geq \bar{x}_i$ and $1 \leq i \leq N_s$. Assume that at time of default $\tau$, the state is $s_\tau = i$. When the country exits financial autarky, its debt-to-GDP ratio is equal to 

$$F_{t+\tau e} = \theta \frac{F_{t+\tau e}}{F_{t+\tau e}} = \theta x_{\tau -}.$$ 

Note that it is possible that $x_{\tau -} > \bar{x}_{s_{\tau -}}$ when the sovereign defaults. This happens upon the occurrence of a “jump-to-default”, in other words a situation where the discrete SDF state jumps from $s_{\tau -} = j$ to $s_\tau = i$ and when $\bar{x}_i < x_{\tau -} < \bar{x}_j$. Thus, I have the following for $x \geq \bar{x}_i$:

$$D^d_i(x) = \hat{E}^i \left[ \exp \left( - \int_0^{\tau e} r_{s\tau + u} du \right) \frac{F_{t+\tau e}}{F_{t-}} D_{s\tau + \tau e} (\theta x) \right]$$

Section A.1.11 establishes the following formula for the defaulted debt price:

$$D^d(x) = \lambda \theta \alpha \Xi^{-1} D(\theta x) \quad (28)$$

In equation (28), $D^d(x)$ is the $N_s \times 1$ vector with $i$th element $D^d_i(x)$, and the $N_s \times N_s$ matrix $\Xi := \text{diag}(r_i + \nu_i \cdot \sigma_i + \lambda - \mu_i) - \hat{\Lambda}$ is well defined thanks to assumption 2. Finally, note that this equation is valid for each coordinate $i$ for $x \geq \bar{x}_i$.

I end this section by discussing three different aspects of the model. First, the existence of a discrete number of SDF regimes leads to two types of defaults: “smooth” defaults, as well as “jumpy” defaults. Both types are illustrated in figure 1, which shows a random realization of output, as well as the government debt path (both in figure 1a), while figure 1b shows the corresponding debt-to-GDP path. In this example, a “smooth” default occurs at $\tau_2$, after a sequence of bad GDP shocks that cause the debt-to-GDP ratio to breach the optimal default boundary that the government has set in such SDF regime. In the same figure, a “jumpy” default occurs at $\tau_1$. At such time, an SDF regime shift occurs, from $s_{\tau_1 -} = 3$ to $s_{\tau_1} = 2$, and the debt-to-GDP ratio satisfies $\bar{x}_3 > x_{\tau_1 -} > \bar{x}_2$. In other words, before the SDF jump, the debt-to-GDP ratio of the sovereign is below the optimal default boundary, but as the SDF regime shifts, the debt-to-GDP ratio is suddenly greater than the new optimal default boundary, causing the sovereign to immediately default. Since the SDF I use will price the sovereign debt of multiple countries, SDF regime shifts induce correlated defaults amongst sovereign governments. Note that jump-to-default risk exists even if the GDP growth rate and GDP growth volatility are not regime-dependent – so long as the SDF exhibits different
risk-prices in different regimes.

Second, note that when $x \searrow 0$, the government debt balance is negligible compared to output. However, the price of any infinitesimally small unit of debt is actually not equal to the risk-free debt price, since the debt price needs to factor in the dilution risk of the government, whose optimal issuance policy will dictate to issue debt to front-load consumption.

Finally, it might be surprising for some readers that a government that is able to adjust its issuance policy continuously (and therefore that is able to buy back debt, when getting close to its default boundary) ends up defaulting in equilibrium. The corporate finance literature (Goldstein, Ju, and Leland (2001) or Leland (1998) to cite a few relevant articles) has concluded that a firm allowed to scale its liability structure downwards (via buying back debt and deleveraging) would not default in equilibrium. Defaults do occur in equilibrium in this paper for the following reason. The debt price is the risk-neutral expected net present value of future debt payments, and is thus always strictly positive. Close to a default boundary, if the government was trying to buy back debt to avoid hitting the default barrier $\bar{x}_s t$, it would need to let its issuance rate $\iota_s t(x_t)$ fall arbitrarily negative\textsuperscript{26}. Focusing on the govern-

\textsuperscript{26}Loosely speaking, the debt issuances and buy-backs are of order “$dt$”, while GDP shocks are of order “$\sqrt{dt}$”. Thus, when $x_t$ is close to $\bar{x}_s t$, negative Brownian shocks can only be compensated by arbitrarily large debt buy-backs.
ment resource constraint equation (12), it means that the consumption rate could end up arbitrarily close to zero and even negative. But the government preferences exhibit Inada conditions: the marginal utility of consumption becomes infinite as the consumption falls to zero, preventing the issuance rate from being arbitrarily negative in equilibrium. Thus, the drift of the state variable remains bounded below on any compact subset of the state space, guaranteeing that the state variable $x_t$ hits any arbitrarily set and finite default boundary in finite time since the volatility term is strictly positive on $(0, \max_i \bar{x}_i)$. Even without the Inada condition mentioned above, DeMarzo and He (2014) show, in a corporate debt model where the firm cannot commit ex-ante to a specific leverage policy, that defaults do occur in equilibrium and that the firm does not find it optimal to buy back debt in distress.

### 3.5 Equilibrium Debt Issuance and Default Policies

Now consider the government’s problem, as described in section 3.3. As a reminder, the government takes the debt price schedule $D(\cdot)$ as given when solving its optimization problem. Thanks to lemma 1, the government value function in state $i$ can be written as follows:

$$V_i(Y, F) := v_i(x) Y^{1-\gamma}$$

In the above, the function $v_i$ will be positive when $\gamma \in (0, 1)$, and negative when $\gamma > 1$. Since $V_i$ is decreasing in $F$, I also have the sign restriction $v_i'(x) < 0$. An appropriate change-in-measure described in section A.1.12 shows that the HJB equation associated with the government problem, in the continuation region $[0, \bar{x}_i)$, is the following:

$$\frac{1-\gamma}{1-\rho} A_i v_i(x) - \sum_{j=1}^{N_i} A_{ij} v_j(x) =$$

$$\sup_{\iota_i} \left[ \delta \left(1 + \iota_i D_i(x) - (\kappa + m) x \right)^{1-\rho} \left[ (1 - \gamma) v_i(x) \right]^{\frac{1-\gamma}{1-\rho}} + \mathcal{L}_i^{(\iota)} v_i(x) \right]$$

In the above, I have used the differential operator $\mathcal{L}_i^{(\iota)}$ defined as follows:

$$\mathcal{L}_i^{(\iota)} := \left[ \iota_i - (\mu_i + m - \gamma |\sigma_i|^2) x \right] \frac{\partial}{\partial x} + \frac{1}{2} |\sigma_i|^2 x^2 \frac{\partial^2}{\partial x^2}$$

The optimal state-contingent issuance policy $\iota_i$ is then given by:

$$\max_{\iota_i} \left[ \frac{\delta}{1-\rho} \left(1 + \iota_i D_i(x) - (\kappa + m) x \right)^{1-\rho} \left[ (1 - \gamma) v_i(x) \right]^{\frac{1-\gamma}{1-\rho}} + \iota_i v_i'(x) \right]$$
This yields the (necessary and sufficient, given the strict concavity of the expression in brackets w.r.t. \(i\)) first order condition:

\[
D_i(x) \delta c_i(x) (1 - \gamma) v_i(x) \int_1^{x_i} = -v'_i(x) \tag{31}
\]

In the above, I have introduced the consumption-to-GDP ratio \(c_i := C/Y\) when the discrete Markov state is \(i\). Focusing on equation (31), I notice that the left-hand side is the product of (a) the marginal utility of consumption \(\delta c_i(x) (1 - \gamma) v_i(x) \int_1^{x_i}\) and (b) the debt price, while the right-hand side is the marginal cost of taking on one extra unit of debt. The optimal Markov issuance policy function \(\iota_i(x)\) is given by:

\[
\iota_i(x) = \frac{1}{D_i(x)} \left[ \left( \frac{\delta D_i(x) [(1 - \gamma) v_i(x)]_{1/x}^\rho}{-v'_i(x)} \right)^{\rho} + (\kappa + m)x - 1 \right] \tag{32}
\]

The expression is well defined since I showed previously that \(v'_i(x) < 0\). The dependence of the issuance policy on the model parameters or on the debt price schedule (which the government takes as given) are ambiguous, since those issuance parameters will also have a feedback effect on the felicity function and its derivative. I can however perform a “partial equilibrium” analysis of the debt price schedule in the unit elasticity of substitution case, i.e. when \(\rho = 1\). In such case, \(\iota_i(x)\) is an increasing function of \(D_i(x)\) whenever the sovereign output \(Y_t\) is greater than the total debt service owed \((\kappa + m)F_t\), which will always be the case in equilibrium (in other words in equilibrium, the sovereign will have defaulted before the sovereign output falls low enough that new debt issuances are required to service the existing debt). For the case where the elasticity of substitution is different from 1, I verify numerically that this comparative static result still holds: when the debt price schedule is more beneficial to the sovereign, the latter takes advantage of it through additional issuances.

For a given set of default thresholds \(\{\bar{x}_i\}_{i \leq N_s}\), additional boundary conditions are needed do solve the system of equations (30). The first set of conditions relates to value matching at the default boundary \(\bar{x}_i\). Let \(V_i^d(Y,F)\) be the government value function in default, if the pre-default output level is \(Y\) and the pre-default debt face value if \(F\). I show in section A.1.13 that \(V_i^d(Y,F) = v_i^d(\bar{x}_i) (\alpha Y)^{1-\gamma}\), which leads to the following value-matching condition:

\[
v_i(\bar{x}_i) = \alpha^{1-\gamma}v_i^d(\bar{x}_i) \tag{33}
\]

\(v_i^d(\bar{x})\) solves a system of non-linear equations discussed in section A.1.13. I also have a set of
Section A.1.13 establishes more formally this optimality condition and shows how \( (v^d_i)'(\bar{x}_i) \)

\[
\frac{1 - \gamma}{1 - \rho} A_i v_i(0) - \sum_{j=1}^{N_s} \Lambda_{ij} v_j(0) = \delta (1 + \nu_i(0)D_i(0))^{1-\rho} \left[ (1 - \gamma) v_i(0) \right]^\frac{\rho - \gamma}{1 - \rho} + \nu_i(0) v'_i(0)
\]

I finally focus on the optimal default policy. Since it is always an option for the government to default, I must have \( V_i(Y, F) - V^d_i(Y, F) \geq 0 \) for all states \((Y, F)\). This leads to a set of \( N_s \) smooth-pasting conditions:

\[
v'_i(\bar{x}_i) = \alpha^{1-\gamma} (v^d_i)'(\bar{x}_i)
\]

Section A.1.13 establishes more formally this optimality condition and shows how \( (v^d_i)'(\bar{x}_i) \)

\[
0 = \max \left[ \sup \left[ \frac{1 - \gamma}{1 - \rho} A_i v_i(x; D) + \sum_{j=1}^{N_s} \Lambda_{ij} v_j(x; D) + \delta (1 + \nu_i D_i(x) - (\kappa + m) x)^{1-\rho} \left[ (1 - \gamma) v_i(x; D) \right]^{\frac{\rho - \gamma}{1 - \rho}} + L(i)v_i(x; D) \right] ; \alpha^{1-\gamma} v^d_i(x; D) - v_i(x; D) \right],
\]

where \( v^d(x; D) \) satisfies (using the \( N_s \times N_s \) matrix \( \Upsilon := \frac{1 - \gamma}{1 - \rho} \text{diag}_i (A_i) + \lambda I - \Lambda)\):

\[
\Upsilon v^d(x; D) - \lambda v(\theta x; D) = \frac{\delta}{1 - \rho} \left[ (1 - \gamma) v^d(x; D) \right]^{\frac{\rho - \gamma}{1 - \rho}},
\]

Then for any state \( i \leq N_s \) and any \( x \in \mathbb{R}^+ \), \( v_i(x; D) \geq J_i(1, x; (\iota, \tau); D) \) for any \( (\iota, \tau) \in \mathcal{I} \times \mathcal{T} \) that satisfy \( \lim_{t \to +\infty} \inf e^{-\int_0^t \frac{\gamma - \rho}{1 - \rho} A_{i\iota} d\iota} \left( x_i^{(\iota, \tau)}; D \right) \leq 0 \). Let the family of thresholds \( \{\bar{x}_i\}_{1 \leq i \leq N_s} \in (\mathbb{R}^+)^{N_s} \) satisfy:

\[
(\nu^d)'(\bar{x}_i) = \lambda \theta \left( \Upsilon + \frac{\gamma - \rho}{1 - \rho} \text{diag}_i \left[ (1 - \gamma) v^d_i(\bar{x}_i) \right]^{\frac{\rho - \gamma}{1 - \rho}} \right)^{-1} \nu'(\bar{x}_i)
\]
Let \((\ell^*, \tau^*)\) be defined as follows:

\[
\ell^*_i(x; D) := \frac{1}{D_i(x)} \left[ \left( \frac{\delta D_i(x) [(1 - \gamma) v_i(x; D)]^{\frac{\gamma}{\gamma - 1}}}{-v'_i(x; D)} \right)^{\frac{1}{\rho}} + (\kappa + m) x - 1 \right]
\]

\[
\tau^*(D) := \inf \{ t \geq 0 : x_t \geq \bar{x}_i \}
\]

Then \(v_i(x; D) = J_i(1, x; (\ell^*, \tau^*); D)\) is the value function.

This proposition, proven in section A.1.14, provides for a characterization of the optimal issuance and default policies given a decreasing debt price schedule \(D\). It does not establish the existence of an equilibrium, which is achieved in the next proposition.

**PROPOSITION 2.** Assume that there exists a set of functions \(\{v_i(\cdot)\}_{i \leq N_s}, \{D_i(\cdot)\}_{i \leq N_s}\), and a set of positive thresholds \(\{\bar{x}_i\}_{i \leq N_s}\) such that the system of nested ordinary differential equations (25), (30) subject to value-matching boundary conditions (26), (27), (33) and (34) are satisfied, where \(\ell_i(\cdot)\) satisfies equation (32) and each threshold \(\bar{x}_i\) satisfies the smooth pasting condition (35). Then a Markov perfect equilibrium exists.

Proving the existence of a Markov perfect equilibrium without relying on the (strong) assumptions of proposition 2 is beyond the scope of this paper, and I leave this proof for future research. I provide in section A.1.15 a discussion of the potential route to pursue to establish such result.

### 3.6 Bonds, CDS Spreads and Expected Excess Returns

The sovereign credit spread \(\varsigma_i(x)\) is the constant margin over the risk-free benchmark that is needed to discount the long-term sovereign bond’s cash flow stream assuming away any default risk. In other words, the credit spread must verify:

\[
D_i(x) := \hat{E}^{i,x} \left[ \int_0^\infty e^{-\int_0^t (r_s + \varsigma_i(x) + m) ds} (\kappa + m) dt \right]
\]  

(36)

The credit spread \(\varsigma_i(x)\) is the unique positive solution to the following equation:

\[
D_i(x) = (m + \kappa) \left[ \left( \text{diag} (r_j + \varsigma_i(x) + m) - \hat{\Lambda} \right)^{-1} 1 \right]_i
\]
Using Itô’s lemma, credit spread innovations under $\mathbb{P}$ take the following form:

$$d\varsigma_t - \mathbb{E}[d\varsigma_t|\mathcal{F}_t] = -\varsigma'_{s_{t-}}(x_t)x_t\sigma_{s_{t-}} \cdot dB_t + \sum_{s'} (\varsigma'_{s'}(x_t) - \varsigma_{s_{t-}}(x_t)) \left( dN^{(s_{t-}-s')} - \Lambda_{s_{t-}-s'}dt \right) \quad (37)$$

This formula will enable me to discuss time-varying sovereign spread volatilities, as well as cross-country sovereign spread correlations. Let me first discuss what happens upon the occurrence of a GDP shock. Section A.1.17 establishes that $\varsigma'_{i} < 0$ in any state $i$. Thus, good GDP shocks translate into decreases in sovereign bond spreads. In other words, credit spreads should be counter-cyclical in this model – a sequence of good GDP shocks will on average lead to lower spreads. I can then leverage equation (37) to compute the instantaneous sovereign bond spread volatility:

$$\sigma^*_t = \sqrt{x^2_t|\sigma_{s_{t-}}|^2\varsigma'_{s_{t-}}(x_t)^2 + \sum_{s'} \Lambda_{s_{t-}-s'} (\varsigma'_{s'}(x_t) - \varsigma_{s_{t-}}(x_t))^2} \quad (38)$$

In a model without SDF regime shifts, sovereign spread volatilities are purely driven by the macroeconomic fundamentals of a country (in the context of this model, the debt-to-GDP ratio $x$), a result that is plainly rejected by the data. Instead, SDF regime shifts in my model induce an additional component to sovereign spread volatilities. Another testable implication emerges from equation (38): spread volatilities tend to be higher when the sovereign government is close to its endogenous default boundary. Indeed, I show in the appendix that the function $x\varsigma'(x)$ is increasing, meaning that the component of sovereign spread volatility stemming from Brownian shocks increases as the sovereign government approaches its default cutoff. Equation (37) also illustrates the crucial importance of the different SDF regimes for cross-sectional spread correlations: absent those regime shifts, pairwise local spread correlation between two different sovereign governments would only stem from output correlation, which is at odds with the data – as I have described in section 2, several pairs of countries in my dataset have modest (and sometimes negative) GDP correlations but still present high levels of spread correlation. If I index by “a” and “b” two countries, the instantaneous spread correlation between those countries takes the following form:

$$\text{corr}_t(\varsigma_{a,t}, \varsigma_{b,t}) = \frac{\varsigma'_{a,s_{t-}} x_{a,t} x_{b,t} \sigma_{a,s_{t-}} \cdot \sigma_{b,s_{t-}} + \sum_{s'} \Lambda_{s_{t-}-s'} (\varsigma_{a,s'} - \varsigma_{a,s_{t-}}) (\varsigma_{b,s'} - \varsigma_{b,s_{t-}})}{\sigma^*_a \sigma^*_b}$$

In the formula above, for all states $i \leq N_s$, the function $\varsigma'_{a,i}$ is evaluated at $x_{a,t}$ and the function $\varsigma'_{b,i}$ is evaluated at $x_{b,t}$. When the SDF state jumps from a low risk-price level $s$ to a high risk-price level $s'$, if both countries’ output processes are positively correlated with
the risk-price vector in all discrete Markov states, spreads for both country “a” and country “b” jump up, meaning that \((\varsigma_{a,s'} - \varsigma_{a,st}) (\varsigma_{b,s'} - \varsigma_{b,st}) > 0\). The same reasoning holds upon a jump from a high risk-price state to a low risk-price state. Thus, the second term in my formula for spread correlations above is positive: spread correlations are induced by SDF regime shifts. This gives my model the ability to reconcile the empirical regularity that many countries have zero or negative output correlations but positive spread correlations – this will be true if both countries of interest have correlations with the risk-price vector that are of the same sign.

I then compute sovereign debt excess returns. Debt excess returns over the time period \((t, t + dt)\) include capital gains \(dD_t\), coupon payments \(\kappa dt\) and principal repayments \(m dt\), while the opportunity cost is \(r_s dt\) and reinvestment costs are equal \(mdt\). Thus, excess returns (under the physical measure \(\mathbb{P}\)) are equal to:

\[
dR_t^e := \frac{dD_t + (\kappa + m) dt}{D_t} - (r_s + m) dt
\]

Using Itô’s lemma and the HJB equation satisfied by the family of debt values \(\{D_i(\cdot)\}_{i \leq N_s}\), I obtain expected excess returns (per unit of time) and return volatilities that are equal to:

\[
\begin{align*}
\mathbb{E}[dR_t^e | \mathcal{F}_t] &= -\left[ x_t D_{st}'(x_t) \nu_{st} \cdot \sigma_{st} + \sum_{s'} \Lambda_{st,s'} \left( \frac{D_{s'}(x_t)}{D_{st}(x_t)} - 1 \right) \left( e^{\nu(st,s')} - 1 \right) \right] dt \\
\text{var}[dR_t^e | \mathcal{F}_t] &= \frac{x_t^2 D_{st}'(x_t)^2}{D_{st}(x_t)^2} |\sigma_{st}|^2 dt + \sum_{s'} \Lambda_{st,s'} \left( \frac{D_{s'}(x_t)}{D_{st}(x_t)} - 1 \right)^2 dt
\end{align*}
\]

Thus, sovereign bond investors are compensated for taking Brownian risk (the first term on the right hand-side of equation (39)), as well as for taking regime jump risk (the second term on the right hand-side of equation (39)). The expected excess return can be read as (minus) the local covariance between (a) sovereign debt returns and (b) the creditors’ pricing kernel. This risk compensation is similar to a standard two-factor asset pricing compensation. Indeed, I can interpret \(-x_t D_{st}(x_t) / D_{st}(x_t)\) as the market beta of sovereign debt w.r.t. the shock \(B_t\), while \(\nu_{st} \cdot \sigma_{st}\) is the sovereign output claim’s risk premium earned in connection with such shock. Similarly, the jump compensation (the second term in equation (39)) can be re-written:

\[
\sum_{s'} \Lambda_{st,s'} \left( e^{\nu(st,s')} - 1 \right) \left( \frac{P_{s'}}{P_{st}} - 1 \right) \left( \frac{D_{s'}(x_t)}{D_{st}(x_t)} - 1 \right)
\]

output claim’s premium for jump risk  
market beta of sovereign debt w.r.t. jump risk

In the above, \(P_i\) is the price of a claim to the output of country \(i\). Using the vector notation,
\( P = \left[ \text{diag}_i (r_i + \nu_i \cdot \sigma_i - \mu_i) - \Lambda \right]^{-1} \mathbf{1} \). Using the language of Hansen (2012a) or Hansen (2012b), the expected excess return in equation (39) is the sum-product of (time-varying) risk-prices \( (\nu_{st}) \) for the Brownian shocks and \( 1 \) for jump risks) and (time-varying) risk-exposures \( -x_i \frac{D_i(x_t)}{D_{st}(x_t)} \sigma_{st} \) for Brownian shocks and \( \left( \frac{D_i}{D_{st}} - 1 \right) \) for jumps). Equation (39) highlights the crucial role of the local covariance between risk-prices and the GDP process for the determination of expected excess returns. Indeed, even if the risk-price vector \( \nu_{st} \) is not (locally) correlated with the country’s output process, expected excess returns can be positive when risk-prices are time-varying and co-move with sovereign debt prices. For this latter effect to “bite”, the pricing kernel must feature jumps (i.e. some of the \( \{\nu(i, j)\}_{1 \leq i, j \leq n_s} \) must be non-zero); the introduction of different SDF regimes only does not suffice in order to produce large model-implied expected excess returns when \( \nu_{st} \cdot \sigma_{st} \leq 0 \) in all states. Note also that the risk-exposure to Brownian shocks (and the corresponding sovereign debt market beta) depends on the elasticity \( \frac{\frac{D_i(x)}{D_{i}(x)}}{D_{i}(x)} \) of the bond price function. It turns out that in all my numerical computations, the debt price function \( D_{i}(\cdot) \) is a concave function, which means that the sovereign debt’s risk exposure to Brownian shocks is increasing in \( x \). This leads to another testable implication of the model: absent SDF jumps, sovereign expected excess return should be increasing in the debt-to-GDP ratio. The properties of sovereign bond return volatilities and cross-country correlations should be identical to those of sovereign spread volatilities and cross-country sovereign spread correlations since realized bond returns between \( t \) and \( t + dt \) are (approximately) proportional to spread changes during that time period.

Finally, I can define \( \varsigma_i(x, T) \), the credit default swap premium for a \( T \) maturity contract. Mathematically, \( \varsigma_i(x, T) \) is defined as follows:

\[
\varsigma_i(x, T) := \frac{\mathbb{E}^{x,i} \left[ 1_{\{\tau<T\}} e^{-\int_0^\tau r_{su} du} \max(0, 1 - D_{s\tau}(x_{\tau})) \right]}{\mathbb{E}^{x,i} \left[ \int_0^{T\wedge\tau} e^{-\int_0^t r_{su} du} dt \right]} = \frac{L_i(x, T)}{P_{i}(x, T)}
\]

\( L_i(x, T) \) is the risk-neutral expected credit loss, while \( P_{i}(x, T) \) is the risk-neutral present-value of CDS premia. Both expected losses and expected CDS premia can be calculated using the Feynman-Kac formula, by solving a set of partial differential equations with boundary conditions discussed in section A.1.18. This will allow me to discuss whether the model-implied term structure of spreads is consistent with the data (see Broner, Lorenzoni, and Schmukler (2013), or section 2). My model with multiple SDF regimes (inducing multiple default boundaries, one per regime) is particularly convenient in analyzing short term CDS premia, and confronting them with the data. Indeed, when the CDS contract maturity is arbitrarily small (i.e. when \( T \to 0 \)), default risk only stems from the risk of regime shifts.
Under the assumption that the discrete SDF states are ordered (i.e. under the assumption that $\bar{x}_1 \leq \ldots \leq \bar{x}_{N_s}$), I then have the following lemma, characterizing short term CDS premia.

**Lemma 3.** When the contract maturity $T$ becomes arbitrarily small, the sovereign CDS premium converges to the following limit:

$$\lim_{T \to 0} \zeta_i(x,T) = \begin{cases} 0 & \text{if } x \leq \min_j \bar{x}_j \text{ or } i = 1 \\ \sum_{j=1}^{i-1} \hat{\Lambda}_{ij} \left(1 - D_d^j(x)\right) & \text{otherwise} \end{cases}$$

Lemma 3 shows that premium compensation for writers of short term sovereign CDS only comes from SDF jump risk, as opposed to output volatility risk. In other words, a model that does not feature multiple discrete regimes will not generate the type of market-implied short term default intensities present in the data. Section A.1.18 also provides formula for expected excess returns and conditional return volatilities of CDS contracts of different maturities, moments that will be used in my model estimation. Specifically, conditional expected excess returns turn out, in all my model estimations, to be increasing in the maturity of the CDS contract of interest, a feature of the data, as I document in section 2.

### 3.7 Numerical Illustration and Comparative Static Results

Before estimating my model for a set of countries of interest, I first provide some comparative static analysis, in order to gain some intuition about the role of certain model parameters. To facilitate this investigation, I shut down for now the multiple SDF regimes, and analyze the effect of certain model parameters in an environment where there is only one SDF regime. I solve the model numerically using a Markov chain approximation method, as described in details in section A.4. Table 4 highlights the base case parameters I use for this comparative static analysis.

I select model parameters that are meant to represent the “average” emerging market economy in my dataset. More specifically, table 15, which I construct using data from the World Bank, shows that the average real GDP growth rate for the countries in my dataset is 3.5% p.a., and the average real GDP growth volatility is 4.1% p.a., leading to the parameters $\mu$ and $\sigma$ in table 4. Table 14 shows summary statistics for a dataset of hard-currency sovereign bonds I collected from Bloomberg$^{27}$; according to such table, the average original maturity

---

$^{27}$For all countries in the data-base I construct, I download all bonds listed on Bloomberg and issued by such country. I only keep in my data-base “hard-currency” bonds, i.e. bonds denominated in either EUR, GBP, USD, JPY or DEM. I also exclude bonds whose original notional amount is less than USD 100mm,
Table 4: Calibration Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1/\rho$</td>
<td>0.5</td>
<td>IES</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>5</td>
<td>Risk aversion</td>
</tr>
<tr>
<td>$\delta$</td>
<td>0.2</td>
<td>Rate of time preference</td>
</tr>
<tr>
<td>$\mu$</td>
<td>0.035</td>
<td>GDP growth rate</td>
</tr>
<tr>
<td>$</td>
<td>\sigma</td>
<td>$</td>
</tr>
<tr>
<td>$1 - \alpha$</td>
<td>0.04</td>
<td>GDP % fall at default</td>
</tr>
<tr>
<td>$1/\lambda$</td>
<td>5</td>
<td>Capital markets’ exclusion (years)</td>
</tr>
<tr>
<td>$\theta$</td>
<td>0.50</td>
<td>Debt-to-GDP upon autarky exit</td>
</tr>
<tr>
<td>$r$</td>
<td>0.05</td>
<td>Creditors’ risk free rate</td>
</tr>
<tr>
<td>$</td>
<td>\nu</td>
<td>$</td>
</tr>
<tr>
<td>$\text{corr}(\nu, \sigma)$</td>
<td>0.50</td>
<td>Business cycle-risk price correlation</td>
</tr>
<tr>
<td>$1/m$</td>
<td>7</td>
<td>Debt average life (years)</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>0.05</td>
<td>Debt coupon rate</td>
</tr>
</tbody>
</table>

The date of sovereign bonds issued by my sample of 27 emerging market countries is 13.8 years. Since a country consistently rolling over 13.8-year original maturity bonds has a debt average life of 6.9 years, I pick $1/m = 7$ years. I select jointly the default punishment parameter $1 - \alpha$ and the rate of time preference $\delta$ to approximately match two moments of the data: the average debt-to-GDP ratio (equal to 50% for the set of countries of focus, following table 15), and the average 5yr CDS premium (equal to 395bps for the set of countries of focus, following table 16). This procedure yields a permanent output drop upon default of $1 - \alpha = 4\%$\textsuperscript{28}, and sovereign rate of time preference $\delta = 20\%$. According to Tomz and Wright (2013), it takes 4.7 years post-default for a country to regain access to capital markets, leading me to pick $1/\lambda = 5$ years. Benjamin and Wright (2009) find a mean creditor haircut following a sovereign default of approximately 40%. Since my model assumes that the face value of debt at exit from financial autarky is equal to $\theta Y_{\tau_+} / Y_{\tau_-}$ times the face value of debt pre-default, the model-implied face value haircut is equal to:

$$1 - \theta \alpha \mathbb{E} \left[ e^{(\mu - \frac{1}{2} \sigma^2) \tau_+} \right] = 1 - \frac{\alpha \theta \lambda}{\lambda - (\mu - \frac{\sigma^2}{2})}$$

I thus pick $\theta = 0.50$, leading to a model-implied average creditor haircut of 42%. I choose whose original term is less than 1 year or greater than 50 years, or bonds with non-fixed coupon rates. The list of remaining bonds is available upon request.

\textsuperscript{28} Note that estimates of output drops following a sovereign default vary vastly across the empirical literature: Hébert and Schreger (2016) for example calculate the cost of Argentina’s sovereign default to correspond to 9.4% permanent reduction in output; Aguiar and Gopinath (2006) use a (transitory) cost of default of 2% of output in their model, citing evidence from Rose (2005), who calculates an 8% decline in international trade following a sovereign default.
an intertemporal elasticity of substitution equal to $1/\rho = 0.5$ that is consistent with the international business cycle literature\textsuperscript{29}, and a risk-aversion parameter $\gamma = 5$ that is consistent with the asset pricing literature. Risk-free rates are set at 5% p.a., and I pick a sovereign debt coupon rate of 5%, which means that (a) the risk-free value of government debt is 1, and (b) the sovereign debt always trades at a discount to par. US equity market mean excess returns of 6% combined with yearly volatility of 15% lead to a Sharpe ratio for US equities of 0.40, which motivate my choice of $|\nu|$. To start with, I assume a correlation between the creditor’s risk-price and the government’s output process of 50%, and will discuss how this correlation affects equilibrium outcomes in section 3.7.4 (of course in this simple model, varying this correlation and keeping risk-prices constant is equivalent to keeping this correlation constant and varying risk-prices). In all the figures I will be discussing, I plot in dotted lines the ergodic distribution of the state variable $x_t$ in order to focus my attention on the sub-interval of the state space where I expect to see most of my model-implied observations. This ergodic distribution is obtained via integrating a set of Kolmogorov forward equations, as section A.1.16 reveals. In the case of a unique SDF regime, the ergodic distribution admits the following semi-closed form expression:

**Lemma 4.** The ergodic distribution $f$ takes the following form:

$$f(x) = \begin{cases} \int_{\theta x}^{x} \exp \left[ \int_{0}^{t} \frac{2}{|\sigma|^2} ((m + \mu)s - \iota(s)) \, ds \right] \frac{2G}{|\sigma|^2T(x)} \, dt & \text{if } x \in [0, \theta \bar{x}) \\ \int_{x}^{\bar{x}} \exp \left[ \int_{0}^{t} \frac{2}{|\sigma|^2} ((m + \mu)s - \iota(s)) \, ds \right] \frac{2G}{|\sigma|^2T(x)} \, dt & \text{if } x \in (\theta \bar{x}, \bar{x}) \end{cases}$$

The constant $G$ is pinned down by the condition $\int_{\theta x}^{\bar{x}} f(x) \, dx = 1 - \frac{1}{1 + \lambda T(\theta \bar{x})}$, where $T(x)$ is the expected default time conditional on the initial debt-to-GDP ratio being equal to $x$.

The base case calibration results in an optimal debt-to-GDP default boundary $\bar{x} = 56\%$, an ergodic mean debt-to-GDP ratio of 52% and an ergodic debt-to-GDP distribution standard deviation of 3.4%. In figure 2, I plot the issuance policy $\iota$ and the resulting trade balance. The issuance policy is a decreasing function of the debt-to-GDP ratio, positive but reaching levels close to zero at the default boundary. It is important to keep in mind that these represent gross issuances, before taking into account any debt amortization. After taking into account debt amortizations and the effect of GDP growth, it is immediate to notice that the debt-to-GDP ratio is a mean-reverting variable – its drift rate is going to be positive for values of $x$ such that $\iota(x) > (\mu + m - |\sigma|^2) x$, while the drift rate will be negative for values of $x$ such that $\iota(x) < (\mu + m - |\sigma|^2) x$. The resulting trade balance (as a fraction of GDP) is

\textsuperscript{29}Aguiar et al. (2016), Aguiar and Gopinath (2006), Arellano (2008) and many others all use an IES of 0.5.
Figure 2: Government Financing Policy and the Trade Balance

(a) Gross Issuance Policy \( \iota(x) \)
(b) Trade Balance \( 1 - c(x) \)

equal to \( 1 - c(x) = (\kappa + m)x - \iota(x)D(x) \). It is negative for low debt-to-GDP levels and positive otherwise. Thus, consistent with the overwhelming data for emerging market economies, the trade balance in this model is countercyclical: after a sequence of good output shocks, both the debt-to-GDP ratio and the trade balance end up below their respective ergodic means. Finally, I show in section A.1.19 that the ratio of (a) consumption growth volatility over (b) output growth volatility can be expressed, in the case of a unique SDF regime, as follows:

\[
\frac{\text{stdev} \left[ \frac{dC_t}{C_t} | F_t \right]}{\text{stdev} \left[ \frac{dY_t}{Y_t} | F_t \right]} = 1 - \frac{x_t c'(x_t)}{c(x_t)}
\]

Since the consumption-to-output ratio \( c(x) \) is a decreasing function of the debt-to-GDP ratio, the model generates immediately a consumption growth volatility that is greater than output growth volatility, an empirical regularity of the data. For the parameters selected, I obtain an ergodic consumption growth vol to output growth vol ratio of 1.98, which is in line with several emerging market economies, as documented in Aguiar and Gopinath (2004). In the case of multiple SDF regimes, section A.1.19 illustrates that the ratio of consumption growth volatility to output growth volatility features an extra term related to SDF regime changes.

\[30\] They find a ratio of 1.38 for Argentina, 2.01 for Brazil, 2.39 for Ecuador, 1.70 for Malaysia, 1.24 for Mexico, 0.92 for Peru, 0.62 for the Philippines, 1.61 for South Africa, 1.09 for Thailand and 1.09 for Turkey.
Note also that consumption growth and output growth are perfectly correlated in a model with one SDF regime, which is obviously counter-factual – the model featuring multiple SDF regimes breaks this result.

I plot the sovereign bond price and the sovereign spread in figure 3. The ergodic mean credit spread of the exponentially amortizing bond is equal to 360bps p.a., and the ergodic credit spread volatility is equal to 143bps p.a. This ergodic mean credit spread volatility is higher than what is obtained by Aguiar et al. (2016) and the difference stems from the much lower average maturity of sovereign debt (2 years) this latter article assumes, compared to the 7-year average life debt in the data for my countries of interest. Note that those model-implied spread volatilities are one order of magnitude smaller than those observed in the data, as computed in table 16.\footnote{To be specific, I use weekly average CDS, compute the standard deviation of the time series, and annualize. If instead I was using monthly or even quarterly averages, the spread volatility computed would be smaller.}

Focusing on figure 3, bond prices decrease with the debt-to-GDP ratio, while bond spreads increase. Even at low debt-to-GDP ratios (i.e. lower than the ergodic debt-to-GDP mean), sovereign spreads are far from negligible. When the sovereign has no debt outstanding, sovereign spreads are strictly positive – a simple manipulation of equation (27) shows that
the sovereign spread at $x = 0$ verifies:

$$ \varsigma(0) = -\frac{\iota(0)D'(0)}{D(0)} $$

This equation highlights the role of future debt issuances (and the implicit dilution risk associated with those future debt issuances): at low debt-to-GDP ratios, creditors perfectly anticipate that the government will be issuing large amounts of debt (since the government is impatient), leading them to price bonds at a discount that reflects such dilution risk.

I then focus on credit default swap premia $\varsigma(x, T)$. While the government uses 7-year bonds to smooth and front-load consumption, my model allows me to compute the premium $\varsigma(x, T)$ of CDS contracts at any time-horizon $T$, as discussed in section 3.6. The numerical procedure to solve the relevant PDEs is described in details at the end of section A.4. I plot CDS premia in figure 4a for 1-year, 3-year and 5-year contracts, and I plot in figure 4b the 5y-1y slope. CDS premia for short-dated contracts are close to zero when the debt-to-GDP ratio is far away from the mean ergodic debt-to-GDP ratio, highlighting the fact that at such low debt-to-GDP levels, the sovereign slowly increases its indebtedness, such that 1-yr credit instruments are almost risk-free. The 5-year CDS premium $\varsigma(x, 5)$ resembles the credit spread of the exponentially amortizing bond $\varsigma(x)$ used by the government to finance itself, since the average life of this bond is 7 years. Lastly, the term structure of credit spreads becomes
inverted as the sovereign approaches its default boundary. This property of my model ends up being a very general property of default models that are structured as “first-hitting-time” models, where the state is mean-reverting and has continuous sample paths. The empirical counterpart of this feature can be found in table 8, columns (4) and (5), which show that countries with bad credit ratings tend to have negative slopes, while countries with good credit ratings tend to have positive slopes. Arellano and Ramanarayanan (2012) highlight a related phenomenon when regressing the slope of the term structure of credit spreads onto the level of spreads.

I finally plot in figure 5 the bond expected returns and expected default times under the measure \( \mathbb{P} \). When the country emerges from financial autarky, it has a debt-to-GDP ratio \( \theta \bar{x} = 32.5\% \), and it takes such country on average 24 years to default on its debt, once it has exited from autarky. The sovereign default frequency under the physical measure is thus equal to 3.4% p.a. Expected default times decrease as the debt-to-GDP ratio gets near the default boundary \( x = \bar{x} \). Expected excess returns are non-zero, since I have used a pricing kernel that co-moves with the output process of the sovereign: the unconditional expected excess return is equal to 1.15% p.a., while the unconditional return volatility is equal to 5.77% p.a. Expected excess returns increase with the debt-to-GDP ratio, since the bond price function’s elasticity increases with the state variable \( x \). After having investigated some of the key outputs of the model in the base case parametrization, I now study the impact of
the model parameters on endogenous quantities of interest: debt spreads, default boundaries and issuance behavior.

### 3.7.1 Comparative Static w.r.t. $\alpha$

It turns out that the GDP drop upon default has a significant impact on government behavior. For GDP drops $1 - \alpha$ of 3%, 4% and 5% of output level pre-default, figure 17 shows that the optimal default boundary increases from 41% to 72%. An increase in the default punishment incentivizes the sovereign government to support higher levels of debt in equilibrium, a result already known in the literature using discrete time models. What is surprising is the large change in behavior for a seemingly low change in the upfront GDP drop post-default. In fact, I can replicate the calculations in section A.1.6, equating the measure of welfare in autarky post default with the measure of welfare of a government that is fully-hedged by rolling over financial claims. Including the upfront GDP drop $1 - \alpha$ leads to a theoretical maximum supportable debt-to-GDP level $x$ in equilibrium that is equal to:

$$x = \frac{1}{r} \left[ 1 - \alpha \left( 1 - \frac{1}{2} \gamma \left( \frac{\sigma^2}{\delta + (\rho - 1) \mu} \right) \right) \right]$$

The impact of a GDP jump upon default on the theoretical maximum debt-to-GDP ratio is large: if I use the same calibration as in table 2, I obtain results as disclosed in table 5 (using the base case IES $1/\rho = 0.5$). Those results are not surprising: my income process

<table>
<thead>
<tr>
<th></th>
<th>Max debt-to-GDP</th>
<th>Max debt-to-GDP</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>$\gamma = \rho$</td>
<td>$\gamma = 5$</td>
</tr>
<tr>
<td>1</td>
<td>21.7%</td>
<td>54.4%</td>
</tr>
<tr>
<td>0.97</td>
<td>81.1%</td>
<td>112.7%</td>
</tr>
<tr>
<td>0.94</td>
<td>140.4%</td>
<td>171.1%</td>
</tr>
</tbody>
</table>

Table 5: Break-even debt-to-GDP ratios – Sensitivity to $\alpha$

being a geometric Brownian motion, any downward GDP drop is a permanent shock that is translated into large welfare losses. Since the default boundary increases with the magnitude of the GDP drop post default, the government issuance policy adjusts upwards, while figure 18 shows that the ergodic credit spread mean decreases slightly, from 405bps (when $1 - \alpha = 97\%$) to 329 bps (when $1 - \alpha = 97\%$).
3.7.2 Comparative Static w.r.t. $\delta$

As the government becomes more impatient, it tends to front-load consumption when it is not significantly indebted. In such case, creditors take into account the dilution risk and price the debt more punitively than in the situation where the government is patient. The impatient government thus has an issuance policy with a steeper slope than the patient government, and the optimal default boundary is lower for the former than for the latter, as figure 19 illustrates. As a side note, it might seem at first glance that the impact of the parameter $\alpha$ on the issuance policy is similar to an increase in the rate of time preference $\delta$ in the government felicity function. However, while an increase in the GDP drop upon default $1 - \alpha$ leads (approximately) to a parallel shift upwards of the issuance policy (and an appropriate increase in the optimal default boundary), an increase of the parameter $\delta$ rotates the issuance policy. Note also that the ergodic mean credit spread is highly sensitive to parameter $\delta$ – from 294bps when $\delta = 0.175$ to 435bps when $\delta = 0.225$, as figure 19 illustrates.

3.7.3 Comparative Static w.r.t. $\gamma$

One of the objectives of this paper is to investigate the separate roles of risk-aversion and intertemporal elasticity of substitution in the government’s decision problem. I illustrate the comparative static w.r.t. $\gamma$ in figure 21, keeping the intertemporal elasticity of substitution at $1/\rho = 0.5$. Increases in risk-aversion tend to marginally decrease the default boundary $\bar{x}$ (from $\bar{x} = 55\%$ for $\gamma = 7$ to $\bar{x} = 59\%$ for $\gamma = 3$). As discussed earlier in the paper, while in theory a greater risk aversion can lead to higher supportable levels of debt-to-GDP ratio in equilibrium, in practice the precautionary savings motive (which increases with risk-aversion) reduces the debt issuance rate and leads to lower default boundaries.

3.7.4 Comparative Static w.r.t. $\nu$

In order to better understand the impact that SDF regime shifts have on equilibrium outcomes, I analyze in this section the comparative static with respect to risk-prices $\nu$, assuming that the GDP shocks are uni-dimensional and assuming that the risk-price vector has 50% correlation with the country’s GDP process. Figure 24 shows that higher risk prices increase sovereign spreads, lower government bond prices, leading the sovereign government to adjust its issuance policy downwards. Higher creditors’ risk prices also cause the sovereign government to default at lower debt-to-GDP levels. This mechanism leads, in a multi-SDF-regime version of my model, to jump-to-default risk, induced by risk prices jumping from one level to another (higher) level. When comparing credit spreads for $\nu = 0$ to credit spreads with
strictly positive risk-prices, one might want to interpret the spread differential between those two parameter configurations as the “spread premium” that a sovereign government is paying to its creditors above and beyond what would be actuarially fair. This is not exactly the case since the government endogenously reacts to those higher risk-prices by altering its issuance and default policies. But this prompts me to create the following credit spread decomposition: starting from the equilibrium without risk-pricing, I first adjust risk-prices to $|\nu| = 0.4$ (and under the assumption that $|\nu| \sigma = 0.50$) and re-compute sovereign debt prices and spreads, keeping the government’s issuance policy constant. I then adjust the government’s issuance policy to reflect such government’s “best response” to this new debt price schedule.

The result of such decomposition is illustrated in figure 6: the credit spread function plotted in red is the spread under the intermediate step where risk-prices are set to 0.40 but where the government issuance policy has not reacted yet. Figure 6 shows that positive risk-prices alone act as a powerful force to push credit spreads wider. Indeed, when $|\nu|$ is increased from zero to $|\nu| = 0.4$ (and the issuance policy is not adjusted), the drift rate of the state variable $x_t$ increases by $x_t \sigma \cdot \nu \approx 0.5 \times 0.04 \times 0.4 \times 0.5 \approx 0.40\%$ per annum (using the ergodic debt-to-GDP ratio for $x_t$ and the assumed 50% correlation between risk-prices and GDP), and the credit spread is wider by 145bps p.a. on average (this can be seen in the plot by the upward shift from the curve $\nu = 0$ to the curve $\nu = 0.40^*$. But the government responds by adjusting its issuance policy downwards, defaulting at a lower debt-to-GDP burden, and the
resulting credit spreads ends up wider by “only” 65bps p.a. on average (using the ergodic distribution with risk-pricing).

3.7.5 Comparative Static w.r.t. $\sigma$

I study the influence of $\sigma$ on the equilibrium outcomes for two reasons. First, I highlight a weakness of the determination of maximum supportable debt-to-GDP levels using approaches similar to Aguiar and Gopinath (2006). Equations (14) or (41) assume that the government rolls over its debt at a constant interest rate $r$. In sovereign default models, that rate is determined endogenously, as the sum of (i) the risk free rate and (ii) a credit spread equating the expected present value of future credit losses with the expected present value of future spread income (both computed under the probability measure $Q$). For a given default boundary and debt issuance policy, the hitting time default probabilities are increasing in the GDP volatility. Thus, credit spreads are increasing in GDP volatility, all else equal. The government is internalizing this effect and reduces its debt issuances accordingly. The other force leading to a downward adjustment in debt issuances stems from a precautionary savings motive, which is stronger as the sovereign government’s risk aversion $\gamma$ is high and the sovereign’s endowment volatility is high. After taking these effects into account, if the ergodic mean credit spread at which the government rolls over its debt in an economy with low GDP volatility is lower than in an economy with high GDP volatility, it could be the case that the economy with low volatility supports a higher level of debt in equilibrium than an economy with higher volatility. This is indeed what happens, as illustrated in figure 26: given the parameter configuration, the ergodic mean of credit spreads decreases with output volatility, and the sovereign default boundary increases, contradicting the intuition from equation (14).

Second, the study of economies with varying levels of output volatility shows that for output volatilities consistent with the data, the ergodic debt-to-GDP distribution is a lot thinner than the one observed in the data – the latter being computed and displayed in table ?? for my countries of interest. This observation is another element motivating the use of a Markov modulated (as opposed to a simple) geometric Brownian motion for output dynamics, since stochastic volatility will increase the dispersion of the ergodic debt-to-GDP distribution. I leave the full investigation of the effects of stochastic output volatility in my model for future research.

4 Structural Estimation

I select a subset of $N = 10$ countries out of the set of $27$ countries discussed in section 2 for which I have the longest time-series data available. My subset consists of Brazil, Bulgaria,
Colombia, Hungary, Indonesia, Mexico, Philippines, South Africa and Turkey. My model with SDF regime shifts has a large number of parameters to determine. I am going to impose restrictions on those parameters as follows. First, I will use pricing kernel specifications widely used in the asset pricing literature to rationalize equity market returns: in a first estimation, I will use the external habit pricing kernel of Campbell and Cochrane (1995), and in a second estimation I will use the long run risk pricing kernel of Bansal and Yaron (2004). Section A.5 provides a detailed description of the properties of each of these SDFs.

Since all these models lead to risk-prices and risk-free rates that are continuous state Markov processes, I use a numerical procedure based on conditional moment matching and described in section A.5 in order to parametrize my discrete state continuous time Markov process for risk-prices \( \{ \nu_t \}_{t \geq 0} \) and risk-free rates \( \{ r_t \}_{t \geq 0} \). This gives me the risk-free rates \( \{ r_i \}_{i \leq N_s} \), risk-prices \( \{ \nu_i \}_{i \leq N_s} \), the matrix of intensities \( \Lambda \) as well as the SDF jumps \( \{ \nu(i,j) \}_{i,j \leq N_s} \).

The remainder of the parameters to estimate are country-specific. For simplicity and due to the small number of GDP data points available, I will assume that for each country of interest, expected consumption growth, consumption growth volatility and the correlation between the country’s output process and the SDF risk-price do not change with SDF regime shifts, in other words for each country \( n \), \( \mu^n = \mu^n_j \), \( \sigma^n_i = \sigma^n_j \) and \( \frac{\sigma^n_i \nu_i}{|\sigma^n_i||\nu_i|} = \frac{\sigma^n_j \nu_j}{|\sigma^n_j||\nu_j|} \) for any pair of SDF states \( i, j \). A few other parameters are calibrated using a-priori evidence. I will leverage the average original maturity of bonds issued by each country (as documented in table 14) in order to calibrate the debt parameter \( m^n \) for country \( n \). The expected time spent in financial autarky \( \frac{1}{\lambda} \), the parameter \( \theta \) governing the debt-to-GDP post-autarky, as well as the preference parameter \( \rho \) governing the intertemporal elasticity of substitution and the parameter \( \gamma \) governing the precautionary savings motive are kept at their values in table 4 and are thus not country-dependent. For each country \( n \), my estimation will then pin down the GDP growth \( \mu^n \), GDP volatility \( |\sigma^n| \), the correlation between GDP and the risk price vector \( \frac{\nu \sigma^n}{|\nu||\sigma^n|} \), the GDP drop upon default \( 1 - \alpha^n \) and the rate of impatience \( \delta^n \). The following 5 moments will be used in my estimation. First, the first difference mean and standard deviation of log output will provide information on \( \mu^n \) and \( |\sigma^n| \). Unconditional expected excess returns on 5y CDS contracts will then provide information on the correlation between country \( n \)’s output and risk-prices. Finally, the level of 5y CDS premia as well as the mean debt-to-GDP ratio will jointly provide information on \( 1 - \alpha^n \) and \( \delta^n \).

My simulated method of moment estimation follows Lee and Ingram (1991). I note \( H = \frac{1}{T} \sum_{t=1}^{T} h_t \) the \( p \times 1 \) vector of target moments in the data, and \( \mathcal{H}_k(\Theta) = \frac{1}{T} \sum_{t=1}^{T} h (\{ x^n_t \}_{n \leq N}, s_t; \Theta) \) the corresponding \( p \times 1 \) vector of moments generated by the \( k \)th simulation of my model. I

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32A country consistently rolling over \( T \)-year original maturity bonds has a debt average life of \( T/2 \) years; for such country I thus use \( \lambda = 2/T \).
use $K = 1000$ simulations, and note $\mathcal{H}(\Theta) = \frac{1}{K} \sum_{k=1}^{K} \mathcal{H}_k(\Theta)$ the sample average of moments generated by my model across the $K$ simulations. For each simulation, I use an identical seed and generate $2T$ years of data, burn-in the first $T$ years, keeping only the simulated data for the last $T$ years. I minimize the criterion function:

$$\hat{\Theta} = \arg \min_{\Theta} (H - \mathcal{H}(\Theta))' W (H - \mathcal{H}(\Theta))$$

I use the weighting matrix $W = \frac{1}{T} \left( \sum_{t=1}^{T} h_t h_t' \right)^{-1}$. As showed in Lee and Ingram (1991), the asymptotic covariance matrix of my estimator is equal to:

$$\text{covar} \left( \hat{\Theta} \right) = \frac{1}{T} \left( 1 + \frac{1}{K} \right) \left( \frac{\partial \mathcal{H}'}{\partial \Theta} W \frac{\partial \mathcal{H}}{\partial \Theta} \right)^{-1} \frac{\partial \mathcal{H}'}{\partial \Theta} W' \hat{\Omega} W \frac{\partial \mathcal{H}}{\partial \Theta} \left( \frac{\partial \mathcal{H}'}{\partial \Theta} W \frac{\partial \mathcal{H}}{\partial \Theta} \right)^{-1}$$

In the above, $\hat{\Omega}$ is a consistent estimator of the long run covariance matrix of the moment conditions $\Omega = \sum_{j=-\infty}^{+\infty} \mathbb{E} \left[ \left( h_t - \mathbb{E} [h_t] \right) \left( h_{t-j} - \mathbb{E} [h_{t-j}] \right)' \right]$ (I pick $\hat{\Omega}$ to be the Newey-West estimator of $\Omega$).

In table 17 and table 18, I first display the parameter estimates, then the target and model-implied moments, and finally additional moments that were not targeted in my estimation. I display in figures 7, 8, 9 and 10 the equilibrium outcome for Brazil, using the external habit model of Campbell and Cochrane (1995), where the SDF was discretized into 5 states (as explained in section A.5), and where state 1 is the highest risk-price state and state 5 the lowest. In the highest risk-price state, Brazil defaults at a debt-to-GDP ratio of 38.7%, whereas in the lowest risk-price state, it defaults at a debt-to-GDP ratio of 39.7%, confirming the relatively small impact of risk-prices on optimal default boundaries. Figure 7 highlights the fact that Brazil adjusts its issuance policy (and therefore its trade balance) such that in the highest risk-price state, since it is facing high credit spreads, the country issues smaller amounts of bonds that in a low risk-price state. Figure 8 then shows that 5yr CDS contract premia jump up with transitions from low risk-price states to high risk-price states – at a 37% debt-to-GDP ratio for example, 5yr CDS jumps by more than 300bps between the lowest and the highest risk-price state. Credit curves may invert for two reasons according to figure 8b: either upon Brazil’s debt-to-GDP ratio approaching the default boundary, or upon an upward shift in risk-prices. Figure 9 illustrate several additional positive aspects of my model: CDS expected excess returns increase with the debt-to-GDP ratio (this was expected given that the elasticity of the bond price function is increasing in the debt-to-GDP ratio), and longer-tenor CDS contracts earn higher excess returns than shorter-term CDS contracts, for a given debt-to-GDP level and a given risk-price level – this latter aspect is a feature of the data, as supported by table 12. Finally, Figure 10 highlights the fact that CDS spread
and return volatilities are time-varying, are increasing with the tenor of the CDS contract, are high in periods of high risk-prices and are high when the sovereign government is close to its optimal default boundary.

On the negative side, two aspects need to be highlighted. First and foremost, the debt-to-GDP ergodic distributions generated by the model have much smaller variances than in the data, even in the presence of time-varying risk-prices. This is due to the fact that the sovereign optimal default boundaries do not strongly react to the presence of high risk-prices, since the government will adjust its issuance policy to reflect worse credit market conditions. Making the sovereign output volatility $\sigma_t$ time varying would help slightly, as suggested by some (unreported) experimentation I ran, but would in no way enable the model-implied distribution variances to match those of the data: as table 15 indicates, the standard deviation of the empirical ergodic debt-to-GDP distribution for my countries of focus is $\%\%\%$, which is an order of magnitude larger than those obtained in my model. The second weakness of the model stems from the level of expected excess returns: in order to hit the target returns from the data, sovereign output processes need to exhibit extremely high levels of correlation with the SDF – for most countries, 100%. If one links shocks to the SDF to US aggregate consumption or US aggregate output, my model-implied correlation estimated are at odds with data on correlations between the GDP of (a) the US and (b) my countries of interest (see table 15).
Figure 8: Brazil - Equilibrium Credit Spreads

(a) 5yr CDS contract premium $\varsigma_s(x,5)$

(b) CDS slope $\varsigma_s(x,5) - \varsigma_s(x,1)$

Figure 9: Brazil - Equilibrium Credit Returns

(a) 1yr CDS expected excess return

(b) 5yr - 1yr CDS expected excess return differential
5 The 1982 Latin American Debt Crisis

For a variety of reasons documented in multiple historical studies (see for example Dooley (1994) or Cline et al. (1995)) Latin American governments borrowed heavily during the 1970s, partly as a consequence of increased lending by US banks, partly as a consequence of the need to finance large current account deficits following the two oil price shocks of 1973 and 1979. Latin American sovereign debt increased by an average of 24% per annum between 1970 and 1979, therefore substantially increasing those countries’ debt-to-GDP ratios. The largest sovereign borrowers during that time period were Mexico and Brazil. The World Bank estimates that two third of that debt was in the form of USD-denominated, long-term, syndicated bank loans whose interest rate was indexed to LIBOR, thereby making sovereign governments’ financing costs directly exposed to the US dollar and US monetary policy. In the early 1980s, the Federal Reserve aggressively increased US short term rates to fight domestic inflation, causing the US dollar to appreciate against most currencies, and causing LIBOR rates to skyrocket. In August 1982, as T-bill rates were approaching 16%, Mexico announced that it could no longer meet its debt service payments; by the end of that year, 40 other nations, including Brazil, Venezuela and Argentina, had defaulted on their sovereign debt.

My model allows me to perform a “lab experiment” on this historical period. Indeed, it is straightforward to change the debt contract structure that the sovereign government enters
into from fixed rate contracts to floating rate contracts. Thus, let me now assume that the sovereign issues floating rate debt indexed to the risk free rate \( r_{st} \), which evolves according to a discrete state Markov process with generator matrix \( \Lambda \). In other words, coupon rates paid by the governments are time varying, and equal to \( \kappa_{st} = r_{st} \) in state \( s_t \). The government resource constraint (out of financial autarky) becomes:

\[
C_t = Y_t + I_t D_t - (\kappa_{st} + m) F_t
\]

Debt prices and the life-time utility function for the government satisfy second order ordinary differential equations similar to those presented in section 3.5, but appropriately modified to account for the floating rate nature of sovereign debt. Default optimality is still obtained by a smooth-pasting condition. It is worthwhile noting that the price of a risk-free debt instrument that amortizes exponentially and pays a coupon of \( r_{st} dt \) for \( t \in [t + dt] \) is equal to:

\[
\hat{E} \left[ \int_0^\infty e^{-\int_0^t (r_{su} + m) du} (r_{st} + m) dt \right] = 1
\]

In other words, the price of risk-free floating rate debt in this set up is always par, and any credit-risky floating rate instrument (where the coupon paid is equal to the floating rate benchmark) will always trade at a discount to par. Using US T-bill rates from Ken French website (starting my time series 1960), I estimate the generator matrix \( \Lambda \), after having discretized the interest rate process into 6-state Markov process taking values (1%, 3%, 5%, 7%, 9%, 12%) (state 1 corresponds to 1% short term rates and state 6 corresponds to 12% short term rates). The time series of actual short rates as well as the discrete state Markov process approximation are plotted in figure 16. On average, \( s_t \) spends 90% of the time in states where US short term rates are 7% or below, 6% of the time in a state where US short rates are 9%, and 4% of the time in a state where US short rates are 12%.

For this exercise, I use model parameters as in table 4; while I assume that creditors’ short rates are stochastic, I assume no risk-premia, in other words \( \nu = 0 \) and \( v(s, s') = 0 \) for all pair of states \( s, s' \). The result of this exercise is striking. With floating rate debt, default boundaries dramatically differ state by state, from \( \bar{x}_1 = 75\% \) (the low US short-rate state) to \( \bar{x}_6 = 63\% \) (the high short rate state). In addition, and more interestingly, issuance policies depend in a negative way on short rates: for a given debt-to-GDP ratio, the lower the interest rate, the higher the issuance rate. This runs against the “partial-equilibrium”
The expression above indicates that for a given debt price schedule $D$ and life-time utility set of functions $v$, the issuance rate increases with the coupon rate paid on the debt – this is a cashflow effect, due to the fact that higher debt servicing costs incentivize the government to issue larger amounts of debt to achieve the same level of consumption than in a low rate environment. But in a low risk free rate environment, adjustments to debt prices and life time utilities counter the cashflow effect and increase the optimal default cutoff, as is seen on figure 11. Notice that at low debt-to-GDP ratios, credit spreads are higher in a low short rate environment than in a high short rate environment: indeed, creditors’ main risk is the dilution risk related to government’s future bond issuances. Instead, with a high debt-to-GDP ratio, credit spreads are higher in a high short rate environment since the main risk faced by creditors is hitting the default boundary.
6 Conclusion

In this paper, I develop a new modeling framework to study sovereign debt quantities and prices by leveraging continuous time technology. This tractable approach leads to pseudo-closed form expressions for key asset pricing moments of interest, which enables me to discuss characteristics of the stochastic discount factor that are needed to explain properties of sovereign spreads, sovereign debt returns and cross-country spread correlations. The approach I propose reduces significantly the number of state variables typically present in discrete time sovereign debt models and opens the door to partial model estimation. My quantitative application shows that this modeling framework has the ability to match many quantities and price moments of the data, but fails on two dimensions: (i) the model-implied dispersion of the debt-to-GDP ergodic distribution remains an order of magnitude smaller than in the data, and (ii) the magnitude of excess returns on sovereign debt requires very high levels of correlation between countries’ output processes and SDFs typically used in the asset pricing literature to explain properties of US equity market returns – if the creditors’ SDF is associated with US consumption or output dynamics, the empirical correlation between the US GDP and the GDP of my countries of interest is too low compared to the model estimates required to rationalize this high level of expected excess returns. My framework can be extended in multiple directions without adding any complexity or state variable. It is straightforward for example to layer a linear production technology and study capital accumulation and the impact of debt overhang on investment decisions. One can also analyze an environment with exogenously specified sudden stops, by adding discrete SDF states where the sovereign government is prevented from issuing any debt. Finally, more theoretical work needs to be done, for example by leveraging viscosity theory and existence results for nested ordinary differential equations, in order to establish the existence of a Markov perfect equilibrium of the game between the sovereign government and its creditors.
Appendix

A.1 Proofs and Detailed Calculations

A.1.1 A Useful Discrete Time Limit

In this section, I study a discrete time counterpart to the continuous-time model developed in this paper. $\Delta t$ will represent a small time interval. I study the limit of a simple sovereign default model as $\Delta t \to 0$, and show heuristically that no debt can be supported in equilibrium at such limit. I focus on a government that has iso-elastic time-separable preferences with rate of time preference $\delta$ and risk-aversion $\gamma$ as follows:

$$E \left[ \sum_{i=0}^{\infty} e^{-\delta_i \Delta t} \frac{C_i (1 - \gamma_i) \Delta t}{1 - \gamma} \right]$$

Government output follows the discrete time equivalent of a geometric Brownian motion:

$$\frac{Y_{(i+1)\Delta t}}{Y_{i\Delta t}} = e^{\omega_{(i+1)\Delta t}}$$

In the above, $\omega_{(i+1)\Delta t}$ is an $\mathcal{F}_{(i+1)\Delta t}$ measurable binomial random variable that can take values $+\sigma \sqrt{\Delta t}$ or $-\sigma \sqrt{\Delta t}$ with respective probabilities $p_u$ and $p_d$:

$$p_u = \frac{1}{2} \left( 1 + \frac{(\mu - \frac{1}{2} \sigma^2) \sqrt{\Delta t}}{\sigma} \right)$$

$$p_d = \frac{1}{2} \left( 1 - \frac{(\mu - \frac{1}{2} \sigma^2) \sqrt{\Delta t}}{\sigma} \right)$$

Thus, $\{\omega_{(i+1)\Delta t}\}_{i \geq 0}$ is a sequence of i.i.d. random variables, and I will use $\omega_{\Delta t}$ for simplicity to denote one of these random variables. When $\Delta t \to 0$, I have the following limits:

$$E[e^{\omega_{\Delta t}}] = 1 + \mu \Delta t + o(\Delta t)$$

$$\text{var}[e^{\omega_{\Delta t}}] = \sigma^2 \Delta t + o(\Delta t)$$

The government has one-period debt at its disposal. Let $B_{i\Delta t}$ be the stock of debt that the government has to repay at time $i\Delta t$. Let $D \left( B_{(i+1)\Delta t}, Y_{i\Delta t} \right)$ be the price of one unit of debt if the government plans to issue, at date $i\Delta t$, $B_{(i+1)\Delta t}$ units of debt maturing at period $(i + 1)\Delta t$. The government resource constraint at time $i\Delta t$ is as follows:

$$C_{i\Delta t} \Delta t = Y_{i\Delta t} \Delta t - B_{i\Delta t} + D \left( B_{(i+1)\Delta t}, Y_{i\Delta t} \right) B_{(i+1)\Delta t}$$
Upon default, GDP suffers a permanent shock of size $1 - \alpha$ and the government is in autarky forever after. Thus, if the pre-default output value is $Y_i \Delta t$, the value function for the government in default is equal to $V_d(Y_i \Delta t)$, which satisfies:

$$V_d(Y) = \frac{(\alpha Y)^{1-\gamma}}{1-\gamma} \Delta t + e^{-\delta \Delta t} \mathbb{E} [V_d(e^{\omega \Delta t} Y)]$$

Guessing that $V_d(Y) = v_d Y^{1-\gamma}$, the constant $v_d$ is equal to:

$$v_d = \frac{\alpha^{1-\gamma} \Delta t}{1 - e^{-\delta \Delta t} \mathbb{E} [e^{(1-\gamma)\omega \Delta t}]} = \frac{\alpha^{1-\gamma} \Delta t}{1 - \delta + (\gamma - 1) (\mu - \frac{1}{2} \gamma \sigma^2)} + o(1)$$

The latter equality is valid when $\Delta t \to 0$. The problem of the government is summarized as follows:

$$V_0(B, Y) = \max \{ V_c(B, Y), V_d(Y) \}$$

$$V_d(Y) = v_d Y^{1-\gamma}$$

$$V_c(B, Y) = \max_{B'} \left[ \frac{(Y + \frac{1}{\Delta t} (D(B', Y) B' - B))^{1-\gamma} \Delta t}{1 - \gamma} + e^{-\delta \Delta t} \mathbb{E} [V_0(B', Y') | Y] \right]$$

The bond price verifies $D(B', Y) = e^{-r \Delta t} \Pr (V_c(B', Y') \geq V_d(Y') | Y)$, where $r$ is the interest rate at which lenders discount risk-free cashflows. This formula assumes that upon default, sovereign creditors recover nothing from their defaulted debt claim. One can show that for any bond price function that is homogeneous of degree zero and decreasing in $B'$, the value function $V_c$ is homogeneous of degree $1 - \gamma$ and the best “default” response by the government is to follow a linear barrier policy of the form $\tau := \inf \{ t : B_t \geq \bar{x} Y_t \}$ for some endogenously determined constant $\bar{x}$ (which depends on the time step $\Delta t$). Noting $x := B/Y$, and using the homogeneity property of the value function, the government life-time utility can be written $V_c(B, Y) = v_c(x) Y^{1-\gamma}$, and the government problem can be simplified as follows:

$$v_c(x) = \max_{x'} \left[ \frac{(1 + \frac{1}{\Delta t} (D(x') x' - x))^{1-\gamma} \Delta t}{1 - \gamma} + e^{-\delta \Delta t} \mathbb{E} [e^{(1-\gamma)\omega \Delta t} \max (v_d, v_c(e^{-\omega \Delta t} x'))] \right]$$

For simplicity, I assume that the choice set for the debt-to-GDP ratio of the government is discrete. In other words, I discretize the state space $[0, \bar{x}]$ into a grid $G_{\Delta t}$ consisting of a countable number of points $x_i = e^{i \sigma \sqrt{\Delta t}}$, for $i = -\infty, \ldots, i_d$, where $i_d$ is the default index:

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33 Of course this statement is assuming the existence of the value function $V_c$, which is the fixed point of a functional equation. Since my discussion on the lack of equilibria with defaultable debt at the limit of my discrete time economies is only heuristic, I side-step the proof of existence of $V_c$. 

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\( x_{id} = e^{i_d \sigma \sqrt{\Delta t}} = \bar{x} \)\(^{34}\). In addition, I assume that at \( x = x_{id} \), at the point where the government is indifferent between defaulting and repaying, the government defaults. This model structure guarantees that the debt-to-GDP ratio stays on the grid \( G_{\Delta t} \) at all times, irrespective of the sequence of GDP shocks and decisions made by the government. In any cutoff equilibrium with default threshold \( \bar{x} = x_{id} \), the debt price must satisfy:

\[
D(x_i) = \begin{cases} 
e^{-r \Delta t} & \text{if } i < i_d - 1 \\ \frac{1}{2} \left( 1 + \frac{(\mu - \frac{1}{2} \sigma^2) \sqrt{\Delta t}}{\sigma} \right) e^{-r \Delta t} & \text{if } i = i_d - 1 \\ 0 & \text{if } i \geq i_d \end{cases}
\]

Now consider the resource constraint for the government if at the beginning of a given period, \( x = x_i \), with \( i = i_d - 1 \). Since \( i < i_d \), it is optimal for the government to continue to perform on its debt obligations. If the government selects a debt-to-GDP ratio \( x_j < x_i \), the consumption-to-output ratio, using the resource constraint, is equal to:

\[
1 + \frac{1}{\Delta t} \left( e^{j \sigma \sqrt{\Delta t} - r \Delta t} - e^{i \sigma \sqrt{\Delta t}} \right) = \frac{(j - i) \sigma}{\sqrt{\Delta t}} + o \left( \frac{1}{\sqrt{\Delta t}} \right) \xrightarrow{\Delta t \to 0} -\infty
\]

If the government selects a debt-to-GDP ratio \( x_j = x_i \), the consumption-to-output ratio is equal to:

\[
1 + \frac{1}{\Delta t} \left( \frac{1}{2} \left( 1 + \frac{(\mu - \frac{1}{2} \sigma^2) \sqrt{\Delta t}}{\sigma} \right) e^{i \sigma \sqrt{\Delta t} - r \Delta t} - e^{i \sigma \sqrt{\Delta t}} \right) = -\frac{1}{2 \Delta t} + o \left( \frac{1}{\Delta t} \right) \xrightarrow{\Delta t \to 0} -\infty
\]

Of course if the government selects a debt-to-GDP ratio \( x_j = x_{id} \), it obtains no proceeds from its debt issuance – this decision cannot be optimal. Thus, as \( \Delta t \to 0 \), the only possibility for government consumption to be positive at all points of the state space \( x_i < \bar{x} \) is for the default boundary to converge to zero. Note that this conclusion stems from (A) the shape of the government bond price schedule (at I take \( \Delta t \to 0 \), such debt price schedule converges to a step function, equal to the risk-free benchmark for \( x < \bar{x} \) and equal to zero at \( x \geq \bar{x} \)), and (B) the continuous sample path of geometric Brownian motions. Had the output process featured jumps, a cutoff equilibrium with strictly positive debt can be supported. \( \Box \)

\(^{34}\)This model structure implicitly assumes that the government is prevented from saving. This can also be achieved endogenously by using a sufficiently high rate of time preference \( \delta \).
When using CDS data, I extract CDS-implied spot default intensities as follows. For country $i$ and time $t$, I observe the credit default swap premium $CDS_{it}(T)$ for a $T$ maturity contract. I also observe its “assumed” recovery rate $R$ – in other words conditioned on a credit event, $1 - R$ is the expected payment that a $1$-notional protection writer owes a protection buyer. I then extract the spot hazard rate implied by this $T$-maturity CDS contract for country $i$ at time $t$ as follows:

$$CDS_{it}(T) = \frac{\mathbb{E}\left[e^{-\tau_i}1\{\tau_i<T\}(1-R)\mid\mathcal{F}_t\right]}{\int_0^{T\wedge\tau_i} e^{-ru}du\mid\mathcal{F}_t} = \frac{L_{it}(T)}{P_{it}(T)}$$

In the above, $\tau_i$ is the (random) default time of country $i$, assumed to follow a Poisson process with a constant arrival rate $\hat{\lambda}_{it}(T)$. Equation (42) can be interpreted as follows: the CDS premium is the ratio of (i) $L_{it}(T)$, the (risk-neutral) expected present value of future losses of the contract over (ii) $P_{it}(T)$, the (risk-neutral) expected present value of future CDS premia paid on the contract. I perform a similar calculation using bond spreads.

When computing returns on CDS contracts, I take advantage of the full term structure of interest rates and credit spreads. Imagine that at a certain time and for a given sovereign government (omitting the subscript $i$ for the country’s identity and the subscript $t$ for the time at which the prices are observed – both for notational simplicity), I observe the spread of CDS contracts $CDS(T_1),...,CDS(T_n)$ and US treasury zero coupon bond prices $B(T_1),...,B(T_n)$. I extract the full term structure of forward default intensities $\{\lambda_k\}_{k\leq n}$ (where $\lambda_k$ is the forward default intensity between $T_{k-1}$ and $T_k$) and forward interest rates $\{f_k\}_{k\leq n}$ (where $f_k$ is the forward interest rate between $T_{k-1}$ and $T_k$) by using the following bootstrapping procedure:

$$B(T) = e^{-\int_0^T f_u du}$$

$$P(T) = \mathbb{E}\left[\int_0^{T\wedge\tau} e^{-\int_0^t f_s ds} dt\right]$$

$$L(T) = \mathbb{E}\left[1\{\tau<T\}e^{-\int_0^{\tau} f_s ds}\right]$$

In the above, by using $T = T_1,...,T_n$, I can extract recursively the risk-neutral forward

---

35 The recovery rate $R$ is provided by Markit. It is unclear whether Markit uses market data on recovery swaps (if any such contracts were to trade at the time) to populate this recovery data set. I verify that my empirical analysis is robust to the recovery rates used to compute market-implied default intensities.
interest rates and forward default intensities. Note for example that for any $k$, I have:

$$B(T_k) = e^{-\sum_{j=1}^{k} f_j(T_j - T_{j-1})}$$

For default intensities, note that the coupon and loss legs $P$ and $L$ satisfy, for $k \geq 1$ and using the convention that $T_0 = 0$:

$$P(T_{k+1}) = P(T_k) + \Pr(\tau \geq T_k) \mathbb{E}\left[ \int_{T_k}^{\tau \land T_{k+1}} e^{-\int_0^t f_s \, ds} \, dt | \tau \geq T_k \right]$$

$$= P(T_k) + \frac{e^{-\sum_{j=1}^{k} (f_j + \lambda_j)(T_j - T_{j-1})}}{f_{k+1} + \lambda_{k+1}} \left(1 - e^{-(f_{k+1} + \lambda_{k+1})(T_{k+1} - T_k)}\right)$$

$$L(T_{k+1}) = L(T_k) + \mathbb{E}\left[ (1 - R)1_{\{T_k < \tau \leq T_{k+1}\}} e^{-\int_0^\tau f_s \, ds} \right]$$

$$= L(T_k) + \frac{(1 - R)\lambda_{k+1} e^{-\sum_{j=1}^{k} (f_j + \lambda_j)(T_j - T_{j-1})}}{f_{k+1} + \lambda_{k+1}} \left(1 - e^{-(f_{k+1} + \lambda_{k+1})(T_{k+1} - T_k)}\right)$$

Excess returns on a $T$ maturity CDS contract for country $i$ between $t$ and $t + dt$ is then computed by repricing at time $t + dt$ both the loss and the coupon legs, using forward default intensities computed using CDS contract prices at time $t + dt$:

$$dR_{it}(T) = CDS_{it}(T) dt + P_{i,t+dt}(T - dt) - L_{i,t+dt}(T - dt)$$

$CDS_{it}(T) dt$ is the carry earned on the contract between $t$ and $t + dt$. $P_{i,t+dt}(T - dt)$ is the price at time $t + dt$ of a coupon leg of $T - dt$ years; $L_{i,t+dt}(T - dt)$ is the price at time $t + dt$ of a loss leg of $T - dt$ years; none of these prices are observed, instead they are computed using the term structure of forward interest rates and default intensities bootstrapped at time $t + dt$. \hfill \Box

A.1.3 Output Process in this Article vs. Aguiar and Gopinath (2006)

The output process in Aguiar and Gopinath (2006) can be recast in continuous time as follows:

$$Y_t = e^{\kappa t \Gamma_t}$$

$$d \ln \Gamma_t = \ln g_t \, dt$$

$$dz_t = -\kappa_z (z_t - \mu_z) \, dt + \sigma_z dB_z^z$$

$$d \ln g_t = -\kappa_g (\ln g_t - \ln \mu_g) \, dt + \sigma_g dB_g^g$$

57
In the above, $B_t^g, B_t^z$ are standard Brownian motions assumed to be independent, $\kappa_z, \kappa_g$ are positive constants that parametrize the speed of mean reversion of the processes $z_t$ and $\ln g_t$ respectively. I use a procedure introduced by Gordin (1969) to extract a martingale component to the logarithm of output. In other words, I can decompose small increments of $\ln Y_t$ as follows:

$$
\begin{align*}
\ln Y_t - \ln Y_0 & = \int_0^t \sigma_g^2 dB_t^g - \frac{1}{\kappa_g} \ln g_t - z_t \, dt + \left( \frac{1}{\kappa_g} \ln g_0 - z_0 \right) - \left( \frac{1}{\kappa_g} \ln g_t - z_t \right) + t \ln \mu_g \\

& = \underbrace{\int_0^t \sigma_g^2 dB_t^g}_{\text{martingale component}} + \underbrace{\left( \frac{1}{\kappa_g} \ln g_0 - z_0 \right)}_{\text{stationary component}} - \underbrace{\left( \frac{1}{\kappa_g} \ln g_t - z_t \right)}_{\text{time trend}} + t \ln \mu_g
\end{align*}
$$

Thus, the permanent component to log output is purely driven by (rescaled) shocks to the state variable $\ln g_t$, while the state variable $z_t$ has a purely transitory role. The long-run time trend in log-output is equal to $\ln \mu_g$. When I focus on the output process driven by equation (6) (and assuming, for the purpose of this section, that $B_t$ is unidimensional), I can compute small increments in logarithms as follows:

$$
\begin{align*}
d \ln Y_t & = \left( \mu_{s_t} - \frac{1}{2} \sigma_{s_t}^2 \right) dt + \sigma_{s_t} dB_t 
\end{align*}
$$

In the above, $s_t$ is the discrete state Markov chain with generator matrix $\Lambda$ and stationary density $\pi$. In such case, note that:

$$
\mathbb{E} \left[ d \ln Y_t | \mathcal{F}_t \right] = \left( \mu_{s_t} - \frac{1}{2} \sigma_{s_t}^2 \right) dt
$$

Let $\kappa_{2,t} := \int_0^t \sigma_{s_u} dB_u$ and note that for $\tau > t$, $\mathbb{E} [\kappa_{2,\tau} | \mathcal{F}_t] = \kappa_{2,t}$, in other words $\kappa_{2,t}$ is a martingale. I then compute the long term average of log output growth:

$$
\ln g_\infty := \pi \cdot \left( \mu - \frac{1}{2} \sigma^2 \right)
$$

I create the function $f(s) := \mu_s - \frac{1}{2} \sigma^2 s - \ln g_\infty = \left( \mu - \frac{1}{2} \sigma^2 - \ln g_\infty \right) \cdot e_{s_t}$, $e_s$ is an $N_s \times 1$ column vector with entry $s$ equal to 1, and all other entries equal to zero, and compute:

$$
\int_0^\infty \mathbb{E} \left[ f(s_{t+u}) | \mathcal{F}_t \right] du = \Lambda^{-1} \left( \mu - \frac{1}{2} \sigma^2 - \ln g_\infty \right) \cdot e_{s_t}
$$

In the above, $\Lambda^{-1}$ is the generalized inverse of the generator matrix $\Lambda$ (since the rows of the matrix $\Lambda$ sum up to zero, $\Lambda$ is not invertible). Finally, I introduce the margingale $\kappa_{1,t}$.
defined as follows:

\[ \kappa_{1,t} := \int_0^t \sum_{i=1}^{N_s} \Lambda^{-1} \left( \mu - \frac{1}{2} \sigma^2 - \ln g_\infty 1 \right) \cdot (e_i - e_{s_{u-}}) \, dN_u^{(s_{u-},i)} + \int_0^t \left( \mu - \frac{1}{2} \sigma^2 - \ln g_\infty 1 \right) \cdot e_{s_{u}} \, du \]

I can then decompose increments in log output growth as follows:

\[ d \ln Y_t = d(\kappa_{1,t} + \kappa_{2,t}) - \Lambda^{-1} \left( \mu - \frac{1}{2} \sigma^2 - \ln g_\infty 1 \right) \cdot (e_{s_{t}} - e_{s_{t-}}) \, dN_{t}^{(s_{t-},s_{t})} + \ln g_\infty \, dt \]

\[ \ln Y_t - \ln Y_0 = \int_0^t d(\kappa_{1,s} + \kappa_{2,s}) + \Lambda^{-1} \left( \mu - \frac{1}{2} \sigma^2 - \ln g_\infty 1 \right) \cdot (e_{s_{0}} - e_{s_{t}}) + t \ln g_\infty \]

Thus, in the case of this article, the martingale component of log output is the sum of a Brownian process \( \kappa_{1,t} \) and a jump process \( \kappa_{2,t} \), while the stationary component is a pure jump process. By carefully parametrizing \( \mu, \sigma, \Lambda \), one can approximate the long run trend, stationary and martingale components in Aguiar and Gopinath (2006) by a stochastic process that follows equation (6).

A.1.4 Epstein-Zin Preferences

For convenience, I introduce the differential operator \( \mathcal{A} \), defined for any stochastic process \( \{Z_t\}_{t \geq 0} \) (belonging to an appropriate class of stochastic processes) as follows:

\[ \mathcal{A}Z_t := \lim_{\epsilon \to 0} \frac{\mathbb{E}[Z_{t+\epsilon} | \mathcal{F}_t] - Z_t}{\epsilon} \quad (43) \]

Assume that preferences are given by the following recursion, for \( \epsilon > 0 \):

\[ \tilde{J}_t = \left[ (1 - e^{-\delta \epsilon}) C_t^{1-\rho} + e^{-\delta \epsilon} \mathcal{R}_t \left( \tilde{J}_{t+\epsilon} \right)^{1-\rho} \right]^{\frac{1}{1-\rho}} \]

In the above, \( \mathcal{R}_t (Z_{t+\epsilon}) := \mathbb{E} \left[ Z_t^{1-\gamma} | \mathcal{F}_t \right]^{\frac{1}{1-\gamma}} \). Some manipulation leads to:

\[ \tilde{J}_t^{1-\rho} - C_t^{1-\rho} = e^{-\delta \epsilon} \left[ \mathcal{R}_t \left( \tilde{J}_{t+\epsilon} \right)^{1-\rho} - C_t^{1-\rho} \right] \]

Taking limits of this expression when \( \epsilon \to 0 \) leads to:

\[ 0 = -\delta \left( \tilde{J}_t^{1-\rho} - C_t^{1-\rho} \right) + \frac{1-\rho}{1-\gamma} \tilde{J}_t^{1-\rho} \mathcal{A} \tilde{J}_t^{1-\gamma} \]
The operator $\mathcal{A}$ is the operator defined via equation (43). Making the change in variable $J_t := \frac{\tilde{J}_t^{1-\gamma}}{1-\gamma}$ and simplifying the above expression leads to:

$$0 = \delta \frac{1 - \gamma}{1 - \rho} J_t \left( \frac{C_t^{1-\rho}}{[(1 - \gamma)J_t]^{1-\gamma}} - 1 \right) + \mathcal{A} J_t$$

For the particular case $\rho = 1, \gamma \neq 1$, the original Epstein-Zin recursion can be written:

$$\ln \tilde{J}_t - \ln C_t = e^{-\delta t} \left[ \ln \mathcal{R}_i \left( \tilde{J}_{t+\epsilon} \right) - \ln C_t \right]$$

Using the same change in variable $J_t := \frac{\tilde{J}_t^{1-\gamma}}{1-\gamma}$, this leads to the functional equation:

$$0 = \delta (1 - \gamma) J_t \left( \ln C_t - \frac{\ln [(1 - \gamma)J_t]}{1 - \gamma} \right) + \mathcal{A} J_t$$

For the particular case $\gamma = 1, \rho \neq 1$, the operator $\mathcal{R}_i$ can be written $\mathcal{R}_i \left( \tilde{J}_{t+\epsilon} \right) = \exp \left( \mathbb{E} \left[ \ln \tilde{J}_{t+\epsilon} | \mathcal{F}_i \right] \right)$, leading to the recursion:

$$\tilde{J}_t^{1-\rho} - C_t^{1-\rho} = e^{-\delta t} \left[ \exp \left( (1 - \rho) \mathbb{E} \left[ \ln \tilde{J}_{t+\epsilon} | \mathcal{F}_i \right] \right) - C_t^{1-\rho} \right]$$

Using the change in variable $J_t := \ln \tilde{J}_t$ leads to the functional equation:

$$0 = \frac{\delta}{1 - \rho} \left( \frac{C_t^{1-\rho}}{\exp((1 - \rho)J_t)} - 1 \right) + \mathcal{A} J_t$$

Finally, for the case $\rho = \gamma$, the recursion takes the following form:

$$0 = \delta \frac{C_t^{1-\gamma}}{1 - \gamma} - \delta J_t + \mathcal{A} J_t$$

This latter expression is the standard HJB equation for an investor with CRRA preferences.

### A.1.5 No-Debt Life-Time Utility

I note $J_i(Y)$ the felicity value when the government cannot issue debt, when the level of output is $Y$ and when the Markov state is $s_t = i$. I postulate that the felicity function can be written $J_i(Y) = K_i Y^{1-\gamma}$ for some constants $\{K_i\}_{i \leq N}$ to be determined. $J_i$ verifies the following HJB equation:

$$0 = \varphi(Y, J_{s_t}(Y_t)) + \mathcal{A} J_{s_t}(Y_t)$$
Injecting my guess function for $J_i$, I obtain the following equation, for $i \leq N_s$:

$$0 = \delta \frac{1 - \gamma}{1 - \rho} K_i Y^{1-\gamma} \left( \frac{1}{[(1 - \gamma)K_i]^{\frac{1-\rho}{1-\gamma}}} - 1 \right) + \mu_i (1 - \gamma) K_i Y^{1-\gamma} - \frac{1}{2} \gamma (1 - \gamma)^2 |\sigma_i| K_i Y^{1-\gamma} + \sum_{j=1}^{N_s} \Lambda_{ij} K_j Y^{1-\gamma}$$

Dividing by $K_i Y^{1-\gamma}$ and simplifying, the set of constants $K := \{K_i\}_{i \leq N_s}$ must satisfy:

$$\left[ \text{diag}(A_i) - \frac{1 - \rho}{1 - \gamma} \Lambda \right] [(1 - \gamma)K] = \delta [(1 - \gamma)K]^{\frac{\rho - 1}{\gamma}}$$

Vector exponentiation in the equation above has to be understood element by element. Under assumption 1, this set of $N_s$ equations in $N_s$ unknowns has a unique solution. □

### A.1.6 Welfare Costs of Business Cycle Fluctuations

In this section, I perform calculations that are similar to Aguiar and Gopinath (2006). For simplicity, I assume that the expected GDP growth rate and the GDP growth volatility are independent of the Markov regime, in other words $\mu_i = \mu$ and $\sigma_i = \sigma$ for all $i \leq N_s$. I compute the life-time utility $V_0^a$ of a government that is in financial autarky forever. I want to compare this “autarky” utility to the life-time utility $V_0^h$ of a government that manages to perfectly hedge its consumption fluctuations via rolling over its debt at some exogenously specified interest cost $r$. In this “fully-hedged” case, I assume that government output, consumption and debt grow at a constant and deterministic rate. I solve for $x = \frac{F}{Y}$ that satisfies $V_0^h = V_0^a$. The “fully-hedged” life-time utility $V_0^h$ verifies:

$$V_0^h = \int_0^\infty \varphi \left( (Y - rF) e^{\mu t}, V_i^h \right) dt = \frac{(Y - rxY)^{1-\gamma}}{(1 - \gamma) \left[ 1 + \frac{\rho - 1}{\delta} \mu \right]^{\frac{1}{1-\gamma}}}$$

The “autarky” life-time utility $V_0^a$ verifies:

$$V_0^a = \mathbb{E} \left[ \int_0^\infty \varphi \left( Y e^{(\mu - \frac{1}{2} |\sigma|^2) t + \beta B_t}, V_i^a \right) dt \right] = \frac{Y^{1-\gamma}}{(1 - \gamma) \left[ 1 + \frac{\rho - 1}{\delta} \left( \mu - \frac{1}{2} \gamma |\sigma|^2 \right) \right]^{\frac{1}{1-\gamma}}}$$

The debt-to-GDP ratio $x$ that equates the two welfare values is:

$$x = \frac{1}{r} \left[ 1 - \left( 1 - \frac{\gamma |\sigma|^2 (\rho - 1)}{2 \delta + (\rho - 1)\mu} \right)^{\frac{1}{1-\gamma}} \right] \approx \frac{1}{2r \delta + (\rho - 1)\mu}$$

61
The approximation above is valid when $|\sigma|^2$ is small, which will be true in my quantitative application. If I factor in the fact that output suffers a downward jump at the time of default, the indifference condition between a fully-hedged economy and an economy in financial autarky leads to a breakeven debt-to-output ratio of:

$$x = \frac{1}{r} \left[ 1 - \alpha \left( 1 - \frac{1}{2} \frac{\gamma |\sigma|^2 (\rho - 1)}{\delta + (\rho - 1) \mu} \right)^{\frac{1}{\tau - \rho}} \right]$$

A.1.7 Creditors’ Stochastic Discount Factor

The discussion below is substantially similar to the discussion in Chen (2010). Assume a representative creditor whose equilibrium consumption $C_{c,t}$ follows, under the physical measure $\mathbb{P}$:

$$\frac{dC_{c,t}}{C_{c,t}} = \mu_{c,s} dt + \sigma_{c,s} dB_t$$

In the above, $\{s_t\}$ follows a discrete state Markov process with generator matrix $\Lambda = (\Lambda_{ij})_{1 \leq i,j \leq N_s}$. $N_s$ is the total number of discrete states. Assume that such creditor ranks consumption streams according to the utility specification:

$$U_t = \mathbb{E} \left[ \int_t^{+\infty} \Psi(C_s, U_s) ds | \mathcal{F}_t \right]$$

In the above, the aggregator $\Psi$ is assumed to be equal to:

$$\Psi(C, U) := \delta_c \frac{1 - \gamma_c}{1 - \rho_c} U \left( \frac{C^{1 - \rho_c}}{(1 - \gamma_c) U^{1 - \rho_c}} - 1 \right)$$

The representative creditor’s life-time utility function can be written $U_t = \frac{(h_i)_{1 \leq i \leq N_s}^{1 - \gamma_c}}{1 - \gamma_c}$, where the constants $\{h_i\}_{1 \leq i \leq N_s}$ satisfy the set of non-linear equations:

$$0 = \delta_c \frac{1 - \gamma_c}{1 - \rho_c} h_i^{\rho_c - 1} + (1 - \gamma_c) \mu_{c,i} - \frac{1}{2} \gamma_c (1 - \gamma_c) |\sigma_{c,i}|^2 - \delta_c \frac{1 - \gamma_c}{1 - \rho_c} + \sum_{j=1}^{N_s} \Lambda_{ij} \left( \frac{h_j}{h_i} \right)^{1 - \gamma_c}$$

When $\gamma_c = \rho_c$, the constants $\{h_i\}_{i \leq N_s}$ can be solved in pseudo-closed form, via:

$$h = \left[ \left( \text{diag} \left( 1 + \frac{\gamma_c}{\delta_c} \left( \mu_{c,i} - \frac{1}{2} \gamma_c |\sigma_{c,i}|^2 \right) \right) - \Lambda \right)^{-1} 1 \right]^{1 \over 1 - \gamma_c}$$
As Duffie and Skiadas (1994) shows, the marginal utility process associated with recursive preferences can be written:

\[ M_t = \exp \left[ \int_0^t \frac{\partial \Psi}{\partial U}(C_{c,z}, U_{c,z}) \, dz \right] \frac{\partial \Psi}{\partial C}(C_{c,t}, U_{c,t}) \]  

(45)

Applying Itô’s lemma then leads to the following dynamic evolution of \( M_t \):

\[ \frac{dM_t}{M_t} = -r_{st} \, dt - \nu_{st} \cdot dB_t + \sum_{s_t \neq s_{t-}} \left( e^{\nu(s_{t-}, s_t)} - 1 \right) \left( dN_t^{(s_{t-}, s_t)} - \Lambda_{s_{t-}, s_t} \, dt \right) \]  

(46)

In the above, \( N_i^{(s_{t-}, s_t)} \) is a Poisson counting process for transitions from state \( s_{t-} \) to state \( s_t \).

For each discrete Markov state \( i \), the state dependent risk price vector \( \nu_i \) is equal to:

\[ \nu_i = \gamma_c \sigma_{c,i} \]  

(47)

For each discrete Markov state \( i \), the state dependent risk free rate \( r_i \) is equal to:

\[ r_i = -\delta_c \left( \frac{1 - \gamma_c}{1 - \rho_c} \right) h_i^{\rho_c - 1} - 1 \]  
\[ + \gamma_c \mu_{c,i} - \frac{1}{2} \gamma_c (1 + \gamma_c) |\sigma_{c,i}|^2 - \sum_{j=1}^{N_s} \Lambda_{ij} e^{\nu(i,j)} \]  

(48)

Finally, the SDF relative jump sizes encoded via \( v(i, j) \) are equal to:

\[ v(i, j) = (\rho_c - \gamma_c) \ln \left( \frac{h_j}{h_i} \right) \]  

(49)

A claim to the creditor’s aggregate consumption \( C_{c,t} = Y_{c,t} \) has a price \( P_{c,t} = P_{c,s_t}(Y_t) \) that solve the following HJB equation:

\[ r_i P_{c,i}(Y) = Y + (\mu_{c,i} - \nu_i \cdot \sigma_{c,i}) Y P'_{c,i}(Y) + \frac{1}{2} |\sigma_{c,i}|^2 Y^2 P''_{c,i}(Y) + \sum_{j=1}^{N_s} \hat{\Lambda}_{ij} P_{c,j}(Y) \]  

The solution to the system above is of the form \( P_{c,i}(Y) = P_{c,i}Y \), for a set of constants \( \{P_{c,i}\}_{i \leq N_s} \) that are equal to:

\[ P_c = \left[ \text{diag}_i (r_i + \nu_i \cdot \sigma_{c,i} - \mu_{c,i}) - \hat{\Lambda} \right]^{-1} 1 \]
Expected excess returns on this claim thus take the following form:

$$\mathbb{E} \left[ \frac{d(P_{c,s}, Y_t) + Y_t dt}{P_{c,s} Y_t} - r_s dt | \mathcal{F}_t \right] = \nu_{s,t} \cdot \sigma_{c,s} dt + \sum_{s'} \Lambda_{s,s'} \left( \frac{P_{c,s'}}{P_{c,s,t-1}} - 1 \right) \left( e^{u(s,s')} - 1 \right) dt$$

A.1.8 A General Equilibrium Interpretation

In this section, I describe how to obtain, in a general equilibrium setting, the stochastic discount factor $M_t$, whose dynamics are given by equation (15). For this, I introduce the subscript “$n$” for a country’s identity, and assume a continuum of countries of measure 1. $Y^n_t$ is re-interpreted as tax collections of country $n$ – imagine for example that country $n$ has a flow output $X^n_t$ and a tax rate $\epsilon^n$, such that $Y^n_t = \epsilon^n X^n_t$. Suppose country $n$’s output evolves as follows:

$$\frac{dX^n_t}{X^n_t} = \mu_{s,t} dt + \sigma_{s,t} \cdot dB^t_t + \sigma^n dB^n_t$$

$B^n_t$ is a country-specific Brownian shock that is independent of $B^t_t$, the vector of aggregate Brownian shocks. Note that all countries need to have the same expected growth rate $\mu_{s,t}$ in all states $s_t$, but not necessarily the same output volatility. The dynamics for aggregate world output is thus:

$$dX_t = \int dX^n_t dn$$

$$= \int [\mu_{s,t} X^n_t dt + X^n_t (\sigma_{s,t} \cdot dB^t_t + \sigma^n dB^n_t)] dn$$

$$= X_t (\mu_{s,t} dt + \sigma_{s,t} \cdot dB^t_t)$$

The latter equality uses a “law of large numbers” and leverages the fact that $X^n_t$ and $dB^n_t$ are not correlated. Given that tax revenues of country $n$ are equal to $Y^n_t = \epsilon^n X^n_t$, the government revenue process $Y^n_t$ follows the same stochastic differential as $X^n_t$. Government $n$ resource constraint is the following:

$$C^n_t = Y^n_t + I^n_t D^n_t - (\kappa + m) F^n_t$$

$C^n_t$ represents government spending, $I^n_t$ represents government debt issuances (debt is issued at price $D^n_t$), and $F^n_t$ represents the face value of government debt. We assume for the moment that $\epsilon^n$ is fixed and exogeneous, and that the government will only choose its financing and for all countries to have the same expected GDP growth rate also guarantees that countries all “survive” as $t \to +\infty$, in other words no country becomes arbitrarily small asymptotically.
default policies. Due to institutional frictions, the government’s utility function is not exactly equal to the utility function of its citizens; more specifically, the government maximizes the following objective function (subject to government resource constraint above):

$$J_t^n = \mathbb{E} \left[ \int_t^{+\infty} \varphi(C_s^n, J_s^n) \, ds \mid \mathcal{F}_t \right]$$

In other words, the government maximizes flow utility over government spending only; moreover, the government will be more impatient than its citizens: $\delta$, the rate of time preference of the government, will verify $\delta > \delta_c$, where $\delta_c$ will be the rate of time preference of the citizens/creditors (to be discussed shortly). The government may elect to default on its debt. In such case, the government is shut down from capital markets for some exponentially distributed time (parametrized by $\lambda$), and suffers a temporary drop in tax collection efficiency: while in autarky, the government only collects $Y_t^n = \alpha \epsilon^n X_t^n$ from its citizens, with $\alpha < 1$. Upon exit from financial autarky, the government from country $n$ emerges with a lower debt burden, and recovers its pre-default tax efficiency $Y_t^n = \epsilon^n X_t^n$. Note that the drop in tax collections suffered by the defaulting government is only temporary, and lasts the time of the capital market’s exclusion.

Government debt is bought by a “representative world investor” – in other words, a citizen whose equilibrium consumption process is proportional to world consumption. Let $\mathcal{D}_t$ be the set of indices of countries in default at time $t$, and $\mathcal{D}_c^t = [0, 1] \setminus \mathcal{D}_t$ its complement. Remember that world citizens enjoy flow consumption $\Gamma_t$ equal to:

$$\Gamma_t := \int_{n \in \mathcal{D}_c^t} [(1 - \epsilon^n) X_t^n + (\kappa + m) F_t^n - I_t^n F_t^n] \, dn + \int_{n \in \mathcal{D}_t} (1 - \alpha \epsilon^n) X_t^n \, dn = X_t - C_t$$

In the above, $C_t$ represents aggregate government spending. Consumption by the “representative world investor” is simply equal to world output minus taxes paid plus income received on its government debt portfolio minus investments in government debt. Assume this “representative world investor” faces dynamically complete markets, and has preferences over consumption $\Gamma_t$ and government spending $C_t$ as follows:

$$U_t = \mathbb{E} \left[ \int_t^{+\infty} \Psi (\Gamma_s + C_s, U_s) \, ds \mid \mathcal{F}_t \right]$$

In other words, consumption and government spending are perfect substitutes in the representative investor’s preferences. Given those assumptions, the investor’s equilibrium consumption (via market clearing) is equal to $\Gamma_t + C_t = X_t$, and the investor’s marginal utility
is given by equation (45), meaning that the pricing kernel $M_t$ evolves as follows:

$$\frac{dM_t}{M_t} = -r_s dt - \nu_s \cdot dB_t + \sum_{s_t \neq s_{t-}} (e^{v(s_{t-}, s_t)} - 1) \left( dN^{(s_{t-}, s_t)}_t - \Lambda_{s_{t-}, s_t} dt \right)$$

$r_s$, $\nu_s$ and the SDF jumps $v(s, s')$ correspond to those described in section A.1.7.

---

A.1.9 Monotonicity of $V_i$

Take a set of debt price functions $\{D_i(\cdot, \cdot)\}_{i \in \mathcal{N}}$ that are homogeneous of degree zero and decreasing in $F$. I focus my attention on two initial levels of output, $Y^{(1)}$ and $Y^{(2)}$, and show that I must have $V_i(Y^{(1)}, F; D) > V_i(Y^{(2)}, F; D)$. First, take any arbitrary policy $(I, \tau) \in \mathcal{I} \times \mathcal{T}$ (not necessarily optimal) followed by the government. Following the issuance policy $I$ and starting in state $(Y^{(2)}, F)$ yields strictly higher flow payoffs at each time $t$. Indeed, $\{Y_i^{(\tau)}|Y_0 = Y^{(2)}\}$ is almost surely greater than $\{Y_i^{(\tau)}|Y_0 = Y^{(1)}\}$. In addition, $\{I_1 D_{s_t} \left( Y_i^{(\tau)}(s_t), F_k^{(I, \tau)} \right) | Y_0 = Y^{(2)} \}$ is almost surely greater than $\{I_1 D_{s_t} \left( Y_i^{(\tau)}(s_t), F_k^{(I, \tau)} \right) | Y_0 = Y^{(1)} \}$, since the debt price conditioned on starting in state $(Y^{(2)}, F)$ is almost surely greater than the debt price conditioned on starting in state $(Y^{(1)}, F)$ (since I assumed that the debt price is decreasing in the debt face value). Thus, $\{C_1^{(I, \tau; D)}|Y_0 = Y^{(2)}\}$ is almost surely greater than $\{C_1^{(I, \tau; D)}|Y_0 = Y^{(1)}\}$, which means that the life-time utility is increasing in $Y$, for any arbitrary issuance and default policy. Thus, the supremum over all feasible issuance and default policies, $V_i(\cdot, \cdot; D)$, is also increasing in output $Y$. The proof for the monotonicity of $V_i(\cdot, \cdot; D)$ in $F$ is identical, since consumption $C_1^{(I, \tau; D)}$ is decreasing in the level of indebtedness $F$ and since $D_i(\cdot, \cdot)$ is decreasing in $F$.

I then show that the optimal issuance policy is homogeneous of degree 1 and the optimal default policy is barrier type. Take an arbitrary $(Y, F)$, and the related optimal Markov issuance and default policies $(I_{Y,F}^*, \tau_{Y,F}^*) = \arg \max J_i(Y, F; (I, \tau); D)$. Take $\epsilon > 0$, and focus on starting output and face value levels $(\epsilon Y, \epsilon F)$. Consider the policy $(I_{\epsilon Y, \epsilon F}, \tau_{\epsilon Y, \epsilon F})$, such that $I_{\epsilon Y, \epsilon F} = \epsilon I_{Y,F}^*$, and $\tau_{\epsilon Y, \epsilon F} = \tau_{Y,F}^*$. Since $(I_{Y,F}^*, \tau_{Y,F}^*)$ is feasible conditioning on $(Y_0, F_0) = (Y, F)$, since the output dynamics are linear in $Y$ and since the debt face value dynamics are homogeneous of degree 1 in $(Y, F)$, it must be the case that $(I_{\epsilon Y, \epsilon F}, \tau_{\epsilon Y, \epsilon F})$ is feasible conditioning on $(Y_0, F_0) = (\epsilon Y, \epsilon F)$. Thus, I have:

$$J_i(\epsilon Y, \epsilon F; (I_{\epsilon Y, \epsilon F}, \tau_{\epsilon Y, \epsilon F}); D) \leq V_i(\epsilon Y, \epsilon F; D)$$

Then assume for a second that the inequality above is strict. If that was the case, then take $(I_{\epsilon Y, \epsilon F}^*, \tau_{\epsilon Y, \epsilon F}^*) = \arg \max J_i(\epsilon Y, \epsilon F; (I, \tau); D)$. Consider the policy $(I_{Y,F}, \tau_{Y,F})$, such that
\( I_{Y,F} = I_{eY,eF}/\epsilon \), and \( \tau_{Y,F} = \tau_{eY,eF}^* \). Then it is immediate to see that this policy is feasible conditioned on starting at \((Y_0, F_0)\), and it is also immediate to see that:

\[
J_i(Y, F; (I_{Y,F}, \tau_{Y,F}); D) > J_i(Y, F; (I_{Y,F}^*, \tau_{Y,F}^*); D) = V_i(Y, F; D)
\]

This is a contradiction. Thus, the optimal issuance policy is homogeneous of degree 1 in \((Y, F)\). Since the value function is decreasing in \(F\) and increasing in \(Y\), the default policy must be a barrier default policy. \(\square\)

**A.1.10 Monotonicity of \(D_i\)**

Let \((I, \tau) \in \mathcal{I} \times \mathcal{T}\) be admissible issuance and default Markov policies. Assume that \(I\) is homogeneous of degree 1 in \((Y, F)\) and that \(\tau\) is barrier. Note \(I_i(Y, F) = \nu_i(F/Y)Y\), where \(\nu_i(x) := I_i(1, x)\), for each Markov state \(i \leq N_s\). Given these assumptions, the default policy can be written:

\[
\tau = \inf\{t \geq 0 : F_t \geq Y_t \bar{x}_{s_t}\} = \inf\{t \geq 0 : x_t \geq \bar{x}_{s_t}\}
\]

Using Itô’s lemma, \(x_t := F_t/Y_t\) follows the following stochastic differential equation under \(Q\):

\[
dx_{t}^{(\nu, \tau)} = \left(1 - 1_{d,t}^{(\tau)}\right) \left[\left(\nu_t - (m + \mu_{s_t} - |\sigma_{s_t}|^2 - \nu_{s_t} \cdot \sigma_{s_t}) x_t^{(\nu, \tau)}\right) dt - x_t^{(\nu, \tau)} \sigma_{s_t} \cdot d\hat{B}_t\right] + x_t^{(\nu, \tau)} \left(\frac{1}{\alpha} - 1\right) dN_{d,t}^{(\tau)} + x_t^{(\nu, \tau)} (\theta \alpha - 1) dN_{e,t}^{(\tau)}
\]

The debt price is an expected present value of flow payoffs. Since such flow payoffs are homogeneous of degree zero in \((Y, F)\), and since the default policy is barrier, the debt price function must be homogeneous of degree zero. With an abuse of notation, I will note \(D_i(Y, F; (\nu, \tau)) = D_i(x; (\nu, \tau))\). I then use a result that will be proven in section A.1.11: the fact that the defaulted debt price must satisfy:

\[
D^d(x) = \lambda \theta \alpha \Xi^{-1} D(\theta x)
\]

The matrix \(N_s \times N_s\) matrix \(\Xi\) is equal to \(\text{diag}_i (r_i + \nu_i \cdot \sigma_i + \lambda - \mu_i)\). I then introduce the operator \(\mathcal{T}\), defined for any \(N_s \times 1\) vector \(f\) of continuous decreasing functions whose \(i^{th}\) coordinate is \(f_i : (0, \max_j \bar{x}_j) \rightarrow [0, D_i^{rf}]\) \((D_i^{rf}\) is the price of risk-free debt is state \(i\), see
equation (23)) as follows. If \( x \geq \bar{x}_i \), set \((\mathbb{T}f)_i(x) = \lambda \theta \alpha [\Xi^{-1}f(\theta x)]_i\). If \( x \leq \bar{x}_i \), then:

\[
(\mathbb{T}f)_i(x) = \hat{E}^{i,x} \int_0^\tau e^{-\int_0^\tau (r_su + m)du} (\kappa + m)dt + \lambda \theta \alpha e^{-\int_0^\tau (r_su + m)du} [\Xi^{-1}f(\theta x_\tau)]_{s_r} \]

\[
= D_i + \hat{E}^{i,x} \left[ e^{-\int_0^\tau (r_su + m)du} \left( \lambda \theta \alpha [\Xi^{-1}f(\theta x_\tau)]_{s_r} - D_{i,s} \right) \right] \quad \text{(51)}
\]

Given assumption 2, given that \( \theta < 1 \) and \( \alpha < 1 \), and given that the function \( f_i \) has an image in \([0, D_i]\), it must be the case that \( \lambda \theta \alpha [\Xi^{-1}f(\cdot)]_i \leq D_i \) for any state \( i \), which means that the term in brackets in equation (51) is negative. Thus, \((\mathbb{T}f)_i(\cdot)\) is a decreasing function of \( x \), positive and bounded above by \( D_i \). The Feynman-Kac theorem also provides for the continuity of the function \((\mathbb{T}f)_i(\cdot)\). Thus, \( \mathbb{T} \) maps \( N_s \times 1 \) vectors of continuous bounded decreasing functions with image in \([0, D_i]\) into itself. For any pair of vectors of functions \( f_1, f_2 \) whose components are continuous and decreasing on the interval \([0, \max_i \bar{x}_i] \), I have:

\[
|| (\mathbb{T}f_2 - \mathbb{T}f_2)_i(x) || = \lambda \theta \alpha \hat{E}^{i,x} \left[ e^{-\int_0^\tau (r_su + m)du} | [\Xi^{-1}(f_2(\theta x_\tau)) - f_1(\theta x_\tau))]_{s_r} | \right] 
\leq \lambda \theta \alpha ||\Xi^{-1}|| \cdot ||f_2 - f_1||_\infty
\]

Since \( \lambda \theta \alpha ||\Xi^{-1}|| < 1 \) (given assumption 2), \( \mathbb{T} \) is a contraction, and the contraction mapping theorem provides for a unique continuous, bounded and decreasing vector of functions \( D(\cdot; (\mathbb{I}, \tau)) \) whose \( i^{th} \) component satisfies the functional equation:

\[
D_i(x) = \hat{E}^{i,x} \left[ \int_0^\tau e^{-\int_0^\tau (r_su + m)du} (\kappa + m)dt + \lambda \theta \alpha e^{-\int_0^\tau (r_su + m)du} [\Xi^{-1}D(\theta x_\tau)]_{s_r} \right]
\]

The function \( D_i \) is decreasing as required. \( \square \)

**A.1.11 Debt Price at Default**

I need to compute the debt price in default \( D_d^i(x) \), for \( x \geq \bar{x}_i \) and \( 1 \leq i \leq n_s \). Assume that at time of default, the state is \( s_r = i \). When the country exits financial autarky, its debt-to-GDP ratio is equal to \( \frac{F_{s+\tau_e}}{Y_{s+\tau_e}} = \theta \frac{F_{x_r}}{Y_{x_r}} = \theta x_\tau \). It is possible that \( x_\tau > \bar{x}_{s_{r-}} \) when the sovereign defaults. This happens upon the occurrence of a “jump-to-default”, in other words a situation where the state jumps from \( s_{r-} = j \) to \( s_r = i \) and when \( \bar{x}_i < x_{r-} < \bar{x}_j \). Thus, I
have the following for $x \geq \bar{x}_i$:

$$D^d_i(x) = \tilde{E} \left[ \exp \left( - \int_0^{\tau_e} r_{st+u} du \right) \frac{F_{t+\tau_e}}{F_t} D_{st+\tau_e} (\theta x) \mid Y_{t-} = Y, s_{t-} = i \right]$$

$$= \theta \alpha \tilde{E} \left[ \exp \left( - \int_0^{\tau_e} r_{st+u} du \right) \frac{Y_{t+\tau_e}}{Y_t} D_{st+\tau_e} (\theta x) \mid Y_{t-} = Y, s_{t-} = i \right]$$

In other words, in order to compute $D^d_i(x)$ (for $x \geq \bar{x}_i$), I need to solve a system of $N_s$ equations in $N_s$ unknowns $\{D^d_i(x)\}_{1 \leq j \leq N_s}$:

$$r_j D^d_j(x) = (\mu_j - \nu_j \cdot \sigma_j) D^d_j(x) + \lambda \left( \theta \alpha D_j(\theta x) - D^d_j(x) \right) + \sum_{k=1}^{N_s} \hat{\Lambda}_{jk} D^d_k(x)$$

If I introduce the $N_s \times N_s$ matrix $\Xi := \text{diag} (r_i + \nu_i \cdot \sigma_i + \lambda - \mu_i) - \hat{\Lambda}$, and if I note $D^d(x)$ the $N_s \times 1$ vector with $i$th element $D^d_i(x)$, I obtain:

$$D^d(x) = \lambda \theta \alpha \Xi^{-1} D(\theta x)$$

Given assumption 2, the quantity above is well defined and finite. Finally, note that this equation is valid for each coordinate $i$ for $x \geq \bar{x}_i$. Indeed, the model with discrete Markov states for the SDF generates default waves, via “jumps-to-default” created when the state $s_t$ jumps from a state $s_{t-} = i$ (for example a state where risk-prices are high) to a state $s_t = j$ (for example a state where risk-prices are low), and when $\bar{x}_i > x_t > \bar{x}_j$. When computing the full set of functions $\{D_i(\cdot)\}_{1 \leq i \leq N_s}$, it is thus essential to compute such function on the interval $[0, \max_i \bar{x}_i]$. □

### A.1.12 Life-Time Utility HJB Equation

Introduce the probability measure $\tilde{P}(A) = E \left[ e^{(1-\gamma) \int_0^t \sigma_{su} dB_u - \frac{1}{2} (1-\gamma)^2 \int_0^t \sigma_{su}^2 du} 1_A \right]$, for some arbitrary Borel set $A \subseteq \mathcal{F}_t$. Under such measure, using Girsanov’s theorem, the variable $x_t$ evolves as follows:

$$dx_t^{(t, \tau)} = 1_{d,t}^{(t, \tau)} \left[ \left( \dot{\theta} - (m + \mu_s - \gamma |\sigma_s|^2) x_t^{(t, \tau)} \right) dt - x_t^{(t, \tau)} \sigma_s \cdot d\tilde{B}_t \right]$$

$$+ x_t^{(t, \tau)} \left( \frac{1}{\alpha} - 1 \right) dN_{d,t}^{(\tau)} + x_t^{(t, \tau)} (\alpha \theta - 1) dN_{e,t}^{(\tau)}$$

In the above, $\tilde{B}_t := B_t + (\gamma - 1) \int_0^t \sigma_{su} du$ is a standard $N_d$-dimensional Brownian motion under this equivalent probability measure. Let $c_t^{(t, \tau; D)}$ be the consumption-to-output ratio.
when the policy used is \((\iota, \tau)\) and when the debt price schedule is \(D\):

\[
c_i^{(\iota, \tau; D)} = \left[ 1 + \iota_t D_s (x^{(\iota, \tau)}_t) - (\kappa + m)x^{(\iota, \tau)}_t \right] \left( 1 - \frac{1(\tau)}{d_t} \right) + 1(\tau)
\]

The value function \(V_i(Y, F)\) in state \(i\) can be written \(V_i(Y, F) = v_i(x)Y^{1-\gamma}\):

\[
V_i(Y, F) = \sup_{(\iota, \tau)} \mathbb{E}^{\iota, \tau; F, Y} \left[ \int_0^\infty \delta \left( 1 - \gamma \right) \frac{v_{st}(x_t)}{1 - \rho} Y_t^{1-\gamma} \left( \frac{(c_i^{(\iota, \tau; D)})^{1-\rho}}{((1 - \gamma)v_{st}(x_t))^{\frac{1-\rho}{1-\gamma}}} - 1 \right) dt \right] = Y^{1-\gamma} \sup_{(\iota, \tau)} \mathbb{E}^{\iota, \tau; F, Y} \left[ \int_0^\infty \delta \left( 1 - \gamma \right) \frac{v_{st}(x_t)}{1 - \rho} Y_t^{1-\gamma} \left( \frac{(c_i^{(\iota, \tau; D)})^{1-\rho}}{((1 - \gamma)v_{st}(x_t))^{\frac{1-\rho}{1-\gamma}}} - 1 \right) e^{(1-\gamma)\int_0^t \left( \mu_s - \frac{1}{2} \gamma |\sigma_s|^2 \right) ds} dt \right] = Y^{1-\gamma} v_i(x)
\]

The corresponding HJB equation for \(v_i\) can thus be written, in the continuation region:

\[
\frac{1 - \gamma}{1 - \rho} \left( \delta + (\rho - 1) \left( \mu_i - \frac{1}{2} \gamma |\sigma_i|^2 \right) \right) v_i(x) - \sum_{j=1}^{N_s} \Lambda_{ij} v_j(x) = \sup_{\iota_t} \left[ \delta \left( 1 + \iota_t D_i(x) - (\kappa + m)x \right) \frac{\rho - \gamma}{1 - \rho} [(1 - \gamma)v_i(x)]^{\frac{\rho - \gamma}{\rho - \gamma}} + [\iota_t - (m + \mu_i - \gamma |\sigma_i|^2) x] v'_i(x) + \frac{1}{2} |\sigma_i|^2 x^2 v''_i(x) \right]
\]

The term in brackets on the right-hand-side of the equal sign is concave in \(\iota_t\), which leads to an optimal issuance policy of the form:

\[
\iota_t(x) = \frac{1}{D_i(x)} \left[ \left( \frac{(\delta D_i(x) [(1 - \gamma)v_i(x)]^{\frac{\rho - \gamma}{\rho - \gamma}}}{-v'_i(x)} \right)^{1/\rho} + (\kappa + m)x - 1 \right]
\]

For the particular case \(\rho = 1\), the guess value function still takes the form \(V_i(Y, F) = v_i(x)Y^{1-\gamma}\), but the HJB equation solved by \(v_i\) is now:

\[
\left( \delta \ln [(1 - \gamma)v_i(x)] - (1 - \gamma) \left( \mu_i - \frac{1}{2} \gamma |\sigma_i|^2 \right) \right) v_i(x) - \sum_{j=1}^{N_s} \Lambda_{ij} v_j(x) = \sup_{\iota_t} \left[ \delta (1 - \gamma)v_i(x) \ln (1 + \iota_t D_i(x) - (\kappa + m)x) + [\iota_t - (m + \mu_i - \gamma |\sigma_i|^2) x] v'_i(x) + \frac{1}{2} |\sigma_i|^2 x^2 v''_i(x) \right]
\]

Optimality of the issuance policy in such case takes the form:

\[
\iota_t(x) = \frac{1}{D_i(x)} \left[ \frac{\delta (1 - \gamma)v_i(x) D_i(x)}{-v'_i(x)} + (\kappa + m)x - 1 \right]
\]
Finally, for the particular case $\gamma = 1$, the guess value function still takes the form $V_i(Y, F) = v_i(x) + \ln Y$, and the HJB equation solved by $v_i$ is now:

$$\frac{1}{1-\rho} \left( \delta + (\rho - 1) \left( \mu_i - \frac{1}{2} |\sigma_i|^2 \right) \right) - \sum_{j=1}^{N_s} v_j(x) =$$

$$\sup_{\iota_i} \left[ \frac{\delta}{1-\rho} \left( 1 + \iota_i D_i(x) - (\kappa + m)x \right)^{1-\rho} \exp \left( (1-\rho)v_i(x) \right) \right] + \left[ \iota_i - (m + \mu_i - |\sigma_i|^2 x) v'_i(x) + 1/2 |\sigma_i|^2 x^2 v''_i(x) \right]$$

Optimality of the issuance policy in such case takes the form:

$$v_i(x) = \frac{1}{D_i(x)} \left[ \left( -\frac{v'_i(x)}{\delta} \right)^{-1/\rho} \exp \left( (1-1/\rho) v_i(x) \right) + (\kappa + m)x - 1 \right]$$

**A.1.13 Life-Time Utility at Default**

Assume that the state at default time $\tau$ is equal to $s_\tau$. The pre-default output is $Y_{\tau-} = Y$, and it falls at the time of default by a factor $\alpha$. The life-time utility in default consists of the flow value of receiving $Y_i$ until the random time interval $\tau_e$, at which point a lump sum value $V_{\tau+\tau_e}(Y_{\tau+\tau_e}, F_{\tau-})$ should be added. The debt-to-GDP ratio at the time the country is exiting from financial autarky is:

$$\frac{F_{\tau+\tau_e}}{Y_{\tau+\tau_e}} = \frac{\theta Y_{\tau+\tau_e}}{Y_{\tau-}} \frac{F_{\tau-}}{Y_{\tau+\tau_e}} = \theta x_{\tau-}$$

Thus the value function at exit from financial autarky can be expressed as $v_{s_{\tau+\tau_e}}(\theta x_{\tau-}) Y_{\tau+\tau_e}^{1-\gamma}$. I thus look for a function $V^{d}_{s_{\tau}}$ of the form $V^{d}_{s_{\tau}}(Y, F) = v^{d}_{s_{\tau}}(x)(\alpha Y)^{1-\gamma}$, for a set of functions $\{v^{d}_{i}(x)\}_{1 \leq i \leq N_s}$ to be determined. For $x \geq \bar{x}_{s_{\tau}}$, the function $V^{d}_{s_{\tau}}$ satisfies the recursive equation:

$$0 = \varphi \left( Y_i, V^{d}_{s_{\tau}} \right) + \mathcal{A}V^{d}_{s_{\tau}}$$

Plugging in my guess function, the HJB equation becomes:

$$0 = \delta \frac{1-\gamma}{1-\rho} v^{d}_{i}(x)(\alpha Y)^{1-\gamma} \left( \frac{1}{(1-\gamma)v'_i(x)} \right)^{\frac{1-\gamma}{\gamma}} - 1 + \sum_{j=1}^{N_s} \Lambda_{ij}(\alpha Y)^{1-\gamma} (v^{d}_{j}(x) - v^{d}_{i}(x))$$

$$+ \mu_i (1-\gamma) v^{d}_{i}(x)(\alpha Y)^{1-\gamma} - \frac{1}{2} \gamma (1-\gamma) |\sigma_i|^2 v^{d}_{i}(x)(\alpha Y)^{1-\gamma} + \lambda (\alpha Y)^{1-\gamma} (v_i(\theta x) - v^{d}_{i}(x))$$

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If I introduce the $N_s \times N_s$ matrix $\Upsilon := \frac{1}{1-\rho} \text{diag}(A_i) + \lambda I - \Lambda$, if I note $v(x)$ the $N_s \times 1$ vector with $i^{th}$ row $v_i(x)$, and if I note $v^d(x)$ the $N_s \times 1$ vector with $i^{th}$ row $v^d_i(x)$, I need to solve the non-linear equation:

$$\Upsilon v^d(x) - \lambda v(\theta x) = \frac{\delta}{1-\rho} [(1-\gamma)v^d(x)]^{\frac{\rho-\gamma}{\rho}}$$  \hspace{1cm} (52)

In the above $[(1-\gamma)v^d(x)]^{\frac{\rho-\gamma}{\rho}}$ is to be understood as an element-by-element power function. Note that $v^d_i(x)$ admits the integral representation (this will be useful in connection with my verification theorem):

$$v^d_i(x) = E^i \left[ \int_0^{\tau_e} e^{-\frac{1-\rho}{\rho} \int_0^u A_s du} \frac{\delta}{1-\rho} [(1-\gamma)v_{st}(x)]^{\frac{\rho-\gamma}{\rho}} dt + e^{-\frac{1-\rho}{\rho} \int_0^{\tau_e} A_s du} v_{st}(\theta x) \right]$$  \hspace{1cm} (53)

A contraction mapping argument can show that equation (52) always has a unique positive solution. Note that for $\gamma = \rho$, I can solve this equation explicitly, obtaining:

$$v^d(x) = \Upsilon^{-1} \left( \lambda v(\theta x) + \frac{\delta}{1-\gamma} 1 \right)$$

In the above, 1 is a $N_s \times 1$ vector of ones. Default optimality can be written $V_i(Y,F) \geq V^d_i(Y,F)$, which, stated in terms of the normalized value functions, can be written:

$$v_{st}(x_t) - \alpha^{1-\gamma} v^d_{st}(x_t) \geq 0 \quad \forall t \geq 0$$

Applying Itô’s lemma to $v_{st}(x_t) - \alpha^{1-\gamma} v^d_{st}(x_t)$, the diffusion term is equal to:

$$-x_t \left( v'_{st}(x_t) - \alpha^{1-\gamma} \left( v^d_{st}(x_t) \right)' \right) \sigma_{st} \cdot dB_t$$

In particular, since $v_{st}(\bar{x}_{st}) = \alpha^{1-\gamma} v^d_{st}(\bar{x}_{st})$, the only way for the inequality $v_{st}(x_t) - \alpha^{1-\gamma} v^d_{st}(x_t) \geq 0$ to be preserved in the presence of Brownian shocks at the default boundary is for the diffusion term above to be identically zero at such boundary. This leads to the smooth-pasting optimality condition:

$$v'_i(\bar{x}_i) = \alpha^{1-\gamma} \left( v^d_i \right)'(\bar{x}_i)$$
Differentiating the implicit equation defining $\bm{v}^d(x)$ w.r.t. $x$, I obtain the following expression for $(\bm{v}^d)'(\bar{x}_i)$:

$$(\bm{v}^d)'(\bar{x}_i) = \lambda \theta \left( \Upsilon + \delta \frac{\gamma - \rho}{1 - \rho} \text{diag}_j \left( \left[ (1 - \gamma) \nu_j^d(\bar{x}_i) \right]^{1 - \frac{\rho}{1 - \rho}} \right) \right)^{-1} \nu'(\theta \bar{x}_i)$$

\[\square\]

### A.1.14 Verification Theorem

Let $\{v_i\}_{i \leq N_z}$ be a family of functions such that for each $i$, $v_i \in C^1(\mathbb{R}^+) \cap C^2(\mathbb{R}^+ \setminus \{\bar{x}_i\})$ satisfies the assumptions of the theorem. Let $(\bm{u}, \bm{\tau}) \in \mathcal{I} \times \mathcal{T}$ be an arbitrary policy, I have the following Itô formula:

$$e^{-\frac{1 - \gamma}{1 - \rho} \int_0^t A_{du} du} v_{s_t}(x_t) = v_i(x) - \int_0^t e^{-\frac{1 - \gamma}{1 - \rho} \int_0^s A_{du} ds} \left( 1 - 1_{d,z}^{(t)} \right) x_z v'_{s_z}(x_z) \sigma_{s_z} \cdot d \tilde{B}_z$$

$$\quad + \int_0^t e^{-\frac{1 - \gamma}{1 - \rho} \int_0^s A_{du} ds} \left[ \sum_{s' \neq s_{z-}} \left( v_{s'}(x_{z-z}) - v_{s_{z-}}(x_z) \right) \left( dN_{s_{z-}}^{s'-s} - \Lambda_{s_{z-},s'} dz \right) \right]$$

$$\quad + \int_0^t e^{-\frac{1 - \gamma}{1 - \rho} \int_0^s A_{du} ds} \left( v_{s_z}(x_z) - v_{s_{z-}}(x_z) \right) dN_{s_z}^{(t)} + \int_0^t e^{-\frac{1 - \gamma}{1 - \rho} \int_0^s A_{du} ds} \left( v_{s_z}(x_z) - v_{s_{z-}}(x_z) \right) dN_{s_z}^{(t)}$$

See for example Protter (2005). For the arbitrary control policy $(\bm{u}, \bm{\tau})$, I note $c_{s_t}^{(x, \tau)}(x_t)$ the resulting consumption-to-output policy. Using Itô’s lemma above, using the variational inequality in the assumption of the theorem, and using equation (53):

$$e^{-\frac{1 - \gamma}{1 - \rho} \int_0^t A_{du} du} v_{s_t}(x_t) \leq v_i(x) - \int_0^t e^{-\frac{1 - \gamma}{1 - \rho} \int_0^s A_{du} ds} \delta c_{s_z}^{(x, \tau)}(x_z) \left( 1 - 1_{d,z}^{(t)} \right) x_z v'_{s_z}(x_z) \sigma_{s_z} \cdot d \tilde{B}_z$$

$$\quad + \int_0^t e^{-\frac{1 - \gamma}{1 - \rho} \int_0^s A_{du} ds} \left[ \sum_{s' \neq s_{z-}} \left( v_{s'}(x_{z-z}) - v_{s_{z-}}(x_z) \right) \left( dN_{s_{z-}}^{s'-s} - \Lambda_{s_{z-},s'} dz \right) \right]$$

$$\quad - \int_0^t e^{-\frac{1 - \gamma}{1 - \rho} \int_0^s A_{du} ds} \left( 1 - 1_{d,z}^{(t)} \right) x_z v'_{s_z}(x_z) \sigma_{s_z} \cdot d \tilde{B}_z$$

73
The terms on the second and third line above are martingales since \( v_i \) and \( v'_i \) are bounded. Thus, taking expectations on both sides of this equality, I obtain:

\[
\mathbb{E}^{i,x}\left[ \int_0^t e^{-\frac{1}{1-\rho} \int_0^s \delta_{s,t}^{(\xi,\tau)}(x_i) (1-\gamma) v_i(x_i) \frac{\rho dz}{1-\rho} dz} \right] + \mathbb{E}^{i,x}\left[ e^{-\frac{1}{1-\rho} \int_0^T \delta_{s,t}^{(\xi,\tau)} v_i(x_t) \frac{\rho dz}{1-\rho} dz} \right] \leq v_i(x) \]

Taking \( t \to +\infty \), using the assumption that \( \lim_{t \to +\infty} \inf e^{-\int_0^t \frac{1}{1-\rho} \delta_{s,t}^{(\xi,\tau)} v_i(x_t) \frac{\rho dz}{1-\rho} dz} \leq 0 \), and using the monotone convergence theorem, I then obtain the desired result: \( v_i(x; D) \geq \tau_i(1, x; (\xi, \tau); D) \) for any admissible control policy. The proof of the second part of the theorem relies on steps identical to those described above, except that inequalities are now replaced by equalities. The uniqueness of \( v_i(\cdot; D) \) as a solution to re-scaled sequence problem equation (22), shows that \( v_i(x; D) = J_i(1, x; (\xi^*, \tau^*); D) \).

\[ \square \]

### A.1.15 Possible Strategy to Establish Existence of Equilibrium

I discuss here a possible route to prove that an equilibrium exists in a simpler environment without discrete SDF states, and where the punishment upon default is financial autarky forever. Upon a sovereign default, creditors’ recovery value is zero. In this simpler environment, the Markov perfect equilibrium features only the debt-to-GDP ratio as a state variable. Take an arbitrary debt price schedule \( D : \mathbb{R}^+ \to [0, \frac{\kappa + m}{r + m}] \) that is continuous and strictly decreasing on that interval. Given this debt price schedule, construct the sovereign’s “best response”, in other words construct the value function \( v(\cdot; D) \) as well as the optimal issuance and default policies \( \xi^*(\cdot; D) \) and \( x^*(D) \). This best response exists: given a debt price schedule \( D \), the function \( v(\cdot; D) \) is simply the optimal life-time utility in a single-agent optimal control, optimal stopping problem, where the control is the issuance rate \( \xi \) and the stopping time is the default time \( \tau \). Using those issuance and default policies, construct a new debt price schedule \( D (\cdot; (\xi^*(\cdot; D), x^*(D))) \).

I have implicitly constructed a functional map \( \mathbb{T} \), which takes a continuous decreasing function \( D : \mathbb{R}^+ \to [0, \frac{\kappa + m}{r + m}] \) and maps it into a continuous decreasing function \( D (\cdot; (\xi^*(\cdot; D), x^*(D))) : \mathbb{R}^+ \to [0, \frac{\kappa + m}{r + m}] \). In fact, by studying the sovereign’s behavior when the debt price is constant and equal to its risk-free value \( \frac{\kappa + m}{r + m} \), I can restrict this functional map to functions defined on the interval \([0, \bar{x}_{eq}] \) (where \( \bar{x}_{eq} \) is the optimal sovereign default boundary when the debt is priced at its risk-free value by creditors). Indeed, it is straightforward to show that the default boundary must be decreasing in the debt price schedule – in other words, if for any \( x, D_1(x) \geq D_2(x) \), it must be the case that \( x^*(D_1) \geq x^*(D_2) \). A Markov perfect equilibrium of my economy is simply a fixed point of the functional map constructed. Schauder’s fixed...
point theorem (appropriate for infinite dimensional spaces) could then be invoked in order
to establish the existence of a fixed point of such functional map. An appropriate space of
functions to use is any subset that is closed, bounded and equicontinuous. Indeed, since
\([0, \bar{x}^I]\) is compact, Arzela-Ascoli’s theorem guarantees that any such subspace of functions
is compact. A good candidate to restrict oneself would be the space of Lipschitz continuous
functions that have the same Lipschitz constant. In order to apply Schauder’s fixed point
theorem, two theoretical hurdles thus have to be overcome. First, one would need to show
that the mapping \( T \) is continuous. Second, one would need to show that the mapping \( T \)
preserves Lipschitz continuity. Once those two conditions are established, existence of a Markov
perfect equilibrium is straightforward.

A.1.16 Expected Default Times and Ergodic Distribution

I note \( T_i(x) \), the risk-natural expected default time conditioned on the debt-to-GDP ratio
being equal to \( x \) and the state \( s_t = i \). Mathematically, the expected default time can be
written \( T_i(x) := \mathbb{E}^{i,x}[\tau] \). Using Feynman-Kac, it is immediate to show that \( T_i(\cdot) \) solves the
following HJB equation, for \( x \in (0, \bar{x}_i) \):

\[
0 = 1 + (\iota_i(x) - (m + \mu_i - |\sigma_i|^2) x) T_i'(x) + \frac{1}{2}|\sigma_i|^2 x^2 T_i''(x) + \sum_{j=1}^{N_s} \Lambda_{ij} T_j(x)
\]

The \( 2 \times N_s \) boundary conditions consist in (i) value matching conditions at the default boundary and (ii) Robin boundary conditions at \( x = 0 \):

\[
0 = T_i(x) \quad \forall x \geq \bar{x}_i
\]

\[
0 = 1 + \iota_i(0) T_i'(0) + \sum_{j=1}^{N_s} \Lambda_{ij} T_j(0)
\]

I then focus on the ergodic measure \( f_i \) of the state variable under the physical measure \( \mathbb{P} \),
conditioned on being in state \( i \) and conditioned on the government being performing under its
debt obligations (I emphasize the word measure as opposed to density since \( f_i \) does not integrate to 1). For \( x \in (0, \bar{x}_i) \) and \( x \notin \{\theta \bar{x}_j\}_{1 \leq j \leq N_s} \), \( f_i \) solves the following Kolmogorov-forward equation:

\[
0 = -\frac{d}{dx} \left[ (\iota_i(x) - (m + \mu_i - |\sigma_i|^2) x) f_i(x) \right] + \frac{1}{2} \frac{d^2}{dx^2} \left[ |\sigma_i|^2 x^2 f_i(x) \right] + \sum_{j=1}^{N_s} \Lambda_{ji} f_j(x)
\]
The equation above is not applicable at the points \( \{\theta \bar{x}_j\}_{1 \leq j \leq N_s} \) (the points of re-entry of the sovereign following a “smooth” default, i.e. a default such that \( x_{\tau-} = \bar{x}_{s_{\tau}} \)), but the measures \( f_i \) are continuous at those points. The following boundary condition holds at \( x = \bar{x}_i \):

\[
f_i(\bar{x}_i) = 0
g\tag{55}
\]

This equation can be obtained heuristically by approximating the continuous time process \( \{x_t\} \) by a discrete state Markov chain, and analyzing transitions in and out of the state \( x = \bar{x}_{s_t} \) between time \( t \) and time \( t + \Delta t \). It is also a standard condition for absorbing boundaries. I note \( g_i \) the fraction of time the sovereign is in autarky in SDF state \( i \) – note that such fraction does not depend on the debt-to-GDP ratio at entry into the default state given the memory-less property of the stochastic process \( s_t \), and given that the autarky time length is exponentially distributed and independent of the process \( \{s_t\} \). In other words, if \( g := \sum_{i=1}^{N_s} g_i \), then I have:

\[
\frac{g_i}{g} = \frac{\mathbb{E}^i[\int_0^{\tau_e} 1_{\{s_t=i\}} dt]}{\mathbb{E}^i[\tau_e]} = \left( I - \frac{1}{\lambda} \Lambda \right)^{-1} \tag{56}
\]

For any Markov state \( s = i \), the integral of the ergodic distribution over the state space \([0, \bar{x}_i]\), in addition to the expected time spent in autarky \( g_i \), need to add up to \( \pi_i \), the stationary measure of the process \( s_t \):

\[
\int_0^{\bar{x}_i} f_i(x) dx + g_i = \pi_i \tag{57}
\]

Thus, equations (55) and (57) gives me \( 2 \times N_s \) “boundary” conditions, allowing me to solve for the \( N_s \) Kolmogorov-forward equations, which are second order ordinary differential equations. However, the constants \( \{g_i\}_{1 \leq i \leq N_s} \) in equation (56) are only determined up to the constant \( g \), which represents the average percentage of time the sovereign spends in autarky post-default. I determine the constant \( g \) numerically via a Markov chain approximation method described in section A.4.

Finally, note that in the particular case where there is only one discrete Markov state, I can derive a pseudo-closed form expression for the stationary measure \( f \). Indeed, in such case, the ergodic measure \( f \) of the state variable under the physical probability measure \( \mathbb{P} \) solves the following Kolmogorov-forward equation, valid for \( x \in (0, \theta \bar{x}) \cup (\theta \bar{x}, \bar{x}) \):

\[
0 = -\frac{d}{dx} \left[ (\iota(x) - (m + \mu - \sigma^2) x) f(x) \right] + \frac{1}{2} \frac{d^2}{dx^2} \left[ \sigma^2 x^2 f(x) \right] \tag{58}
\]

\( f \) is continuous at \( x = \theta \bar{x} \) (the point of re-entry of the sovereign post-autarky). At \( x = \bar{x}, \)
the ergodic distribution must satisfy the absorbing boundary condition:

\[ f(\bar{x}) = 0 \]

Equation (58) can be integrated out as follows. For \( x \in (\theta \bar{x}, \bar{x}) \), I have:

\[ 0 = G + ((m + \mu) x - \iota(x)) f(x) + \frac{1}{2} |\sigma|^2 x^2 f'(x) \]  

(59)

The constant \( G \) is the “flow” of the density in the positive direction. Using \( f(\bar{x}) = 0 \), I can integrate equation (59) to obtain, for \( x \in (\theta \bar{x}, \bar{x}) \):

\[ f(x) = \int_x^{\bar{x}} \exp \left[ \int_x^t \frac{2}{\sigma^2 s^2} ((m + \mu) s - \iota(s)) ds \right] \frac{2G}{\sigma^2 t^2} dt \]

I also know that the density \( f \) is continuous at \( \theta \bar{x} \) (even though it is not differentiable at that point). At \( x = 0 \), the density must be zero. Indeed, note that in a neighborhood of \( x = 0 \), the stochastic process \( \{x_t\} \) behaves similarly to a geometric brownian motion to which a constant strictly positive drift \( \iota(0) \) has been added, and it is straightforward to show that the stationary distribution of a geometric Brownian motion which, at \( x = \bar{x} \), is “reset” to \( x = \theta \bar{x} \), admits a stationary density with value zero at \( x = 0 \). Thus, on \([0, \theta \bar{x}]\), the density \( f \) takes the following form:

\[ f(x) = \exp \left[ \int_x^{\theta \bar{x}} \frac{2}{\sigma^2 s^2} ((m + \mu) s - \iota(s)) ds \right] f(\theta \bar{x}) \]

This integration provides for the continuity of \( f \) at \( x = \theta \bar{x} \). Finally, the integral of the ergodic measure over the state space \([0, \bar{x}]\), in addition to the expected percentage of time spent in autarky, need to add up to 1:

\[ \int_0^{\bar{x}} f(x) dx + \frac{1/\lambda}{1/\lambda + T(\theta \bar{x})} = 1 \]

This pins down the unknown constant \( G \). \( \square \)

A.1.17 Credit Spreads

I leverage the equation that the credit spread \( \varsigma_i(x) \) satisfies:

\[ D_i(x) = (m + \kappa) \left[ \left( \text{diag}_j (r_j + \varsigma_i(x) + m) - \hat{\Lambda} \right)^{-1} 1 \right]_i \]
Some algebra can show that for any state \( i \), I have:

\[
\varsigma_i'(x) = -\frac{D_i'(x)}{(\kappa + m)\left(\text{diag}(r_j + \varsigma_i(x) + m) - \hat{\Lambda}\right)^{-2}_i}
\]

Since the debt price function \( D_i \) is decreasing in the debt-to-GDP ratio \( x \) and since the denominator in the expression above is positive, \( \varsigma_i' < 0 \). Some algebra also shows that the function \( x\varsigma_i'(x) \) can be expressed as follows:

\[
\frac{d}{dx} (x\varsigma_i'(x)) = \varsigma_i'(x) + \frac{(\kappa + m)x (\varsigma_i'(x))^2}{(\kappa + m)\left(\text{diag}(r_j + \varsigma_i(x) + m) - \hat{\Lambda}\right)^{-2}_i} - xD_i''(x) - \frac{(\kappa + m)x (\varsigma_i'(x))^2}{(\kappa + m)\left(\text{diag}(r_j + \varsigma_i(x) + m) - \hat{\Lambda}\right)^{-2}_i} - \frac{D_i''(x)}{D_i'(x)}
\]

I have showed previously that \( \varsigma_i' > 0 \). The second term is positive, and the third term is also positive since the debt price function \( D_i \) is concave.

\[\square\]

### A.1.18 Credit Default Swap Premia

As specified in the main text, I define the risk-neutral present value of future credit losses and the risk-neutral present value of future CDS premia as follows:

\[
L_i(x,T) := \hat{E}^{x,i}[1_{\{\tau<T\}}e^{-\int_0^\tau r_s du} \max(0, 1 - D_s, (x_\tau))]
\]

\[
P_i(x,T) := \hat{E}^{x,i}[\int_0^{T\wedge \tau} e^{-\int_0^t r_s du} dt]
\]

The CDS premium is simply the ratio of those two quantities: \( \varsigma_i(x,T) = \frac{L_i(x,T)}{P_i(x,T)} \). An application of Feynman-Kac leads to the following partial differential equations satisfied by \( L_i \) and \( P_i \), for \( 1 \leq i \leq N_s \):

\[
r_iL_i(x,t) = -\frac{\partial L_i}{\partial t}(x,t) + \hat{L}_iL_i(x,t) + \sum_{j=1}^{N_s} \hat{\Lambda}_{ij}L_j(x,t)
\]

\[
r_iP_i(x,t) = 1 - \frac{\partial P_i}{\partial t}(x,t) + \hat{L}_iP_i(x,t) + \sum_{j=1}^{N_s} \hat{\Lambda}_{ij}P_j(x,t)
\]
The boundary conditions are as follows, for \( t \in [0, T) \):

\[
\begin{align*}
L_i(x, 0) &= 0 & \forall x < \bar{x}_i \\
P_i(x, 0) &= 0 & \forall x < \bar{x}_i \\
L_i(x, t) &= 1 - D_i(x) & \forall x \geq \bar{x}_i \\
P_i(x, t) &= 0 & \forall x \geq \bar{x}_i
\end{align*}
\]

I can then compute the expected excess return and the return volatility on a \( T \)-maturity CDS contract. Imagine that at time \( t \), an investor sells protection on the specific sovereign credit, for \$1\) dollar notional amount and using a \( T \)-maturity contract. At time \( t \), no cash-flow is exchanged, the value of the CDS contract is zero and the premium agreed upon between the buyer and the seller is equal to \( \zeta_{st}(x_t, T) = L_{st}(x_t, T)/P_{st}(x_t, T) \). At time \( t + dt \), the protection seller has accrued \( \zeta_{st}(x_t, T)dt \) of premium income. The value of the “premium leg” of his CDS contract is now equal to \( \zeta_{st}(x_t, T)P_{st+dt}(x_t + dt, T - dt) \) while the value of the “default leg” of his CDS contract is now equal to \( L_{st+dt}(x_t + dt, T - dt) \). In other words, his excess return (computed based on a \$1\) notional risky investment) is equal to:

\[
dR^e_{t,T} = \zeta_{st}(x_t, T)dt + \zeta_{st}(x_t, T)P_{st+dt}(x_t + dt, T - dt) - L_{st+dt}(x_t + dt, T - dt)
\]

This return is viewed as an excess return since the protection seller did not put any money upfront to enter into his contract. To compute CDS expected excess returns and return volatilities, I use Ito’s lemma and the relationship \( dR^e_{t,T} = \zeta_t dt + \zeta_t P_{st+dt}(x_t + dt, T - dt) - L_{st+dt}(x_t + dt, T - dt) \):

\[
dR^e_{t,T} = \zeta_t dt + \zeta_t \left( P_{st} + \mathcal{L}_{st} P_{st} dt - \frac{\partial P_{st}}{\partial t} dt - x_t \frac{\partial P_{st}}{\partial x} \sigma_{st} \cdot dB_t + \sum_{s'} (P_{s'} - P_{st}) dN_{t}^{(s,s')}\right)
- \left( L_{st} + \mathcal{L}_{st} L_{st} dt - x_t \frac{\partial L_{st}}{\partial x} \sigma_{st} \cdot dB_t + \sum_{s'} (L_{s'} - L_{st}) dN_{t}^{(s,s')}\right)
\]

I then use the relationship between the operators \( \mathcal{L}_{st} \) and \( \hat{\mathcal{L}}_{st} \):

\[
\hat{\mathcal{L}}_{st} = \mathcal{L}_{st} + x_t \nu_{st} \cdot \sigma_{st} \frac{\partial}{\partial x}
\]
Using \( \varsigma_t = L_{st}/P_{st} \) and the HJB equation satisfied by \( L_{st} \) and \( P_{st} \), I have:

\[
dR_{t,T}^e = -L_{st} \sum_{s'} \Lambda_{st,s'} \left( e^{\psi(st,s')} - 1 \right) \left( \frac{P_{st}}{P_{s't}} - \frac{L_{st}}{L_{s't}} \right) \, dt - L_{st} \left( \frac{x_t \partial P_{st}}{P_{st}} - \frac{x_t \partial L_{st}}{L_{st}} \right) \nu_{st} \cdot \sigma_{st} \, dt \\
+ L_{st} \sum_{s'} \left( \frac{P_{s'}}{P_{s't}} - \frac{L_{s'}}{L_{st}} \right) \left( dN^t_{(st,s')} - \Lambda_{st,s'} \, dt \right) - L_{st} \left( \frac{x_t \partial P_{st}}{P_{st}} - \frac{x_t \partial L_{st}}{L_{st}} \right) \sigma_{st} \cdot dB_t
\]

This leads to the following expression for conditional expected excess returns and conditional return volatilities:

\[

\mathbb{E} \left[ dR_{t,T}^e | \mathcal{F}_t \right] = - \left[ \left( \frac{x_t \partial P_{st}}{P_{st}} - \frac{x_t \partial L_{st}}{L_{st}} \right) \nu_{st} \cdot \sigma_{st} + \sum_{s'} \Lambda_{st,s'} \left( e^{\psi(st,s')} - 1 \right) \left( \frac{P_{st}}{P_{s't}} - \frac{L_{st}}{L_{s't}} \right) \right] L_{st} \, dt \\
\text{var} \left[ dR_{t,T}^e | \mathcal{F}_t \right] = \left[ \left( \frac{x_t \partial P_{st}}{P_{st}} - \frac{x_t \partial L_{st}}{L_{st}} \right)^2 \right] |\sigma_{st}|^2 + \sum_{s'} \Lambda_{st,s'} \left( \frac{P_{s'}}{P_{s't}} - \frac{L_{s'}}{L_{st}} \right)^2 \right] L_{st}^2 \, dt

A.1.19 Consumption Growth vs. Output Growth Volatility

Let me note \( \mu_i^x(x_t) \) the drift rate of \( x_t \) in SDF regime \( i \), and \( \sigma_i^x(x_t) \) its volatility vector:

\[

\mu_i^x(x_t) := \nu_i(x_t) - (\mu_i + m - \gamma |\sigma_i|^2) x_t \\
\sigma_i^x(x_t) := -x_t \sigma_i

\]

Using Itô’s lemma, I can compute consumption growth volatility as follows:

\[

\frac{dC_t}{C_t} = \left[ \frac{c_{st}(x_t)}{c_{s't}(x_t)} \mu_{st}(x_t) + \frac{1}{2} |\sigma_{st}|^2 \frac{c''_{st}(x_t)}{c_{s't}(x_t)} + \mu_{st} \right] \, dt \\
+ \sum_{s'} \left( \frac{c_{st}(x_t)}{c_{s't}(x_t)} - 1 \right) \, dN^t_{(st,s')} + \left( \frac{c'_{st}(x_t)}{c_{s't}(x_t)} \sigma_{st}(x_t) \cdot \sigma_{st} \right) \, dB_t
\]

In other words, conditioned on being in SDF regime \( s_t \), the ratio of consumption growth volatility to output growth volatility has the following simple expression:

\[

\frac{\text{var} \left[ \frac{dC_t}{C_t} | \mathcal{F}_t \right]}{\text{var} \left[ \frac{dy_{it}}{Y_t} | \mathcal{F}_t \right]} = \left( 1 - \frac{x_t c'_{st}(x_t)}{c_{s't}(x_t)} \right)^2 + \frac{1}{|\sigma_{st}|^2} \sum_{s'} \Lambda_{st,s'} \left( \frac{c'_{st}(x_t)}{c_{st}(x_t)} - 1 \right)^2
\]

Thus, the ratio of consumption growth volatility to output growth volatility crucially depends on the elasticity of the consumption function w.r.t. the debt-to-GDP ratio. Moreover, since the consumption function \( c_{st}(\cdot) \) is decreasing in the debt-to-GDP ratio, it turns out
that consumption growth volatility is greater than output growth volatility. Consumption
volatility is also enhanced by the SDF shocks. It is also immediate to verify that the presence
of SDF shocks breaks the unit correlation between consumption growth and output growth.
Such correlation is equal to:

\[
\text{corr} \left[ \frac{dC_t}{C_t}, \frac{dY_t}{Y_t} \bigg| F_t \right] = \frac{1}{\sqrt{1 + \sum_{s,t,s'} \Lambda_{s,t,s'} \left( \frac{c_{st}(x_t) - c_{st}(x_t)}{c_{st}(x_t) - x_{t1} c_{st}(x_t)} \right)^2}} < 1
\]
A.2 Tables and Plots

Figure 12: Historical vs. Market-Implied Default Intensities ("A" and "Baa" countries)

(a) A-rated Countries

(b) Baa-rated Countries

Figure 13: Historical vs. Market-Implied Default Intensities ("Ba" and "B" countries)

(a) Ba-rated Countries

(b) B-rated Countries
Table 6: Market-Implied Intensities vs. Ratings and US-based Factors

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Note: *p<0.1; **p<0.05; ***p<0.01
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<td>0.265*</td>
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<td>(0.055)</td>
<td>(0.042)</td>
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<td></td>
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*Note:* *p<0.1; **p<0.05; ***p<0.01
Table 8: Short Term Intensities and Intensity Slope

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<th>( \hat{\lambda}_{it}(1) )</th>
<th>( \hat{\lambda}<em>{it}(5) - \hat{\lambda}</em>{it}(1) )</th>
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<td>(0.000)</td>
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<td>(0.012)</td>
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Note: *p<0.1; **p<0.05; ***p<0.01
Table 9: CDS and EMBI Data Availability

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<tr>
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<td>4.2%* (2.4%)</td>
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<tr>
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<td>14.8%</td>
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<tr>
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<tr>
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<td>6.9%</td>
</tr>
<tr>
<td>Peru</td>
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<td>8%</td>
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<tr>
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<td>4%* (2.2%)</td>
<td>8%</td>
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<tr>
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<td>1% (1.1%)</td>
<td>4%</td>
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<td>Russia</td>
<td>3.7% (2.7%)</td>
<td>10%</td>
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<tr>
<td>Serbia</td>
<td>2% (2.2%)</td>
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<td>6.3%</td>
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*Note:  
*p<0.1; **p<0.05; ***p<0.01
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<th>$\hat{\beta}_{ZC}$</th>
<th>s-e</th>
<th>$\hat{\alpha}$</th>
<th>s-e</th>
<th>$\hat{\alpha}$ s-e</th>
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<td>(0.002)</td>
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<td>(0.07)</td>
<td>0.53***</td>
<td>(0.07)</td>
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<td>(0)</td>
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<td>1.5%</td>
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<td>(0.08)</td>
<td>0.27***</td>
<td>(0.05)</td>
<td>0.001*</td>
<td>(0)</td>
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<td>0.96***</td>
<td>(0.04)</td>
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<td>0.59***</td>
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<td>(0.08)</td>
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<td>(0.001)</td>
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<td>0.45***</td>
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<td>(0)</td>
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<td>(0.001)</td>
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<td>5.5%</td>
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<td>(0.08)</td>
<td>0.51***</td>
<td>(0.06)</td>
<td>0.001**</td>
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<td>0.58***</td>
<td>(0.07)</td>
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<td>5.3%</td>
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<td>(0.08)</td>
<td>0.45***</td>
<td>(0.08)</td>
<td>0.001**</td>
<td>(0)</td>
<td>70%</td>
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<td>(0.001)</td>
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<td>(0.05)</td>
<td>0.27**</td>
<td>(0.12)</td>
<td>0</td>
<td>(0.001)</td>
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<td>(0.001)</td>
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*Note:*  
*p<0.1; **p<0.05; ***p<0.01
Table 12: Expected Excess Returns, Distance-to-Default, Time-to-Maturity

**Dependent variable: Excess Returns (annualized)**

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<th>Moody’s “A”</th>
<th>Moody’s “Baa”</th>
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<th>Moody’s “B”</th>
<th>Moody’s “Caa”</th>
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<td>0.003</td>
<td>0.010***</td>
<td>0.009***</td>
<td>0.150***</td>
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<td>0.132***</td>
<td>0.969***</td>
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<td></td>
<td>0.016***</td>
<td>2.148***</td>
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<td></td>
<td></td>
<td>(0.003)</td>
<td>(1.69)</td>
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Note: *p<0.1; **p<0.05; ***p<0.01
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<th>s-e</th>
<th>$\hat{\beta}$</th>
<th>s-e</th>
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<td></td>
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<td>(0.12)</td>
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<td>Brazil</td>
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<td>0.257***</td>
<td>(0.064)</td>
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<tr>
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<td>0% (2.1%)</td>
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<td>(0.04)</td>
<td></td>
</tr>
<tr>
<td>Chile</td>
<td>-0.8% (0.9%)</td>
<td>0.142***</td>
<td>(0.033)</td>
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</tr>
<tr>
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<td>0% (1.3%)</td>
<td>0.264***</td>
<td>(0.061)</td>
<td></td>
</tr>
<tr>
<td>Dominican Republic</td>
<td>2.7% (5.1%)</td>
<td>0.141**</td>
<td>(0.066)</td>
<td></td>
</tr>
<tr>
<td>Egypt</td>
<td>0.6% (2.9%)</td>
<td>0.139***</td>
<td>(0.026)</td>
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</tr>
<tr>
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<td>0.132***</td>
<td>(0.036)</td>
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</tr>
<tr>
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<td>0.363***</td>
<td>(0.115)</td>
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<td>(0.084)</td>
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<td>(0.069)</td>
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<td>(0.136)</td>
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*Note: $^*$p<0.1; $^{**}$p<0.05; $^{***}$p<0.01*
Table 14: Bond Issuance Average Maturities

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<th>Original Weighted Term (years)</th>
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Table 15: Country-Specific Macro Moments

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<th>Time Period</th>
<th>GDP Growth Rate (% p.a.)</th>
<th>GDP Growth Volatility (% p.a.)</th>
<th>Correl with US GDP Growth (%)</th>
<th>Avg. Debt GDP (%)</th>
<th>Stdev. Debt GDP (%)</th>
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<td>30.1</td>
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*Average* 3.5 4.1 49.9 24.5
Table 16: Country-Specific Debt Price Moments

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<th>Country</th>
<th>Time Period</th>
<th>1-year CDS (% p.a.)</th>
<th>5-year CDS (% p.a.)</th>
<th>5-year CDS Vol. (% p.a.)</th>
<th>5-year CDS Excess Return (% p.a.)</th>
</tr>
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<td>139.98</td>
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<td>9.02</td>
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<td>13.42</td>
<td>3.14</td>
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Table 17: Estimation Result (part 1) – External Habit SDF

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<th>Bulgaria Data</th>
<th>Bulgaria Model</th>
<th>Hungary Data</th>
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<td>15</td>
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<tr>
<td>$1 - \alpha$ (%)</td>
<td>2.5</td>
<td>5.0</td>
<td>5.0</td>
<td>3.0</td>
<td></td>
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<tr>
<td>$\text{corr} (\sigma_t, \nu_t)$ (%)</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>GDP growth rate (% p.a.)</td>
<td>4.2</td>
<td>4.2</td>
<td>1.7</td>
<td>1.7</td>
<td>1.8</td>
<td>4.2</td>
<td>5.4</td>
<td>5.4</td>
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<tr>
<td>GDP vol (% p.a.)</td>
<td>3.8</td>
<td>3.8</td>
<td>4.7</td>
<td>4.7</td>
<td>2.8</td>
<td>2.1</td>
<td>3.7</td>
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<tr>
<td>Avg. debt-to-GDP ratio (%)</td>
<td>28.4</td>
<td>36.7</td>
<td>71.1</td>
<td>65.9</td>
<td>93.8</td>
<td>61.9</td>
<td>50.6</td>
<td>57.2</td>
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<tr>
<td>Avg. 5y CDS premium (bps p.a.)</td>
<td>391</td>
<td>432</td>
<td>177</td>
<td>186</td>
<td>273</td>
<td>196</td>
<td>237</td>
<td>284</td>
</tr>
<tr>
<td>Avg. 5y CDS excess return (bps p.a.)</td>
<td>643</td>
<td>336</td>
<td>291</td>
<td>182</td>
<td>834</td>
<td>276</td>
<td>274</td>
<td>248</td>
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<tr>
<td>Stddev. debt-to-GDP ratio (%)</td>
<td>10</td>
<td>2.0</td>
<td>34.0</td>
<td>2.9</td>
<td>43.7</td>
<td>3.0</td>
<td>24.7</td>
<td>2.8</td>
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<tr>
<td>$\text{vol}(\ln C_t)/\text{vol}(\ln Y_t)$</td>
<td>2.0</td>
<td>1.6</td>
<td>1.8</td>
<td>1.9</td>
<td>1.7</td>
<td>1.3</td>
<td>1.73</td>
<td>1.7</td>
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Table 18: Estimation Result (part 2) – External Habit SDF

<table>
<thead>
<tr>
<th>Moment/Parameter</th>
<th>Malaysia Data</th>
<th>Malaysia Model</th>
<th>Mexico Data</th>
<th>Mexico Model</th>
<th>Philippines Data</th>
<th>Philippines Model</th>
<th>South Africa Data</th>
<th>South Africa Model</th>
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<tbody>
<tr>
<td>$\delta$ (% p.a.)</td>
<td>10</td>
<td>10</td>
<td>15</td>
<td>15</td>
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<tr>
<td>$1 - \alpha$ (%)</td>
<td>3.5</td>
<td>3.0</td>
<td>4.5</td>
<td>3.5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\text{corr} (\sigma_t, \nu_t)$ (%)</td>
<td>40</td>
<td>75</td>
<td>100</td>
<td>80</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>GDP growth rate (% p.a.)</td>
<td>6.1</td>
<td>6.1</td>
<td>3.9</td>
<td>3.9</td>
<td>4.1</td>
<td>4.1</td>
<td>3.1</td>
<td>3.1</td>
</tr>
<tr>
<td>GDP vol (% p.a.)</td>
<td>3.3</td>
<td>3.3</td>
<td>3.5</td>
<td>3.5</td>
<td>3.0</td>
<td>3.0</td>
<td>2.5</td>
<td>2.5</td>
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<tr>
<td>Avg. debt-to-GDP ratio (%)</td>
<td>42.0</td>
<td>73.9</td>
<td>32.9</td>
<td>56.0</td>
<td>54.9</td>
<td>70.8</td>
<td>23.0</td>
<td>36.6</td>
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<tr>
<td>Avg. 5y CDS premium (bps p.a.)</td>
<td>91</td>
<td>200</td>
<td>133</td>
<td>228</td>
<td>241</td>
<td>311</td>
<td>155</td>
<td>202</td>
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<tr>
<td>Avg. 5y CDS excess return (bps p.a.)</td>
<td>93</td>
<td>486</td>
<td>185</td>
<td>178</td>
<td>391</td>
<td>270</td>
<td>281</td>
<td>148</td>
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<tr>
<td>Stddev. debt-to-GDP ratio (%)</td>
<td>16.4</td>
<td>5.0</td>
<td>15.1</td>
<td>2.8</td>
<td>20.2</td>
<td>3.8</td>
<td>65.9</td>
<td>1.9</td>
</tr>
<tr>
<td>$\text{vol}(\ln C_t)/\text{vol}(\ln Y_t)$</td>
<td>2.0</td>
<td>1.7</td>
<td>1.9</td>
<td>1.9</td>
<td>1.59</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>Bond spread vol (bps p.a.)</td>
<td>37</td>
<td>183</td>
<td>56</td>
<td>174</td>
<td>102</td>
<td>217</td>
<td>71</td>
<td>157</td>
</tr>
<tr>
<td>Avg. 1y CDS premium (bps p.a.)</td>
<td>53</td>
<td>17</td>
<td>78</td>
<td>54</td>
<td>139</td>
<td>95</td>
<td>84</td>
<td>45</td>
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<tr>
<td>Avg. 5y-1y slope (bps p.a.)</td>
<td>507</td>
<td>486</td>
<td>636</td>
<td>495</td>
<td>759</td>
<td>582</td>
<td>589</td>
<td>392</td>
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</tbody>
</table>
Figure 14: Predicted vs. Actual Expected Excess Returns
Figure 15: 1st Principal Component 5y CDS vs. US Risks

(a) 1st PC of Sovereign CDS and the VIX
(b) 1st PC of Sovereign CDS and CDX
Figure 16: T-bill rates since 1960 and discrete state Markov process approximation
A.3 Comparative Static Plots

Figure 17: Comparative Static w.r.t. $\alpha$ – Quantities

(a) Issuance Policy $\iota(x)$
(b) Trade Balance $1 - c(x)$

Figure 18: Comparative Static w.r.t. $\alpha$ – Prices

(a) Bond Spread $\varsigma(x)$
(b) Expected Excess Return $\mathbb{E}_t [dR_t - r_t dt]$
Figure 19: Comparative Static w.r.t. $\delta$ – Quantities

(a) Issuance Policy $\iota(x)$

(b) Trade Balance $1 - c(x)$

Figure 20: Comparative Static w.r.t. $\delta$ – Prices

(a) Bond Spread $\varsigma(x)$

(b) Expected Excess Return $\mathbb{E}_t [dR_t - r_t dt]$
Figure 21: Comparative Static w.r.t. $\gamma$ – Quantities

(a) Issuance Policy $\iota(x)$

(b) Trade Balance $1 - c(x)$

Figure 22: Comparative Static w.r.t. $\gamma$ – Prices

(a) Bond Spread $\varsigma(x)$

(b) Expected Excess Return $\mathbb{E}_t [dR_t - r_t dt]$
Figure 23: Comparative Static w.r.t. $\nu$ – Quantities

(a) Issuance Policy $\iota(x)$

(b) Trade Balance $1 - c(x)$

Figure 24: Comparative Static w.r.t. $\nu$ – Prices

(a) Bond Spread $\varsigma(x)$

(b) Expected Excess Return $\mathbb{E}_t [dR_t - r_t dt]$
Figure 25: Comparative Static w.r.t. $\sigma$ – Quantities

(a) Issuance Policy $\iota(x)$

(b) Trade Balance $1 - c(x)$

Figure 26: Comparative Static w.r.t. $\sigma$ – Prices

(a) Bond Spread $\varsigma(x)$

(b) Expected Excess Return $E_t [dR_t - r_t dt]$
A.4 Numerical Algorithm – Markov Chain Approximation

I will use a Markov Chain approximation method, as explained in Kushner and Dupuis (2001). I select this method over the more traditional finite difference schemes: indeed, to guarantee convergence of both methods to the solution of the differential equations of interest, the latter requires using numerical schemes that are consistent, monotone and stable, and the last two requirements are not always simple to verify 37, while the former requires the construction of approximating Markov chains whose transition probabilities need to satisfy consistency properties that are extremely simple to verify in practice. I also favor the Markov Chain approximation method over collocation methods: the latter methods require inverting non-sparse matrices, while the former involves the inversion of sparse matrices, making my algorithm significantly faster.

I compute the functions \( \{(v_s, D_s)\}_{s \leq N_s} \) numerically over the compact set \([0, \max_s \bar{x}_s]\), by determining their values on an equally-spaced grid \(G_h\), where \(h > 0\) is my scalar approximation parameter. I will note \( \{x_k = kh\}_{0 \leq k \leq N_h}\) the grid points of \(G_h\). I start with a guess equilibrium cutoff \(N_s \times 1\) vector \(\bar{x}^{(1)}\), and a guess issuance policy vector \(\ell^{(1,1)} = \{\ell^{(1,1)}(x_k)\}_{0 \leq k \leq N_h, 0 \leq s \leq N_s}\). My algorithm has an outer-loop, which updates the equilibrium default cutoff vector \(\bar{x}^{(i)}\), and an inner loop, which, for a given vector of cutoffs \(\bar{x}^{(i)}\), updates the functions \(\{v^{(i,j)}_s, D^{(i,j)}_s, \iota^{(i,j)}_s\}\). In the inner loop, I calculate the functions \(v_s(\cdot; \bar{x}^{(i)}), D_s(\cdot; \bar{x}^{(i)}\) and \(\iota_s(\cdot; \bar{x}^{(i)}\) as follows.

I first describe how to compute \(\{D^{(i,j)}_s\}_{s \leq N_s}\) given an issuance schedule \(\ell^{(i,j)}\) and a default policy \(\bar{x}^{(i)}\). To simplify notation, I omit the superscript \((i, j)\) when possible. Given a set of cutoffs \(\{\bar{x}_s\}_{s \leq N_s}\) and issuance policies \(\{\ell_s\}_{s \leq N_s}\), the dynamic evolution of the state variables \(x_t\) and \(s_t\) under \(Q\) can be written as follows, for \(x_t \in [0, \bar{x}_{s_t}]\):

\[
\begin{align*}
    dx_t &= (\ell_s(x_t) - (m + \mu + |\sigma_{s_{t}}|^2 - \nu_{s_{t}} \cdot \sigma_{s_{t}}) x_t) dt - x_t \sigma_{s_{t}} \cdot d\hat{B}_t \\
    ds_t &= \sum_{s' \neq s_{t-}} (s'_t - s_{t-}) d\hat{N}_{t}^{s_{t-} \ldots s'_t}
\end{align*}
\]

In the inner loop, I create a Markov Chain \(X^{h}_{Q,n} := (x^{h}_{Q,n}, s^{h}_{Q,n})\) that approximates the process \(\{(x_t, s_t)\}_{t \geq 0}\) under \(Q\). I introduce \(Q^{h}_{Q}(x, s)\) and \(\Delta t^{h}_{Q}(x, s)\) as follows:

\[
\begin{align*}
    Q^{h}_{Q}(x, s) &= x^2 |\sigma_s|^2 + h |\mu_Q(x, s)| \\
    \Delta t^{h}_{Q}(x, s) &= \frac{h^2}{Q^{h}_{Q}(x, s)}
\end{align*}
\]

37 I note however that upwinding has been used as a classic tool to implement monotone schemes.
Since $\mu_Q(0, s) > 0$ for all discrete SDF state $s$ (in equilibrium the sovereign will be borrowing when it is not indebted, i.e. $\tau_s(0) > 0$), $\inf_{x,s} Q^h_Q(x, s) > 0$, which means that $\Delta t^h_Q(x, s)$ is well defined. For all $x$ and all $s \leq N_s$, I have:

$$\lim_{h \to 0} \Delta t^h_Q(x, s) = 0$$

I then define the following transition probabilities:

$$\Pr \left( X^h_{t,n+1} = (x + h, s) | X^h_{t,n} = (x, s) \right) = \frac{e^{\hat{\Lambda}_{ss'}}(x, s)}{Q^h_Q(x, s)} \left( \frac{x^2 | \sigma_s |^2}{2} + h \max \left( 0, \mu_Q(x, s) \right) \right)$$

$$\Pr \left( X^h_{t,n+1} = (x - h, s) | X^h_{t,n} = (x, s) \right) = \frac{e^{\hat{\Lambda}_{ss'}}(x, s)}{Q^h_Q(x, s)} \left( \frac{x^2 | \sigma_s |^2}{2} + h \max \left( 0, -\mu_Q(x, s) \right) \right)$$

$$\Pr \left( X^h_{t,n+1} = (x, s') | X^h_{t,n} = (x, s) \right) = \left( \frac{\hat{\Lambda}_{ss'}}{-\Lambda_{ss}} \right) \left( 1 - e^{\hat{\Lambda}_{ss'}}(x, s) \right)$$

Notice that these transition probabilities are all greater than zero, less than 1, and they add up to 1. Noting $\Delta x^h_{t,n} := x^h_{t,n+1} - x^h_{t,n}$ and $\Delta s^h_{t,n} := s^h_{t,n+1} - s^h_{t,n}$, the Markov chain created satisfies the local consistency condition:

$$\mathbb{E}^{x,s} [\Delta x^h_{t,n}] = \mu_Q(x, s) \Delta t^h_Q(x, s) \quad (60)$$

$$\text{var}^{x,s} [\Delta x^h_{t,n}] = x^2 | \sigma_s |^2 \Delta t^h_Q(x, s) + o \left( \Delta t^h_Q(x, s) \right) \quad (61)$$

$$\mathbb{E}^{x,s} [\Delta s^h_{t,n}] = \sum_{s'} \hat{\Lambda}_{ss'}(s' - s) \Delta t^h_Q(x, s) \quad (62)$$

$$\text{var}^{x,s} [\Delta s^h_{t,n}] = \sum_{s'} \hat{\Lambda}_{ss'}(s' - s)^2 \Delta t^h_Q(x, s) + o \left( \Delta t^h_Q(x) \right) \quad (63)$$

$$\text{cov}^{x,s} [\Delta x^h_{t,n}, \Delta s^h_{t,n}] = 0 \quad (64)$$

For $\bar{x}_s \geq x_k \geq 0$, the sovereign government is performing and I compute $D^{(i,j)}_s(x_k)$ as follows:

$$D^{(i,j)}_s(x_k) = (\kappa + m) \Delta t^h_Q(x_k, s) + e^{-(r_s+m)\Delta t^h_Q(x_k, s)} \sum_{x'_s, s'_h} \Pr \left( X^h_{t} | x_k, s \right) D^{(i,j)}_s(x'_s)$$

For $x_k \geq \bar{x}_s$, the sovereign government is in default and I compute $D^{(i,j)}_s(x_k)$ as follows:

$$D^{(i,j)}_s(x_k) = \lambda \theta \alpha \left( \Xi^{-1} D^{(i,j)}(\theta x_k) \right)_s$$

This is a linear system of $N_s \times (N_h+1)$ equations in $N_s \times (N_h+1)$ unknown $\{D^{(i,j)}_s(x_k)\}_{0 \leq k \leq N_h, s \leq N_s}$ which can be solved easily via a simple matrix inversion. Note that the matrix to be inverted
is sparse, which greatly reduces computing time.

I then describe how to compute \( v_s^{(i,j)} \) in each discrete Markov state \( s \), given an issuance schedule \( \iota^{(i,j)} \), a debt price schedule \( D^{(i,j)} \) and a default policy \( \bar{\alpha}^{(i)} \). Once again I omit the superscript \((i,j)\) when possible. Given a vector of cutoffs \( \bar{\alpha} \), an issuance policy \( \iota \) and a debt price schedule \( D \), the dynamic evolution of the state variables \( x_t \) and \( s_t \) under the probability measure \( \tilde{\Pr} \). The transition probabilities of this Markov chain will be expressed as follows:

\[
\begin{align*}
\text{dx}_t &= (\iota_s(x_t) - (m + \mu_s) - \gamma|\sigma_{st}|^2) \text{d}t - x_t \sigma_{st} \cdot \text{d}\tilde{\mathbf{B}}_t \\
&= \mu_{\bar{\iota}}(x_t, s_t) \text{d}t - x_t \sigma_{st} \cdot \text{d}\tilde{\mathbf{B}}_t \\
ds_t &= \sum_{s' \neq s_{-1}} (s' - s_{-1}) \text{d}N_{t_{s_{-1}}:s'}
\end{align*}
\]

By introducing \( Q^h_{\tilde{\mathbf{P}}}(x, s) \) and \( \Delta t^h_{\tilde{\mathbf{P}}}(x, s) \), computed in a similar fashion to \( Q^h_{\mathbf{P}}(x, s) \) and \( \Delta t^h_{\mathbf{P}}(x, s) \), I can construct a new Markov Chain \( \{X^n_{\tilde{\mathbf{P}}}, \tau\}_{n \geq 0} \) that approximates the process \( \{(x_t, s_t)\}_{t \geq 0} \) under the probability measure \( \tilde{\Pr} \). The transition probabilities of this Markov chain will satisfy consistency conditions similar to those of equations (60), (61), (62), (63) and (64).

For \( \bar{x}_s > x_k \geq 0 \), the sovereign government is performing and I compute \( v_s^{(i,j)}(x_k) \) as follows:

\[
v_s^{(i,j)}(x_k) = \frac{\delta}{1 - \rho} (1 + \iota_s(x_k)D_s(x_k) - (\kappa + m)x_k) \left[ (1 - \gamma) v_s^{(i,j)}(x_k) \right]^{\frac{\bar{\alpha}_s}{1 - \gamma}} \Delta t^h_{\tilde{\mathbf{P}}}(x_k, s) + e^{-\frac{1 - \gamma}{1 - \rho} A \Delta t^h_{\tilde{\mathbf{P}}}(x_k, s)} \sum_{x'_s, s'_h} \Pr(X'_{\tilde{\mathbf{P}}} | x_k, s) v_s^{(i,j)}(x'_s)
\]

For \( x_k \geq \bar{x}_s \), the sovereign government is in default and I compute \( v_s^{(i,j)}(x_k) = \alpha^{1-\gamma} (v_s^{d})^{(i,j)}(x_k) \) as follows:

\[
(v^{d})^{(i,j)}(x_k) = \left[ \gamma^{-1} \left( \frac{\delta}{1 - \rho} (1 - \gamma) (v^{d})^{(i,j)}(x_k) \right)^{\frac{\bar{\alpha}_s}{1 - \gamma}} + \lambda (v^{i,j})(\theta x_k) \right]_{s}
\]

Note that the resulting system of \( N_s \times (N_h + 1) \) equations in \( N_s \times (N_h + 1) \) unknown \( \{v_s^{(i,j)}(x_k)\}_{0 \leq k \leq N_h, s \leq N_s} \) is not linear. In order to solve such system, I use a simple procedure: starting with a guess \( \{v^{(i,j,m)}(x_k)\}_{0 \leq k \leq N_h} \), I iterate, for \( x_k < \bar{x}_s \), on the following:

\[
v_s^{(i,j,m+1)}(x_k) = \frac{\delta}{1 - \rho} (1 + \mu_s(x_k)D_s(x_k) - (\kappa + m)x_k) \left[ (1 - \gamma) v_s^{(i,j,m)}(x_k) \right]^{\frac{\bar{\alpha}_s}{1 - \gamma}} \Delta t^h_{\tilde{\mathbf{P}}}(x_k, s) + e^{-\frac{1 - \gamma}{1 - \rho} A \Delta t^h_{\tilde{\mathbf{P}}}(x_k, s)} \sum_{x'_s, s'_h} \Pr(X'_{\tilde{\mathbf{P}}} | x_k, s) v_s^{(i,j,m+1)}(x'_s)
\]

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For \( x_k \geq \bar{x}_s \), the iteration becomes:

\[
(t_s^{d(i,j,m+1)}(x_k)) = \left[ Y^{-1} \left( \frac{\delta}{1 - \rho} \left( (1 - \gamma) (v_d^{(i,j,m+1)}(x_k))^{\frac{\rho - \gamma}{1 - \gamma}} + \lambda v_t^{(i,j,m)}(\theta x_k) \right) \right) \right]_s
\]

The iterative procedure is stopped once \( ||v^{(i,j,m+1)} - v^{(i,j,m)}||_\infty \) is sufficiently small. Once \( v^{(i,j)} \) and \( D^{(i,j)} \) are computed, I can update the issuance policy as follows, in each state \( s \):

\[
l_s^{(i,j+1)}(x_k) = \varpi l_s^{(i,j)}(x_k) + \frac{1 - \varpi}{D_s^{(i,j)}(x_k)} \left[ \left( \frac{\delta D^{(i,j)}(x_k) \left( (1 - \gamma) v_s^{(i,j)}(x_k) \right)^{\frac{\rho - \gamma}{1 - \gamma}}}{1 - \gamma} \right) \right]^{1/\rho} + (\kappa + m)x_k - 1
\]

In the above \( \varpi \in (0, 1) \) is a dampering parameter that “smoothes” the transition from \( l_s^{(i,j)} \) to \( l_s^{(i,j+1)} \) and prevents infinite loops between debt price and issuance policy.\(^{38}\) The derivative \( (v_s^{(i,j)})^{'}(x_k) \) is computed by using a centered finite difference approximation. I iterate on the inner loop until \( ||l^{(i,j+1)} - l^{(i,j)}||_\infty \) is sufficiently small.

At the conclusion of the inner loop, I have obtained \( v^{(i)}, D^{(i)}, l^{(i)} \), all assuming a default policy \( \bar{x}^{(i)} \). I then set \( \bar{x}^{(i+1)} \) by checking the smooth pasting condition at \( \bar{x}_s^{(i)} \) for all discrete states \( s \):

\[
(v^{(i)})^{'}(\bar{x}_s^{(i)}) \geq \alpha^{1-\gamma} \lambda \theta \left[ \left( Y + \delta \frac{\gamma - \rho}{1 - \rho} \text{diag} \left( \left( (1 - \gamma) v_s^{d(i)}(\bar{x}_s) \right)^{\frac{\rho - \gamma}{1 - \gamma}} \right) \right)^{-1} v^{'}(\theta \bar{x}_s) \right]_s
\]

Depending on whether the left handside is greater or less than the right handside, I update \( \bar{x}_s^{(i+1)} \) using a binomial search method.

Once the optimal default boundary \( \bar{x} \) and the optimal issuance policy \( I \) are known, I can compute the expected default time and the ergodic density of \( x \). The computation of the expected default time \( T(x) \) follows the same logic as the computation of the debt price, except that the Markov transition probabilities are adjusted to reflect the stochastic evolution of \( x_t \) and \( s_t \) under the physical measure \( P \). Finally, the ergodic density of \( x_t, s_t \) under \( P \) is constructed by finding the eigen-vector (associated with the eigen-value 1) of the transpose of a Markov matrix whose elements correspond to transition probabilities in and out of performing states \( (x_k, s) \) (for \( x_k \leq \bar{x}_s \)), as well as in and out of default states \( (x_k, s) \) (for \( x_k \geq \min_s \bar{x}_s \)). Once again those transition probabilities are constructed in an identical

\(^{38}\)For most parameter configurations of interest, the issuance policy is increasing in the debt price schedule, and the debt price is a decreasing function of the issuance schedule. Thus, without dampening, the algorithm ends up frequently in an infinite loop: a high debt price at the end of iteration \( j \) leads to a high issuance policy in iteration \( j + 1 \); such high issuance policy feeds back into a low debt price at iteration \( j + 1 \), which leads to a low issuance policy in iteration \( j + 2 \), thus creating the infinite loop.
way to those described previously.

In order to compute CDS prices, I use a slightly modified procedure. As described previously, I need to compute the risk-neutral expected loss \( L_s(x,T) \) and the risk-neutral expected present value of CDS premia \( P_s(x,T) \). Introduce the constant \( \epsilon > 0 \). \( Q_Q^h(x,s) \) and \( \Delta t_Q^h(x,s) \) are now defined as follows:

\[
Q_Q^h(x,s) := x^2|\sigma_s|^2 + h|\mu_Q(x,s)| + h/\epsilon\\
\Delta t_Q^h(x,s) := \frac{h^2}{Q_Q^h(x,s)}
\]

I still have \( \inf_{x,s} Q_Q^h(x,s) > 0 \), which means that \( \Delta t_Q^h(x,s) \) is well defined. For all \( x, s \), I have:

\[
\lim_{h \to 0} \Delta t_Q^h(x,s) = 0
\]

The state space now includes time-to-maturity \( T \), and the approximating Markov chain is now the three-dimensional process \( X_{Q,n}^h := (x_{Q,n}, s_{Q,n}, T_{Q,n}) \). Given a starting state \( X_{Q,n}^h = (x, s, T) \), I define the following transition probabilities:

\[
\Pr \left( X_{Q,n+1}^h = (x + h, s, T) \mid X_{Q,n}^h \right) = e^{\hat{\Lambda}_ss \Delta t_Q^h(x,s)} \left( \frac{x^2|\sigma_s|^2}{2} + h \max(0, \mu_Q(x,s)) \right)
\]

\[
\Pr \left( X_{Q,n+1}^h = (x - h, s, T) \mid X_{Q,n}^h \right) = e^{\hat{\Lambda}_ss \Delta t_Q^h(x,s)} \left( \frac{x^2|\sigma_s|^2}{2} + h \max(0, -\mu_Q(x,s)) \right)
\]

\[
\Pr \left( X_{Q,n+1}^h = (x, s, T - \epsilon h) \mid X_{Q,n}^h \right) = e^{\hat{\Lambda}_ss \Delta t_Q^h(x,s)} \frac{h/\epsilon}{Q_Q^h(x,s)}
\]

\[
\Pr \left( X_{Q,n+1}^h = (x, s', T) \mid X_{Q,n}^h \right) = \left( \frac{\hat{\Lambda}_ss'}{-\hat{\Lambda}_{ss}} \right) \left( 1 - e^{\hat{\Lambda}_ss \Delta t_Q^h(x,s)} \right)
\]

Notice that these transition probabilities are all greater than zero, less than 1, and they add up to 1. The Markov chain created satisfies local consistency conditions similar to those in equations (60), (61), (62), (63) and (64), in addition to:

\[
\mathbb{E}^{s,x} \left[ \Delta T_{Q,n}^h \right] = -\Delta t_Q^h(x,s)\\
\text{var}^{s,x} \left[ \Delta T_{Q,n}^h \right] = o(\Delta t_Q^h(x,s))
\]

The state space grid \( G_h \) is three-dimensional, and is of the form \( \{(x_i, s_j, T_k)\}_{1 \leq i \leq N_x, 1 \leq s \leq N_s, 1 \leq j \leq N_T} \), where the grid points \( \{x_i\} \) and the grid points \( \{T_j\} \) are equally spaced with distance \( h \) and \( \epsilon h \) respectively. For \( \bar{x}_s > x_i \geq 0 \) and for \( T \geq T_i > 0 \), I compute \( L_s(x_i, T_j) \) and \( P_s(x_i, T_j) \) on
the grid $G_h$ as follows:

$$L_s(x_i, T_j) = e^{-r_s \Delta t_h(x_i, s)} \sum_{(x'_Q, s'_Q, T'_Q)} \Pr(X'_Q|(x_i, s, T_j)) L_{s'}(x'_Q, T'_Q)$$

$$P_s(x_i, T_j) = \Delta t_h(x_i, s) + e^{-r_s \Delta t_h(x_i, s)} \sum_{(x'_Q, s'_Q, T'_Q)} \Pr(X'_Q|(x_i, s, T_j)) P_{s'}(x'_Q, T'_Q)$$

I note $x_{N_{x,h,s}}$ the grid point at which the sovereign government defaults optimally in state $s$, in other words $N_{x,h,s} = \bar{x}_s/h$. The boundary conditions at $T = 0$ are $L_s(x_i, 0) = 0$ for $i < N_{x,h,s}$, $L_s(x, 0) = (1 - D(x))$ for $i \geq N_{x,h,s}$, and $P_s(x_i, 0) = 0$ for all $i$. The boundary conditions at $T > 0$ are $L_s(x_i, T) = (1 - D(x))$ and $P_s(x_i, T) = 0$ for $i \geq N_{x,h,s}$ and for any $T$. This system of linear equations is solved recursively: starting from $\{L_s(x_i, 0), P_s(x_i, 0)\}_{1 \leq i \leq N_{x,h,s}} \leq N_s$, I can compute $\{L_s(x_i, \epsilon h), P(x_i, \epsilon h)\}_{1 \leq i \leq N_{x,h,s}} \leq N_s$ via the system of linear equations above, and progress backwards.
A.5 Approximating Continuous State Markov Processes

Assume a discrete time continuous state stationary Markov process with transition density \( f(s'; s) \), with related transition distribution function \( F(\cdot; s) \). I discretize the state space by choosing a set of points \( \{s_1, ..., s_N\} \). I then create two Markov chains, using the following two-steps procedure. In the first step, I create a Markov Chain characterized by the matrix \( P = (p_{ij})_{1 \leq i, j \leq N} \) using Tauchen (1986). If \( \{s_t\} \) is one-dimensional for example and the discretization points \( \{s_1, ..., s_N\} \) are equally spaced by \( \Delta s \), this method leads to transition probabilities:

\[
\begin{align*}
p_{ij} &= F(s_j + \Delta s/2; s_i) - F(s_j - \Delta s/2; s_i) \quad 1 < j < N \\
p_{iN} &= 1 - F(s_N - \Delta s/2; s_i) \\
p_{i1} &= F(s_1 + \Delta s/2; s_i)
\end{align*}
\]

At the end of this step, I obtain an approximating Markov chain, whose conditional moments will usually not match the conditional moments of the original stochastic process, specially when such stochastic process is highly persistent (which will be the case for the asset pricing models considered here). Using Gaussian quadrature methods as in Tauchen and Hussey (1991) as opposed to the simple procedure described in Tauchen (1986) does not solve this issue. Instead, in a second step, for each state \( s_i \), using the constructed “prior” conditional distribution \( p_i \), I construct an approximating conditional distribution \( \pi_i \) by leveraging the minimum entropy method introduced by Tanaka and Toda (2013) and Tanaka and Toda (2015). This method consists in minimizing the Kullback-Leibler distance between my conditional distribution \( \pi_i \) and my initial prior \( p_i \), subject to moment constraints \( \tilde{M}_i = \int M(s') f(s'; s_i) ds' \) as follows:

\[
\begin{align*}
\min_{\{\pi_{ij}\}_{j \leq N}} & \sum_{j=1}^{N} \pi_{ij} \ln \frac{\pi_{ij}}{p_{ij}} \\
\text{s.t.} & \sum_{j=1}^{N} \pi_{ij} M(s_j) = \tilde{M}_i \\
& \sum_{j=1}^{N} \pi_{ij} = 1
\end{align*}
\]

I use the first and second conditional moments, in other words \( \tilde{M}(x) := (x, x^2) \). As Tanaka and Toda (2015) show, the dual maximization problem is simpler to tackle than the original
minimization problem (the following being a monotone transformation of the dual problem):

$$\min_q \sum_{j=1}^{N} p_{ij} e^{q'\left(\mathbb{T}(s_j)-\bar{T}_i\right)}$$

For a given state $s_i$, the dual problem above has a solution if and only if $\bar{T}_i$ belongs to the convex hull of $\{\mathbb{T}(s_j)\}_{1 \leq j \leq N}$. If $q$ is the minimizing vector, the corresponding transition probabilities are equal to $\pi_{ij} = \frac{p_{ij} e^{q'\left(\mathbb{T}(s_j)-\bar{T}_i\right)}}{\sum_{j=1}^{N} p_{ij} e^{q'\left(\mathbb{T}(s_j)-\bar{T}_i\right)}}$. The transition matrix $\Pi = (\pi_{ij})_{1 \leq i,j \leq n_s}$ depends on the specific time step assumed for the discrete time continuous state Markov process. In order to convert the discrete time discrete state Markov Chain into a continuous time discrete state Markov process, I need to compute the generator matrix $\Lambda$. $\Lambda$ verifies $\Pi = \exp(\Lambda)$, where $\Lambda$ is expressed as an intensity per time period used for the discrete time continuous state Markov process. In order to recover the intensity matrix, I use an observation from Jarrow, Lando, and Turnbull (1997): if at most one transition can occur within a time period, the elements of the generator matrix $\Lambda$ can be written:

$$e^{\Lambda_{ii}} = \pi_{ii}$$

$$\frac{\Lambda_{ij}}{-\Lambda_{ii}}(1 - e^{\Lambda_{ii}}) = \pi_{ij} \quad \text{for} \quad i \neq j$$

Inverting this system of equations yields:

$$\Lambda_{ii} = \ln \pi_{ii}$$

$$\Lambda_{ij} = \frac{-\pi_{ij} \ln \pi_{ii}}{1 - \pi_{ii}} \quad \text{for} \quad i \neq j$$
A.5.1 External Habit Model

Take the external habit model of Campbell and Cochrane (1995) for example, where the state of the discount factor is uniquely characterized by the surplus consumption ratio $H_t := 1 - \frac{C_{c,t}}{C_{c,t}^*}$, where $C_{c,t}$ is the investor’s equilibrium consumption and $C_{c,t}^*$ an external habit level. Investors in this external habit model have a marginal utility process $e^{-\delta_c} (C_{c,t}^* H_t)^{-\gamma_c}$, where $\delta_c$ is the investor’s rate of time preference and $\gamma_c$ is a measure of risk-aversion when the surplus consumption ratio is unity. The logarithm $h_t$ of the surplus consumption ratio follows the dynamic equation:

$$h_{t+1} - \bar{h} = \phi(h_t - \bar{h}) + \lambda(h_t)\epsilon_{t+1}$$

In the above, $\epsilon_{t+1} \sim \mathcal{N}(0,\sigma_c^2)$ is a shock to the level of (log) surplus consumption ratio (this shock is fully correlated with the shock to the investor’s consumption process, i.e. $\log C_{c,t+1} - \log C_{c,t} = \mu_c + \epsilon_{t+1}$), $\bar{H} := \sigma_c (\frac{\gamma_c}{1-\delta})^{1/2}$ is the steady-state surplus consumption ratio, $h_{\text{max}} = \bar{h} + \frac{1}{2} (1 - H^2)$ is the maximum attainable log surplus consumption ratio and the sensitivity function $\lambda$ can be written:

$$\lambda(h) = \begin{cases} \frac{1}{H} \left(1 - 2(h - \bar{h})\right)^{1/2} - 1 & \text{if } h \leq h_{\text{max}} \\ 0 & \text{if } h \geq h_{\text{max}} \end{cases}$$

The functional form for $\lambda(\cdot)$ was picked by Campbell and Cochrane (1995) so that risk-free rates are constant, and so that the habit level $C_{t}^*$ is pre-determined at and near the steady state surplus consumption level $\bar{H}$. In this model, since shocks to the surplus consumption ratio are perfectly correlated with shocks to the investor’s consumption process, only one shock is priced. In that case, sovereign bonds’ expected excess returns will be entirely dependent on the correlation between the investor’s consumption (and thus output) process and the output process of the government “$i$” of interest $\text{corr}(dC_{c,t},dY^i_t)$. I approximate the dynamics of the state variable $h_t$ by choosing an equally-spaced grid $H_1, ..., H_N$, and I note $h_i = \ln H_i$. I will use the first and second non-centered moments in the entropy minimization step, in other words $\mathbb{M}(h) = (h, h^2)$. For each grid point $h_i$, the conditional moments $\mathbb{M}_i$ have closed form expressions:

$$\mathbb{M}_i = \begin{pmatrix} \mathbb{E}[h'|h_i] \\ \mathbb{E}[(h')^2|h_i] \end{pmatrix} = \begin{pmatrix} \bar{h} + \phi(h_i - \bar{h}) \\ \sigma_c^2 \lambda(h_i)^2 + [\bar{h} + \phi(h_i - \bar{h})]^2 \end{pmatrix}$$
In the model of Campbell and Cochrane (1995), the risk-free rate $r$ and the uni-dimensional risk price vector $\nu(h)$ (which is function of the log surplus consumption ratio) are equal to:

$$r = \delta_c + \gamma_c \mu_c - \frac{\gamma_c}{2} (1 - \phi)$$

$$\nu(h) = \left[ e^{\gamma_c^2 \sigma_c^2 (1 + \lambda(h))^2} - 1 \right]^{1/2} \approx \gamma_c \sigma_c (1 + \lambda(h))$$

Table 19: Parameters for External Habit Model

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Variable</th>
<th>Value (annualized)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Consumption growth rate</td>
<td>$\mu_c$</td>
<td>1.89%</td>
</tr>
<tr>
<td>Consumption volatility</td>
<td>$\sigma_c$</td>
<td>1.50%</td>
</tr>
<tr>
<td>Risk free rate</td>
<td>$r$</td>
<td>0.94%</td>
</tr>
<tr>
<td>Habit persistence</td>
<td>$\phi$</td>
<td>0.87</td>
</tr>
<tr>
<td>Utility curvature</td>
<td>$\gamma_c$</td>
<td>2</td>
</tr>
<tr>
<td>Rate of time preference</td>
<td>$\delta_c$</td>
<td>11.7%</td>
</tr>
<tr>
<td>Steady-state surplus consumption</td>
<td>$\bar{H}$</td>
<td>0.057</td>
</tr>
<tr>
<td>Max surplus consumption</td>
<td>$H_{max}$</td>
<td>0.094</td>
</tr>
</tbody>
</table>

Campbell and Cochrane (1995) use the parametrization as disclosed in Table 19. Using these parameters leads to a risk-free rate that is constant and equal to $r = 0.016$ per annum. In my numerical application, I pick a 5-state approximation, which leads to an ergodic mean risk price of 0.51 (compared to an ergodic average risk-price under the “true” model of 0.53) and an ergodic risk price volatility of 0.09 (compared to an ergodic risk-price volatility under the “true” model of 0.09). The discretized values of the risk-prices, as well as the ergodic distribution of the Markov process and the transition intensities, are showed in Table 20.

Table 20: Discretized External Habit Model

<table>
<thead>
<tr>
<th>Risk-price $\nu$</th>
<th>Ergodic Density</th>
<th>$\Lambda_{1,.}$</th>
<th>$\Lambda_{2,.}$</th>
<th>$\Lambda_{3,.}$</th>
<th>$\Lambda_{4,.}$</th>
<th>$\Lambda_{5,.}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5213</td>
<td>0.0348</td>
<td>-0.1918</td>
<td>0.1251</td>
<td>0.0372</td>
<td>0.0172</td>
<td>0.0123</td>
</tr>
<tr>
<td>0.9338</td>
<td>0.1138</td>
<td>0.0643</td>
<td>-0.3863</td>
<td>0.2771</td>
<td>0.0392</td>
<td>0.0059</td>
</tr>
<tr>
<td>0.6597</td>
<td>0.2077</td>
<td>0</td>
<td>0.2015</td>
<td>-0.561</td>
<td>0.2992</td>
<td>0.0602</td>
</tr>
<tr>
<td>0.4526</td>
<td>0.2765</td>
<td>0</td>
<td>0.0022</td>
<td>0.2941</td>
<td>-0.6003</td>
<td>0.304</td>
</tr>
<tr>
<td>0.2393</td>
<td>0.3673</td>
<td>0</td>
<td>0.0021</td>
<td>0.2219</td>
<td>-0.224</td>
<td></td>
</tr>
</tbody>
</table>
A.5.2 Long Run Risk Model

Consider the model of Bansal and Yaron (2004), where the investor’s log consumption $c_{c,t}$, the latent state variable $z_t$ and the stochastic volatility $\eta_t$ evolve as follows:

\[
\begin{align*}
    c_{c,t+1} - c_{c,t} &= \mu_c + z_t + \sqrt{\eta_t} \epsilon_{c,t+1} \\
    z_{t+1} &= \kappa_z z_t + \sigma_z \sqrt{\eta_t} \epsilon_{z,t+1} \\
    \eta_{t+1} &= \bar{\eta} + \kappa_\eta (\eta_t - \bar{\eta}) + \sigma_\eta \epsilon_{\eta,t+1}
\end{align*}
\]

I approximate the dynamics of the state variable $(\eta_t, z_t)$ by choosing an equally-spaced grid $(\eta_i, z_j)_{1 \leq i \leq N_\eta, 1 \leq j \leq N_z}$. I will use the first and second non-centered moments in the entropy minimization step, in other words $\mathbb{M}(\eta, z) = (\eta, z, \eta^2, \eta z, z^2)$. For each grid point $(\eta_i, z_j)$ on the grid, the conditional moments $\bar{\mathbb{M}}_{ij}$ have closed form expressions:

\[
\bar{\mathbb{M}}_{ij} = \begin{pmatrix}
    \mathbb{E}[\eta'| (\eta_i, z_j)] \\
    \mathbb{E}[z' | (\eta_i, z_j)] \\
    \mathbb{E}[(\eta')^2 | (\eta_i, z_j)] \\
    \mathbb{E}[\eta' z' | (\eta_i, z_j)] \\
    \mathbb{E}[(z')^2 | (\eta_i, z_j)]
\end{pmatrix} = \begin{pmatrix}
    \bar{\eta} + \kappa_\eta (\eta_i - \bar{\eta}) \\
    \kappa_z z_j \\
    \sigma_\eta^2 + [\bar{\eta} + \kappa_\eta (\eta_i - \bar{\eta})]^2 \\
    \bar{\eta} + \kappa_\eta (\eta_i - \bar{\eta}) \kappa_z z_j \\
    \sigma_z^2 \eta_i + (\kappa_z z_j)^2
\end{pmatrix}
\]

Under the discrete state approximation, the investor’s expected consumption growth rate takes $N_\eta \times N_z$ possible values $\{\mu_{c,ij} := \mu_c + z_{ij}\}_{1 \leq i \leq N_\eta, 1 \leq j \leq N_z}$, while consumption growth volatility takes $N_\eta$ possible values $\{\sigma_{c,i} := \sqrt{\eta_i}\}_{1 \leq i \leq N_\eta}$. Bansal and Yaron (2004) also assume that the representative investor has Epstein-Zin preferences with risk-aversion coefficient $\gamma_c$, coefficient of intertemporal substitution $1/\rho_c$, and rate of time preference $\delta_c$. In such model, three shocks are priced: shocks to consumption growth, shocks to the latent long-run variable $z_t$, and stochastic volatility shocks. I will use the authors’ parametrization of their model, as disclosed in table 21.

In my numerical application, I pick a $2 \times 3$-state approximation (2 consumption volatility levels, and for each volatility level, 3 consumption growth levels). The discretized values of consumption growth, consumption volatility, the implied risk-free rates, risk-prices and the ergodic distribution are showed in table 22. The conditional expected excess return on a claim to aggregate consumption varies between 0.45% and 0.75% per annum depending on the state.

---

While this expected excess return seems small, remember that in Bansal and Yaron (2004), the market is viewed as a levered claim, with a leverage factor between 3 and 4.5.
Table 21: Parameters for Long Run Risk Model

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Variable</th>
<th>Value (annualized)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Consumption growth rate</td>
<td>$\mu_c$</td>
<td>1.80%</td>
</tr>
<tr>
<td>Long-run consumption vol.</td>
<td>$\sqrt{\eta}$</td>
<td>2.70%</td>
</tr>
<tr>
<td>Consumption vol. of vol.</td>
<td>$\sigma_\eta$</td>
<td>0.0027%</td>
</tr>
<tr>
<td>Consumption vol. persistence</td>
<td>$\kappa_\eta$</td>
<td>0.987</td>
</tr>
<tr>
<td>Latent variable persistence</td>
<td>$\kappa_z$</td>
<td>0.979</td>
</tr>
<tr>
<td>Latent variable vol. ratio</td>
<td>$\sigma_z$</td>
<td>0.152</td>
</tr>
<tr>
<td>Investor risk aversion</td>
<td>$\gamma_c$</td>
<td>10</td>
</tr>
<tr>
<td>Investor IES</td>
<td>$1/\rho_c$</td>
<td>1.5</td>
</tr>
<tr>
<td>Rate of time preference</td>
<td>$\delta_c$</td>
<td>2.4%</td>
</tr>
</tbody>
</table>

Table 22: Discretized Long Run Risk Model

<table>
<thead>
<tr>
<th>state 1</th>
<th>state 2</th>
<th>state 3</th>
<th>state 4</th>
<th>state 5</th>
<th>state 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_{c,s}$</td>
<td>-0.0022</td>
<td>0.018</td>
<td>0.0382</td>
<td>-0.0022</td>
<td>0.018</td>
</tr>
<tr>
<td>$\sigma_{c,s}$</td>
<td>0.0245</td>
<td>0.0245</td>
<td>0.0245</td>
<td>0.0293</td>
<td>0.0293</td>
</tr>
<tr>
<td>$r_s$</td>
<td>0.0171</td>
<td>0.0303</td>
<td>0.044</td>
<td>0.0149</td>
<td>0.0279</td>
</tr>
<tr>
<td>$\nu_s$</td>
<td>0.2452</td>
<td>0.2452</td>
<td>0.2452</td>
<td>0.2931</td>
<td>0.2931</td>
</tr>
<tr>
<td>$\pi_s$</td>
<td>0.1442</td>
<td>0.2105</td>
<td>0.1486</td>
<td>0.1592</td>
<td>0.1773</td>
</tr>
</tbody>
</table>
References


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Chatterjee, Satyajit and Burcu Eyigungor. 2010. “Maturity, indebtedness, and default risk.”


